set-A solutions

PART-A

1\. a) Axiom-1: The probability of any event is always a non-zero.

$P(A) \geq 0.$

Axiom-2: The probability of sample space is 1.

$P(S) = 1.$

Axiom-3: The probability of mutually exclusive event is

$$P\left(\bigcup_{n=1}^{N} A_n\right) = \sum_{n=1}^{N} P(A_n)$$

b)

$F_X(-\infty) = 0$

$F_X(\infty) = 1$

$0 \leq F_X(x) \leq 1$

$F_X(x_1) \leq F_X(x_2)$ if $x_1 < x_2$

$P(x_1 < X \leq x_2) = F_X(x_2) - F_X(x_1)$

$F_X(x^+) = F_X(x).$

c) $F_X(x) = 1 - e^{-ax}.$

pdf $f_X(x) = \frac{\partial}{\partial x} F_X(x) = \frac{\partial}{\partial x}\left[1 - e^{-ax}\right] = -\frac{\partial}{\partial x}\left(e^{-ax}\right) = a \cdot e^{-ax}.$

d) Uniform Random Variable

density function $f_X(x) = \begin{cases} \frac{1}{b-a} & \text{for } a \leq x \leq b \\ 0 & \text{otherwise.} \end{cases}$

Distribution function $F_X(x) = \begin{cases} 0 & \text{for } x > a \\ \frac{x-a}{b-a} & \text{for } a \leq x < b \\ 1 & \text{for } x > b \end{cases}$

e) Central Limit Theorem states that probability distribution of sum of large number of independent random variables approaches normal gaussian function.

$$X = X_1 + X_2 + \cdots + X_N.$$

$$F(x) = G\left(\frac{x-\mu}{\sigma}\right) \text{ as } N \text{ increases.}$$

f) Covariance $C_{xy} = E[XY] - E[X].E[Y]$

If X & Y are independent $E[XY] = E[X].E[Y]$

$$\Rightarrow C_{xy} = \bar{X}\bar{Y} - \bar{X}\bar{Y} = 0.$$

g) Mean Ergodic Process:- A random process $X(t)$ is mean ergodic if time average of any sample function $x(t)$ is equal to its statistical average.

i.e. $E[X(t)] = \bar{X} = A\langle x(t)\rangle = \bar{x}.$

Correlation Ergodic Process: A random process is correlation ergodic if the time autocorrelation is equal to statistical autocorrelation.

i.e. $A\langle x(t).x(t+\tau)\rangle = E[x(t).x(t+\tau)]$

$$\mathcal{R}_{xx}(\tau) = R_{xx}(\tau).$$

h) $E[X(t)] = \bar{X} = \text{Constant}.$

$R_{xx}(t, t+\tau) = R_{xx}(\tau)$ (depends only on τ but not on t).

i) A sample function of a wide sense stationary random process is called white noise if its PSD is constant for all frequencies.

$$S_{NN}(\omega) = \frac{N_0}{2}$$ where N_0 – real positive constant.

$$R_{NN}(\tau) = \frac{N_0}{2} \cdot \delta(\tau).$$

A colored noise has only a portion of white light frequencies in its spectrum.

j) $R_{xx}(\tau) = a \cdot e^{-b|\tau|}$

$$S_{xx}(\omega) = \int_{-\infty}^{\infty} R_{xx}(\tau) \cdot e^{-j\omega\tau} \cdot d\tau.$$

$$= \int_{-\infty}^{\infty} a \cdot e^{-b|\tau|} \cdot e^{-j\omega\tau} \cdot d\tau.$$

$$= \int_{-\infty}^{0} a \cdot e^{b\tau} \cdot e^{-j\omega\tau} \cdot d\tau + \int_{0}^{\infty} a \cdot e^{-b\tau} \cdot e^{-j\omega\tau} \cdot d\tau.$$

$$= a \cdot \frac{e^{(b-j\omega)\tau}}{(b-j\omega)} \Bigg|_{-\infty}^{0} + a \cdot \frac{e^{-(b+j\omega)\tau}}{-(b+j\omega)} \Bigg|_{0}^{\infty}$$

$$= a\left[\frac{1-0}{b-j\omega}\right] + a\left[\frac{0-1}{-(b+j\omega)}\right]$$

$$= \frac{a}{b-j\omega} + \frac{a}{b+j\omega} = \frac{2ab}{b^2+\omega^2}.$$

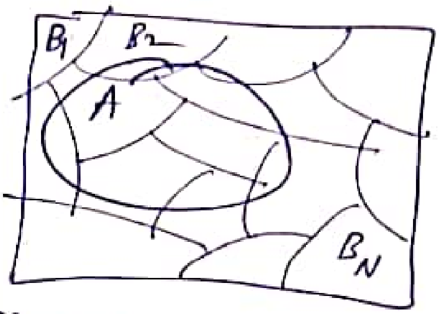
PART-B

2.(a) statement:- The probability of any event A, defined on the Sample space s which has N mutually exclusive events $B_1, B_2 \cdots B_N$ is expressed as

$$\boxed{P(A) = \sum_{n=1}^{N} P\left(\frac{A}{B_n}\right) \cdot P(B_n)}$$

Proof:-

Consider Sample Space 's' with N-mutually exclusive events.



$$B_m \cap B_n = \{\phi\} \text{ for } m \neq n.$$

$$\bigcup_{n=1}^{N} B_n = 1.$$

let $A \cap S = A$

$$A \cap \left(\bigcup_{i=1}^{N} B_i\right) = A$$

$$\bigcup_{i=1}^{N} (A \cap B_i) = A$$

Apply Probability.

$$P\left[\bigcup_{i=1}^{N} (A \cap B_i)\right] = P(A)$$

$$\Rightarrow P(A) = \sum_{i=1}^{N} P(A \cap B_i) \quad (\because \text{by axioms of prob.})$$

but $P(A \cap B_i) = P\left(\frac{A}{B_i}\right) \cdot P(B_i)$ from joint Probability

$$\Rightarrow \boxed{P(A) = \sum_{i=1}^{N} P\left(\frac{A}{B_i}\right) \cdot P(B_i)}$$

2.(b) Let 'D' be the event of defective R.

$$P(D) = P(A).P\left(\frac{D}{A}\right) + P(B).P\left(\frac{D}{B}\right).$$ ($\because$ form total probability)

$P(D/A) = 15\% = 0.15$

$P(D/B) = 5\% = 0.05$

$$P(A) = \frac{1,00,000}{2,50,000} = \frac{10}{25}.$$

$$P(B) = \frac{1,50,000}{2,50,000} = \frac{15}{25}.$$

$$P(D) = \frac{10}{25} \times \frac{15}{100} + \frac{15}{25} \times \frac{5}{100} = \frac{9}{100} = 9\%$$

3.(a) Mean $E[x] = \sum_{x=0}^{\infty} x.p(x) = \sum_{x=0}^{\infty} x.\frac{e^{-\lambda}.\lambda^x}{x!}$

$$= e^{-\lambda}.\sum_{x=0}^{\infty} \frac{x.\lambda^x}{x!}$$

$x=0$ is not valid $\Rightarrow$

$$E[x] = e^{-\lambda}.\sum_{x=1}^{\infty} \frac{\cancel{x}.\lambda^x}{(x-1)!} = \sum_{x=1}^{\infty} \frac{\lambda.\lambda^{x-1}.e^{-\lambda}}{(x-1)!} = \lambda.\sum_{x=1}^{\infty} \frac{e^{-\lambda}.\lambda^{x-1}}{(x-1)!}$$

$$= \lambda * 1$$

$$= \lambda.$$

Mean $\boxed{\therefore E[x] = \lambda}$

$$E[x^2] = \sum_{x=0}^{\infty} x^2.p(x) = \sum_{x=0}^{\infty} x^2.\frac{e^{-\lambda}.\lambda^x}{x!}$$

$x^2 = x + x(x-1)$

$$E[x^2] = \sum_{x=0}^{\infty} \frac{x(x-1)}{x(x-1)(x-2)!} \cdot e^{-\lambda} \cdot \lambda^x + \sum_{x=0}^{\infty} \frac{x \cdot e^{-\lambda} \cdot \lambda^x}{x!}$$

$$= \sum_{x=0}^{\infty} \frac{\lambda^2 \cdot e^{-\lambda} \cdot \lambda^{x-2}}{(x-2)!} + \lambda$$

$$E[x^2] = \lambda^2 + \lambda$$

Variance $\sigma_x^2 = E[x^2] - [E(x)]^2$

$$= \lambda^2 + \lambda - \lambda^2$$

$$\boxed{\text{Variance } \sigma_x^2 = \lambda}$$

3. (b) $f_X(x) = 6x(1-x)$ for $0 \le x \le 1$

$\int_{-\infty}^{\infty} f_X(x) \cdot dx = 1$ for valid pdf

$$\Rightarrow \int_0^1 6x(1-x) \cdot dx = 6\int_0^1 (x - x^2) \cdot dx$$

$$= 6\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1$$

$$= 6\left[\frac{(1-0)}{2} - \frac{(1-0)}{3}\right] = 6\left[\frac{1}{2} - \frac{1}{3}\right]$$

$$= 6\left[\frac{3-2}{6}\right] = 6\left[\frac{1}{6}\right] = 1.$$

$\therefore$ Valid pdf.

4. (a) Joint density function is defined as second derivative of joint distribution function.

$$f_{XY}(x,y) = \frac{\partial^2}{\partial x \partial y} F_{XY}(x,y).$$

Marginal density of X is

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y).dy.$$

Margidnal density of Y is

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y).dx.$$

4. (b) $E(x) = 2$

$E(y) = 3$

$E[xy] = 10$

$E(x^2) = 9$

$E(y^2) = 16$.

Covariance = ___.

Correlation Coeff = ___.

Covoriance $C_{XY} = E(xy) - E(x).E(y) = 10 - 2\times3 = 4$.

Corrdation Coefficient $\rho = \frac{C_{XY}}{\sigma_X \cdot \sigma_Y}$

$\sigma_x^2 = E(x^2) - (E(x))^2$

$\sigma_x^2 = 9 - 4 = 5$

$\sigma_x = \sqrt{5}$

$\sigma_y^2 = E(y^2) - (E(y))^2$

$\sigma_y^2 = 16 - 9 = 7.$

$\sigma_y = \sqrt{7}.$

$$\rho = \frac{4}{\sqrt{5} \times \sqrt{7}} = 0.676$$

5.(a) $x(t) = a.\sin(\omega_0 t+\theta)$

density function $f_\theta(\theta) = \begin{cases} \frac{1}{2\pi} & \text{for } 0<\theta<2\pi \\ 0 & \text{otherwise.} \end{cases}$

Mean $E[x(t)] = \int x(t).f_\theta(\theta).d\theta.$

$$= \int_0^{2\pi} a.\cos(\omega_0 t+\theta).\frac{1}{2\pi}.d\theta.$$

$$= \frac{a}{2\pi}\int_0^{2\pi} \cos(\omega_0 t+\theta).d\theta.$$

$$= \frac{a}{2\pi}\left[\sin(\omega_0 t+\theta)\right]_0^{2\pi}$$

$$= \frac{a}{2\pi}\left[\sin(2\pi+\omega_0 t) - \sin(\omega_0 t)\right] = 0.$$

5. (b) Auto Correlation of $x(t)$:

$$R_{xx}(t, t+\tau) = E[x(t).x(t+\tau)] = E[a.\sin(\omega_0 t+\theta).a\sin(\omega_0 t+\omega_0\tau+\theta)]$$

$$= \frac{a^2}{2}.E[\cos\omega_0\tau - \cos(2\omega_0 t+2\theta+\omega_0\tau)]$$

$$= \frac{a^2}{2}.E[\cos\omega_0\tau] - \frac{a^2}{2}E[\cos(2\omega_0 t+2\theta+\omega_0\tau)]$$

$$= \frac{a^2}{2}.\cos\omega_0\tau - \frac{a^2}{2}.0.$$

$$= \frac{a^2}{2}.\cos\omega_0\tau.$$

$\therefore$ $x(t)$ Is wide Sense stationary process.

6. Consider a WSS random process $x(t)$

the time average autocorrelation $A[R_{xx}(t,t+\tau)] = A\{E[x(t).x(t+\tau)]\}$

For WSS random process $A[R_{xx}(t,t+\tau)] = R_{xx}(\tau)$

PSD, $S_{xx}(\omega) = \lim_{T\to\infty} \frac{E[|X_T(\omega)|^2]}{2T}$

$$|X_T(\omega)|^2 = X_T(\omega).\, X_T^*(\omega).$$

$$X_T(\omega) = \int_{-T}^{T} x(t_1).\, e^{-j\omega t_1}.\, dt_1$$

$$X_T^*(\omega) = \int_{-T}^{T} x(t).\, e^{+j\omega t}.\, dt$$

$$S_{xx}(\omega) = \lim_{T\to\infty} \frac{1}{2T} E\left[\int_{-T}^{T} x(t_1).e^{-j\omega t_1}.dt_1 \int_{-T}^{T} x(t).e^{j\omega t}.dt\right]$$

$$= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T}\int_{-T}^{T} E[x(t).x(t_1)].\, e^{-j\omega(t_1-t)}.\, dt.\, dt_1$$

but $E[x(t).x(t_1)] = R_{xx}(t,t_1)$

$$\therefore S_{xx}(\omega) = \lim_{T\to\infty} \frac{1}{2T} \int\int R_{xx}(t,t_1).\, e^{-j\omega(t_1-t)}.\, dt.\, dt_1$$

Taking Inverse Fourier Transform.

$$\frac{1}{2\pi}\int S_{xx}(\omega).\, e^{j\omega\tau}.\, d\omega = \lim_{T\to\infty} \frac{1}{2T} \int\int_{-T}^{T} R_{xx}(t,t_1).\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j\omega\tau}.\, d\omega.$$

$$= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T}\int_{-T}^{T} R_{xx}(t,t_1) \cdot \frac{1}{2\pi} \int e^{-j\omega(t_1-t-\tau)} \cdot dt \cdot dt_1 \cdot d\omega .$$

$$= \frac{1}{2\pi} \int S_{xx}(\omega) \cdot e^{j\omega\tau} \cdot d\omega = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} R_{xx}(t,t_1) \cdot \int_{-T}^{T} \delta(t_1-t-\tau)\, dt\, dt_1$$

let $t_1 = t+\tau$.

$$\frac{1}{2\pi} \int S_{xx}(\omega) \cdot e^{j\omega\tau} \cdot d\omega = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} R_{xx}(t,t+\tau)\, dt .$$

$$= A\left[R_{xx}(t,t+\tau)\right]$$

$$\therefore\ S_{xx}(\omega) = \int A\left[R_{xx}(t,t+\tau)\right] \cdot e^{-j\omega\tau} \cdot d\tau .$$

$\therefore$ PSD & time average auto Correlation forms F.T. pair

7. (a) Bayes Theorem:

Statement:- If a sample space has 'N' mutually exclusive events & any event A defined on this space, then Conditional probability of B_n & A is

$$P\left(\frac{B_n}{A}\right) = \frac{P\left(\frac{A}{B_n}\right) \cdot P(B_n)}{\sum_{n=1}^{N} P\left(\frac{A}{B_n}\right) \cdot P(B_n)}$$

proof:-

Conditional probability

$$P\left(\frac{B_n}{A}\right) = \frac{P(B_n \cap A)}{P(A)}$$

$$P(B_n \cap A) = P\left(\frac{A}{B_n}\right) \cdot P(B_n).$$

From total probability theorem, $P(A) = \sum_{n=1}^{N} P(B_n \cap A)$

$$P(B_n / A) = \frac{P(B_n \cap A)}{\sum_{n=1}^{N} P(B_n \cap A)}$$

$$P\left(\frac{B_n}{A}\right) = \frac{P\left(\frac{A}{B_n}\right) \cdot P(B_n)}{\sum_{n=1}^{N} P\left(\frac{A}{B_n}\right) \cdot P(B_n)}$$

7. (b) $f_X(x) = \begin{cases} \frac{1}{2} & \text{for } -1 \le x \le 1 \\ 0 & \text{otherwise.} \end{cases}$

$$\phi_X(\omega) = \int_{-1}^{1} e^{j\omega x} \cdot \frac{1}{2} \cdot dx$$

$$= \frac{1}{2} \cdot \frac{e^{j\omega x}}{j\omega} \Big|_{-1}^{1}$$

$$= \frac{1}{2j\omega}\left[e^{j\omega} - e^{-j\omega}\right] = \frac{1}{\omega}\left[\frac{e^{j\omega} - e^{-j\omega}}{2j}\right]$$

$$= \frac{\sin\omega}{\omega} = Sa(\omega).$$

8. (a) Joint probability density function

$$f_{XY}(x,y) = \frac{\partial^2}{\partial x \partial y} F_{XY}(x,y)$$

properties:-

1. $f_{XY}(x,y) \ge 0.$
2. $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y) \cdot dx\, dy = 1.$
3. $F_{XY}(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f_{XY}(x,y) \cdot dx \cdot dy$
4. $P\{x_1 < X \le x_2, y_1 < Y \le y_2\} = \int_{x_1}^{x_2}\int_{y_1}^{y_2} f_{XY}(x,y) \cdot dx\, dy$
5. $f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y) \cdot dy.$
6. $f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y) \cdot dx.$

8. (b) First order stationary process:

$$f_X(x_1, t_1) = f_X(x_1, t_1 + \Delta t) \text{ for any } t_1.$$

$$E[x(t)] = \bar{x} = \text{Constant}.$$

Second order stationary process:

$$f_X(x_1, x_2; t_1, t_2) = f_X(x_1, x_2; t_1 + \Delta t, t_2 + \Delta t)$$

$$R_{XX}(t_1, t_2) = R_{XX}(\tau).$$

Nth order stationary process:

$$f_X(x_1, x_2 \dots x_N; t_1, t_2 \dots t_N) = f_X(x_1, x_2 \dots x_N; t_1 + \Delta t, t_2 + \Delta t \dots t_N + \Delta t)$$

9. (a)

$$S_{XY}(\omega) = F\{R_{XY}(\tau)\} = \int_{-\infty}^{\infty} R_{XY}(\tau) \cdot e^{-j\omega\tau} \cdot d\tau.$$

properties:-

1. $S_{XY}(\omega) = S_{YX}(-\omega) = S^{*}_{YX}(\omega)$.
2. Realpart of $S_{XY}(\omega)$ & $S_{YX}(\omega)$ are evenfunctions of ω.
3. Imaginary part of $S_{XY}(\omega)$ & $S_{YX}(\omega)$ are odd functions of ω.
4. $S_{XY}(\omega) = 0$ if X & Y are orthogonal.

9. (b) $N = 4$

$p = \frac{1}{2}$

$q = \frac{1}{2}$

Using bienomial distributon.

Variance $\sigma_X^2 = NPq = 4 \times \frac{1}{2} \times \frac{1}{2} = 1.$