Set-B Solutions.

PART-A

1. (a) $P(\text{defective}) = \frac{4}{25}$.

$N=5$.

$$P(x=1) = {}^{N}C_x \cdot p^x \cdot (1-p)^{N-x}$$

$$= {}^{5}C_1 \cdot \left(\tfrac{4}{25}\right)^1 \left(1-\tfrac{4}{25}\right)^{5-1}$$

$$= 5 \times 0.16 \times (0.84)^4 = 0.398$$

(b) <u>Mutually Exclusive events:-</u>

If any two events has no Common outcome then the two events are "mutually exclusive".

Ex:- In tossing a coin, head & tail are mutually exclusive.

<u>Independent events:-</u>

If the occurance of one event doesnot effect the occurance of another event then they are known as "Independent events."

Ex:- tossing a coin & Rolling a die are independent.

(c) Moment Generating function $M_X(t) = E[e^{tx}] = \int_{-\infty}^{\infty} e^{tx} \cdot f_X(x)\, dx$

Characteristic function $\phi_X(\omega) = E[e^{j\omega x}] = \int_{-\infty}^{\infty} e^{j\omega x} \cdot f_X(x) \cdot dx$.

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(d) $F_X(x) = P\{X \leq x\}$.

properties:- $F_X(-\infty) = 0$

$F_X(\infty) = 1$

$0 \leq F_X(x) \leq 1$.

$F_X(x_1) \leq F_X(x_2)$ if $x_1 < x_2$

$P\{x_1 < X \leq x_2\} = F_X(x_2) - F_X(x_1)$

$F_X(x^+) = F_X(x)$

(e) $Var(X+Y) = E[(x+y)^2] - [E(x+y)]^2$

$= E[x^2 + y^2 + 2xy] - (\bar{x} + \bar{y})^2$

$= \overline{x^2} + \overline{y^2} + \cancel{2\bar{x}\bar{y}} - \bar{x}^2 - \bar{y}^2 \cancel{- 2\bar{x}\bar{y}}$

$= \overline{x^2} - \bar{x}^2 + \overline{y^2} - \bar{y}^2$

$= Var(X) + Var(Y)$.

(f) Covariance $C_{XY} = E(XY) - E(X)E(Y)$

If X & Y are independent $E(XY) = E(X) \cdot E(Y)$.

$\Rightarrow$ $C_{XY} = \bar{X}\bar{Y} - \bar{X}\bar{Y} = 0$.

(g) For wide sense stationary process

$$\text{Mean} = E[x(t)] = \bar{x} = \text{Constant}.$$

AutoCorrelation function $R_{xx}(t, t+\tau) = R_{xx}(\tau)$ Independent of t.

(h) Time average = statistical average ⇒ Mean Ergodic.

$$E[x(t)] = \bar{x} = A\langle x(t)\rangle = \bar{x}.$$

Time AutoCorrelation = statistical AutoCorrelation ⇒ Correlation Ergodic.

$$A\langle x(t).x(t+\tau)\rangle = E[x(t).x(t+\tau)].$$

(i)

$$S_{xx}(\omega) = \frac{\omega}{j\omega^6 + \omega^2 + 3}.$$

$$S_{xx}(-\omega) = \frac{-\omega}{j\omega^6 + \omega^2 + 3} \neq S_{xx}(\omega).$$

∴ It is not a valid PSD.

(j) For white noise. power spectral density is Constant.

$$S_{NN}(\omega) = \frac{N_0}{2}. \text{ where } N_0 - \text{Constant}.$$

A Colored noise has only a portion of white noise frequencies in its spectrum.

2. (a) $P(B_1) = 0.30$

$P(B_2) = 0.45$

$P(B_3) = 0.25$

$P(D/B_1) = 0.02$

$P(D/B_2) = 0.03$

$P(D/B_3) = 0.02$

$$P(\text{Defective}) = P(B_1).P\left(\frac{D}{B_1}\right) + P(B_2).P\left(\frac{D}{B_2}\right) + P(B_3).P\left(\frac{D}{B_3}\right).$$

$$= 0.3 \times 0.02 + 0.45 \times 0.03 + 0.25 \times 0.02$$

$$= 0.0245$$

2. (b) $N = 6$

$P(\text{power drawn ON}) = 0.4$

$P(0w) = {}^6C_0\,(0.4)^0\,(0.6)^6 = 0.0467$

$P(0.5w) = {}^6C_1\,(0.4)^1\,(0.6)^5 = 0.1866$

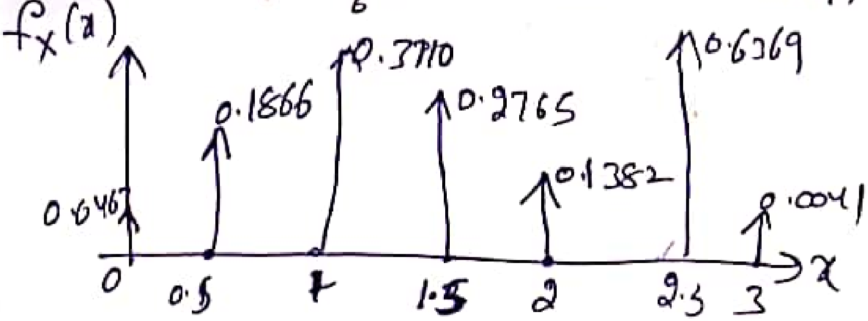
$P(1w) = {}^6C_2\,(0.4)^2\,(0.6)^4 = 0.3110$

$P(1.5w) = {}^6C_3\,(0.4)^3\,(0.6)^3 = 0.2765$

$P(2w) = {}^6C_4\,(0.4)^4\,(0.6)^2 = 0.1382$

$P(2.5w) = {}^6C_5\,(0.4)^5\,(0.6)^1 = 0.6369.$

$P(3w) = {}^6C_6\,(0.4)^6\,(0.6)^6 = 0.0041$



3- (a) Mean $E[X] = \sum_{x=0}^{\infty} x \cdot p(x) = \sum_{x=0}^{\infty} x \cdot \frac{e^{-\lambda} \cdot \lambda^x}{x!}$

$$= e^{-\lambda} \cdot \sum_{x=0}^{\infty} \frac{x \cdot \lambda^x}{x!}$$

$x=0$ is not valid $\Rightarrow$

$$E[X] = e^{-\lambda} \cdot \sum_{x=1}^{\infty} \frac{x \cdot \lambda^x}{(x-1)!} = \sum_{x=1}^{\infty} \frac{\lambda \cdot \lambda^{x-1} \cdot e^{-\lambda}}{(x-1)!} = \lambda \cdot \sum_{x=1}^{\infty} \frac{e^{-\lambda} \cdot \lambda^{x-1}}{(x-1)!}$$

$$= \lambda * 1$$

$$= \lambda.$$

Mean $\boxed{\therefore E[X] = \lambda}$

$$E[x^2] = \sum_{x=0}^{\infty} x^2 \cdot p(x) = \sum_{x=0}^{\infty} x^2 \cdot \frac{e^{-\lambda} \cdot \lambda^x}{x!}$$

$$x^2 = x + x(x-1)$$

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$$E[x^2] = \sum_{x=0}^{\infty} \frac{x(x-1)}{x(x-1)(x-2)!} \cdot e^{-\lambda} \cdot \lambda^{x} + \sum_{x=0}^{\infty} \frac{x \cdot e^{-\lambda} \cdot \lambda^{x}}{x!}$$

$$= \sum_{x=0}^{\infty} \frac{\lambda^2 \cdot e^{-\lambda} \cdot \lambda^{x-2}}{(x-2)!} + \lambda$$

$$E[x^2] = \lambda^2 + \lambda$$

Variance $\sigma_x^2 = E[x^2] - [E(x)]^2$

$$= \lambda^2 + \lambda - \lambda^2$$

$$\boxed{\text{Variance } \sigma_x^2 = \lambda}$$

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3. (b) X is uniform RV in $[-5, -15]$

$$f_X(x) = \frac{1}{-15-(-5)} = \frac{-1}{10} \quad \text{in } [-5, -15]$$

$$y = e^{-x/5}$$

$$E(y) = E[e^{-x/5}] = \int_{-5}^{-15} e^{-x/5} \cdot \left(\frac{-1}{10}\right) . dx$$

$$= \frac{-1}{10} \cdot \left[\frac{e^{-x/5}}{-1/5}\right]_{-5}^{-15} = \frac{1}{2}\left(e^{15/5} - e^{5/5}\right) = \frac{1}{2}(e^3 - e^1)$$

$$= 8.6836$$

$$E(y^2) = E\left[e^{-2x/5}\right] = \int e^{-2x/5} \cdot \left(\frac{-1}{10}\right) . dx$$

$$= \frac{-1}{10} \int_{-5}^{-15} e^{-2x/5} . dx = \frac{-1}{10}\left[\frac{e^{-2x/5}}{-2/5}\right]_{-5}^{-15}$$

$$= \frac{1}{4}\left[e^6 - e^2\right] = 99.0099.$$

$$Var(y) = E(y^2) - [E(y)]^2 = 99.0099 - (8.6836)^2 = 23.6049$$

4. (a) Central Limit Theorem states that the probability distribution function of sum of large number of independent Random Variables approaches normal Gaussian function.

i.e. Given N independent Random Variables $x_1, x_2 \ldots x_N$.

$$X = x_1 + x_2 + \cdots + x_N \;\; \& \;\; \sigma^2 = \sigma_1^2 + \sigma_2^2 + \cdots + \sigma_N^2.$$

$$F(x) \equiv G\left(\frac{x-\mu}{\sigma}\right) \text{ as } N \text{ increases.}$$

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-(x-\mu)^2/2\sigma^2}$$

where μ is mean

σ^2 is Variance.

4. (b)

$E(x) = 4$

$E(y) = 9$

$E(xy) = 100$

$E(x^2) = 81$

$E(y^2) = 256$

$$\rho = \frac{C_{xy}}{\sigma_x \sigma_y}$$

$$C_{xy} = E(xy) - E(x)E(y) = 100 - 4 \times 9 = 64$$

$\sigma_x^2 = E(x^2) - [E(x)]^2 = 81 - 4^2 = 65 \Rightarrow \sigma_x = 8.06$

$\sigma_y^2 = E(y^2) - [E(y)]^2 = 256 - 9^2 = 256 - 81 = 175$

$\Rightarrow \sigma_y = 13.228$

$$\rho = \frac{64}{8.06 \times 13.228} = 0.60011$$

5.(a) $x(t) = A \cdot \cos(\omega_0 t + \theta)$.

θ is uniform RV in $(0, \pi)$.

$f_\theta(\theta) = \frac{1}{\pi}$. in $(0, \pi)$.

$$E[x(t)] = \int_0^\pi A \cdot \cos(\omega_0 t + \theta) \cdot \frac{1}{\pi} \cdot d\theta.$$

$$= \frac{A}{\pi} \cdot \left[\sin(\omega_0 t + \theta\right]_0^\pi$$

$$= \frac{A}{\pi}\left[\sin(\omega_0 t + \pi) - \sin(\omega_0 t)\right]$$

$$= \frac{A}{\pi}\left[-\sin \omega_0 t - \sin \omega_0 t\right] = -\frac{2A}{\pi} \sin(\omega_0 t).$$

$\therefore E[x(t)]$ is not Constant.

$\therefore x(t)$ is not WSS.

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5. (b) $S_{XX}(\omega) = \lim_{T\to\infty} \dfrac{E\left[|X_T(\omega)|^2\right]}{2T}$.

$$X_T(\omega) = \int_{-T}^{T} A_0 \cdot \cos(\omega_0 t + \theta) \cdot e^{-j\omega t} \cdot dt.$$

$$= \frac{A_0}{2} \cdot e^{j\theta} \int_{-T}^{T} e^{j(\omega_0-\omega)t} \cdot dt + \frac{A_0}{2} \cdot e^{-j\theta} \int_{-T}^{T} e^{-j(\omega_0+\omega)t} \cdot dt.$$

$$= A_0 T e^{j\theta} \cdot \frac{\sin[(\omega-\omega_0)T]}{(\omega_0-\omega)T} + A_0 T e^{-j\theta} \cdot \frac{\sin[(\omega_0+\omega)T]}{(\omega_0+\omega)T}$$

$$|X_T(\omega)|^2 = X_T(\omega) \cdot X_T^*(\omega).$$

$$\frac{E\left[|X_T(\omega)|^2\right]}{2T} = \frac{A_0^2 T}{2}\left[\frac{T}{\pi} \cdot \frac{\sin^2[(\omega-\omega_0)T]}{[(\omega-\omega_0)T]^2} + \frac{T}{\pi} \cdot \frac{\sin^2[(\omega+\omega_0)T]}{[(\omega+\omega_0)T]^2}\right]$$

$$\lim_{T\to\infty} \frac{T}{\pi}\left[\frac{\sin(\alpha T)}{\alpha T}\right]^2 = \delta(\alpha).$$

$$\therefore S_{XX}(\omega) = \frac{A_0^2}{2}\pi\left[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)\right]$$

6. (a) $S_{xx}(\omega) = \dfrac{\omega^2}{\omega^4 + 13\omega^2 + 36}$

$$R_{xx}(\tau) = F^{-1}\left\{ S_{xx}(\omega) \right\} = F^{-1}\left\{ \frac{\omega^2}{(\omega^2+9)(\omega^2+4)} \right\}.$$

$$= F^{-1}\left\{ \frac{-4/5}{\omega^2+4} + \frac{9/5}{\omega^2+9} \right\}.$$

$$= F^{-1}\left\{ \frac{-4}{5} \cdot \frac{4}{4(\omega^2+4)} + \frac{9}{5} \cdot \frac{6}{6(\omega^2+9)} \right\}.$$

we know that $\dfrac{2a}{a^2+\omega^2} \Longleftrightarrow e^{-a|\tau|}$.

$$\Rightarrow R_{xx}(\tau) = F^{-1}\left\{ -\frac{1}{5} \cdot \frac{4}{\omega^2+4} + \frac{9}{30} \cdot \frac{6}{\omega^2+9} \right\}.$$

$$R_{xx}(\tau) = -\frac{1}{5} \cdot e^{-2|\tau|} + \frac{9}{30} \cdot e^{-3|\tau|}$$

6. (b) Average Power $P_{xx} = R_{xx}(\tau)\Big|_{\tau=0}$.

$$= -\frac{1}{5} \cdot e^{-2|0|} + \frac{9}{30} \cdot e^{-3|\tau|}$$

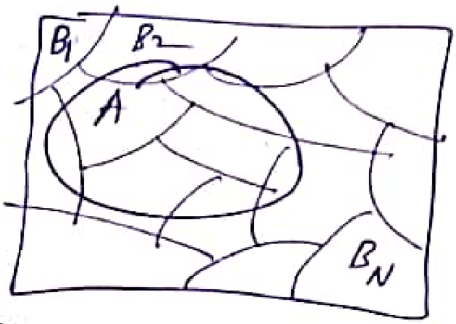
$$= -\frac{1}{5} + \frac{9}{30}$$

$$= 0.1 \text{ watts.}$$

7.(a) <u>statement</u>:- The probability of any event A, defined on the Sample space s which has N mutually exclusive events $B_1, B_2 \cdots B_N$ is expressed as $\boxed{P(A) = \sum_{n=1}^{N} P\left(\frac{A}{B_n}\right) \cdot P(B_n)}$

<u>Proof</u>:-

Consider Sample Space 's' with N-mutually exclusive events.



$$B_m \cap B_n = \{\phi\} \text{ for } m \neq n.$$

$$\bigcup_{n=1}^{N} B_n = 1.$$

let $A \cap S = A$

$$A \cap \left(\bigcup_{i=1}^{N} B_i\right) = A$$

$$\bigcup_{i=1}^{N} (A \cap B_i) = A$$

Apply Probability.

$$P\left[\bigcup_{i=1}^{N} (A \cap B_i)\right] = P(A)$$

$$\Rightarrow P(A) = \sum_{i=1}^{N} P(A \cap B_i) \quad (\because \text{by axioms of prob.})$$

but $P(A \cap B_i) = P\left(\frac{A}{B_i}\right) \cdot P(B_i)$ from joint Probability

$$\Rightarrow \boxed{P(A) = \sum_{i=1}^{N} P\left(\frac{A}{B_i}\right) \cdot P(B_i)}$$

7.(b) $f_X(x) = \frac{3}{4} x(2-x)$ for $0 \le x \le 2$

$$E[x] = \int_0^2 x \cdot f_X(x) \cdot dx$$

$$E[x] = \int_0^2 x \cdot \frac{3}{4} \cdot x(2-x) \cdot dx$$

$$= \frac{3}{4} \int_0^2 \left[2x^2 - x^3\right] \cdot dx$$

$$= \frac{3}{4}\left[\frac{2x^3}{3} - \frac{x^4}{4}\right]_0^2 .$$

$$= \frac{3}{4}\left[\frac{2}{3}(8-0) - \frac{1}{4}(16-0)\right]$$

$$= \frac{3}{4}\left[\frac{16}{3} - 4\right] = \frac{3}{4}\left[\frac{16-12}{3}\right]$$

$$= \frac{3}{4} \times \frac{4}{3} = 1$$

8.(a) Joint distribution function

$$F_{XY}(x,y) = P\{X \le x, Y \le y\}.$$

properties:–

1. $F_{XY}(-\infty, -\infty) = 0$

$F_{XY}(-\infty, y) = 0$

$F_{XY}(x, -\infty) = 0.$

proof:- $F_{XY}(-\infty,-\infty) = P(X \le -\infty \cap Y \le -\infty) = P(\phi) = 0.$

2. $F_{XY}(\infty,\infty) = 1$

$F_{XY}(\infty,\infty) = P(X \le \infty \cap Y \le \infty) = P(S \cap S) = P(S) = 1.$

3. $0 \le F_{XY}(x,y) \le 1.$

4. $F_{XY}(x,y)$ is monotonic and non-decreasing function of both x and y.

5. $P\{x_1 < X \le x_2,\ Y \le y\} = F_{XY}(x_2,y) - F_{XY}(x_1,y)$

$P\{X \le x,\ y_1 < Y \le y_2\} = F_{XY}(x,y_2) - F_{XY}(x,y_1).$

6. $P\{x_1 < X \le x_2,\ y_1 < Y \le y_2\} = F_{XY}(x_2,y_2) + F_{XY}(x_1,y_1) - F_{XY}(x_1,y_2) - F_{XY}(x_2,y_1).$

7. $F_X(x) = F_{XY}(x,\infty).$

$F_Y(y) = F_{XY}(\infty,y).$

8. (b) First order stationary process:

$$f_X(x_1, t_1) = f_X(x_1, t_1 + \Delta t) \text{ for any } t_1.$$

$$E[x(t)] = \bar{X} = \text{Constant}.$$

Second order stationary process:

$$f_X(x_1, x_2; t_1, t_2) = f_X(x_1, x_2; t_1 + \Delta t, t_2 + \Delta t)$$

$$R_{XX}(t_1, t_2) = R_{XX}(\tau).$$

Nth order stationary process:

$$f_X(x_1, x_2 \ldots x_N; t_1, t_2 \ldots t_N) = f_X(x_1, x_2 \ldots x_N; t_1 + \Delta t, t_2 + \Delta t \ldots t_N + \Delta t)$$

9. (a)

1. $R_{xy}(\tau) = R_{yx}(-\tau)$ is symmetric property.

2. $R_{xy}(\tau) \leq \sqrt{R_{xx}(0) \cdot R_{yy}(0)}$.

3. ~~$R_{xy}(\tau)$~~ $\sqrt{R_{xx}(0)\, R_{yy}(0)} \leq \dfrac{R_{xx}(0) + R_{yy}(0)}{2}$.

4. $\lim_{\tau \to \infty} R_{xy}(\tau) = \bar{x} \cdot \bar{y}$ if x & y are independent.

5. $\lim_{\tau \to \infty} R_{xy}(\tau) = 0$ if x & y has zero mean.

6. If $x(t)$ & $y(t)$ are uncorrelated then
$R_{xy}(\tau) = E[x(t)] \cdot E[y(t+\tau)]$.

7. Correlation matrix

$$R(\tau) = \begin{bmatrix} R_{xx}(\tau) & R_{xy}(\tau) \\ R_{yx}(\tau) & R_{yy}(\tau) \end{bmatrix}$$

9.

(b) Given $m_1 = 1$

$m_2 = 16$

$m_3 = 40$

$$\text{Mean} = E(x) = m_1 = 1.$$

$$\text{Variance} = E(x^2) - [E(x)]^2$$

$$= m_2 - m_1^2$$

$$= 16 - 1 = 15.$$

$$E[(x-\bar{x})^3] = m_3 - 3\sigma_x^2 m_1 - m_1^3.$$

$$\mu_3 = 40 - 3 \times 1 \times 15 - 1$$

$$\mu_3 = 40 - 45 - 1 = -6.$$