SIGNALS & SYSTEMS

(Common to EC/CS/IT)

Sub Code : ES215EC

UNIT-I

Some useful operations on signals: Time shifting, Time scaling, Time inversion. Signal models: Impulse function, Unit step function, Exponential function, Even and odd signals. Systems: Linear and Non-linear systems, Constant parameter and time varying parameter systems, Static and dynamic systems, Causal and Non-causal systems, Lumped Parameter and distributed parameter systems, Continuous-time and discretetime systems, Analog and digital systems.

UNIT-II

Fourier series: Signals and Vectors, Signal Comparison: correlation, Signal representation by orthogonal signal set, Trigonometric Fourier Series, Exponential Fourier Series, LTI system response to periodic inputs.

UNIT-III

Continuous-Time Signal Analysis: Fourier Transform: Aperiodic signal representation by Fourier integral, Fourier Transform of some useful functions, Properties of Fourier Transform, Signal transmission through LTI Systems, ideal and practical filters, Signal energy. Laplace transform: Definition, some properties of Laplace transform, solution of differential equations using Laplace transform.

UNIT-IV

Discrete-time signals and systems: Introduction, some useful discrete-time signal models, Sampling continuous-time sinusoids and aliasing, Useful signal operations, examples of discrete-time systems. Fourier analysis of discrete-time signals, periodic signal representation of discrete-time Fourier series, aperiodic signal representation by Fourier integral.

UNIT-V

Discrete-time signal analysis: Z-Transform, some properties of Z-Transform, Solution to Linear difference equations using Z-Transform, System realization. Relation between Laplace transform and Z-Transform.
DTFT: Definition, Properties of DTFT, comparison of continuous-time signal analysis with discrete-time signal analysis.

Suggested Readings:

1. B. P. Lathi, *Linear Systems and Signals,* Oxford University Press, 2nd Edition, 2009
2. Alan V O P Penheim, A. S. Wlisky, *Signals and Systems,* 2nd Edition, Prentice Hall
3. Rodger E. Ziemer, William H Trenter, D. Ronald Fannin, *Signals and Systems,* 4th Edition

UNIT 1

INTRODUCTION

Learning Objectives: **Introduction**: Definitions of a signal and a system, classification of signals, basic Operations on signals, elementary signals, Systems viewed as Interconnections of operations, properties of systems.

Signal:

A signal is a function representing a physical quantity or variable, and typically it contains information about the behaviour or nature of the phenomenon.
For instance, in a RC circuit the signal may represent the voltage across the capacitor or the current flowing in the resistor. Mathematically, a signal is represented as a function of an independent variable 't'. Usually 't'represents time. Thus, a signal is denoted by x(t).

System:

A system is a mathematical model of a physical process that relates the input (or excitation) signal to the output (or response) signal.
Let x and y be the input and output signals, respectively, of a system. Then the system is viewed as a transformation (or mapping) of **x** into y. This transformation is represented by the mathematical notation

y= T(x)--(1.1)

where**T** is the operator representing some well-defined rule by which **x** is transformed into y. Relationship (1.1) is depicted as shown in Fig. 1-1(a). Multiple input and/or output signals are possible as shown in Fig. 1-1(b). We will restrict our attention for the most part in this text to the single-input, single-output case.

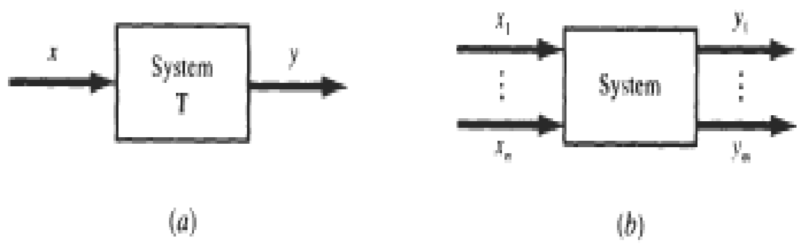


Fig1.1 :System with single or multiple input and output signals

Classification of signals :

Basically seven different classifications are there:

1. Continuous-Time and Discrete-Time Signals
2. Analog and Digital Signals
3. Real and Complex Signals
4. Deterministic and Random Signals
5. Even and Odd Signals
6. Periodic and Nonperiodic Signals
7. Energy and Power Signals

1.Continuous-Time and Discrete-Time Signals

A signal x(t) is a continuous-time signal if t is a continuous variable. If t is a discrete variable, that is, x(t) is defined at discrete times, then x(t) is a discrete-time signal. Since a discrete-time signal is defined at discrete times, a discrete-time signal is often identified as a sequence of numbers, denoted by {x,) or x[n], where n = integer. Illustrations of a continuous- time signal x(t) and of a discrete-time signal x[n] are shown in Fig. 1-2.

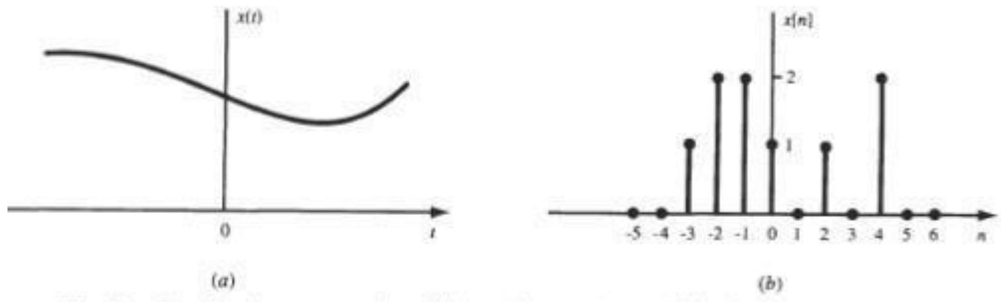


Fig 1.2 Graphical representation of (a) continuous-time and (b) discrete-time signals

2. Analog and Digital Signals

If a continuous-time signal x(t) can take on any value in the continuous interval (a, b), where a may be $-\infty$ and b may be $+\infty$ then the continuous-time signal x(t) is called an analog signal. If a discrete-time signal x[n] can take on only a finite number of distinct values, then we call this signal a digital signal.

3. Real and Complex Signals

A signal x(t) is a real signal if its value is a real number, and a signal x(t) is a complex signal if its value is a complex number. A general complex signal x(t) is a function of the form

$x(t) = x_1(t) + jx_2(t)$ --- (1.2)

where $x_1(t)$ and $x_2(t)$ are real signals and $j = \sqrt{-1}$
Note that in Eq. (1.2) **'t'**represents either a continuous or a discrete variable.

Deterministic and Random Signals:

Deterministic signals are those signals whose values are completely specified for any given time. Thus, a deterministic signal can be modelled by a known function of time **'t'.**
Random signals are those signals that take random values at any given time and must be characterized statistically.

Even and Odd Signals:

A signal ***x (t)*** or ***x[n]*** is referred to as an ***even*** signal if x (- t) = x(t)

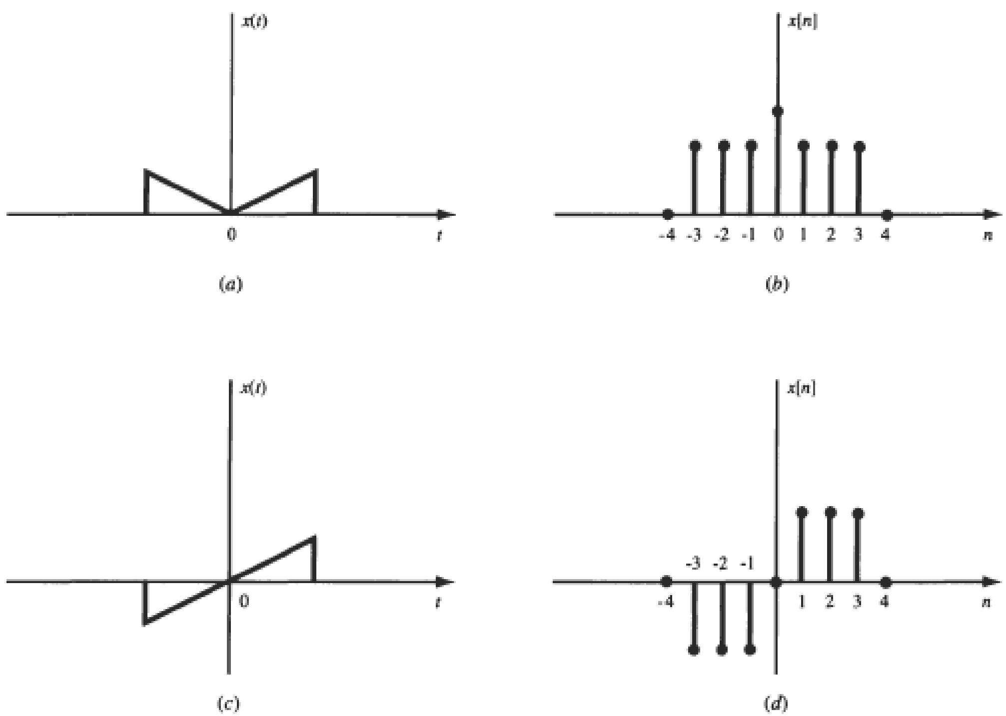
x [-n] = x [n]--- (1.3)

A signal ***x (t)*** or ***x[n]*** is referred to as an ***odd*** signal if x(-t) = - x(t)

x[- n] = - x[n]

--------------(1.4)

Examples of even and odd signals are shown in Fig. 1.3.



Figuer1.3 Examples of even signals (a and b) and odd signals (c and d).

Any signal x(t) or x[n] can be expressed as a sum of two signals, one of which is even and one of which is odd. That is,

$$x(t) = x_o(t) + x_e(t)$$
-------(1.5)

Where,

$$x_e(t) = \frac{1}{2}\left(x(t) + x(-t)\right)$$

$$x_o(t) = \frac{1}{2}\left(x(t) - x(-t)\right)$$

-----(1.6)

Similarly for x[n],

$X[n] = X_e[n] + X_o[n]$ -------(1.7)

Where,

$X_e[n]=1/2\{x[n] +x[-n]\}$

$X_o[n]=1/2\{x[n] -x[-n]\}$ -------(1.8)

Note that the product of two even signals or of two odd signals is an even signal and that the product of an even signal and an odd signal is an odd signal.

Periodic and Nonperiodic Signals :

A continuous-time signal x (t) is said to be periodic with period T if there is a positive nonzero value of T for which

$$x(t+T) = x(t) \qquad \text{all } t$$ ………(1.9)

An example of such a signal is given in Fig. 1-4(a). From Eq. (1.9) or Fig. 1-4(a) it follows that

$$x(t+mT) = x(t)$$ ---------------------------(1.10)

for all t and any integer m. The fundamental period T, of x(t) is the smallest positive value of T for which Eq. (1.9) holds. Note that this definition does not work for a constant

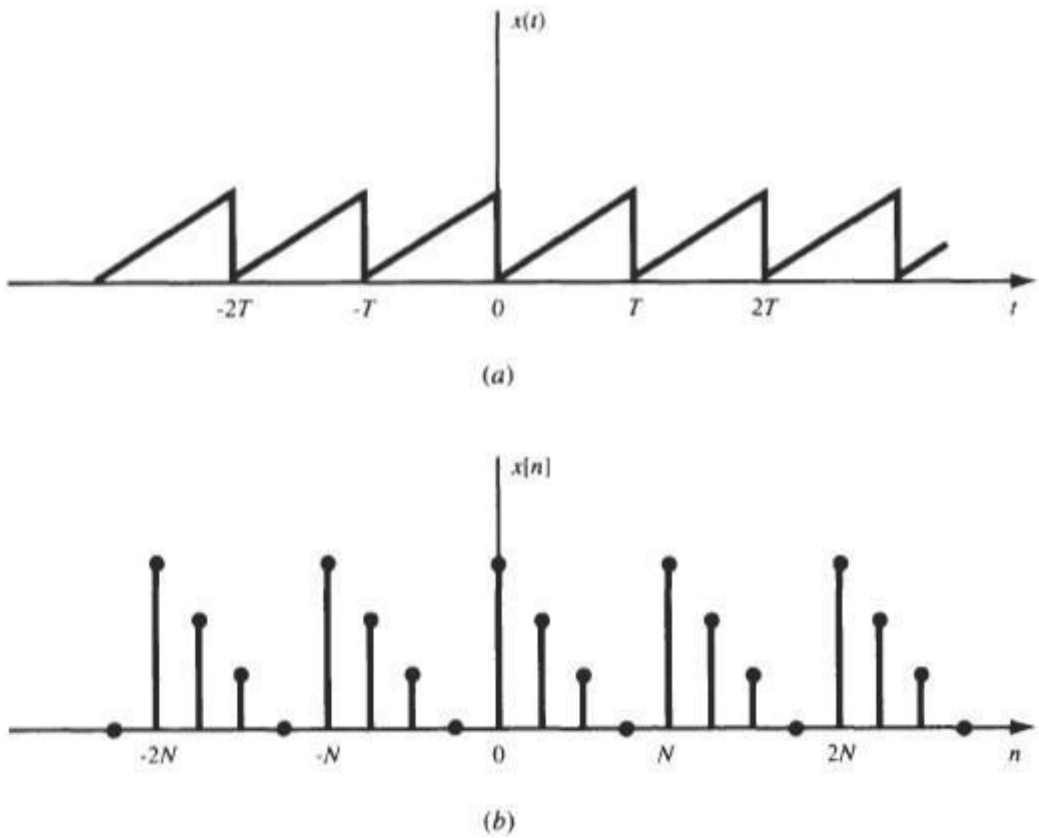


Figure 1.4 Examples of periodic signals.

signal x(t) (known as a dc signal). For a constant signal x(t) the fundamental period is undefined since x(t) is periodic for any choice of T (and so there is no smallest positive value). Any continuous-time signal which is not periodic is called a nonperiodic (or aperiodic) signal.

Periodic discrete-time signals are defined analogously. A sequence (discrete-time signal) x[n] is periodic with period N if there is a positive integer N for which

$$x[n+N] = x[n] \qquad \text{all } n \qquad \text{……….(1.11)}$$

An example of such a sequence is given in Fig. 1-4(b). From Eq. (1.11) and Fig. 1-4(b) it follows that

$$x[n+mN] = x[n] \qquad \text{…………..(1.12)}$$

for all n and any integer m. The fundamental period N_0 of x[n] is the smallest positive integer N for which Eq.(1.11) holds. Any sequence which is not periodic is called a nonperiodic (or aperiodic sequence).

Note that a sequence obtained by uniform sampling of a periodic continuous-time signal may not be periodic. Note also that the sum of two continuous-time periodic signals may not be periodic but that the sum of two periodic sequences is always periodic.

Energy and Power Signals :

Consider v(t) to be the voltage across a resistor R producing a current i(t). The instantaneous power p(t) per ohm is defined as

$$p(t) = \frac{v(t)i(t)}{R} = i^2(t)$$

… ………(1.13)

Total energy E and average power P on a per-ohm basis are

$$E = \int_{-\infty}^{\infty} i^2(t)\,dt \quad \text{joules}$$

$$P = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} i^2(t)\,dt \quad \text{watts}$$

……(1.14)

For an arbitrary continuous-time signal x (t), the normalized energy content E of x(t) is defined as

$$E = \int_{-\infty}^{\infty} |x(t)|^2\,dt$$

…………………(1.15)

The normalized average power P of x(t) is defined as

$$P = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} |x(t)|^2\,dt$$

…………….(1.16)

Similarly, for a discrete-time signal x[n], the normalized energy content ***E*** of x[n] is defined as

$$E = \sum_{n=-\infty}^{\infty} |x[n]|^2$$

…………….(1.17)

The normalized average power P of x[n] is defined as

$$P = \lim_{N\to\infty} \frac{1}{2N+1}\sum_{n=-N}^{N} |x[n]|^2$$

…………….(1.18)

Based on definitions (1.15) to (1.18), the following classes of signals are defined:

1. x(t) (or x[n]) is said to be an energy signal (or sequence) if and only if 0 < E < m, and so P = 0.
2. x(t) (or x[n]) is said to be a power signal (or sequence) if and only if 0 < P < m, thus implying that E = m.
3. Signals that satisfy neither property are referred to as neither energy signals nor power signals.

Note that a periodic signal is a power signal if its energy content per period is finite, and then the average power of this signal need only be calculated over a period

Basic Operations on signals

The operations performed on signals can be broadly classified into two kinds

- Operations on dependent variables
- Operations on independent variables

Operations on dependent variables

The operations of the dependent variable can be classified into five types: amplitude scaling, addition, multiplication, integration and differentiation.

Amplitude scaling

Amplitude scaling of a signal $x(t)$ given by equation 1.19, results in amplification of $x(t)$ if $a > 1$, and attenuation if $a < 1$.

$y(t) = ax(t)$ .... (1.20)

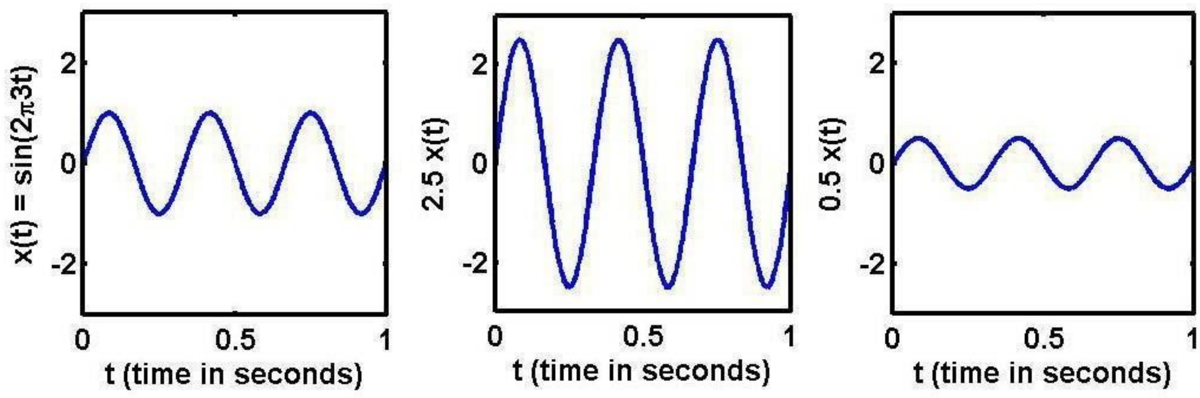


Figure 1.5 Amplitude scaling of sinusoidal signal

Addition

The addition of signals is given by equation of 1.21.

$y(t) = x1(t) + x2\,(t)$ ……(1.21)

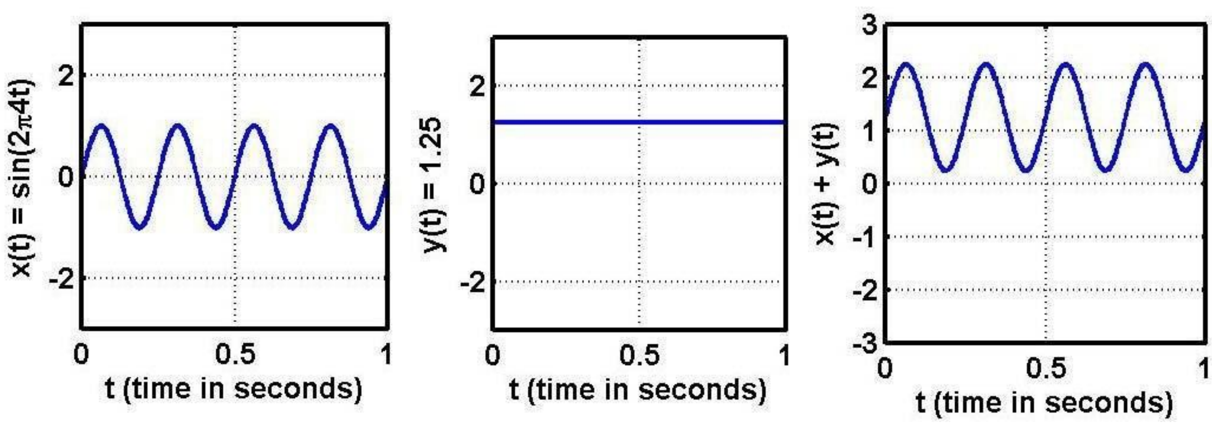


Figure 1.6 Example of the addition of a sinusoidal signal with a signal of constant amplitude

Physical significance of this operation is to add two signals like in the addition of the background music along with the human audio. Another example is the undesired addition of noise along with the desired audio signals.

Multiplication

The multiplication of signals is given by the simple equation of 1.22.

$y(t) = x1(t).x2\,(t)$ ....(1.22)

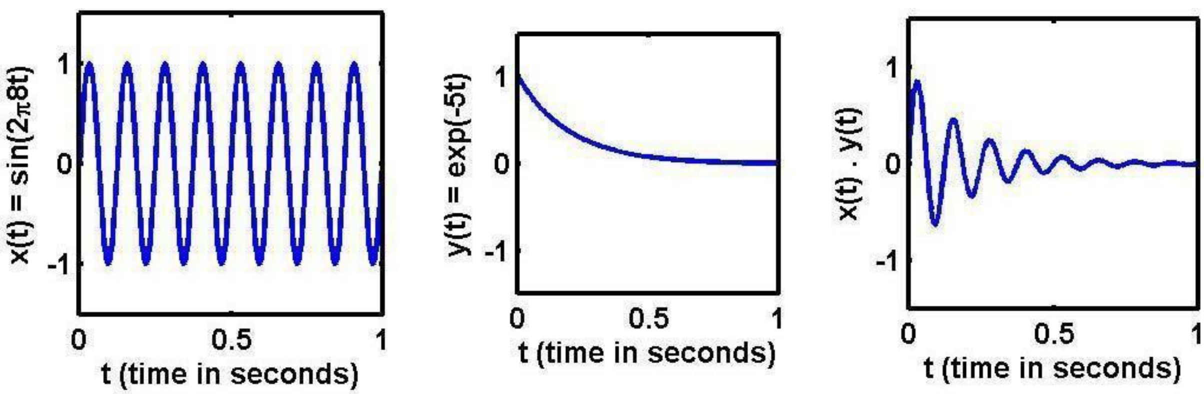


Figure 1.7 Example of multiplication of two signals

Differentiation

The operation of differentiation gives the rate at which the signal changes with respect to time, and can be computed using the following equation, with Δt being a small interval of time.
If a signal doesnt change with time, its derivative is zero, and if it changes at a fixed rate with time, its derivative is constant. This is evident by the example given in figure 1.8.

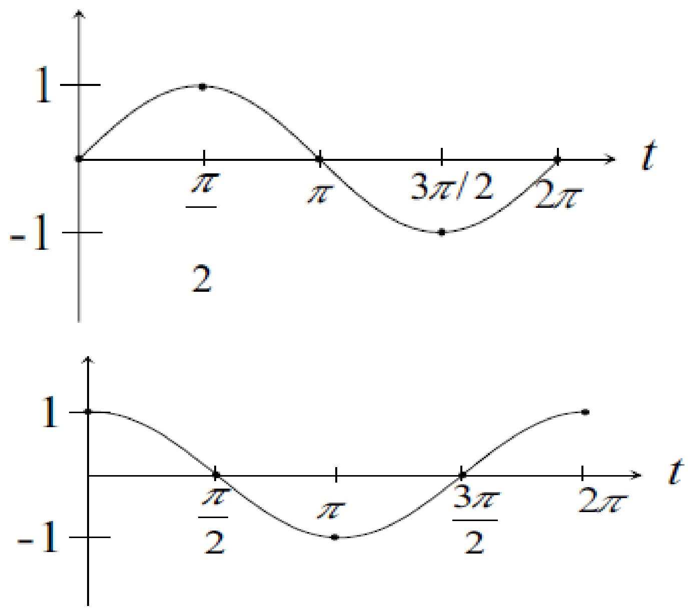


Figure 1.8 Differentiation of Sine – Cosine

Operations on independent variables

Time scaling

Time scaling operation is given by equation 1.26

y(f) = x(af) 1.26

This operation results in expansion in time for a<l and com pressionintime for a>1.as evident from the ex amp1es of figure 1.10.

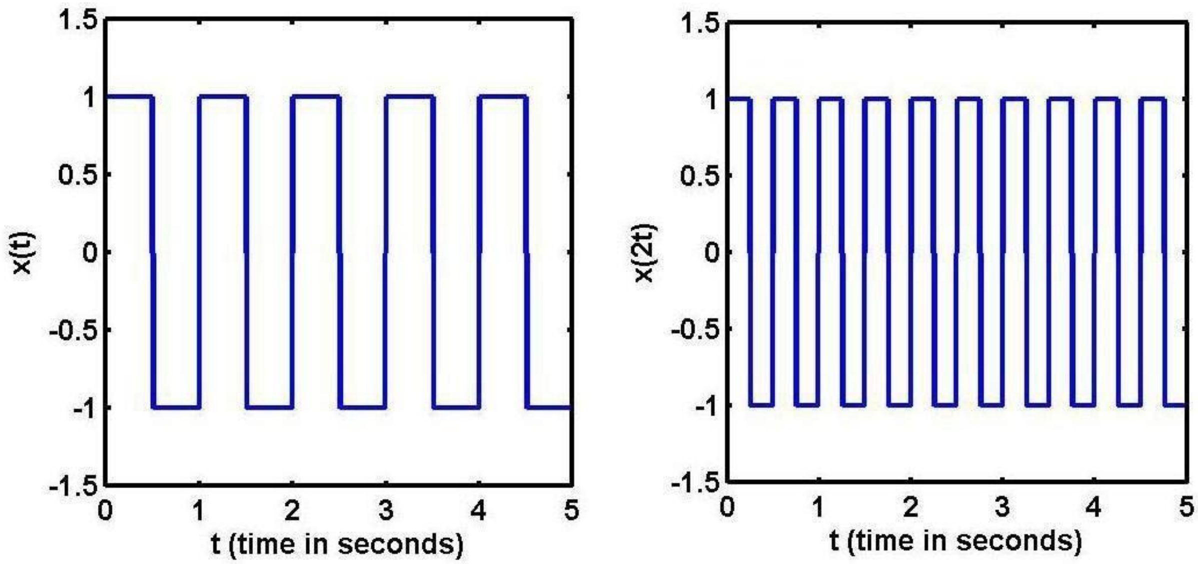


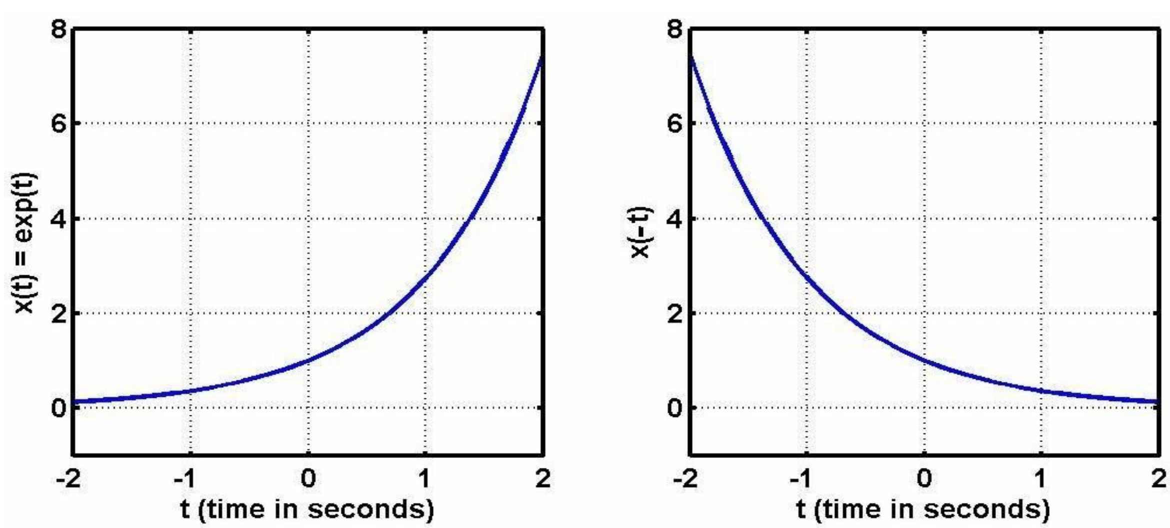
Figure 1.10 Examples of time scaling of a continuous time signal

An example of this operation is the compression or expansion of the time scale that results in the „*fast-forward'* or the „*slow motion'* in a video, provided we have the entire video in some stored form.

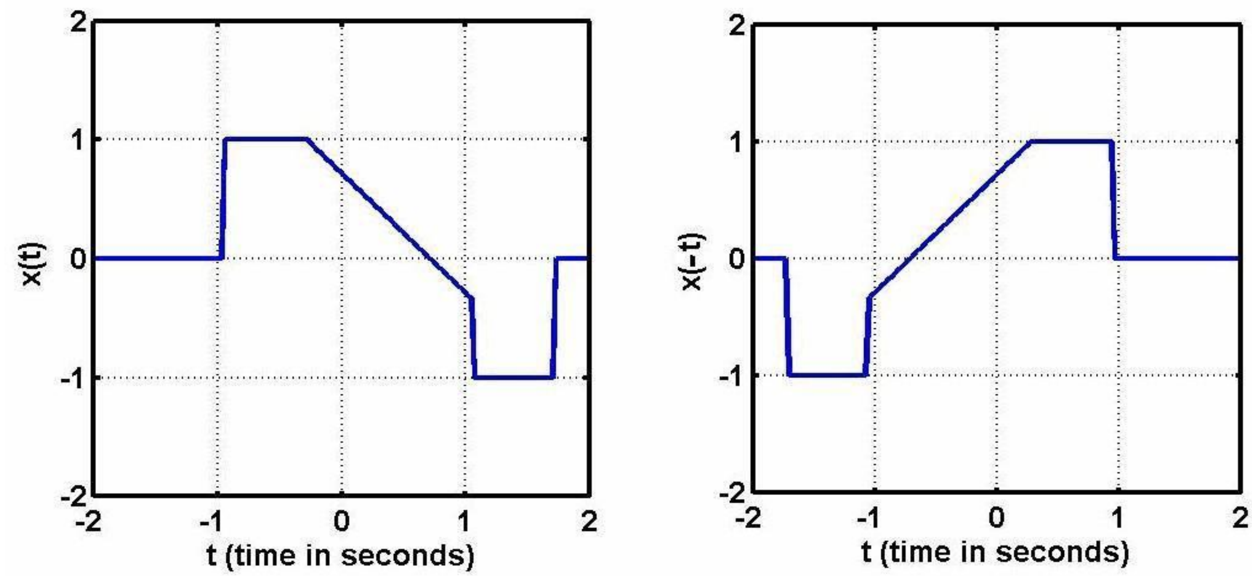
Time reflection

Time reflection is given by equation (1.27), and some examples are contained in fig1.11.

$y(t) = x(-t)$ 1.27



(a)



(b)

Figure 1.11 Examples of time reflection of a continuous time signal

Time shifting

The equation representing time shifting is given by equation (1.28), and examples of this operation are given in figure 1.12.

$y(t)$ $x(t- t_0)$..1.28

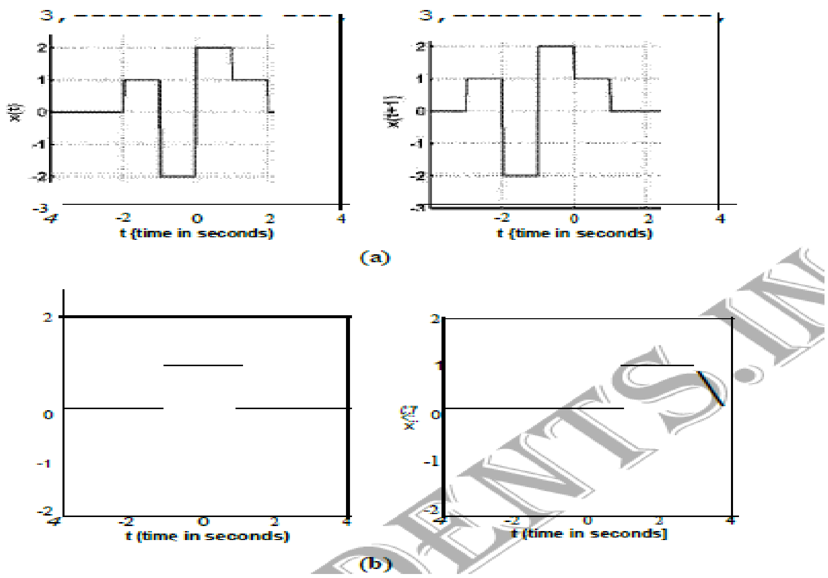
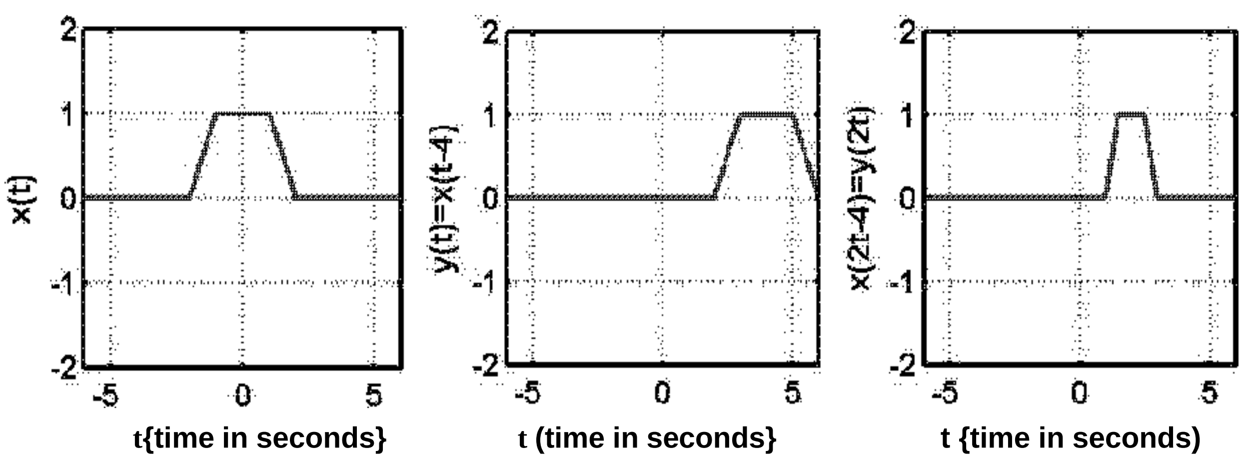


Figure 1.12 Examples of time shift of a continuous time signal

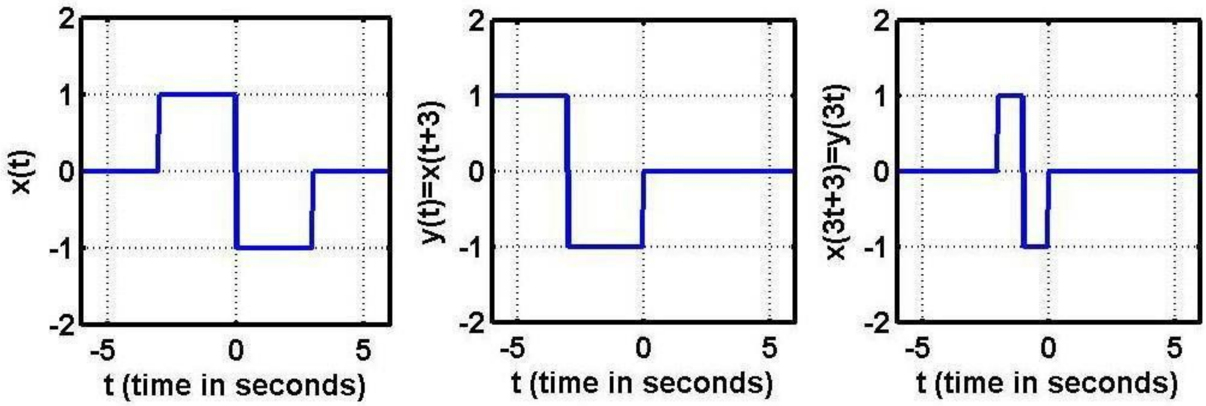
Time shifting and scaling

The combined transformation of shifting and scaling is contained in equation (1.29), along with examples in figure 1.13. Here, time shift has a higher precedence than time scale.

$y(t)\ x(at - t_0)$..1.29



(a)



(b)

Figure 1.13 Examples of simultaneous time shifting and scaling. The signal has to be shifted first and then time scaled.

Elementary signals

Exponential signals:

The exponential signal given by equation (1.29), is a monotonically increasing function if $a > 0$, and is a decreasing function if $a < 0$.

$$x(t) = e^{at} \qquad \textbf{……………………(1.29)}$$

It can be seen that, for an exponential signal,

$$x(t + a^{-1}) = e.x(t)$$

$$x(t - a^{-1}) = e^{-1}.x(t) \qquad \textbf{…………………..(1.30)}$$

Hence, equation (1.30), shows that change in time by $\pm 1/a$ seconds, results in change in magnitude by $e^{\pm 1}$. The term $1/a$ having units of time, is known as the time-constant. Let us consider a decaying exponential signal

$$x(t) = e^{-at} \qquad \textbf{……………(1.31)}$$

This signal has an initial value $x(0) = 1$, and a final value $x(\infty) = 0$. The magnitude of this signal at five times the time constant is,

$$x(5/a) = 6.7x10^{-3} \qquad \text{...............(1.32)}$$

while at ten times the time constant, it is as low as,

$$x(10/a) = 4.5x10^{-5} \qquad \text{...............(1.33)}$$

It can be seen that the value at ten times the time constant is almost zero, the final value of the signal. Hence, in most engineering applications, the exponential signal can be said to have reached its final value in about ten times the time constant. If the time constant is 1 second, then final value is achieved in 10 seconds!! We have some examples of the exponential signal in figure 1.14.

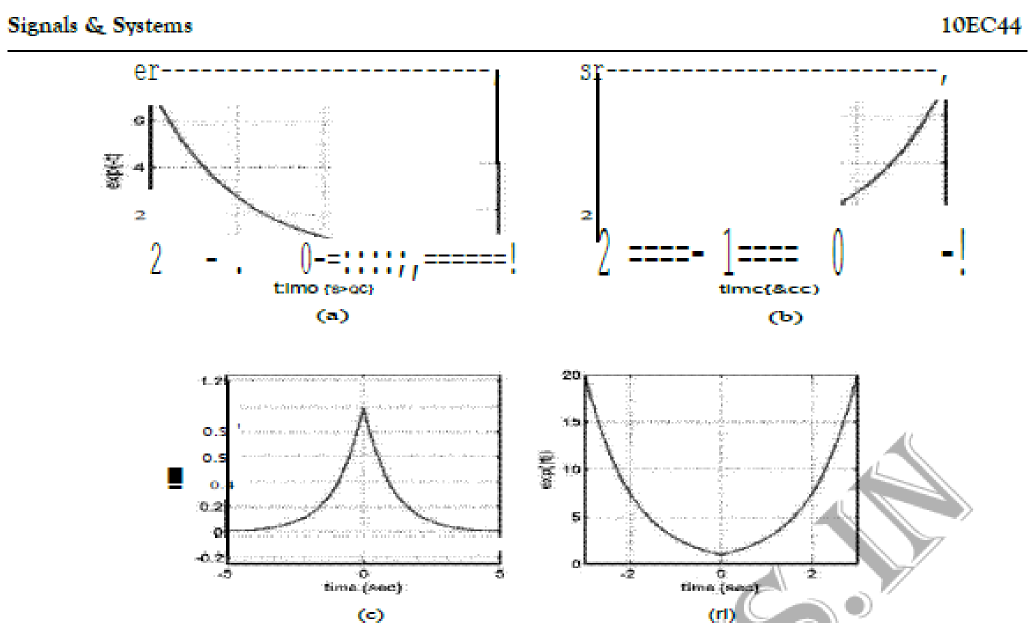


Fig 1.14 The continuous time exponential signal (a) e^{-t}, (b) e^{t}, (c) $e^{-|t|}$, and (d) $e^{|t|}$

The sinusoidal signal:

The sinusoidal continuous time periodic signal is given by equation 1.34, and examples are given in figure 1015

$$x(t) = A\sin(2\pi ft) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1.34)$$

The different parameters are: Angular frequency co *2nfin* radians,

Frequency *fin* Hertz, (cycles per second) Amplitude *A* in Volts (or Amperes) Period *Tin* seconds

The complex exponential:

We now represent the complex exponential using the Euler's identity

$$=(\cos\theta + j\sin\theta) \qquad \ldots\ldots\ldots\ldots(1.35)$$

to represent sinusoidal signals. We have the complex exponential signal given by equation (1.36)

$$e^{j\omega t} = \left(\cos(\omega t) + j\sin(\omega t)\right)$$
$$e^{-j\omega t} = \left(\cos(\omega t) - j\sin(\omega t)\right) \qquad \ldots\ldots\ldots(1.36)$$

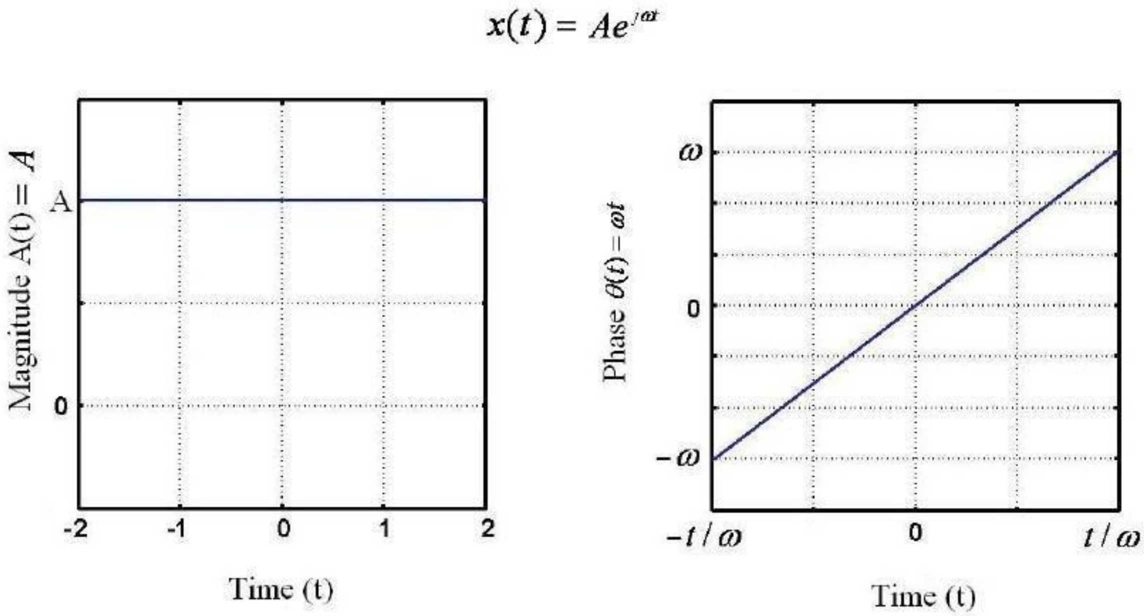
Since sine and cosine signals are periodic, the complex exponential is also periodic with the same period as sine or cosine. From equation (1.36), we can see that the real periodic sinusoidal signals can be expressed as:

Let us consider the signal $x(t)$ given by equation (1.38). The sketch of this is given in fig 1.15

$$\cos(\omega t) = \left(\frac{e^{j\omega t} + e^{-j\omega t}}{2}\right)$$

$$\sin(\omega t) = \left(\frac{e^{j\omega t} - e^{-j\omega t}}{2j}\right) \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1.37)$$

$$x(t) = A(t)e^{j\theta(t)} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1.38)$$

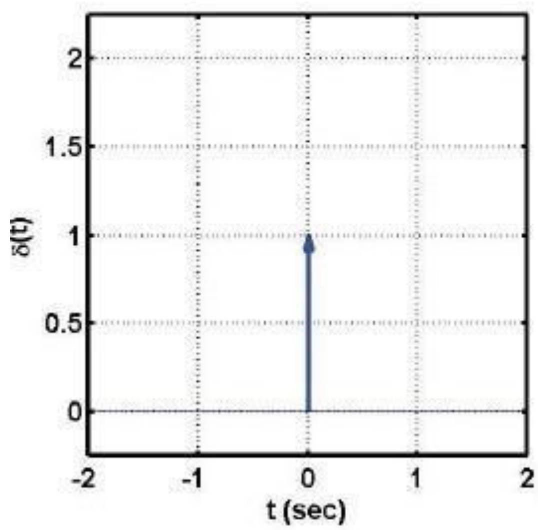


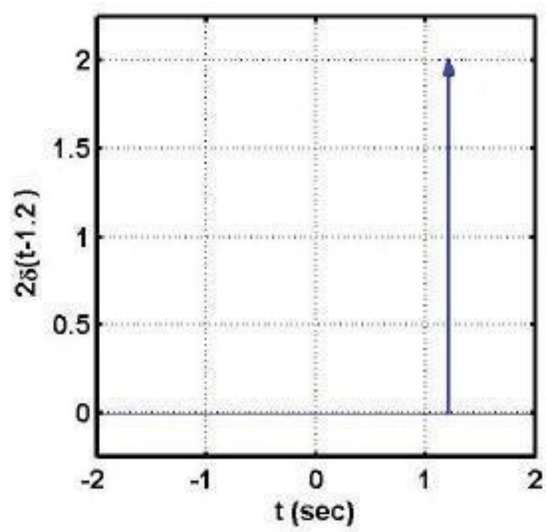
The unit impulse:

The unit impulse usually represented as $\delta(t)$, also known as the dirac delta function, is given by,

$$\delta(t) = 0 \quad for \quad t \neq 0; \quad and \quad \int_{-\infty}^{\infty} \delta(t)dt = 1 \qquad \text{.......(1.38)}$$

From equation (1.38), it can be seen that the impulse exists only at $t = 0$, such that its area is 1. This is a function which cannot be practically generated. Figure 1.16, has the plot of the impulse function



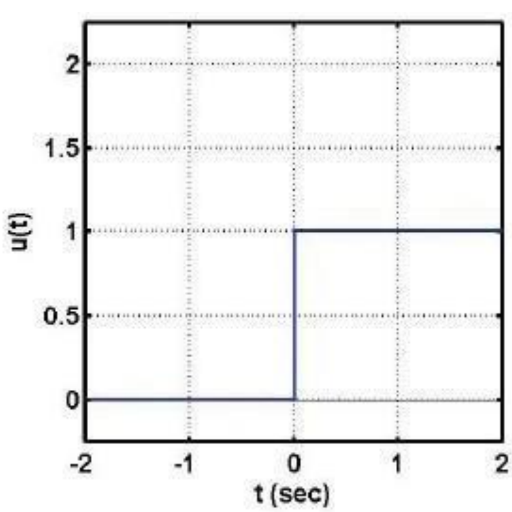


The unit step:

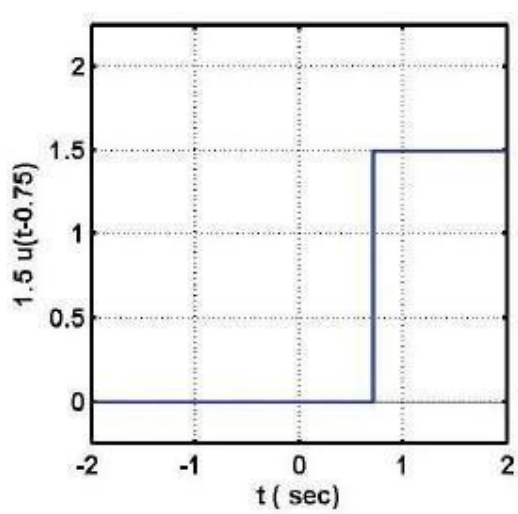
The unit step function, usually represented as $u(t)$, is given by,

$$u(t)=\begin{cases}1 & t\geq 0\\ 0 & t<0\end{cases}$$

……………….(1.39)



(a)



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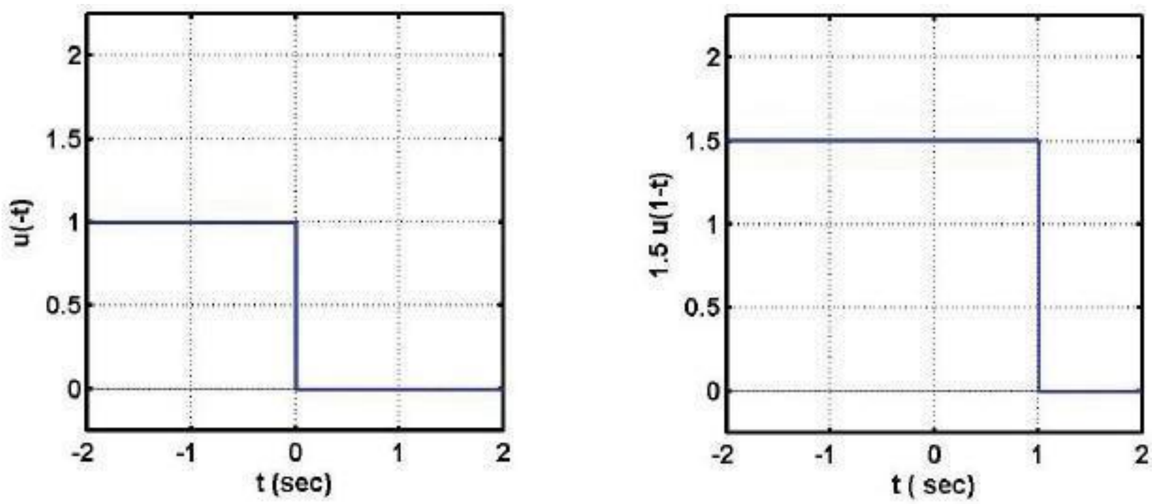


(c) (d)

Fig 1.17 Plot of the unit step function along with a few of its transformations

The unit ramp:

The unit ramp function, usually represented as $r(t)$, is given by,

$$r(t) = \begin{cases} t & t \geq 0 \\ 0 & t < 0 \end{cases}$$

……………(1.40)

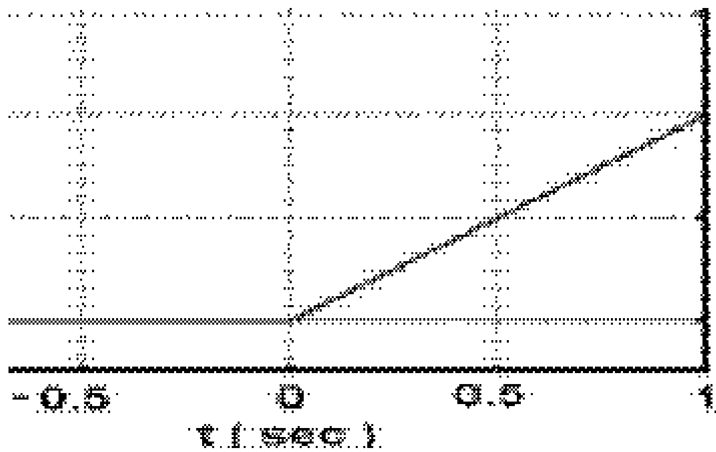


Figure1.18 Plot of the unit ramp function

The signum function:

The signum function, usually represented as sgn(t) , is given by

$$\text{sgn}(t) = \begin{cases} 1 & t \\ 0 & t- \\ 1 & t \end{cases}$$ ....(1.41)

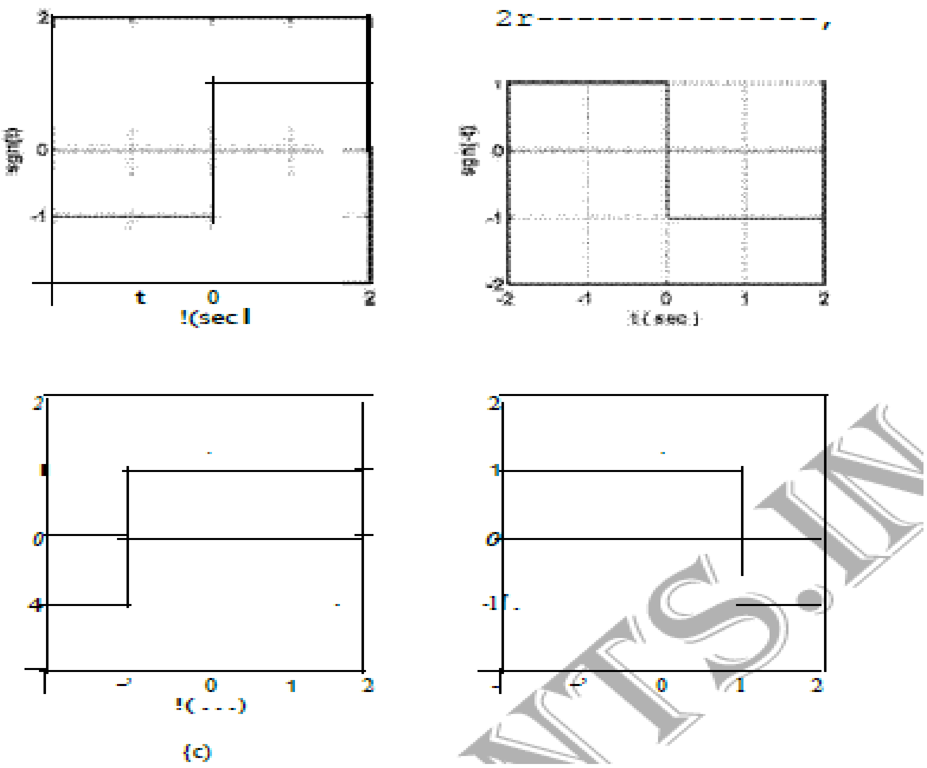


Figure 1.19 Plot of the unit signum function along with a few of its transformations

System viewed as interconnection of operation:

This article is dealt in detail again in chapter 2/3. This article basically deals with system connected in series or paralleL Further these systems are connected with adders/subtractor, multipliers etc.

Properties of system:

In this article discrete systems are taken into account. The same explanation stands for continuous time systems also.

The discrete time system:

The discrete time system is a device which accepts a discrete time signal as its input, transforms it to another desirable discrete time signal at its output.

Stability

A system is stable if 'bounded input results in a bounded output'. This condition, denoted by BIBO, Hence, a finite input should produce a finite output, if the system is stable. Some examples of stable and unstable systems are given in figure 1.21

Memory

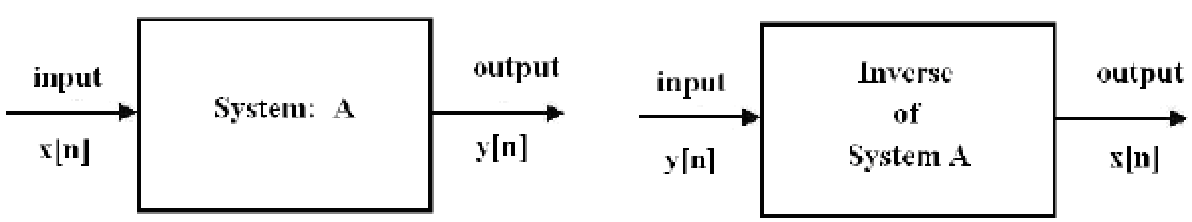
The system is memory-less if its instantaneous output depends only on the current input. In memory-less systems, the output does not depend on the previous or the future input

Causality:

A system is causal, if its output at any instant depends on the current and past values of input. The output of a causal system does not depend on the future values of input.

Invertibility:

A system is invertible if,



Linearity:

The system is a device which accepts a signal, transforms it to another desirable signal, and is available at its output. We give the signal to the system, because the output is s

Time invariance:

A system is time invariant, if its output depends on the input applied, and not on the time of application of the input. Hence, time invariant systems, give delayed outputs for delayed inputs.

UNIT 2

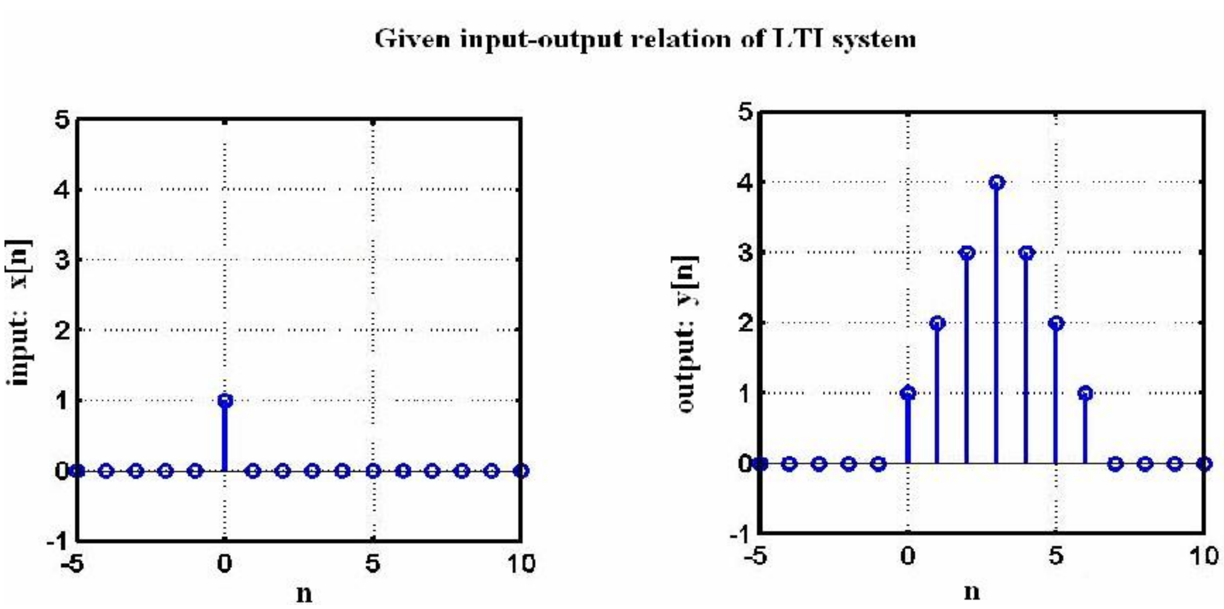
Time-domain representations for LTI systems – 1

Learning Objectives: Time-domain representations for LTI systems – 1: Convolution, impulse response representation, Convolution Sum and Convolution Integral.

Introduction:

The Linear time invariant (LTI) system:

Systems which satisfy the condition of linearity as well as time invariance are known as linear time invariant systems. Throughout the rest of the course we shall be dealing with LTI systems. If the output of the system is known for a particular input, it is possible to obtain the output for a number of other inputs. We shall see through examples, the procedure to compute the output from a given input-output relation, for LTI systems.



Convolution:

A continuous time system as shown below, accepts a continuous time signal x(t) and gives out a transformed continuous time signal y(t).

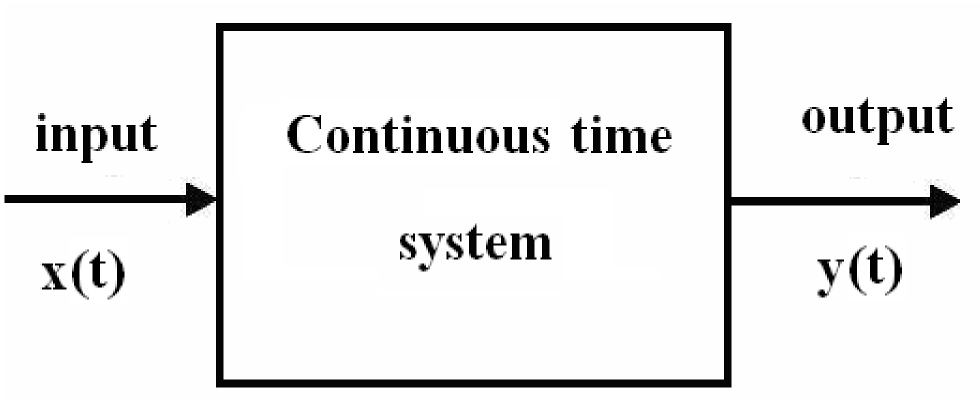


Figure 1: The continuous time system

Some of the different methods of representing the continuous time system are:

- i) Differential equation
- ii) Block diagram
- iii) Impulse response
- iv) Frequency response
- v) Laplace-transform
- vi) Pole-zero plot

It is possible to switch from one form of representation to another, and each of the representations is complete. Moreover, from each of the above representations, it is possible to obtain the system properties using parameters as: stability, causality, linearity, invertibility etc. We now attempt to develop the convolution integral.

Impulse Response

The impulse response of a continuous time system is defined as the output of the system when its input is an unit impulse, $\delta(t)$. Usually the impulse response is denoted by $h(t)$.

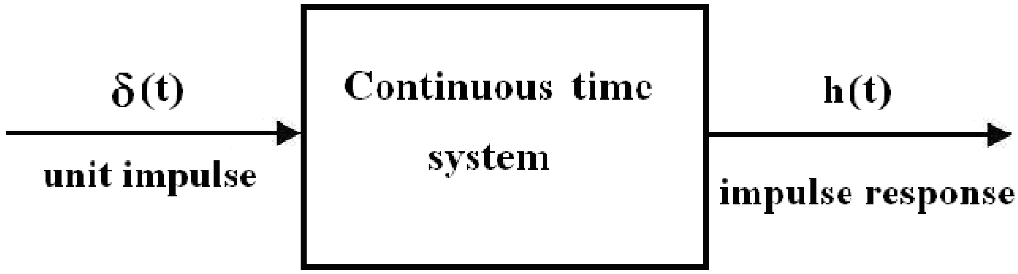
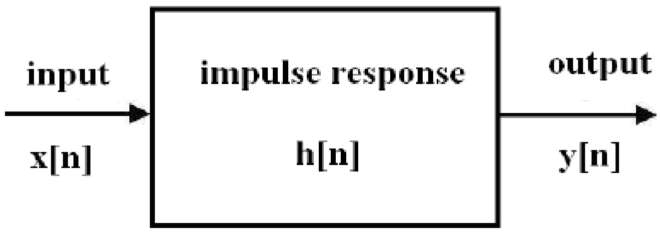


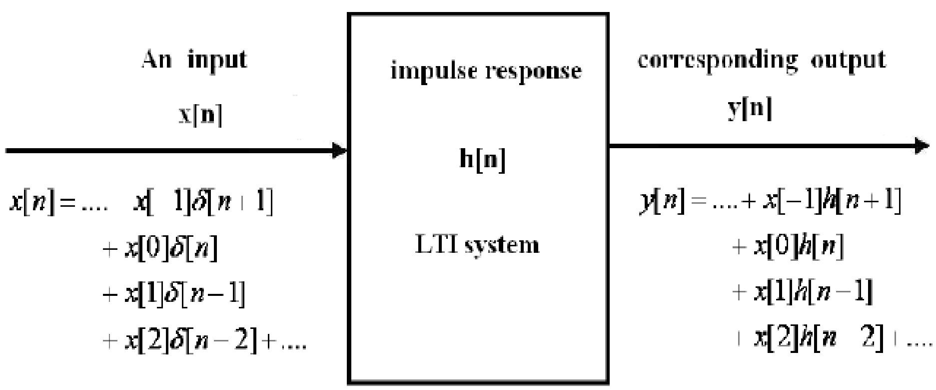
Figure 2: The impulse response of a continuous time system

Convolution Sum:

We now attempt to obtain the output of a digital system for an arbitrary in input x[n], from the knowledge of the system impulse response h[n].



y[n] = x[n]* h[n]



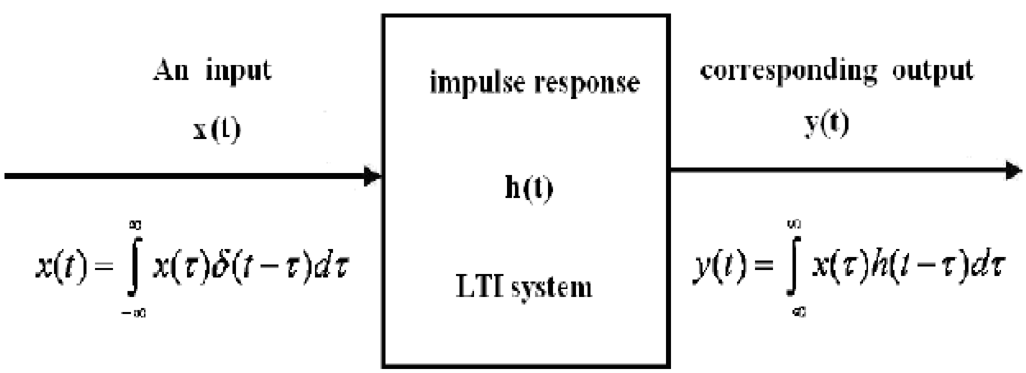
$$x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

Convolution Integral:

We now attempt to obtain the output of a continuous time/Analog digital system for an arbitrary input x(t), from the knowledge of the system impulse response h(t), and the properties of the impulse response of an LTI system.

The output y(t) is given by, using the notation, y(t)=R{x(t)}.

$$
\begin{aligned}
y(t) &= R\{x(t)\} \\
&= R\left\{\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau\right\} \\
&= \int_{-\infty}^{\infty} x(\tau)R\{\delta(t-\tau)\}d\tau \\
&= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \\
&= x(t) * h(t)
\end{aligned}
$$



Methods of evaluating the convolution integral: (Same as Convolution sum)

Given the system impulse response h(t), and the input x(t), the system output y(t), is given by the convolution integral:

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

Some of the different methods of evaluating the convolution integral are: Graphical representation, Mathematical equation, Laplace-transforms, Fourier Transform, Differential equation, Block diagram representation, and finally by going to the digital domain.

Unit 3

Time-Domain Representations For LTI Systems – 2

Learning Objectives: **Time-domain representations for LTI systems – 2:** Properties of impulse response representation, Differential and difference equation Representations, Block diagram representations.

Time domain representation of LTI Systems

- Impulse response: characterizes the behavior of any LTI system
- Linear constant coefficient differential or difference equation: input output behavior
- Block diagram: as an interconnection of three elementary operations

Differential and Difference equation

- General form of differential equation is

$$\sum_{k=0}^{N} a_k \frac{d^k}{dt^k} y(t) = \sum_{k=0}^{M} b_k \frac{d^k}{dt^k} x(t)$$

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

$$Ry(t) + L\frac{d}{dt}y(t) + \frac{1}{C}\int_{-\infty}^{t} y(\tau)d\tau = x(t)$$

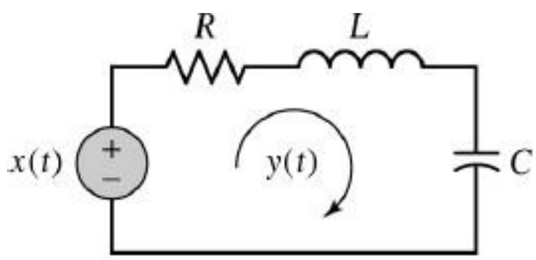


Figure 1.1: RLC circuit

$$y[n] + \sum_{k=1}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k], \qquad n = 0, 1, 2, \ldots$$

$$y[n] + \sum_{k=1}^{N} a_k y[n-k] = 0 \qquad n = 0, 1, 2, \ldots$$

$$y[n] = \sum_{k=0}^{M} b_k x[n-k] \qquad n = 0, 1, 2, \ldots$$

$$y[n] = -\sum_{k=1}^{N} a_k y[n-k] + \sum_{k=0}^{M} b_k x[n-k] \qquad n = 0, 1, 2, \ldots$$

$$y[0] = \underbrace{-\sum_{k=1}^{N} a_k y[-k]}_{depends\ on\ ICs} + \underbrace{\sum_{k=0}^{M} b_k x[-k]}_{depends\ on\ input\ x[0]\to x[-M]}$$

$$y[1] = \sum_{k=1}^{N} a_k y[1-k] + \sum_{k=0}^{M} b_k x[1-k]$$

$$y[1] = -a_1 y[0] - \underbrace{\sum_{k=1}^{N-1} a_{k+1} y[-k]}_{depends\ on\ ICs} + \underbrace{\sum_{k=0}^{M} b_k x[1-k]}_{depends\ on\ input x[1]...x[1-M]}$$

$$y[2] = \sum_{k=1}^{N} a_k y[2-k] + \sum_{k=0}^{M} b_k x[2-k]$$

$$y[2] = -a_1 y[1] - a_2 y[0] - \underbrace{\sum_{k=1}^{N-1} a_{k+1} y[-k]}_{depends\ on\ ICs} + \underbrace{\sum_{k=0}^{M} b_k x[1-k]}_{depends\ on\ input\ x[2]...x[2-M]}$$

$$Ry(t) + L\frac{d}{dt}y(t) + \frac{1}{C}\int_{-\infty}^{t} y(\tau)d\tau = x(t)$$

$$\frac{1}{C}y(t) + R\frac{d}{dt}y(t) + L\frac{d^2}{dt^2}y(t) = \frac{d}{dt}x(t)$$

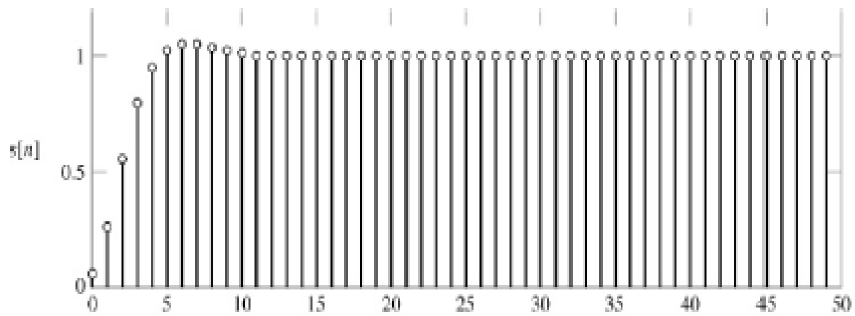
$$y[n] + y[n-1] + \frac{1}{4}y[n-2] = x[n] + 2x[n-1]$$

$$y[-N], y[-N+1], \ldots, y[-1]$$

$$y(t)|_{t=0}, \frac{d}{dt}y(t)|_{t=0}, \frac{d^2}{dt^2}y(t)|_{t=0}, \ldots \frac{d^{N-1}}{dt^{N-1}}y(t)|_{t=0}$$

$$\begin{aligned}
y[0] &= -ay[-1] + x[0] \\
y[1] &= -ay[0] + x[1] \\
&= -a(-ay[-1] + x[0]) + x[1] \\
&= a^2 y[-1] - ax[0]) + x[1] \\
y[2] &= -ay[1] + x[2] \\
&= -a(-a^2 y[-1] - ax[0] + x[1]) + x[2] \\
&= a^3 y[-1] + a^2 x[0] - ax[1] + x[2]
\end{aligned}$$

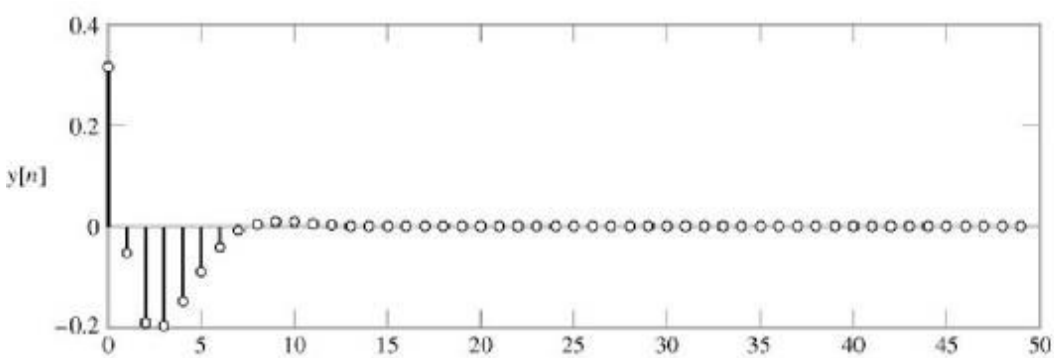
$$y[n] = (-a)^{n+1}y[-1] + \sum_{i=0}^{n}(-a)^{n-i}x[i], \qquad n = 0, 1, 2, \ldots$$



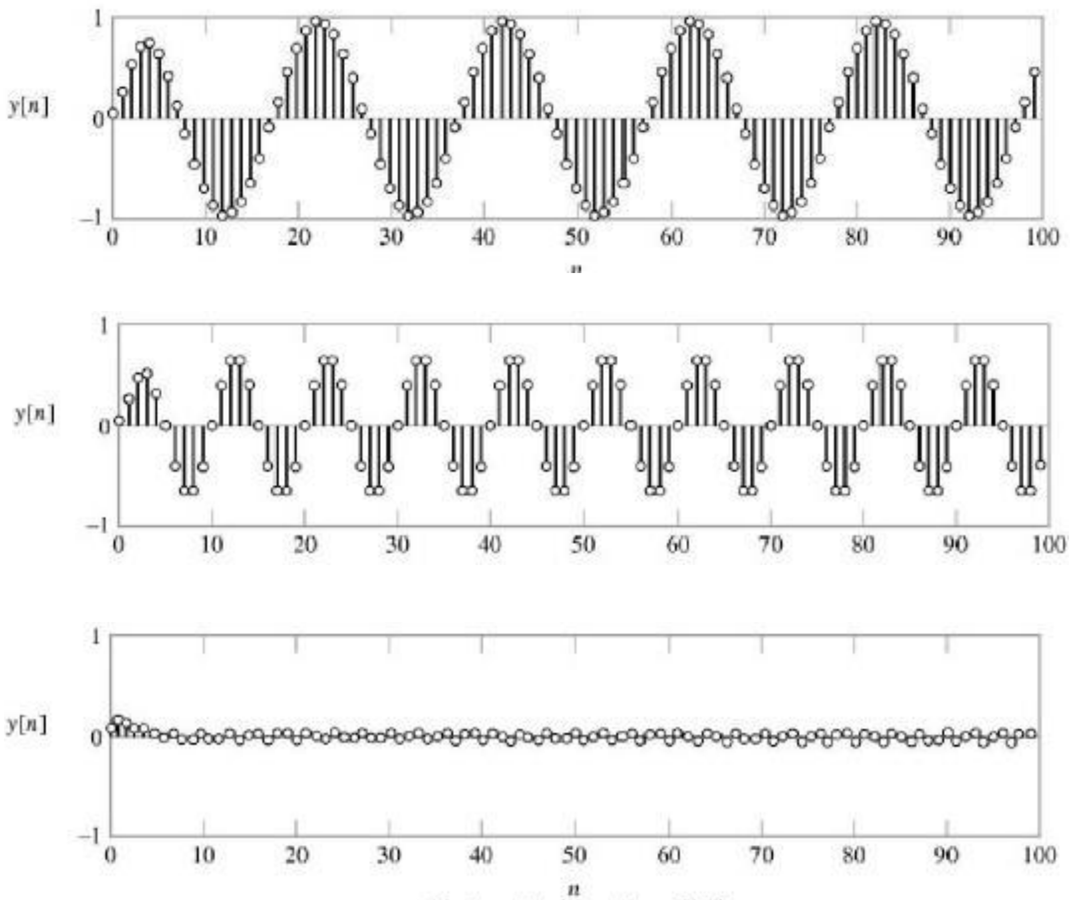
$$y[n] = -\sum_{i=1}^{N} a_i y[n-i] + \sum_{i=0}^{M} b_i x[n-i], \qquad n = 0, 1, 2, \ldots$$

Example of difference equation

- example: A system is described by $y[n] - 1.143y[n-1] + 0.4128y[n-2] = 0.0675x[n] + 0.1349x[n-1] + 0.675x[n-2]$
- Rewrite the equation as $y[n] = 1.143y[n-1] - 0.4128y[n-2] + 0.0675x[n] + 0.1349x[n-1] + 0.675x[n-2]$



Output due to initial condition with zero input



Output due to x[n]

Block diagram representations

· A block diagram is an interconnection of elementary operations that act on the input signal

· This method is more detailed representation of the system than impulse response or differential/difference equation representations

· The impulse response and differential/difference equation descriptions represent only the input-output behavior of a system, block diagram representation describes how the operations are ordered

· Each block diagram representation describes a different set of internal computations used to determine the system output

·Block diagram consists of three elementary operations on the signals:

– Scalar multiplication: $y(t) = cx(t)$ or $y[n] = x[n]$, where c is a
scalar

– Addition: $y(t) = x(t)+w(t)$ or $y[n] = x[n]+w[n]$.

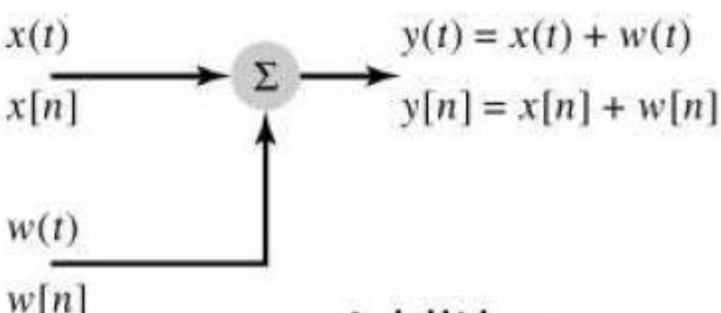
·Block diagram consists of three elementary operations on the signals

Integration for continuous time LTI system: $y(t) = \int_{-\infty}^{t} x(\tau)d\tau$

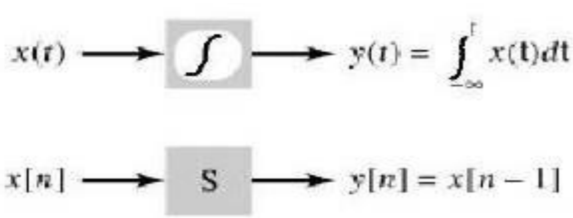
Time shift for discrete time LTI system: $y[n] = x[n-1]$

·Scalar multiplication: $y(t) = cx(t)$ or $y[n] = x[n]$, where c is a scalar

Scalar Multiplication

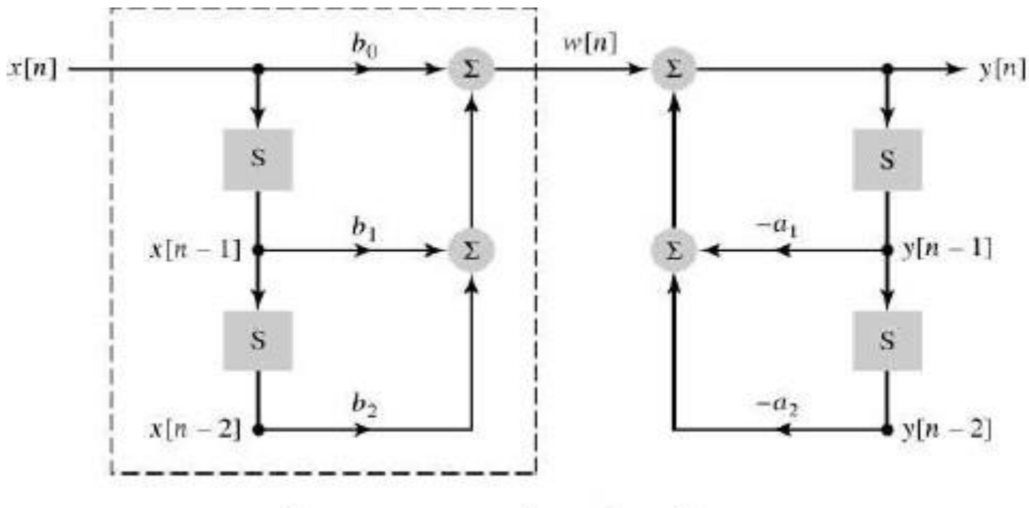


Addition



Integration and timeshifting

Direct form1



Example 1: Direct form I

Examples

Example 1

·Consider the system described by the block diagram as in Figure 1.10

·Consider the part within the dashed box

·The input $x[n]$ is time shifted by 1 to get $x[n-1]$ and again time shifted

by one to get $x[n-2]$. The scalar multiplications are carried out and they are added to get $w[n]$ and is given by

$w[n] = b0x[n]+b1x[n-1]+b2x[n-2]$.

·Write $y[n]$ in terms of $w[n]$ as input $y[n]=w[n]-a1y[n-1]-a2y[n-2]$

·Put the value of $w[n]$ and we get $y[n]=-a1y[n-1]-a2y[n-2]+b0x[n]$

$+b1x[n-1]+b2x[n-2]$

and $y[n]+a1y[n-1]+a2y[n-2]= b0x[n]+b1x[n-1]+b2x[n-2]$

·The block diagram represents an LTI system

Example 2

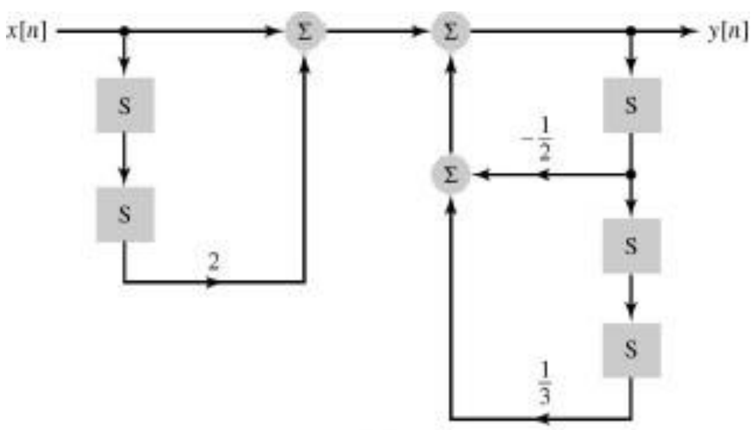
·Consider the system described by the block diagram and its difference

equation is $y[n]+(1/2)y[n-1]-(1/3)y[n-3]= x[n]+2x[n-2]$

Example 3

·Consider the system described by the block diagram and its difference

equation is $y[n]+(1/2)y[n-1]+(1/4)y[n-2]= x[n-1]$



(a)

Block diagram representation is not unique, direct form II structure of Example 1

·We can change the order without changing the input output behavior Let the output of a new system be $f[n]$ and given input $x[n]$ are related by

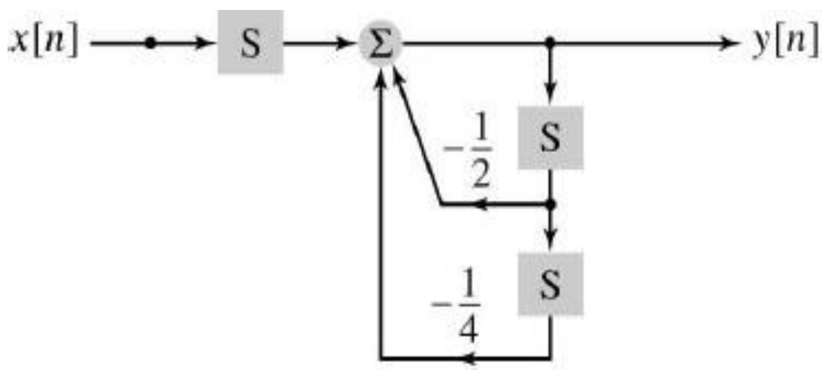
$f[n] = -a1\, f[n-1]-a2\, f[n-2]+x[n]$

·The signal $f[n]$ acts as an input to the second system and output of second system is

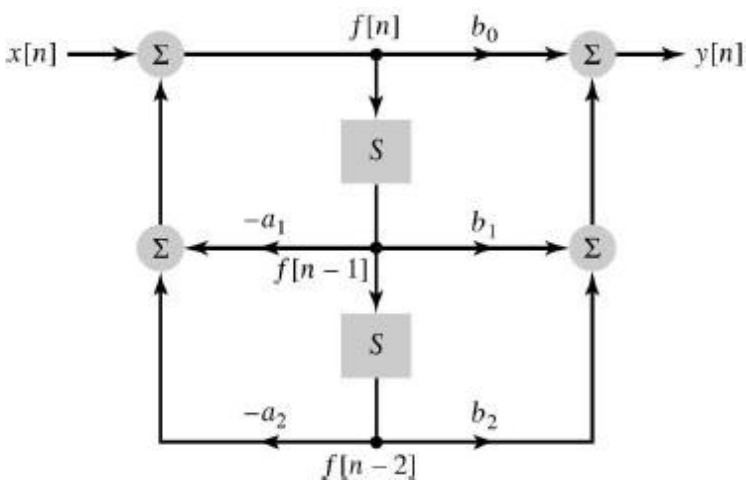
$y[n] = b0\, f[n]+b1\, f[n-1]+b2\, f[n-2]$.

·The block diagram representation of an LTI system is not unique

Example 2: Direct form I



Direct form II



UNIT 4

Fourier Representation for Signals – 1

Learning Objectives: Introduction, Discrete time and continuous time Fourier series (derivation of series excluded) and their properties .

INTRODUCTION

In 1807, Jean Baptiste Joseph Fourier Submitted a paper of using trigonometric series to represent "any" periodic signal. But Lagrange rejected it!

• In 1822, Fourier published a book "The Analytical Theory of Heat"

• Fourier's main contributions: Studied vibration, heat diffusion, etc. and found that a series of harmonically related sinusoids is useful in representing the temperature distribution through a body.

• He also claimed that "any" periodic signal could be represented by Fourier series.

• These arguments were still imprecise and it remained for P.L.Dirichlet in 1829 to provide precise conditions under which a periodic signal could be represented by a FS.

• He however obtained a representation for aperiodic signals i.e., Fourier integral or transform

• Fourier did not actually contribute to the mathematical theory of Fourier series.

• Hence out of this long history what emerged is a powerful and cohesive framework for the analysis of continuous- time and discrete-time signals and systems

• And an extraordinarily broad array of existing and potential application.

• Let us see how this basic tool was developed and some important Applications

Key Properties: for Input to LTI System

1. To represent signals as linear combinations of basic signals.

2. Set of basic signals used to construct a broad class of signals.

3. The response of an LTI system to each signal should be simple enough in structure.

4. It then provides us with a convenient representation for the response of the system.

5. Response is then a linear combination of basic signal

Eigenfunctions and Values

• One of the reasons the Fourier series is so important is that it represents a signal in terms of eigenfunctions of LTI systems.

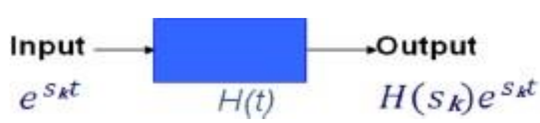
• When I put a complex exponential function like $x(t) = e^{j\omega t}$ through a linear time-invariant system, the output is $y(t) = H(s)x(t) = H(s)\, e^{j\omega t}$ where $H(s)$ is a complex constant (it does not depend on time).

• The LTI system scales the complex exponential $e^{j\omega t}$

The Response of LTI Systems to Complex Exponentials

Let us analyse how an LTI system responds to complex signals

The Response of an LTI System:



$$y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau$$

For CT (Continuous Times) and DT (Discrete Times) we can say that

$$\mathrm{CT}: e^{st} \rightarrow H(s)e^{st} \longrightarrow \textbf{Eigenfunction}$$
$$\mathrm{DT}: z^{n} \rightarrow H(z)z^{n} \longrightarrow \textbf{Eigenvalue}$$

Where the complex amplitude factor H(s), H(z) is called the frequency response of the system. The complex exponential $e^{st}$is called an eigenfunctionof the system, as the output is of the same form, differing by a scaling factor.

The Response of LTI Systems to Complex Exponentials We know for LTI System Output and for CT Signals,

Let $x(t) = e^{st}$, $y(t) = \int_{-\infty}^{+\infty} h(\tau)x(t-\tau)d\tau$

$$= \int_{-\infty}^{+\infty} h(\tau)e^{s(t-\tau)}d\tau = e^{st}\int_{-\infty}^{+\infty} h(\tau)e^{s(\tau)}d\tau$$

$y(t) = H(s)x(t) = H(s)e^{st}$ where

$$H(s) = \int_{-\infty}^{+\infty} h(\tau)e^{s(t)}d\tau$$

For Discrete time signals $\quad Let\ x[n] = z^n$

We know for LTI System Output $\quad y[n] = \sum_{k=-\infty}^{+\infty} h[k]x[n-k]$

$$= \sum_{k=-\infty}^{+\infty} h[k]z^{n-k} = z^n \sum_{k=-\infty}^{+\infty} h[k]z^{-k}$$

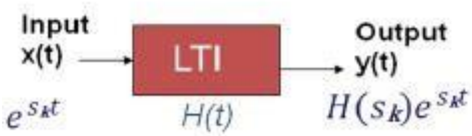
$$y[n] = H(z)x[n] = H(z)z^n$$

Eigenvalue

Eigenfunction

$$where\ \ H(z) = \sum_{k=-\infty}^{+\infty} h[k]z^{-k}$$

Eigenfunction and Superposition Properties



$$a_1 e^{s_1 t} \rightarrow a_1 H(s_1)e^{s_1 t}$$
$$a_2 e^{s_2 t} \rightarrow a_2 H(s_2)e^{s_2 t}$$
$$a_3 e^{s_3 t} \rightarrow a_3 H(s_3)e^{s_3 t}$$

$$x(t) = a_1 e^{s_1 t} + a_2 e^{s_2 t} + a_3 e^{s_3 t}$$

$$y(t) = a_1 H(s_1)e^{s_1 t} + a_2 H(s_2)e^{s_2 t} + a_3 H(s_3)e^{s_3 t}$$

$$\rightarrow x(t) = \sum_k a_k e^{s_k t} \rightarrow \mathrm{y(t)} = \sum_k a_k H(s_k)e^{s_k t}$$

$$\rightarrow x[n] = \sum_k a_k {z_k}^n \rightarrow \mathrm{y}[n] = \sum_k a_k H(z_k){z_k}^n$$

Hence for both continuous time and discrete time, each coefficient in this representation of the output is obtained as the product of the corresponding coefficient a_k of the input and the system's eigenvalue or $H(z_k)$ associated with the eigenfunction $e^{s_k t}$ or ${z_k}^n$ respectively.

• Each system has its own constant H(s) that describes how it scales eigenfunctions. It is called the frequency response.

• The frequency response H(ω)=H(s) does not depend on the input.

• If we know H(ω), it is easy to find the output when the input is an eigenfunction. y(t)=H(ω)x(t) true when x is eigenfunction.

• So, given the system response to an eigenfunction, H(s), we can compute the magnitude response |H(s)| and the phase response H(s).

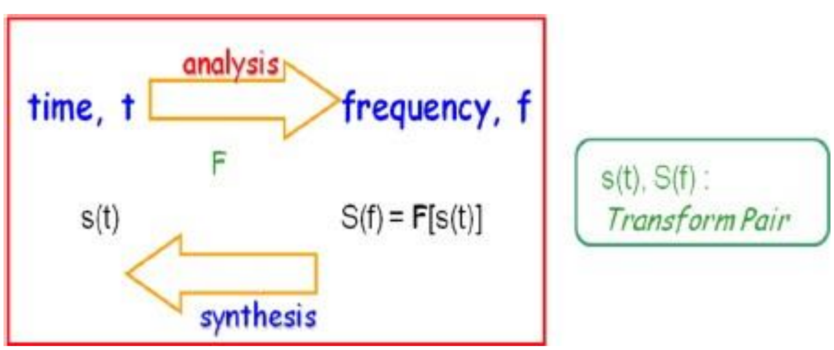
• These form the scaling factor and phase shift in the output, respectively.

• The frequency of the output sinusoid will be the same as the frequency of the input sinusoid in any LTI system.

• The LTI system scales and shifts sinusoids for both continuous and discrete signals and systems.

Need for Frequency Analysis

- Fast & efficient insight on signal's building blocks.
- Simplifies original problem - ex.: solving Part. Diff. Eqns.
- Powerful & complementary to time domain analysis techniques.
- Several transforms in DSPing: Fourier, Laplace, z, etc.



Fourier Analysis : T

The following are its Applications Telecommunication- GSM/cellular phones, Electronics/IT - most DSP-based applications, Entertainment - music, audio, multimedia, Accelerator control (tune measurement for beam steering/control), Imaging, image processing, Industry/research - X-ray spectrometry, chemical analysis (FT spectrometry), PDE solution, radar design, Medical - (PET scanner, CAT scans & MRI interpretation for sleep disorder & heart malfunction diagnosis, Speech analysis (voice activated "devices", biometry, …).

Orthogonality of the Complex exponentials

Definition : Two signals are orthogonal if their inner product is zero. The inner product is defined using complex conjugation when the signals are complex valued. For continuous-time signals with period T, the inner product is defined in terms of an integral as

$$I_{k,m} = \int_{(T)} \emptyset_k(t)\emptyset_m^*(t)\ dt$$

For discrete-time signals with period N, their inner product is defined as

$$I_{k,m} = \sum_{n=\langle N\rangle} \emptyset_k(n)\emptyset_m^*(n)$$

Orthogonality of the Complex exponentials

- For continuous-time signals with period T, the complex exponentials $e^{jn\omega_o t}$ must satisfy the orthogonality condition

$$\frac{1}{T}\int_{-\frac{T}{2}}^{+\frac{T}{2}} e^{jn\omega_o t}\; e^{jm\omega_o t*}\, dt \quad \begin{cases} 0 \text{ if } n\neq m \\ 1 \text{ if } n=m \end{cases}$$

Where $T = \frac{2\pi}{\omega_o}$

and f*(t) denotes the complex conjugate of f(t)

- **Theorem :**

$$\frac{1}{T}\int_{-\frac{T}{2}}^{+\frac{T}{2}} e^{jn\omega_o t}\; e^{jm\omega_o t*}\, dt \quad \frac{1}{T}\int_{-\frac{T}{2}}^{+\frac{T}{2}} e^{jn\omega_o t}\; \bar{e}^{jm\omega_o t} dt$$

$$= \frac{1}{T}\int_{-\frac{T}{2}}^{+\frac{T}{2}} e^{j(n-m)\omega_o t}$$

When n≠m, let n-m=p. Then

$$\frac{1}{T}\int_{-\frac{T}{2}}^{+\frac{T}{2}} e^{jn\omega_o t}\; e^{jm\omega_o t*}\, dt \quad \frac{1}{T}\int_{-\frac{T}{2}}^{+\frac{T}{2}} e^{jp\omega_o t}\, dt$$

$$= \frac{e^{jp\omega_o t}}{Tjp\omega_o}\Bigg|_{-\frac{T}{2}}^{+\frac{T}{2}} \quad \frac{e^{jp\pi} - e^{-jp\pi}}{Tjp\omega_o} = 0$$

On the other hand when n=m, then equation becomes

$$\frac{1}{T}\int_{-\frac{T}{2}}^{+\frac{T}{2}} e^{jn\omega_o t}\; e^{jm\omega_o t*}\, dt = \frac{1}{T}\int_{-\frac{T}{2}}^{+\frac{T}{2}} e^{0}\, dt = 1$$

Hence the theorem may be stated in general

$$\frac{1}{T}\int_{-\frac{T}{2}}^{+\frac{T}{2}} e^{jn\omega_o t}\; e^{jm\omega_o t*}\, dt \quad \begin{cases} 0 \text{ if } n\neq m \\ 1 \text{ if } n=m \end{cases}$$

Harmonically Related Complex Exponentials

$$\emptyset(t) = e^{jk\omega_o t} = e^{jk\left(\frac{2\pi}{T}\right)t}, k = 0, \pm 1, \pm 2 \ldots.. \quad \text{where } \omega_o = \left(\frac{2\pi}{T}\right)$$

Fourier Series Representation

$$x(t) = \sum_{k=-\infty}^{+\infty} ak\ \emptyset(t) = \sum_{k=-\infty}^{+\infty} ak\ e^{jk\omega_o t}$$

$$= \sum_{k=-\infty}^{+\infty} ak\, e^{jk\left(\frac{2\pi}{T}\right)t}$$

Fourier Series representation

Where, k=+1,-1; the first harmonic components or the fundamental Component and k=+2,-2; the second harmonic components or the fundamental Component ….. etc.

Convergence for Fourier Fourier maintained that "any" periodic signal could be represented by a Fourier series The truth is that Fourier series can be used to represent an extremely large class of periodic signals. The question is that When a periodic signal x(t) does in fact have a Fourier series representation? Convergence One class of periodic signals: Which have finite energy over a single period. One class of periodic signals: Which have finite energy over a single period. The other class of periodic signals which satisfy Dirichlet conditions. Dirichlets Condition Condition 1: Krupa Over any period, x(t) must be absolutely integrable, i.e each coefficient is to be finite. Condition 2: In any finite interval, x(t) is of bounded variation; i.e., – There are no more than a finite number of maxima and minima during any single period of the signal Condition 3: In any finite interval, x(t) has only finite number of discontinuities. Furthermore, each of these discontinuities is finite.

Gibbs phenomenon:

When a sudden change of amplitude occurs in a signal and the attempt is made to represent it by a finite number of terms (N) in a Fourier series, the overshoot at the corners (at the points of abrupt change) is always found. As the number of terms is increased, the overshoot is still found; this is called the Gibbs phenomenon.

Properties of Fourier Representation The following are the Properties for the fourier Series

1. Linearity Properties
2. Translation or Time Shift Properties
3. Frequency Shift Properties
4. Scaling Properties
5. Time Differentiation
6. Time Domain Convolution
7. Modulation or Multiplication theorem
8. Parsevals Relationships

1) Linearity Properties

- Linearity: $$a_k = \frac{1}{T}\int_T x(t)e^{-jkw_0t}dt$$
  - $x(t), y(t)$: periodic signals with period T

$$x(t) \overset{\mathcal{FS}}{\longleftrightarrow} a_k \qquad x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jkw_0t}$$

$$y(t) \overset{\mathcal{FS}}{\longleftrightarrow} b_k \qquad y(t) = \sum_{m=-\infty}^{+\infty} b_m e^{jmw_0t}$$

$$\Rightarrow \; z(t) = Ax(t)+By(t) \overset{\mathcal{FS}}{\longleftrightarrow} c_k = Aa_k+Bb_k$$

$$z(t) = \sum_{k=-\infty}^{+\infty} c_k e^{jkw_0t}$$

The Fourier series coefficient ck are given by the same linear combination of FS coefficients for x(t) and y(t)

2) Frequency Shift Properties :

In other words frequency shift applied to a continuous-time signal results in a time shift of the corresponding sequence of Fourier series coefficients

$x(t) \xleftrightarrow{\mathcal{FS}} a_k$ Let $z(t) = e^{-jk_0 w_0 t} x(t)$

Then $z(t) \xleftrightarrow{\mathcal{FS}} z_k = a_{k-k_0}$

Proof: $$z_k = \frac{1}{T}\int_T z(t)\, e^{-jkw_0 t} dt$$

$$= \frac{1}{T}\int_T e^{-jk_0 w_0 t} x(t)\; e^{-jkw_0 t} dt$$

$$= \frac{1}{T}\int_T x(t)\; e^{-j(k-k_0) w_0 t} dt$$

$$z_k = a_{k-k_0}$$

3) Scaling Properties :

- $x(\alpha t)$: periodic signals with period $\frac{T}{\alpha}$ and fundamental frequency αw_0
- $x(t)$: periodic signals with period T and fundamental frequency $w_0 = \frac{2\pi}{T}$

$$= \sum_{k=-\infty}^{+\infty} a_k e^{jk\left(\frac{2\pi}{T}\right)t} \qquad x(\alpha t) = \sum_{k=-\infty}^{+\infty} a_k e^{jkw_0(\alpha t)}$$

$$= \sum_{k=-\infty}^{+\infty} a_k e^{jk\alpha\left(\frac{2\pi}{T}\right)t} = \sum_{k=-\infty}^{+\infty} a_k e^{jk(\alpha w_0)t}$$

$$= \sum_{k=-\infty}^{+\infty} a_k e^{jk\left(\frac{2\pi}{\left(\frac{T}{\alpha}\right)}\right)t} \qquad x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jkw_0 t}$$

4) Time Differentiation

- $x(t)$: periodic signals with period T

$$x(t) \overset{\mathcal{FS}}{\longleftrightarrow} a_k$$

$$\frac{d}{dt}x(t) \overset{\mathcal{FS}}{\longleftrightarrow} jkw_0\, a_k$$

Proof: $x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jkw_0 t}$

then $\frac{d}{dt}x(t) = \sum_{k=-\infty}^{+\infty} a_k\, jkw_0 e^{jkw_0 t} = \sum_{k=-\infty}^{+\infty} (jkw_0\, a_k) e^{jkw_0 t}$

$= jkw_0\, a_k$ ------- **Proved**

5) Modulation or Multiplication theorem

- $x(t), y(t)$: periodic signals with period T

$$x(t) \stackrel{\mathcal{FS}}{\longleftrightarrow} a_k \qquad x(t) = \sum_{l=-\infty}^{+\infty} a_l e^{jlw_0 t}$$

$$y(t) \stackrel{\mathcal{FS}}{\longleftrightarrow} b_k \qquad y(t) = \sum_{m=-\infty}^{+\infty} b_m e^{jmw_0 t}$$

$\Rightarrow x(t)y(t)$: also periodic with T $\qquad z(t) = \sum_{k=-\infty}^{+\infty} c_k e^{jkw_0 t}$

$$z(t) = x(t)y(t) \stackrel{\mathcal{FS}}{\longleftrightarrow} c_k = \sum_{l=-\infty}^{\infty} a_l b_{k-l}$$

- $x(t)$: periodic signal with period T

$x(t) \xleftrightarrow{\mathcal{FS}} a_k$ $\quad$ $y(t) \xleftrightarrow{\mathcal{FS}} b_k$ $\quad$ **Then** $z(t) = x(t)\, y(t) \xleftrightarrow{\mathcal{FS}} a_k * b_k = c_k$

$$= \frac{1}{T}\int_T x(t)\, y(t)\, e^{-jkw_0 t}\,dt$$

$$x(t) = \sum_{l=-\infty}^{+\infty} a_l\, e^{jlw_0 t}$$

$$= \frac{1}{T}\int_T \sum_{l=-\infty}^{+\infty} a_l e^{jlw_0 t}\, y(t)\, e^{-jkw_0 t}\,dt$$

$$= \frac{1}{T}\sum_{l=-\infty}^{+\infty} a_l \int_T y(t)\, e^{-j(k-l)w_0 t}\,dt = \frac{1}{T}\sum_{l=-\infty}^{+\infty} a_l\, b_{(k-l)}$$

$c_k = a_k * b_k$ **Then** $z(t) = x(t)\, y(t) \xleftrightarrow{\mathcal{FS}} a_k * b_k = c_k$

$$c_k = \frac{1}{T}\int_T z(t)\, e^{-jkw_0 t}\,dt$$

$$z(t) = \sum_{k=-\infty}^{+\infty} c_k e^{jkw_0 t}$$

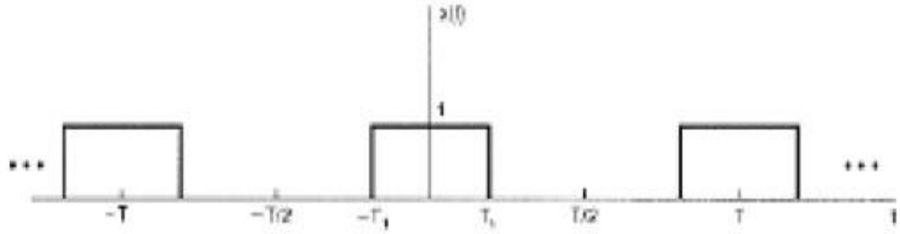
$$z(t) = x(t)y(t) \xleftrightarrow{\mathcal{FS}} c_k = \sum_{l=-\infty}^{\infty} a_l b_{k-l}$$

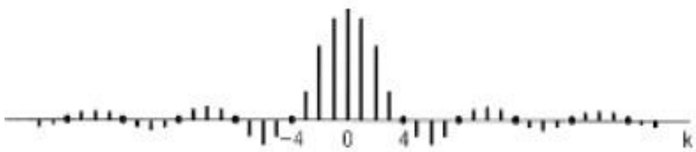
When we multiply the fourier series representation of x(t) and y(t), note that the *k*th harmaonic component in the product will have a coefficient which is the sum of terms of the form $a_l b_{k-l}$

6) Parsevals Relationships:

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jkw_0 t}$$

$$a_k = \frac{1}{T}\int_T x(t) e^{-jkw_0 t} dt$$

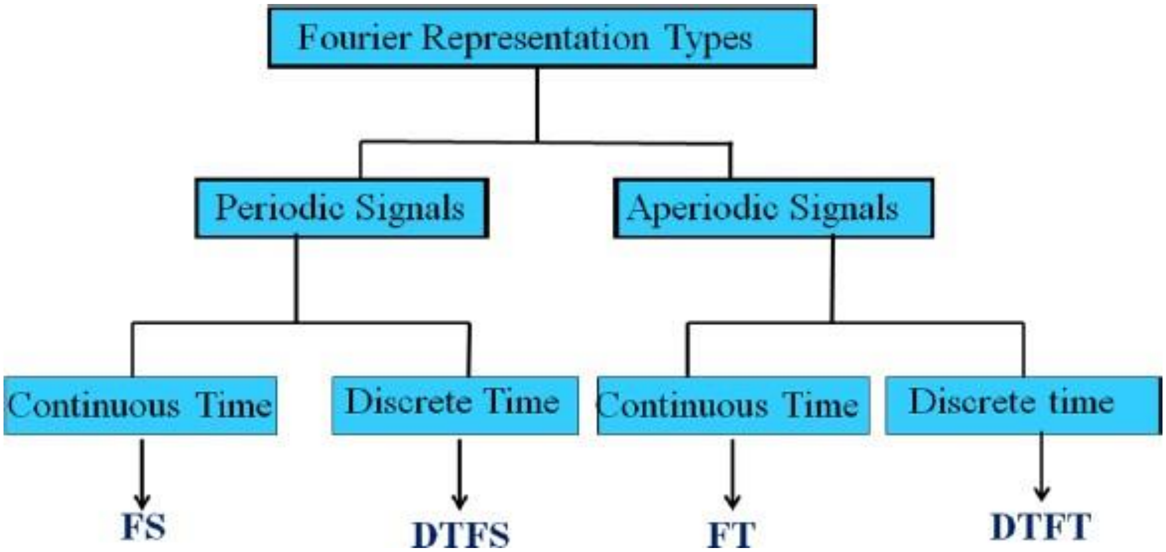




(b)

$$\frac{1}{T}\int_T \left|x(t)\right|^2 dt = \sum_{k=-\infty}^{+\infty} \left|a_k\right|^2$$

- Parseval's relation states that
  the total average power in a periodic signal
  equals
  the sum of the average powers
  in all of its harmonic components



Fourier Representation for Continuous Time Signals

– The Synthesis Equation

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jkw_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{jk(2\pi/T)t}$$

– The Analysis Equation

$$a_k = \frac{1}{T}\int_T x(t) e^{-jkw_0 t} dt = \frac{1}{T}\int_T x(t) e^{-jk(2\pi/T)t} dt$$

Introduction Fourier Representation for Continuous Time Vs Discrete Time Signals

Some Important Differences

• DTFS is a finite series while FS is an infinite series representation. Hence mathematical convergence issues are not there in DTFS.

• Discrete-time signal x[n] is periodic with period N. i.ex[n] = x[n+N]

• The fundamental period is the smallest positive integer **N** for which the above holds and $\omega o = 2\pi/N$ and $\boldsymbol{\varphi_k[n] = e^{jk\omega on} = e^{jk(2\pi/N)n}}$, $k = 0, \pm1, \pm2, \ldots$ Etc.

Discrete time Fourier Series

- The Synthesis Equation

$$x[n] = \sum_{k=\langle N\rangle} a_k e^{jkw_0 n} = \sum_{k=\langle N\rangle} a_k e^{jk\left(\frac{2\pi}{N}\right)n}$$

- The Analysis Equation

$$a_k = \frac{1}{N}\sum_{n=\langle N\rangle} x[n]e^{-jkw_0 n} = \frac{1}{N}\sum_{n=\langle N\rangle} x[n]e^{-jk\left(\frac{2\pi}{N}\right)n}$$

$$a_k = a_{k+N}$$

- $x[n] \overset{\mathcal{FS}}{\longleftrightarrow} a_k$: DT Fouries series pair
- $\{a_k\}$: the Fourier series coefficients or the spectral coefficients of $x[n]$

These Equations play the same role for discrete time periodic signals as the Synthesis and Analysis Equations for Continuous time signals. akare referred to as the spectral coefficient of x[n]. These coefficients specify a decomposition of x[n] into a sum of N harmonically related complex exponentials. We also observe that the graph nature both in Time domain and frequency domain are both discrete unlike in Fourier Series for continuous times

Unit 5

Fourier Representation for Signals

Learning Objectives: **Fourier representation for signals – 2:** Discrete and continuous Fourier transforms(derivations of transforms are excluded) and their properties.

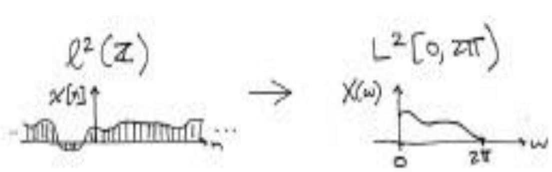
Discrete-Time Fourier Transform (DTFT)

Discrete-Time Fourier Transform

$$X(\omega) = \sum_{n=-\infty}^{\infty} \left(x(n)\, e^{-(j\omega n)} \right)$$

Inverse Discrete-Time Fourier Transform

$$x(n) = \frac{1}{2\pi} \int_0^{2\pi} X(\omega)\, e^{j\omega n} d\omega$$



Mapping $l^2(\mathbb{Z})$ in the time domain to $L^2([0, 2\pi))$ in the frequency domain.

Discrete-Time Fourier Transform Pair

When we obtain the discrete-time signal via sampling an analog signal, the Nyquist frequency corresponds to the discrete-time frequency 1/2 . To show this, note that a sinusoid atthe Nyquist frequency 1/2Tshas a sampled waveform that equalsSinusoid at Nyquist Freqency 1/2T

The exponential in the DTFT at frequency 1/2 equals $e^{\frac{-(j2\pi n)}{2}} = e^{-(j\pi n)} = (-1)^n$
meaning that the correspondence between analog and discrete-time frequency is established:

Analog, Discrete-Time Frequency Relationship

$$f_D = f_A T_s$$

Where fD and fA represent discrete-time and analog frequency variables, respectively. The aliasing figure (pg??) provides another way of deriving this result. As the duration of each pulse in the periodic sampling signal pTs(t) narrows, the amplitudes of the signal's spectral repetitions, which are governed by the Fourier series coefficients (pg??) of pTs(t) , become increasingly equal.Thus, the sampled signal's spectrum becomes periodic with period1/ Ts Thus, the Nyquist frequency 1/2Ts orresponds to the frequency 1/2 . The inverse discrete-time Fourier transform is easily derived from the following relationship:

$$\int_{-\left(\frac{1}{2}\right)}^{\frac{1}{2}} e^{-(j2\pi fm)} e^{+j\pi fn} df = \begin{cases} 1 \text{ if } m = n \\ 0 \text{ if } m \neq n \end{cases}$$

	Sequence Domain	Frequency Domain
Linearity	$a_1 s_1(n) + a_2 s_2(n)$	$a_1 S_1\left(e^{j2\pi f}\right) + a_2 S_2\left(e^{j2\pi f}\right)$
Conjugate Symmetry	$s(n)$ real	$S\left(e^{j2\pi f}\right) = S\left(e^{-(j2\pi f)}\right)^*$
Even Symmetry	$s(n) = s(-n)$	$S\left(e^{j2\pi f}\right) = S\left(e^{-(j2\pi f)}\right)$
Odd Symmetry	$s(n) = -(s(-n))$	$S\left(e^{j2\pi f}\right) = -\left(S\left(e^{-(j2\pi f)}\right)\right)$
Time Delay	$s(n - n_0)$	$e^{-(j2\pi f n_0)} S\left(e^{j2\pi f}\right)$
Complex Modulation	$e^{j2\pi f_0 n} s(n)$	$S\left(e^{j2\pi(f-f_0)}\right)$
Amplitude Modulation	$s(n)\cos(2\pi f_0 n)$	$\frac{S\left(e^{j2\pi(f-f_0)}\right)+S\left(e^{j2\pi(f+f_0)}\right)}{2}$
	$s(n)\sin(2\pi f_0 n)$	$\frac{S\left(e^{j2\pi(f-f_0)}\right)-S\left(e^{j2\pi(f+f_0)}\right)}{2j}$
Multiplication by n	$ns(n)$	$\frac{1}{-(2j\pi)}\frac{d}{df}\left(S\left(e^{j2\pi f}\right)\right)$
Sum	$\sum_{n=-\infty}^{\infty}(s(n))$	$S\left(e^{j2\pi 0}\right)$
Value at Origin	$s(0)$	$\int_{-\left(\frac{1}{2}\right)}^{\frac{1}{2}} S\left(e^{j2\pi f}\right) df$
Parseval's Theorem	$\sum_{n=-\infty}^{\infty}\left((\lvert s(n)\rvert)^2\right)$	$\int_{-\left(\frac{1}{2}\right)}^{\frac{1}{2}} \left(\lvert S\left(e^{j2\pi f}\right)\rvert\right)^2 df$

Figure:Discrete-Time Fourier Transform Properties

Therefore, we find that

Fourier Transform Pairs in Discrete Time

$$\begin{aligned}\int_{-\left(\frac{1}{2}\right)}^{\frac{1}{2}} S\left(e^{j2\pi f}\right) e^{+j2\pi fn} df &= \int_{-\left(\frac{1}{2}\right)}^{\frac{1}{2}} \sum_m \left(s(m)\, e^{-(j2\pi fm)} e^{+j2\pi fn}\right) df \\ &= \sum_m \left(s(m) \int_{-\left(\frac{1}{2}\right)}^{\frac{1}{2}} e^{(-(j2\pi f))(m-n)} df\right) \\ &= s(n)\end{aligned}$$

The Fourier transform pairs in discrete-time are

$$S\left(e^{j2\pi f}\right) = \sum_n \left(s(n)\, e^{-(j2\pi fn)}\right)$$

Fourier Transform Pairs in Discrete Time

$$s(n) = \int_{-\left(\frac{1}{2}\right)}^{\frac{1}{2}} S\left(e^{j2\pi f}\right) e^{+j2\pi fn} df$$

Continuous-Time Fourier Transform (CTFT)

Introduction

Due to the large number of continuous-time signals that are present, the Fourier series provided us the first glimpse of how me we may represent some of these signals in a general manner: as a superpostion of a number of sinusoids. Now, we can look at a way to represent continuous-time nonperiodic signals using the same idea of superposition. Below we will present the Continuous-Time Fourier Transform (CTFT), also referred to as just the Fourier Transform (FT). Because the CTFT now deals with nonperiodic signals, we must now find a way to include all frequencies in the general equations.

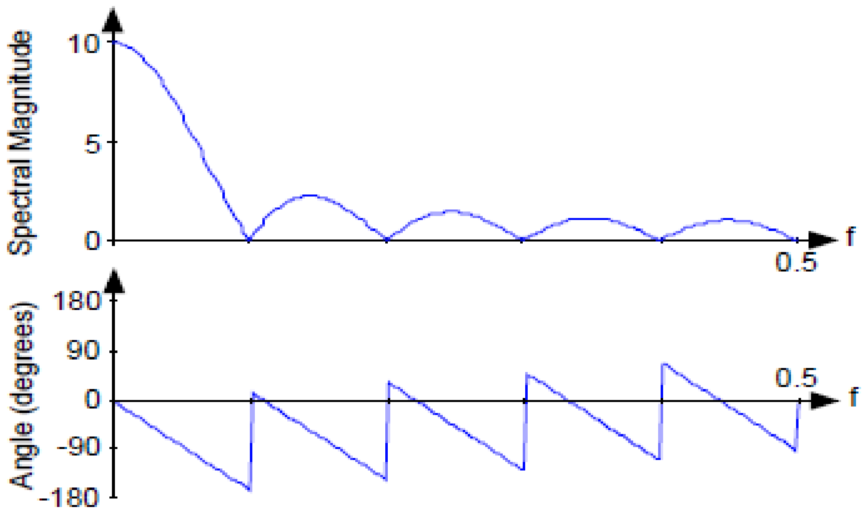


Figure: The spectrum of a length-ten pulse is shown. Can you explain the rather complicated appearance of the phase?

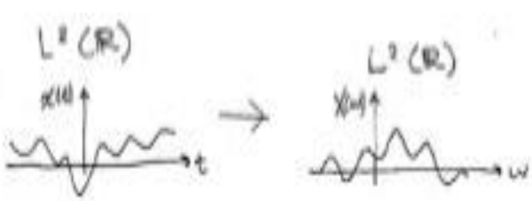
Equations

Continuous-Time Fourier Transform

$$F(\Omega) = \int_{-\infty}^{\infty} f(t)\, e^{-(j\Omega t)}\, dt$$

Inverse CTFT

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\Omega)\, e^{j\Omega t}\, d\Omega$$



Mapping $L^2(\mathbb{R})$ in the time domain to $L^2(\mathbb{R})$ in the frequency domain.

The above equations for the CTFT and its inverse come directly from the Fourier series and our understanding of its coefficients. For the CTFT we simply utilize integration rather than summation to be able to express the aperiodic signals. This should make sense since for the CTFT we are simply extending the ideas of the Fourier series to include nonperiodic signals, and thus the entire frequency spectrum. Look at the Derivation of the Fourier Transform for a more in depth look at this.

Properties of the Continuous-Time Fourier Transform

This module will look at some of the basic properties of the Continuous-Time Fourier Transform (CTFT). The first section contains a table that illustrates the properties, and the sections following it discuss a few of the more interesting properties in more depth. In the table, click on the operation name to be taken to the properties explanation found later on this page. Look at this module for an expanded table of more Fourier transform properties. note: We will be discussing these properties for aperiodic, continuous-time signals but understand that very similar properites hold for discrete-time signals and periodic signals as well

Table of CTFT Properties

Operation Name	Signal ($f(t)$)	Transform ($F(\omega)$)
Addition (Section 9.9.2.1)	$f_1(t) + f_2(t)$	$F_1(\omega) + F_2(\omega)$
Scalar Multiplication (Section 9.9.2.1)	$\alpha f(t)$	$\alpha F(t)$
Symmetry (Section 9.9.2.2)	$F(t)$	$2\pi f(-\omega)$
Time Scaling (Section 9.9.2.3)	$f(\alpha t)$	$\frac{1}{\|\alpha\|} F\left(\frac{\omega}{\alpha}\right)$
Time Shift (Section 9.9.2.4)	$f(t-\tau)$	$F(\omega) e^{-(j\omega\tau)}$
Modulation (Frequency Shift) (Section 9.9.2.5)	$f(t) e^{j\omega\phi}$	$F(\omega - \phi)$
Convolution in Time (Section 9.9.2.6)	$(f_1(t), f_2(t))$	$F_1(t) F_2(t)$
Convolution in Frequency (Section 9.9.2.6)	$f_1(t) f_2(t)$	$\frac{1}{2\pi}(F_1(t), F_2(t))$
Differentiation (Section 9.9.2.7)	$\frac{d^n}{dt^n} f(t)$	$(j\omega)^n F(\omega)$

Fourier Transform Properties

Linearity

The combined addition and scalar multiplication properties in the table above demonstrate the basic property of linearity. What you should see is that if one takes the Fourier transform of a linear combination of signals then it will be the same as the linear combination of the Fourier transforms of each of the individual signals. This is crucial when using a table of transforms to find the transform of a more complicated signal.

We will begin with the following signal:

$$z(t) = \alpha f_1(t) + \alpha f_2(t)$$

Now, after we take the Fourier transform, shown in the equation below, notice that the linear combination of the terms is unaffected by the transform.

$$Z(\omega) = \alpha F_1(\omega) + \alpha F_2(\omega)$$

Symmetry

Symmetry is a property that can make life quite easy when solving problems involving Fourier transforms. Basically what this property says is that since a rectangular function in time is a sinc function in frequency, then a sincfucntion in time will be a rectangular function in frequency. This is a direct result of the similarity between the forward CTFT and the inverse CTFT. The only difference is the scaling by 2_ and a frequency reversal.

Time Scaling

This property deals with the effect on the frequency-domain representation of a signal if the time variable is altered. The most important concept to understand for the time scaling property is that signals that are narrow in time will be broad in frequency and vice versa. The simplest example of this is a delta function, a unit pulse (pg??) with a very small duration, in time that becomes an infinite-length constant function in frequency. The table above shows this idea for the general transformation from the time-domain to the frequency-domain of a signal. You should be able to easily notice that these equations show the relationship mentioned previously: if the time variable is increased then the frequency range will be decreased.

Time Shifting

Time shifting shows that a shift in time is equivalent to a linear phase shift in frequency. Since the frequency content depends only on the shape of a signal, which is unchanged in a time shift, then only the phase spectrum will be altered. This property can be easily proved using the Fourier Transform, so we will show the basic steps below:

We will begin by letting z (t) = f (t −τ). Now let us take the Fourier transform
with the previous expression substitued in for z (t).

$$Z(\omega) = \int_{-\infty}^{\infty} f(t-\tau)\, e^{-(j\omega t)} dt$$

Now let us make a simple change of variables, where _ = t −τ_ . Through the calculations below, you can see that only the variable in the exponential are altered thus only changing the phase in the frequency domain.

$$\begin{aligned} Z(\omega) &= \int_{-\infty}^{\infty} f(\sigma)\, e^{-(j\omega(\sigma+\tau)t)} d\tau \\ &= e^{-(j\omega\tau)} \int_{-\infty}^{\infty} f(\sigma)\, e^{-(j\omega\sigma)} d\sigma \\ &= e^{-(j\omega\tau)} F(\omega) \end{aligned}$$

Modulation (Frequency Shift)

Modulation is absolutely imperative to communications applications. Being able to shift a signal to a different frequency, allows us to take advantage of different parts of the electromagnetic spectrum is what allows us to transmit television, radio and other applications through the same space without significant interference. The proof of the frequency shift property is very similar to that of the time shift however, here we would use the inverse Fourier transform in place of the Fourier transform. Since we went through the steps in the previous, time-shift proof, below we will just show the initial and final step to this proof:

$$z(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega - \phi)\, e^{j\omega t} d\omega$$

Now we would simply reduce this equation through another change of variables and simplify
the terms. Then we will prove the property experssed in the table above

$$z(t) = f(t)\, e^{j\phi t}$$

Convolution

Convolution is one of the big reasons for converting signals to the frequency domain, since convolution in time becomes multiplication in frequency. This property is also another excellent example of symmetry between time and frequency. It also shows that there may be little to gain by changing to the frequency domain when multiplication in time is involved. We will introduce the convolution integral here, but if you have not seen this before or need to refresh your memory, then look at the continuous-time convolution module for a more in depth explanation and derivation

$$\begin{aligned} y(t) &= (f_1(t), f_2(t)) \\ &= \int_{-\infty}^{\infty} f_1(\tau)\, f_2(t-\tau)\, d\tau \end{aligned}$$

Time Differentiation

Since LTI systems can be represented in terms of differential equations, it is apparent with this property that converting to the frequency domain may allow us to convert these complicated differential equations to simpler equations involving multiplication and addition. This is often looked at in more detail during the study of the Laplace Transform.

Unit 6

Applications Of Fourier Representations

Learning Objectives: **Applications of Fourier representations:** Introduction, Frequency response of LTI systems, Fourier transform representation of periodic signals, Fourier transform representation of discrete time signals. Sampling theorm and Nyquist rate.

FT representation of continuous periodic pulses

To use Fourier methods to analyse interaction between **continuous times** and **discrete-time signals**, we need a bridge between Fourier representation for different signal classes

FT representation of periodic signals

Consider a continuous-time signal x(t) with a fundamental period equal to 'To'. This signal can be represented using exponential fourier series as

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_o t} \;;\; \omega o = \frac{2\pi}{To}$$

We know that $x(t) = e^{j\omega_o t} \xleftrightarrow{FT} X(j\omega) = 2\pi\delta(\omega - \omega_o)$

If we represent the transform in a periodic way we get

$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega o) = 2\pi \sum_{k=-\infty}^{\infty} a_k \delta(\omega - k\omega o)$$

Example 1

Show that the FT of a Dirac comb is also Dirac comb.

Soln: A periodic impulse train is known as Dirac comb

Example 1

- The Complex Fourier Coefficient is given by

$$a_k = \frac{1}{T}\int_T x(t)\, e^{-jk\omega_o t} dt \;=\; a_k = \frac{1}{To}\int_0^{To} \delta(t)\, e^{-jk\omega_o t} dt = \frac{1}{To}$$

- The FT of x(t) is given by

$$X(e^{j\omega}) = 2\pi \sum_{k=-\infty}^{\infty} a_k \delta(\omega - k\omega o) = \frac{2\pi}{To} \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega o)$$

$$= \omega o \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega o)$$

Example2

Find the FT, $X(e^{j\omega})$ of the periodic signal x(t) and sketch the magnitude spectrum and the phase spectrum $x(t) = 3 + 2\cos(10\pi t)$

Soln:

$x(t) = 3 + 2\cos(10\pi t)$

$$x(t) = 3 + 2\left[\frac{1}{2}e^{j10\pi t} + \frac{1}{2}e^{-j10\pi t}\right] = 3 + e^{j10\pi t} + e^{-j10\pi t}$$

where $\Omega_0 = 10\pi$ rads

Comparing Eqn 1 with the FS synthesis eqn

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$$

We get, X(0)=3, X(1)=1, X(-1)=1, and X(k)=0 for all other k.

Using the FT of a periodic signal x(t) is given by

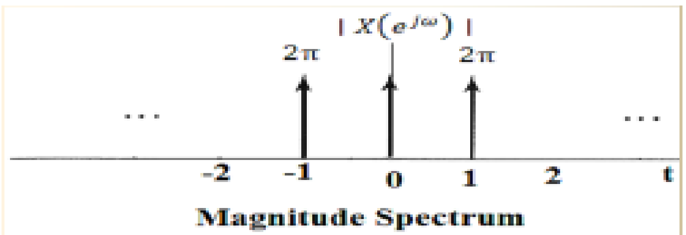
$$X(e^{j\omega}) = 2\pi \sum_{k=-\infty}^{\infty} a_k \delta(\omega - k\omega o)$$

Expanding the above Eqn

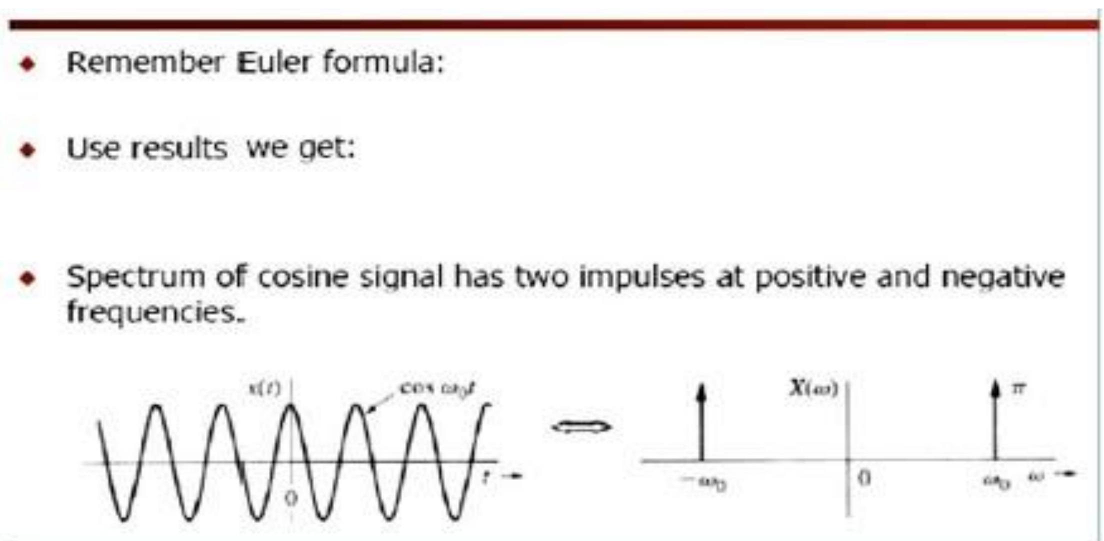
$$X(e^{j\omega}) = 2\pi[X(-1)\delta(\omega + \omega o) + X(0)\delta(\omega) + X(1)\delta(\omega - \omega o)]$$

Substituting the coefficient values, we get

$$X(e^{j\omega}) = 2\pi\delta(\omega + \omega o) + 6\pi\delta(\omega) + 2\pi\delta(\omega - \omega o)$$



Fourier Transform of everlasting sinusoidal cosωot



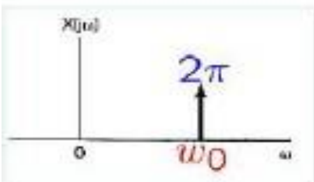
FT FOR PERIODIC SIGNALS: For analysing Discrete time periodic and a periodic signals DTFT is used. As in continuous time case, DT periodic signals can be incorporated within the framework of the DTFT, by interpreting the transform of periodic signal as an impulse train in the frequency domain

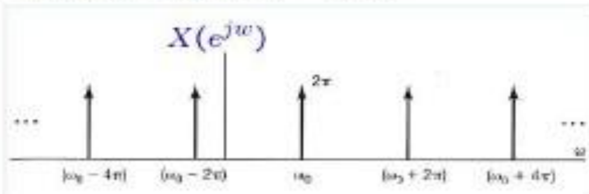
• Fourier Transform from Fourier series

$$x[n] \xleftrightarrow{DTFT} X(e^{j\omega})$$

$$\text{where } X(e^{j\omega}) = \sum_{l=-\infty}^{+\infty} 2\pi\delta(\omega - \omega_o - 2\pi l)$$

$$x(t) = e^{j\omega_o t} \xleftrightarrow{FT} X(j\omega) = 2\pi\delta(\omega - \omega_o)$$





$$\text{and } \omega_o = \frac{2\pi}{N}$$

Fourier Transform from Fourier series In order to check the validity of the above expression let us use the synthesys equation.

We know $$x[n] = \frac{1}{2\pi}\int_{\omega=-\pi}^{+\pi} X(e^{j\omega})\, e^{j\omega n} d\omega$$

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{+\pi} \sum_{l=-\infty}^{+\infty} 2\pi\delta(\omega - \omega_o - 2\pi l)\, e^{j\omega n}\, d\omega$$

Fourier Transform from Fourier series Note that any interval of length 2π includes exactly one pulse in the above analysis eqn. Hence if the integral interval is chosen includes one pulse located at $\omega o+2\pi r$, then

$$\frac{1}{2\pi}\int_{\omega=-\pi}^{+\pi} X(e^{j\omega})\, e^{j\Omega n} d\omega = e^{j(\omega o + 2\pi r)n} = e^{j\omega o n}$$

Now consider a periodic sequence x[n] with period N its DTFS representation is

$$\frac{1}{2\pi}\int_{\omega=-\pi}^{+\pi} X(e^{j\omega})\, e^{j\Omega n} d\omega = e^{j(\omega o+2\pi r)n} = e^{j\omega o n}$$

So that the FS can be directly constructed from its coefficient. To verify the above equation is correct note that x[n] in the above is a linear combination of and thus must be a combination of transforms

In this case, the Fourier transform is

$$X(e^{j\omega}) = \frac{2\pi}{N}\sum_{k=-\infty}^{+\infty} 2\pi a k \delta\left(\omega - \frac{2\pi k}{N}\right)$$

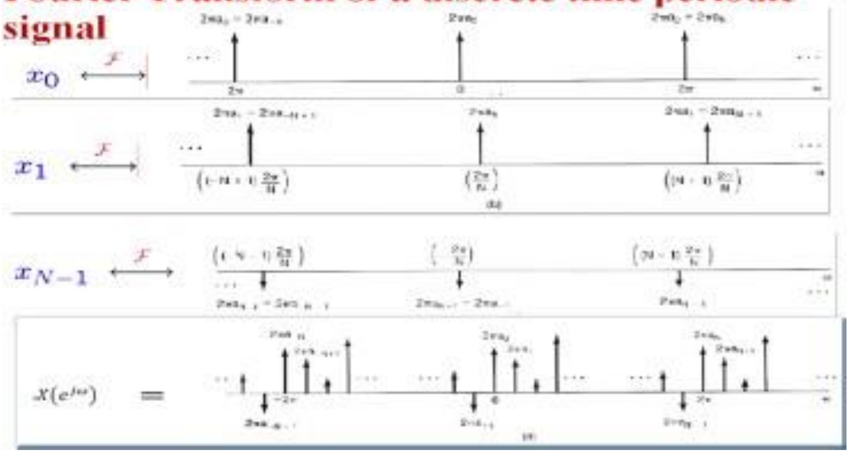
So that the FS can be directly constructed from its coefficient. To verify the above equation is correct note that x[n] in the above is a linear combination of and thus must be a combination of transforms Fourier Transform from Fourier series

Suppose we chose the summation of interval as k=0,1……N, so that

$$x[n] = a_0 + a_1 e^{j\left(\frac{2\pi}{N}\right)n} + a_2 e^{j2\left(\frac{2\pi}{N}\right)n} a_3 e^{j3\left(\frac{2\pi}{N}\right)n} + \cdots \ldots\ldots a_{N-1} e^{j(N-1)\left(\frac{2\pi}{N}\right)n}$$
$$= x_0 + x_1 + x_3 + \cdots \ldots \ldots\ldots + x_{(N-1)}$$

Hence x[n] is a linear combination of signals as initial x[n]. With ω0 =0,2π/N, 4π/N,……. (N-1)2π/N. The waveforms are depicted as

Fourier Transform of a discrete time periodic signal



Example 1:

Consider the periodic signal

$$x[n] = \cos \omega_0 n = \frac{1}{2}e^{j\omega_0 n} + \frac{1}{2}e^{-j\omega_0 n}, \qquad with\ \omega_0 = \frac{2\pi}{5}$$

$If, x[n] \xleftrightarrow{DTFT} X(e^{j\omega})$ **Using** **Fourier Transform from Fourier series equation**

periodic

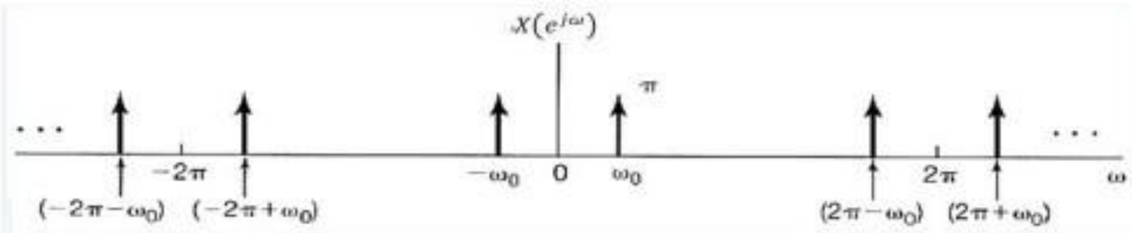
$$X\left(e^{j\omega}\right) = \sum_{i=-\infty}^{+\infty} 2\pi\delta(\omega - \omega_0 - 2\pi l) \qquad and\ \omega_0 = \frac{2\pi}{N}$$

$$X\left(e^{j\omega}\right) = \sum_{i=-\infty}^{+\infty} \pi\delta\left(\omega - \frac{2\pi}{5} - 2\pi l\right) + \sum_{i=-\infty}^{+\infty} \pi\delta\left(\omega + \frac{2\pi}{5} - 2\pi l\right)$$

$$i.e\ X(e^{j\omega}) = \pi\delta\left(\omega - \frac{2\pi}{5}\right) + \pi\delta\left(\omega + \frac{2\pi}{5}\right)$$

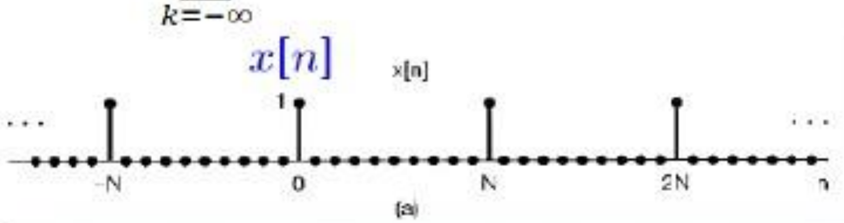
$$with - \pi \le \omega < \pi$$

With repeats periodically with a period 2π, as illustrated below



Example 2:
The discrete counter part of the periodic impulse train is the sequence

$$x[n] = \sum_{k=-\infty}^{+\infty} \delta[n - kN]$$



The Fourier coefficients for this signal can be calculated directly as

$$a_k = \frac{1}{N}\sum_{n=N} x[n]e^{-jk(2\pi/N)n}$$

Choosing the interval of summation as $0 \le n \le N-1$, we have

$$a_k = \frac{1}{N} \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

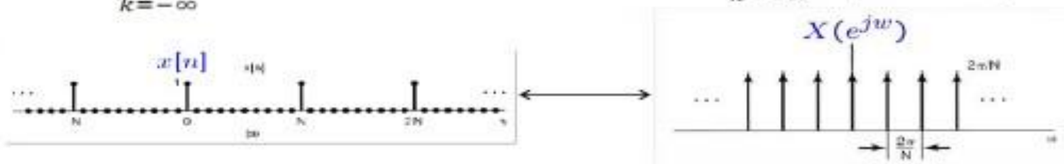
We know the FT for the periodic signal is

$$X(e^{j\omega}) = \sum_{k=-\infty}^{+\infty} 2\pi a k\delta\left(\omega - \frac{2\pi k}{N}\right)$$

Using Equation (1) and (2) we can then represent the FT of signal as

$$X(e^{j\omega}) = \frac{2\pi}{N} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{N}\right)$$

$$x[n] = \sum_{k=-\infty}^{+\infty} \delta[n - kN] \xleftrightarrow{DTFT} X(e^{j\omega}) = \frac{2\pi}{N} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{N}\right)$$



FT for Periodic Signals (Summarize)

- To represent both continuous-time and Discrete-time periodic signals using Fourier Transform Coefficients ($X(e^{j\omega})$, $X(j\omega)$) in relation with the Fourier series Coefficient (ak).
- This allows us to consider both periodic and aperiodic signals within a unified context.
- FT of a periodic sequence x(t)

$$X(e^{j\omega}) = 2\pi \sum_{k=-\infty}^{\infty} a_k \delta(\omega - k\omega o)$$

- DTFT of a periodic sequence x(n)

$$X(e^{j\omega}) = \frac{2\pi}{N} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{N}\right)$$

- **Application of Fourier transform**

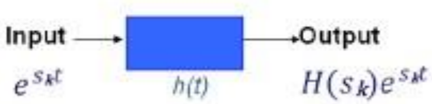
The Response of LTI Systems to Complex Exponentials

- The set of complex exponential signals

  Signals of the form e^{st} in CT

  Signals of the form z^n in DT

- The Response of an LTI System:



$$y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau$$

CT : $e^{st} \rightarrow H(s)e^{st}$ ⟶ **Eiigenfunction**

DT : $z^n \rightarrow H(z)z^n$ ⟶ **Eiigenvalue**

Example 1 :

As an illustration, consider an LTI system for which the input x(t) and output y(t) are related by a time shift of 3, i.e y(t)=x(t-3). If the input to this system is the complex exponential signal $x(t) = e^{j2t}$

- Soln:

$$y(t) = e^{j2(t-3)} = e^{-j6}e^{j2t}$$
$$= H(s)x(t)$$
$$where, H(s) = e^{-j6}(an\ eigen\ value)$$
$$and\ x(t) = e^{j2t}\ (an\ eigen\ function)$$

The impulse response is

$$and\ so\ H(j\omega) = \int_{-\infty}^{\infty} \delta(\tau - 3)\, e^{-s\tau} d\tau$$
$$= e^{-3s}$$
$$so\ that\ H(j2) = e^{-j6} \qquad h(t) = \delta(\tau - 3)d\tau$$

Example 1.b

Consider the input signal x(t)=cos(4t)+cos(7t). Then, if as in example 1, y(t)=x(t-3), then $y(t) = \cos(4(t-3)) + \cos(7(t-3))$.

Expanding x(t), using Euler's relation

$$x(t) = \frac{1}{2}e^{j4t} + \frac{1}{2}e^{-j4t} + \frac{1}{2}e^{j7t} + \frac{1}{2}e^{-j7t}$$

Representing in the above as the LTI System, we get the output as

$$y(t) = \frac{1}{2}e^{-j12t}e^{j4t} + \frac{1}{2}e^{j12t}e^{-j4t} + \frac{1}{2}e^{-j21t}e^{j7t} + \frac{1}{2}e^{j12t}e^{-j7t}$$

<u>Fourier Series and LTI System</u>

- Fourier series representation can be used to construct any periodic signals in discrete as well as continuous-time signals of practical importance.
- We have also seen the response of an LTI system to a linear combination of complex exponentials taking a simple form.
- Now, let us see how Fourier representation is used to analyze the response of LTI System.

Example 1

Consider the CTFS synthesis equation for x(t) given by
Suppose we apply this signal as an input to an LTI System with impulse respose h(t). Then, since each of the complex exponentials in the expression is an eigen function of the system. Then, with $sk = jk\omega_o$, it follows that the output is

$$y(t) = \sum_{k=-\infty}^{+\infty} a_k H(e^{jk\omega o})e^{jk\omega ot}$$

Thus y(t) is periodic with frequency as x(t). Further, if ak is the set of Fourier series coefficients for the input x(t), then $\{a_k H(e^{jk\omega o})\}$ is the set of coefficient for the y(t). Hence in LTI, modify each of the Fourier coefficient of the input by multiplying by the frequency response at the corresponding frequency.

Consider a periodic signal x(t), with fundamental frequency 2π, that is expressed in the form

$$x(t) = \sum_{k=-3}^{+3} a_k e^{jk2\pi t} \qquad \ldots\ldots(1)$$

$$where, \quad a_0=1, \quad a_1=a_{-1}=1/4\ , \quad a_2=a_{-2}=1/2, \quad a_3=a_{-3}=1/3,$$

Suppose that the this periodic signal is input to an LTI system with impulse response
To calculate the FS Coeff. Of o/p y(t), lets compute the frequency response.The impulse response is therefore,

$$H(j\omega) = \int_0^{\infty} e^{-\tau} e^{-j\omega\tau} d\tau \qquad = -\frac{1}{1+j\omega} e^{-\tau} e^{-j\omega\tau} \Big|_0^{\infty}$$

and

$$H(j\omega) = \frac{1}{1+j\omega}$$

Y(t) at ωo = 2π . We obtain,

$$y(t) = \sum_{k=-3}^{+3} b_k e^{jk2\pi t} \quad with\ b_k = a_k H(jk2\pi), so\ that$$

$$b_1 = \frac{1}{4}\left(\frac{1}{1+j2\pi}\right) b_2 = \frac{1}{2}\left(\frac{1}{1+j4\pi}\right) b_3 = \frac{1}{3}\left(\frac{1}{1+j6\pi}\right)$$

$$b_{-1} = \frac{1}{4}\left(\frac{1}{1-j2\pi}\right) b_{-2} = \frac{1}{2}\left(\frac{1}{1-j4\pi}\right) \quad b_{-3} = \frac{1}{3}\left(\frac{1}{1-j6\pi}\right)$$

$$b_o = 1$$

The above o/p coefficients. Could be substituted in

$$y(t) = \sum_{k=-3}^{+3} b_k e^{jk2\pi t}$$

Example 2

Consider an LTI system with impulse response $h[n]=\alpha^n u[n]$, $-1<\alpha<1$, and with the input $x[n] = \cos\left(\frac{2\pi}{N}\right)$

Soln:

Let us write in the Fourier Series form as

$$x[n] = \frac{1}{2}e^{j\left(\frac{2\pi}{N}\right)n} + \frac{1}{2}e^{-j\left(\frac{2\pi}{N}\right)n}$$

The frequency response as earlier

$$H(e^{j\omega}) = \sum_{n=0}^{\infty} \alpha^n e^{-j\omega n} = \sum_{n=0}^{\infty}\left(\alpha e^{-j\omega}\right)^n = \frac{1}{1-\alpha e^{-j\omega}}$$

We know the o/p y[n] for discrete-time signal is

$$y[n] = \sum_{k=\langle N\rangle} a_k H\left(e^{j2\pi/N}\right) e^{jk\left(\frac{2\pi}{N}\right)n}$$

Substituting in the above we get the Fourier series for the output:

$$y[n] = \frac{1}{2}\left(\frac{1}{1-\alpha e^{-j\frac{2\pi}{N}}}\right)e^{j\left(\frac{2\pi}{N}\right)n} + \frac{1}{2}\left(\frac{1}{1-\alpha e^{j\frac{2\pi}{N}}}\right)e^{-j\left(\frac{2\pi}{N}\right)n}$$

If we write

$$\left(\frac{1}{1-\alpha e^{-j\frac{2\pi}{N}}}\right) = re^{j\theta}$$

then y[n] reduces to

$$y[n] = r\cos\left(\frac{2\pi}{N}n + \theta\right)$$

Finding the Frequency Response

We can begin to take advantage of this way of finding the output for any input once we have $H(\omega)$.

To find the frequency response $H(\omega)$ for a system, we can:

1. Put the input $x(t) = e^{i\omega t}$ into the system definition
2. Put in the corresponding output $y(t) = H(\omega)\, e^{i\omega t}$
3. Solve for the frequency response $H(\omega)$. (The terms depending on t will cancel.)

Example:

Consider a system with impulse response

$$h(t) = \begin{cases} \frac{1}{5} & \text{for } t \in [0,5] \\ 0 & \text{otherwise} \end{cases}$$

Find the output corresponding to the input $x(t) = \cos(10\,t)$.

$$y(t) = \int_{\tau=-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau = \int_{\tau=0}^{5} \frac{1}{5} \cos(10(t-\tau))\, d\tau$$

$$y(t) = \frac{1}{5}\left(-\frac{1}{10}\sin(10(t-\tau))\right)\Bigg|_{\tau=0}^{5} = \frac{1}{50}\left(\sin(10t) - \sin(10(t-5))\right)$$

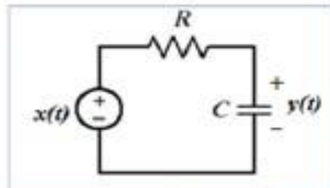
- Ex 3 The impulse response is $h(t) = \frac{1}{RC}e^{-t/RC}u(t)$ for the circuit as shown below. Plot the magnitude response of this system on a linear scale characterize this system as a filter.
- Soln: we know,

$$H(j\omega) = \int_{-\infty}^{+\infty} h(\tau)e^{-j\omega\tau}d\tau$$

$$H(j\omega) = \int_{0}^{+\infty} \frac{1}{RC}e^{-\frac{\tau}{RC.}}\, e^{-j\omega\tau}d\tau$$

$$H(j\omega) = \int_{-\infty}^{+\infty} \frac{1}{RC}e^{-\frac{\tau}{RC}}u(\tau)e^{-j\omega\tau}d\tau$$

$$H(j\omega) = \frac{1}{RC}\int_{0}^{+\infty} e^{-\tau\left(\frac{1}{RC.}+j\omega\right)}d\tau$$



$$\boxed{H(j\omega) = \frac{1}{j\omega RC + 1}}$$

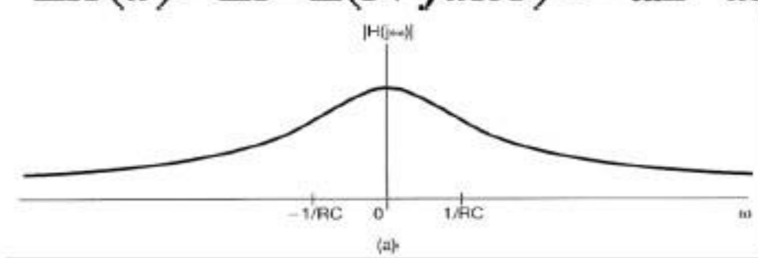
Consider our RC circuit from last time,where we found

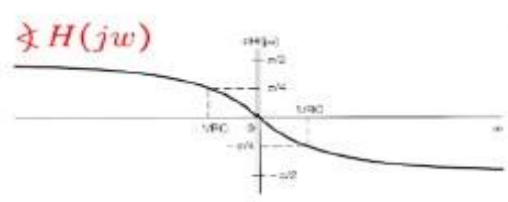
$$H(\omega) = \frac{1}{1 + j\omega RC}$$

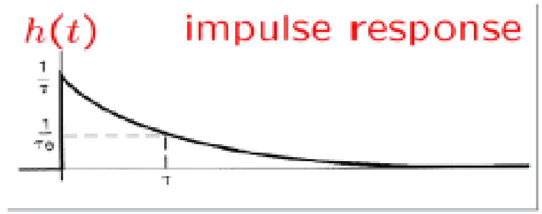
To compute the voltage over the capacitor, y(t), for a sinusoidal input voltage x(t), I simply need to find the magnitude and phase of H(ω) and plug in:

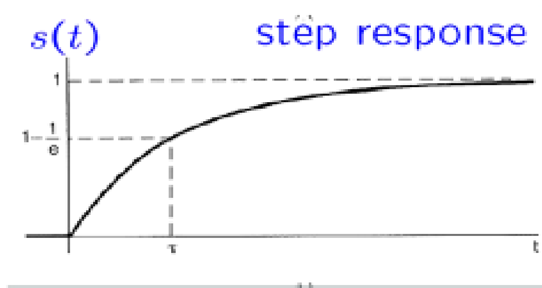
$$|H(\omega)| = \frac{1}{|1 + j\omega RC|} = \frac{1}{\sqrt{1 + (\omega RC)^2}}$$

$$\angle H(\omega) = \angle 1 - \angle(1 + j\omega RC) = -\tan^{-1}\omega RC$$









The output of a system in response to an input $x(t) = e^{-2t}u(t)$ is $y(t) = e^{-t}u(t)$.
Find the frequency response and the impulse response of this system.

- Soln:

$$X(j\omega) = \frac{1}{j\omega + 2} \qquad Y(j\omega) = \frac{1}{j\omega + 1} \qquad H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} \qquad H(j\omega) = \frac{j\omega + 2}{j\omega + 1}$$

$$H(j\omega) = \left(\frac{j\omega + 2}{j\omega + 1}\right) + \frac{1}{j\omega + 1} \qquad H(j\omega) = 1 + \frac{1}{j\omega + 1}$$

$$\boxed{h(t) = \delta(t) + e^{-t}u(t)}$$

Summaries Fourier in LTI

• The LTI system scales the complex exponential eiωt.

• Each system has its own constant H(ω) that describes how it scales eigenfunctions. It is called the **frequency response**.

• The frequency response H(ω) does not depend on the input. It is another way to describe a system, like (A, B, C, D), h, etc.

• If we know H(ω), it is easy to find the output when the input is an eigenfunction. y(t)=H(ω)x(t) true **when x is eigenfunction**!

Differential and Difference Equation Descriptions Frequency Response is the system"s steady state response to a sinusoid. In contrast to differential and difference-equation descriptions for a system, the frequency response description cannot represent initial conditions, it can only describe a system in a steady state condition. The differential-equation representation for a continuous-time system is

$$\sum_{k=0}^{N} a_k \frac{d^k}{dt^k} y(t) = \sum_{k=0}^{N} b_k \frac{d^k}{dt^k} x(t)$$

$$since, \frac{d}{dt} g(t) \xleftrightarrow{FT} j\omega G(j\omega)$$

Rearranging the equation we get

$$\frac{Y(j\omega)}{X(j\omega)} = \frac{\sum_{k=0}^{M} b_k (j\omega)^k}{\sum_{k=0}^{N} a_k (j\omega)^k}$$

The frequency of the response is

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{\sum_{k=0}^{M} b_k (j\omega)^k}{\sum_{k=0}^{N} a_k (j\omega)^k}$$

Hence, the equation implies the frequency response of a system described by a linear constant-coefficient differential equation is a ratio of polynomials in jω.

The difference equation representation for a discrete-time system is of the form.

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

Take the DTFT of both sides of this equation, using the time-shift property.

$$g[n-k] \xleftrightarrow{DTFT} e^{-jk\omega} G(e^{j\omega})$$

To obtain

$$\sum_{k=0}^{N} a_k \left(e^{-j\omega}\right)^k Y(e^{j\omega}) = \sum_{k=0}^{N} a_k \left(e^{-j\omega}\right)^k X(e^{j\omega})$$

- Rewrite this equation as the ratio

$$\frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{\sum_{k=0}^{M} b_k \left(e^{j\omega}\right)^k}{\sum_{k=0}^{N} a_k \left(e^{j\omega}\right)^k}$$

- The frequency response is the polynomial in $e^{j\omega}$

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{\sum_{k=0}^{M} b_k \left(e^{j\omega}\right)^k}{\sum_{k=0}^{N} a_k \left(e^{j\omega}\right)^k}$$

<u>Differential Equation Descriptions</u>

Ex: Solve the following differential Eqn using FT.

$$\frac{d^2}{dt^2}y(t) + 4\frac{d}{dt}y(t) + 5y(t) = 3\frac{d}{dt}x(t) + x(t)$$

For all t where, $x(t) = (1 + e^{-t})u(t)$

Soln:we have

$$\frac{d^2}{dt^2}y(t) + 4\frac{d}{dt}y(t) + 5y(t) = 3\frac{d}{dt}x(t) + x(t)$$

FT gives,

$$[(j\omega)^2 + 4(j\omega) + 5]Y(j\omega) = (3j\omega + 1)X(j\omega)$$

$$and\ x(t) = (1 + e^{-t})u(t) \qquad x(t) = u(t) + (e^{-t})u(t)$$

$$X(j\omega) = \left(\frac{1}{j\omega} + \pi\delta(\omega)\right) + \frac{1}{(j\omega + 1)}\ since\ u(t) \xleftrightarrow{FT} \pi\delta(\omega) + \frac{1}{j\omega}$$

$$and (e^{-t})u(t) \xleftrightarrow{FT} \frac{1}{j\omega + 1}$$

Hence we have

$$X(j\omega) = \left(\frac{1}{j\omega} + \pi\delta(\omega)\right) + \frac{1}{(j\omega + 1)}$$

And $[(j\omega)^2 + 4(j\omega) + 5]Y(j\omega) = (3j\omega + 1)X(j\omega)$

i.e

$$Y(j\omega) = \frac{(3j\omega + 1)}{[(j\omega)^2 + 4(j\omega) + 5]}X(j\omega)$$

$$Y(j\omega) = \frac{(3j\omega + 1)}{[(j\omega + 2)^2 + 1]}\left[\frac{1}{j\omega} + \pi\delta(\omega) + \frac{1}{(j\omega + 1)}\right]$$

$$Y(j\omega) = \frac{(3j\omega + 1)}{[(j\omega)^2 + 4(j\omega) + 5]}\left[\left(\frac{1}{j\omega} + \pi\delta(\omega)\right) + \frac{1}{(j\omega + 1)}\right]$$

$$\boxed{Y(j\omega) = Y(1) + Y(2) + Y(3)}$$

$$Y(j\omega) = \frac{(3j\omega+1)}{[(j\omega+2)^2+1]j\omega} + \frac{\pi}{5}\delta(\omega) + \frac{(3j\omega+1)}{[(j\omega+2)^2+1](j\omega+1)}$$

$$Y(j\omega) = \frac{(3j\omega+1)}{[(j\omega+2)^2+1]j\omega} + \frac{(3j(\omega=0)+1)\pi[\delta(0)=1]}{[(j(\omega=0)+2)^2+1]j(\omega=0)} + \frac{(3j\omega+1)}{[(j\omega+2)^2+1](j\omega+1)}$$

$$Y(1) = \frac{(3j\omega+1)}{[(j\omega+2)^2+1]j\omega} \qquad Y(1) = \frac{A}{j\omega} + \frac{Bj\omega+C}{[(j\omega+2)^2+1]}$$

Performing partial fraction we get $A = \frac{1}{5}, B = -\frac{1}{5}, C = \frac{11}{5}$

$$\boxed{Y(1) = \frac{1/5}{j\omega} + \frac{-1/5j\omega+11/5}{[(j\omega+2)^2+1]}}$$

Similarly

$$Y(3) = \frac{(3j\omega+1)}{[(j\omega+2)^2+1](j\omega+1)}$$

$$Y(3) = \frac{R}{(j\omega+1)} + \frac{Pj\omega+Q}{[(j\omega+2)^2+1]}$$

Performing partial fraction we get $R = -1, P = 1, Q = 6$

$$\boxed{Y(3) = \frac{-1}{(j\omega+1)} + \frac{j\omega+6}{[(j\omega+2)^2+1]}}$$

$$Y(3) = \frac{-1}{(j\omega+1)} + \frac{j\omega+6}{[(j\omega+2)^2+1]} \; Y(j\omega) = Y(1) + Y(2) + Y(3)$$

Hence, we have

$$Y(1) = \frac{1/5}{j\omega} + \frac{-1/5j\omega + 11/5}{[(j\omega+2)^2+1]}$$

$$Y(2) = \frac{\pi}{5}\delta(\omega)$$

Readjusting

$$Y(j\omega) = \frac{1/5}{j\omega} + \frac{-1/5j\omega + 11/5}{[(j\omega+2)^2+1]} + \frac{\pi}{5}\delta(\omega) + \frac{-1}{(j\omega+1)} + \frac{j\omega+6}{[(j\omega+2)^2+1]}$$

$$\boxed{Y(j\omega) = \frac{1}{5}\left[\frac{1}{j\omega} + \pi\delta(\omega)\right] - \frac{1}{(j\omega+1)} + \frac{1}{5}\left[\frac{4j\omega+41}{[(j\omega+2)^2+1]}\right]}$$

$$Y(j\omega) = \frac{1/5}{j\omega} + \frac{\pi}{5}\delta(\omega) + \frac{11/5 - 1/5j\omega}{[(j\omega+2)^2+1]} + \frac{j\omega+6}{[(j\omega+2)^2+1]} - \frac{1}{(j\omega+1)}$$

we know that,

$$e^{-\beta t}\cos\omega_o t u(t) \xleftrightarrow{FT} \frac{\beta + j\omega}{[(\beta+j\omega)^2 + \omega_o{}^2]}$$

$$e^{-\beta t}\sin\omega_o t u(t) \xleftrightarrow{FT} \frac{\omega_o}{[(\beta+j\omega)^2 + \omega_o{}^2]}$$

Readjusting the last term, we get

$$Y(j\omega) = \frac{1}{5}\left[\frac{1}{j\omega} + \pi\delta(\omega)\right] - \frac{1}{(j\omega+1)} + \frac{4}{5}\left[\frac{j\omega+2}{[(j\omega+2)^2+1]}\right] + \frac{33}{5}\left[\frac{1}{[(j\omega+2)^2+}\right.$$

Now, taking the inverse Fourier Transform, we get

$$\boxed{y(t) = \frac{1}{5}u(t) - e^{-t}u(t) + \frac{4}{5}e^{-2t}\cos t\, u(t) + \frac{33}{5}e^{-2t}\sin t\, u(t)}$$

Differential Equation Descriptions

- Ex: Find the frequency response and impulse response of the system described by the differential equation.

$$\frac{d^2}{dt^2}y(t) + 3\frac{d}{dt}y(t) + 2y(t) = 2\frac{d}{dt}x(t) + x(t)$$

Here we have N=2, M=1. Substituting the coefficients of this differential equation in

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{\sum_{k=0}^{M} b_k (j\omega)^k}{\sum_{k=0}^{N} a_k (j\omega)^k}$$

Differential Equation Descriptions

We obtain

$$H(j\omega) = \frac{2j\omega + 1}{(j\omega)^2 + 3j\omega + 2}$$

The impulse response is given by the inverse FT of H(jω). Rewrite H(jω) using the partial fraction expansion.

$$H(j\omega) = \frac{A}{j\omega + 1} + \frac{B}{j\omega + 2}$$

Solving for A and B we get, A=-1 and B=3. Hence

$$H(j\omega) = \frac{-1}{j\omega + 1} + \frac{3}{j\omega + 2}$$

The inverse FT gives the impulse response

$$\boxed{h(t) = 3e^{-2t}u(t) - e^{-t}u(t)}$$

Difference Equation

Ex: Consider an LTI system characterized by the following second order linear constant coefficient difference equation.

$$y[n] = 1.3433y[n-1] - 0.9025y[n-2] + x[n] - 1.4142x[n-1] + x[n-2]$$

Find the frequency response of the system.

Soln:

$$y[n] = 1.3433y[n-1] - 0.9025y[n-2] + x[n] - 1.4142x[n-1] + x[n-2]$$

$$Y(e^{j\omega}) = 1.3433(e^{-j\omega})Y(e^{j\omega}) - 0.9025(e^{-j2\omega})Y(e^{j\omega}) + X(e^{j\omega}) - 1.4142(e^{-j\omega})X(e^{j\omega}) + (e^{-j2\omega})X(e^{j\omega})$$

$$we\ know, y[n-k] \xleftrightarrow{DTFT} e^{-jk\omega}Y(e^{j\omega})$$

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})}$$
$$= \frac{1 - 1.4142e^{-j\omega} + e^{-j2\omega}}{1 - 1.3433e^{-j\omega} + 0.9025e^{-j2\omega}}$$

Ex: If the unit impulse response of an LTI System is $h(n)=\alpha^n u[n]$, find the response of the system to an input defined by $x[n] = \beta^n u[n]$, where $\beta, \alpha < 1$ *and* $\alpha \neq \beta$

Soln:

$$y[n] = h[n] * x[n]$$

Taking DTFT on both sides of the equation, we get

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega}) \qquad Y(e^{j\omega}) = \frac{1}{1 - \alpha e^{-j\omega}} \text{x} \frac{1}{1 - \beta e^{-j\omega}}$$

$$Y(e^{j\omega}) = \frac{1}{1 - \alpha e^{-j\omega}} \text{x} \frac{1}{1 - \beta e^{-j\omega}} = \frac{A}{1 - \alpha e^{-j\omega}} \text{x} \frac{B}{1 - \beta e^{-j\omega}}$$

where A and B are constants to be found by using partial fractions

$$Let, \; e^{-j\omega} = v \qquad Then, Y(e^{j\omega}) = \frac{A}{1 - \alpha v} \text{x} \frac{B}{1 - \beta v}$$

By performing partial fractions, we get $A = \frac{\alpha}{\alpha - \beta}$, B$= \frac{-\beta}{\alpha - \beta}$

$$Therefore, Y(e^{j\omega}) = \frac{\frac{\alpha}{\alpha - \beta}}{1 - \alpha\, e^{-j\omega}} \text{x} \frac{\frac{-\beta}{\alpha - \beta}}{1 - \beta\, e^{-j\omega}}$$

Taking inverse DTFT, we get

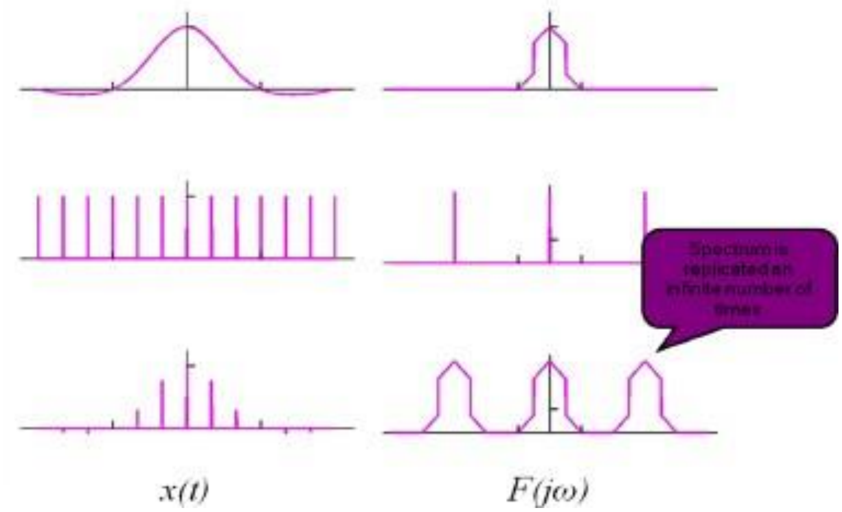
$$\boxed{y[n] = \left[\frac{\alpha}{\alpha - \beta}\alpha^n - \frac{\beta}{\alpha - \beta}\alpha^n\right]u[n]}$$

Sampling

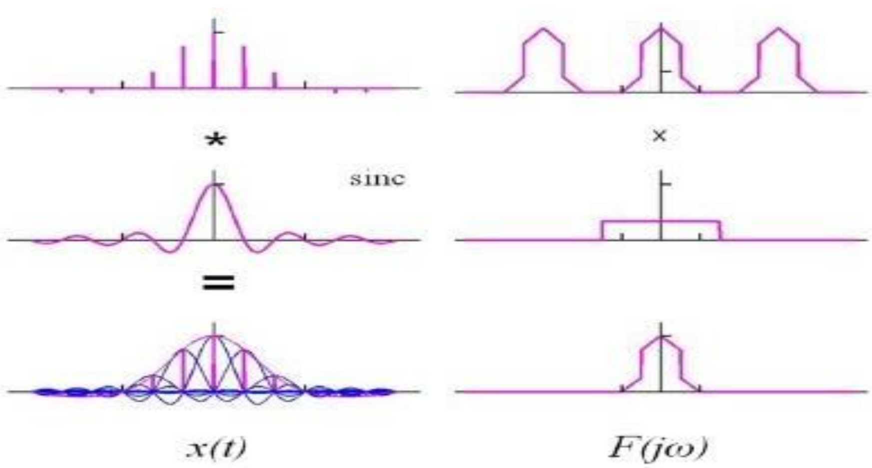
In this chapter let us understand the meaning of sampling and which are the different methods of sampling. There are the two types. Sampling Continuous-time signals and **Sub-sampling**. In this again we have *Sampling Discrete-time signals*. **Sampling Continuous-time signals** Sampling of continuous-time signals is performed to process the signal using digital processors. The sampling operation generates a discrete-time signal from a continuous-time signal.DTFT is used to analyze the effects of uniformly sampling a signal.Let us see, how a DTFT of a sampled signal is related to FT of the continuous-time signal.

• Sampling: Spatial Domain: A continuous signal x(t) is measured at fixed instances spaced apart by an interval „T‟. The data points so obtained form a discrete signal x[n]=x[nT]. Here, ΔT is the sampling period and 1/ ΔT is the sampling frequency.Hence, sampling is the multiplication of the signal with an impulse signal.

- **Sampling theory**



• **Reconstruction theory**



Sampling: Spatial Domain

From the Figure we can see

Where x[n] is equal to the samples of x(t) at integer multiples of a sampling interval T

$$x_\delta(t) = \sum_{n=-\infty}^{+\infty} x(n)\delta(t - n\tau)$$

Now substitute $x(nT)$ *for* $x[n]$ *to obtain*

$$x_\delta(t) = \sum_{n=-\infty}^{+\infty} x(n\tau)\ \delta(t - n\tau)$$

$$since\ x(t)\delta(t - n\tau) = x(n\tau)\delta(t - n\tau)$$

we may rewrite $x_\delta(t)$ *as a product of time functions*

$$x_\delta(t) = x(t)p(t) \qquad where, \qquad p(t) = \delta(t - n\tau)$$

Hence, Sampling is the multiplication of the signal with an impulse train.

The effect of sampling is determined by relating the FT of $x_\delta(t)$ to the FT of $x(t)$. Since Multiplication in the time domain corresponds to convolution in the frequency domain, we have

$$X_\delta(j\omega) = \frac{1}{2\pi} X(j\omega) * P(j\omega)$$

Substituting the value of $P(j\omega)$ as the FT of the pulse train i.e

$$p(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT)$$

We get,

$$P(j\omega) = \frac{2\pi}{\tau} \sum_{n=-\infty}^{+\infty} \delta(\omega - k\omega_s)$$

$$where, \omega_s = \frac{2\pi}{\tau}, is\ the\ sampling\ frequency. Now$$

$$X_\delta(j\omega) = \frac{1}{2\pi} X(j\omega) * \frac{2\pi}{\tau} \sum_{n=-\infty}^{+\infty} \delta(\omega - k\omega_s)$$

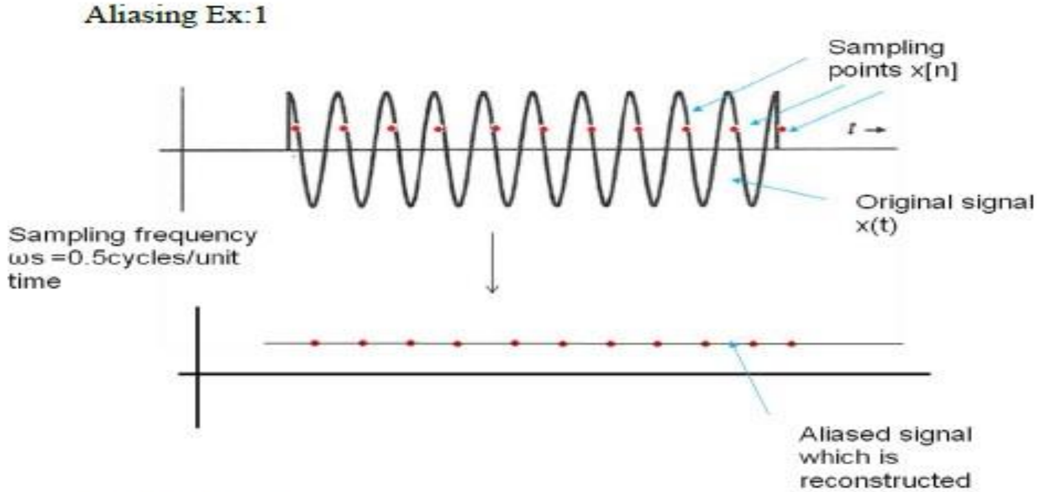
$$\boxed{X_\delta(j\omega) = \frac{1}{\tau} \sum_{n=-\infty}^{+\infty} X(j(\omega - k\omega_s))}$$

The FT of the sampled signal is given by an infinite sum of shifted version of the original signals FT and the offsets are integer multiples of ω_s.

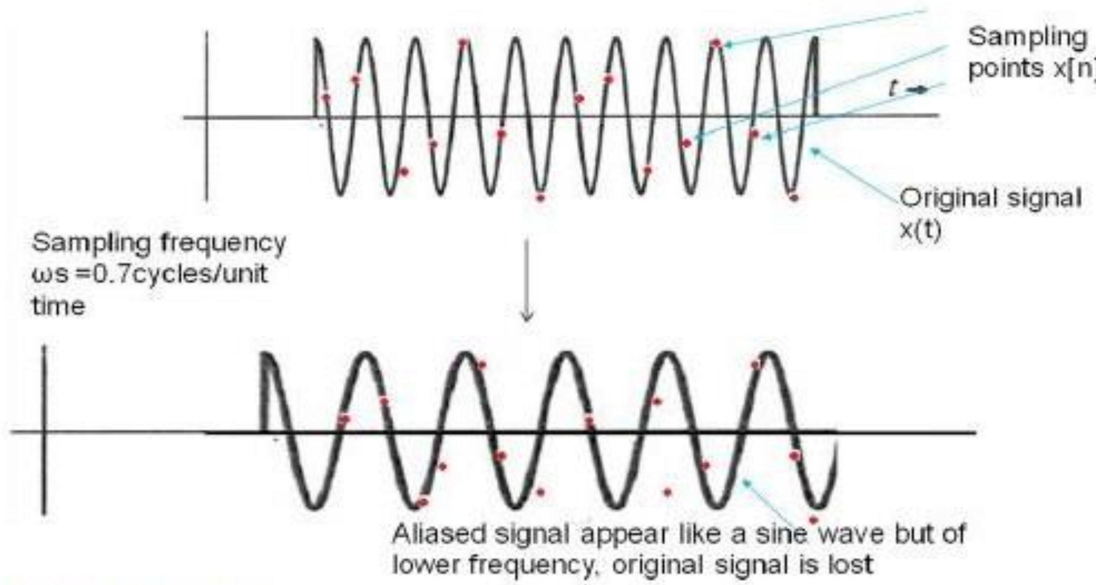
Aliasing : an example

Frequency of original signal is 0.5 oscillations per time unit). Sampling frequency is also 0.5 oscillations per time unit). Original signal cannot be recovered.

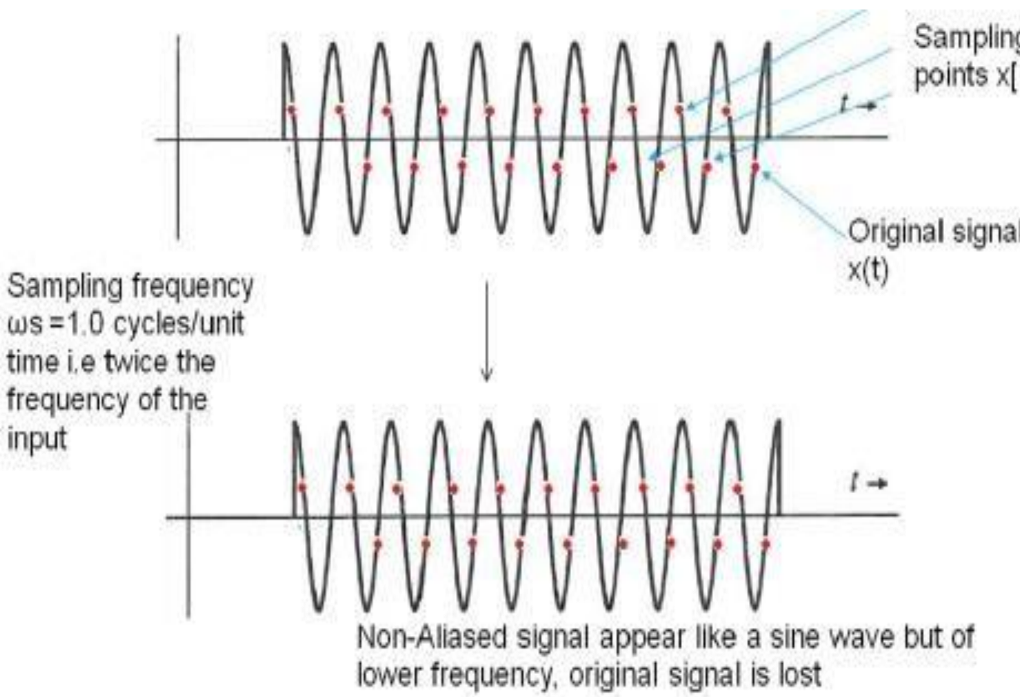
Aliasing Ex:1



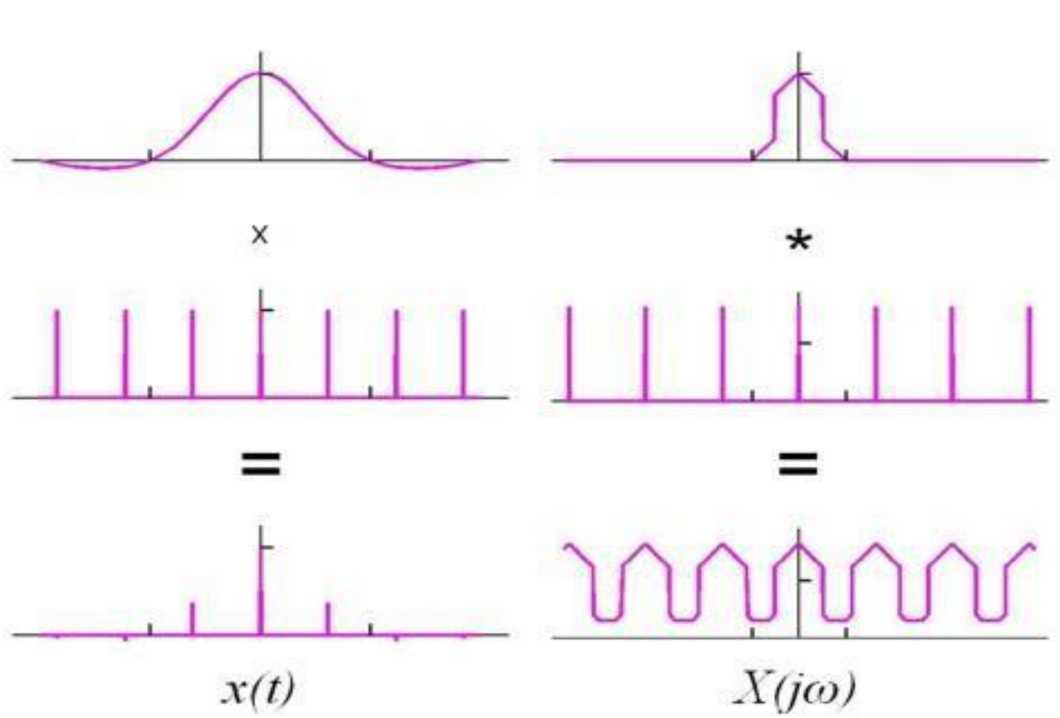
Aliasing Ex:2



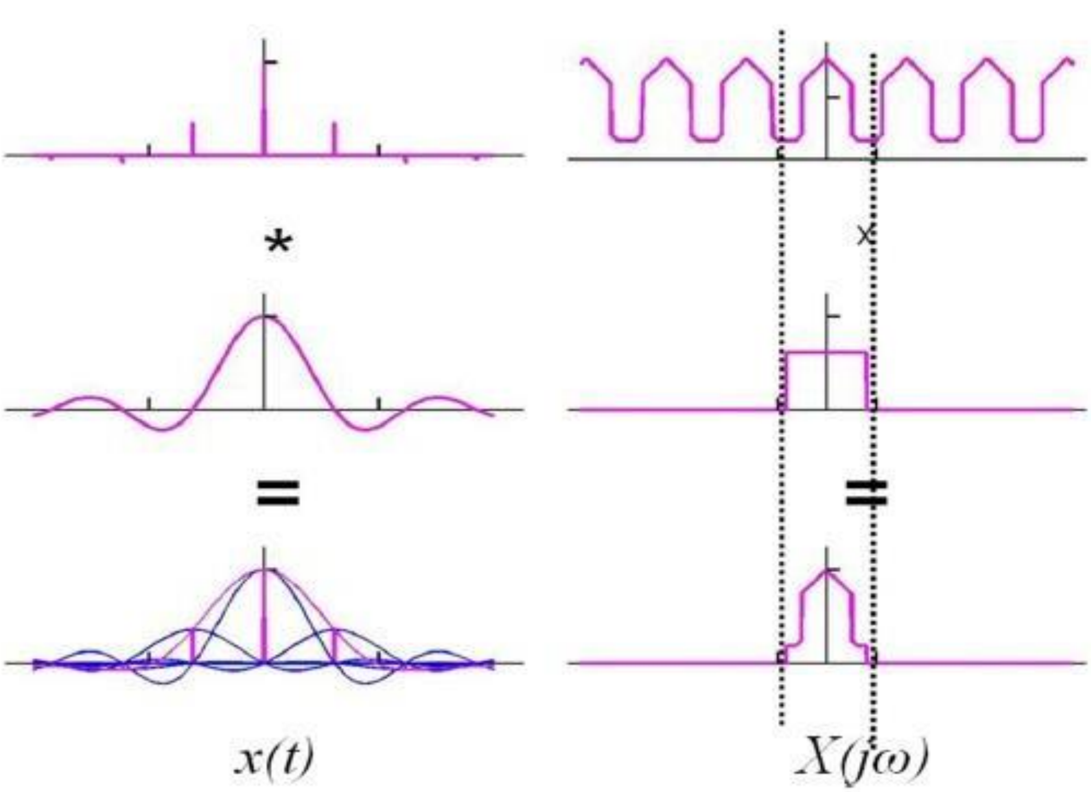
Non-Aliasing: Ex 3



Sampling below the Nyquist rate



Reconstruction below the Nyquist rate



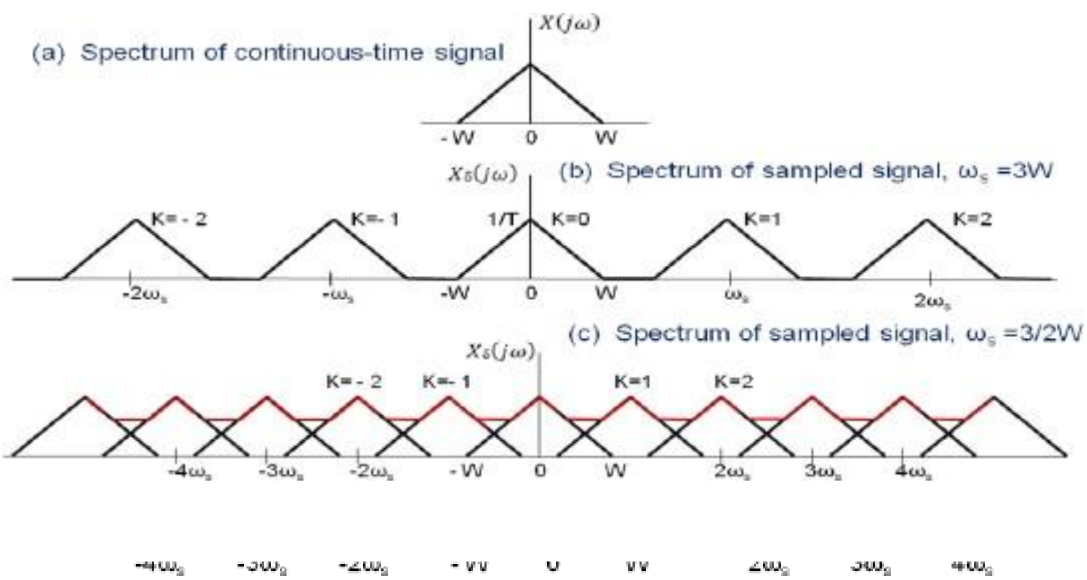
Aliasing summary

• We learned that we need to sample each oscillation period of the input signal ≥ two times for good reconstruction.(**Nyquist Criteria**)

• The shifted version of X(jω) may overlap with each other if ωs(sampling frequency) is not large enough compared to the frequency content of X(jω).

• Overlap in the shifted replicas of the original spectrum is termed **Aliasing**, which refers to the phenomenon of a high frequency component taking on the identity of a low-frequency one.

FT of sampled signal for different sampling frequency



- Reconstruction problem is addressed as follows.
- Aliasing is prevented by choosing the sampling interval T so that ω_s>2W, where W is the highest frequency component in the signal.
- This implies we must satisfy T<π/W.
- Also, DTFT of the sampled signal is obtained from $X_\delta(j\omega)$ using the relationship Ω= ωT, that is

$$x[n] \xleftrightarrow{DTFT} X(e^{j\omega}) = X_\delta(j\omega)\,|_{\omega = \Omega/\tau}$$

- This scaling of the independent variable implies that $\omega=\omega_s$ corresponds to Ω=2π

Subsampling: Sampling discrete-time signal

• FT is also used in discrete sampling signal.

• Let be a subsampled version x[n], where q is a positive integer.

• Relating DTFT of y[n] to the DTFT of x[n], by using FT to represent x[n] as a sampled versioned of a continuous time signal x(t).

• Expressing now y[n] as a sampled version of the sampled version of the same underlying CT x(t) obtained using a sampling interval q that associated with x[n]

• We know to represent the sampling version of x[n] as the impulse sampled CT signal with sampling interval T.

$$x_\delta(t) = \sum_{n=-\infty}^{+\infty} x(n)\,\delta(t - n\tau)$$

- Suppose, x[n] are the samples of a CT signal x(t), obtained at integer multiples of T. That is, x[n]=x[nT]. Let $x(t) \xleftrightarrow{FT} X(j\omega)$ and applying it to obtain

$$\boxed{X_\delta(j\omega) = \frac{1}{\tau}\sum_{k=-\infty}^{+\infty} X(j(\omega - k\omega_s))}$$

- Since y[n] is formed using every qth sample of x[n], we may also express y[n] as a sampled version of x(t).we have $y[n] = x[qn] = x(nq\tau)$
- Hence, active sampling rate for y]n] is T'=qT. Hence

$$y_\delta(t) = x(t)\sum_{n=-\infty}^{\infty} \delta(t - n\tau') \xleftrightarrow{FT} Y_\delta(j\omega) = \frac{1}{\tau'}\sum_{k=-\infty}^{+\infty} X(j(\omega - k\omega_s'))$$

- Hence substituting T'=q T, and ω_s'= ω_s/q

$$Y_\delta(j\omega) = \frac{1}{q\tau}\sum_{k=-\infty}^{+\infty} X(j(\omega - \frac{k}{q}\omega_s))$$

- We have expressed both $Y_\delta(j\omega)$ and $X_\delta(j\omega)$ as a function of .
- Expressing $X(j\omega)$ as a function of $X_\delta(j\omega)$. Let us write k/q as a proper function, we get

$$\frac{k}{q} = l + \frac{m}{q},$$

$$\text{where } l \text{ is the integer portion of } \frac{k}{q}, \text{and } m \text{ is the remainder}$$

$$\text{allowing } k \text{ to range from } -\infty \text{ to } +\infty \text{ corresponds}$$

$$\text{to having } l \text{ range from } -\infty \text{ to } +\infty \text{ and } m \text{ from } 0 \text{ to } q-1$$

$$Y_\delta(j\omega) = \frac{1}{q}\sum_{m=0}^{q-i}\left\{\frac{1}{\tau}\sum_{l=-\infty}^{+\infty} X_\delta\left(j\left(\omega - l\omega_s - \frac{m}{q}\omega_s\right)\right)\right\}$$

$$Y_\delta(j\omega) = \frac{1}{q}\sum_{m=0}^{q-i} X_\delta\left(j\left(\omega - \frac{m}{q}\omega_s\right)\right)$$

$$\text{which represents a sum of shifted versions of}$$

$$X_\delta(j\omega) \text{ normalized by } q.$$

$$\text{Converting from the FT representation back to DTFT}$$

$$\text{and substituting } \Omega = \omega\tau' \text{ above}$$

$$\text{and also } X\left(e^{j\Omega}\right) = X_\delta(j\Omega/\tau) \text{ , we write this result as}$$

$$\boxed{Y_\delta\left(e^{j\Omega}\right) = \frac{1}{q}\sum_{m=0}^{q-i} X_q\left(e^{j(\Omega - m2\pi)}\right)}$$

$$\text{where,} \quad X_q\left(e^{j\Omega}\right) = X\left(e^{j\Omega/q}\right) - \text{a scaled DTFT version}$$

UNIT 7

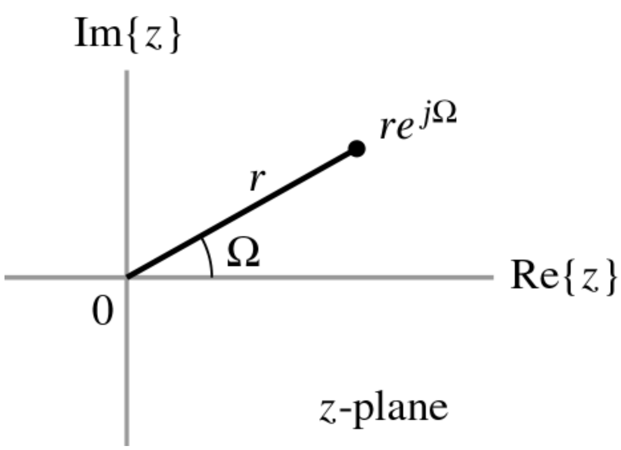
Z TRANSFORMS-1

Learning Objectives: Introduction, Z – transform, properties of ROC, Properties of Z – transforms, inversion of Z – transforms.

Introduction

The *z*-transform is a transform for sequences. Just like the Laplace transformtakes a function of *t* and replaces it with another function of an auxiliaryvariable *s*. The *z*-transform takes a sequence and replaces it with afunction of an auxiliary variable, *z*. The reason for doing this is that itmakes difference equations easier to solve, again, this is very like what happenswith the Laplace transform, where taking the Laplace transform makesit easier to solve differential equations. A difference equation is an equationwhich tells you what the *k*+2th term in a sequence is in terms of the *k*+1thand *k*th terms, for example. Difference equations arise in numerical treatmentsof differential equations, in discrete time sampling and when studyingsystems that are intrinsically discrete, such as population models in ecologyand epidemiology and mathematical modelling of mylinated nerves.

- Generalizes the complex sinusoidal representations of DTFT to more generalized representation using complex exponential signals
- It is the discrete time counterpart of Laplace transform



The z-Plane

• Complex number $z = re^{j\Omega}$ is represented as a location in a complex plane (z *plane*)

The z-transform

• Let $z = re^{j\Omega}$ be a complex number with magnitude r and angle Ω.

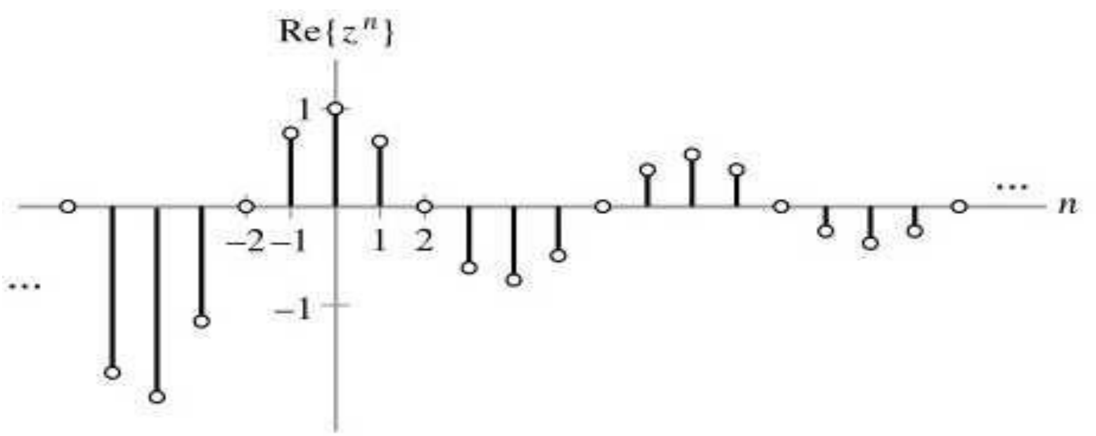
• The signal $x[n] = z^n$ is a complex exponential and $x[n] = r^n\cos(\Omega n) + jr^n\sin(\Omega n)$

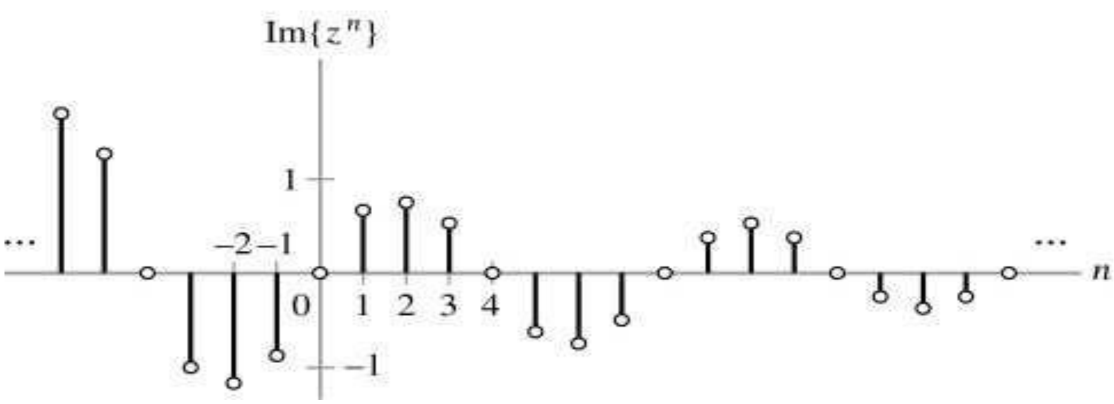
• The real part of $x[n]$ is exponentially damped cosine.

• The imaginary part of $x[n]$ is exponentially damped sine.

• Apply $x[n]$ to an LTI system with impulse response $h[n]$, Then

$y[n] = H\{x[n]\} = h[n] * x[n]$





$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$

· If $x[n] = z^n$

we get

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]z^{n-k}$$

$$y[n] = z^n \sum_{k=-\infty}^{\infty} h[k]z^{-k}$$

The *z*-transform is defined as

$$H(z) = \sum_{k=-\infty}^{\infty} h[k]z^{-k}$$

we may write as

$$H(z^n) = H(z)z^n$$

You can see that when you do the *z*-transformit sums up all the sequence, and so the individual terms affect the dependence on *z*, but the resultingfunction is just a function of *z*, it has no *k* in it. It will become clearer later why we might do this

- This has the form of an eigen relation, where z^n is the eigen function and $H(z)$ is the eigenvalue.

- The action of an LTI system is equivalent to multiplication of the input by the complex number $H(z)$.

- If $H(z) = |H(z)|e^{j\phi(z)}$ then the system output is

$$y[n] = |H(z)|e^{j\phi(z)}z^n$$

- Using $z = re^{j\Omega}$ we get

$$y[n] = |H(re^{j\Omega})|r^n\cos(\Omega n+\phi(re^{j\Omega})+ j|H(re^{j\Omega})|r^n\sin(\Omega n+\phi(re^{j\Omega})$$

- Rewriting $x[n]$

$$x[n] = z^n = r^n\cos(\Omega n)+ jr^n\sin(\Omega n)$$

- If we compare $x[n]$ and $y[n]$, we see that the system modifies

– the amplitude of the input by $|H(re^{j\Omega})|$ and

– shifts the phase by $\phi(re^{j\Omega})$

DTFT and the *z*-transform

- Put the value of z in the transform then we get

$$H(re^{j\Omega}) = \sum_{n=-\infty}^{\infty} h[n](re^{j\Omega})^{-n}$$

$$= \sum_{n=-\infty}^{\infty} (h[n]r^{-n})e^{-j\Omega n}$$

- We see that $H(re^{j\Omega})$ corresponds to DTFT of $h[n]r^{-n}$.
- The inverse DTFT of $H(re^{j\Omega})$ must be $h[n]r^{-n}$.
- We can write

$$h[n]r^{-n} = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(re^{j\Omega})e^{j\Omega n}d\Omega$$

- Multiplying $h[n]r^{-n}$ with r^n gives

$$h[n] = \frac{r^n}{2\pi}\int_{-\pi}^{\pi} H(re^{j\Omega})e^{j\Omega n}d\Omega$$

$$h[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(re^{j\Omega})(re^{j\Omega})^n d\Omega$$

- We can convert this equation into an integral over z by putting $re^{j\Omega} = z$
- Integration is over Ω, we may consider r as a constant
- We have

$$dz = jre^{j\Omega}d\Omega = jzd\Omega$$

$$d\Omega = \frac{1}{j}z^{-1}dz$$

• Consider limits on integral

– Ω varies from $-\pi$ to π

– *z*traverses a circle of radius *r* in a counterclockwise direction

We can write *h*[*n*] as

$$h[n] = \frac{1}{2\pi j} \oint H(z) z^{n-1} dz$$

where H is integration around the circle of radius |*z*| = *r* in a counter clockwise direction

• The z-transform of any signal x[n] is

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

• The inverse z-transform of is

$$x[n] = \frac{1}{2\pi j} \oint X(z) z^{n-1} dz$$

• Inverse z-transform expresses x[n] as a weighted superposition of complex exponentialsZ^n

• The weights are

$$\left(\frac{1}{2\pi j}\right) X(z) z^{-1} dz$$

• This requires the knowledge of complex variable theory.

Convergence

• Existence of *z-transform*: exists only if

$\sum_{n=-\infty}^{\infty} x[n] z^{-n}$ converges

Necessary condition: absolute summability of $x[n]z^{-n}$, since $|x[n]z^{-n}| = |x[n]r^{-n}|$, the condition is

$$\sum_{n=-\infty}^{\infty} |x[n]r^{-n}| < \infty$$

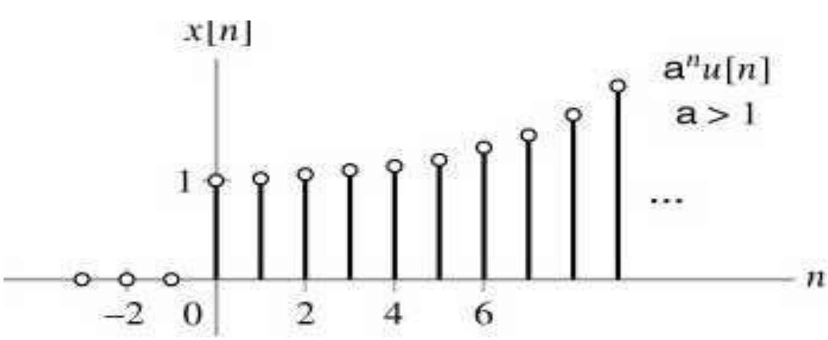
- The range r for which the condition is satisfied is called the *range of convergence* (ROC) of the *z-transform*
- ROC is very important in analyzing the system stability and behavior
- We may get identical *z-transform* for two different signals and only ROC differentiates the two signals
- The *z-transform* exists for signals that do not have DTFT.
- existence of DTFT: absolute summability of $x[n]$
- By limiting restricted values for r we can ensure that $x[n]r^{-n}$ is absolutely summable even though $x[n]$ is not

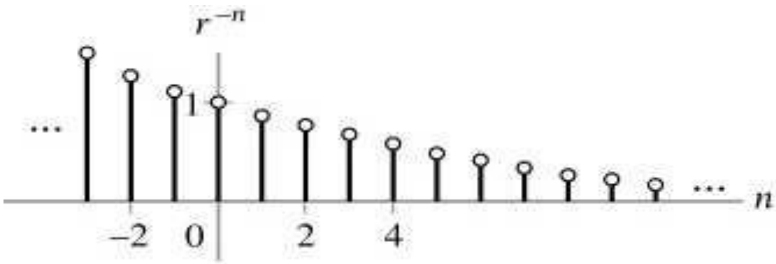
- Consider an example: the DTFT of $x[n] = \alpha^n u[n]$ does not exists for

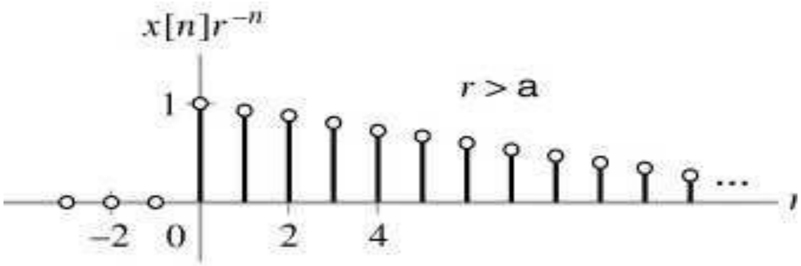
$|\alpha| > 1$

- If $r > \alpha$, then r^{-n} decays faster than $x[n]$ grows

- Signal $x[n]r^{-n}$ is absolutely summable and *z-transform* exists







The *z*-Plane and DTFT

• If $x[n]$ is absolutely summable, then DTFT is obtained from the *ztransform*

by setting $r = 1$ ($z = e^{j\Omega}$), ie. $X(e^{j\Omega}) = X(z)|_{z=e^{j\Omega}}$ as shown

in Figure

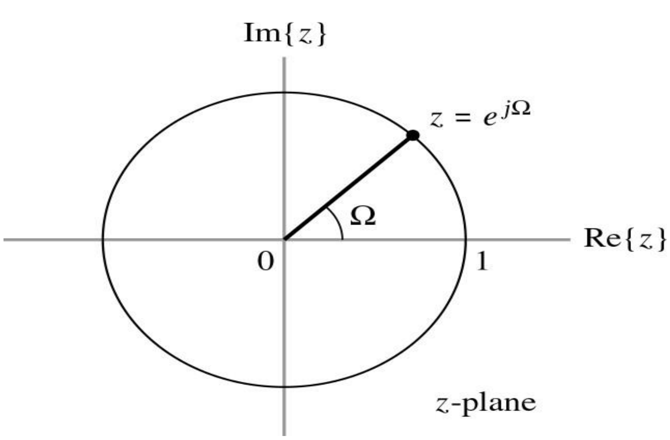


Figure : DTFT and *z*-transform

Poles and Zeros

·Commonly encountered form of the z-transform is the ratio of two polynomials in z^{-1}

$$X(z) = \frac{b_0 + b_1 z^{-1} + \ldots + b_M z^{-M}}{a_0 + a_1 z^{-1} + \ldots + b_N z^{-N}}$$

·It is useful to rewrite $X(z)$ as product of terms involving roots of the numerator and denominator polynomials

$$X(z) = \frac{\tilde{b} \prod_{k=1}^{M} (1 - c_k z^{-1})}{\prod_{k=1}^{N} (1 - d_k z^{-1})}$$

where ˜b = b0/a0

- Zeros: The $c_k$are the roots of numerator polynomials
- Poles: The $d_k$are the roots of denominator polynomials
- Locations of zeros and poles are denoted by ”o” and ”×” respectively

Region of convergence (ROC)

Properties of convergence

- ROC is related to characteristics of $x[n]$
- ROC can be identified from $X(z)$ and limited knowledge of $x[n]$
- The relationship between ROC and characteristics of the $x[n]$ is used to find inverse z-transform

Property 1: ROC can not contain any poles

·ROC is the set of all z for which z-transform converges

·$X(z)$ must be finite for all z

·If p is a pole, then $|H(p)| = \infty$ and z-transform does not converge at

the pole

·Pole can not lie in the ROC

Property 2: The ROC for a finite duration signal includes entire z-plane except $z = 0$

or/and $z = \infty$

·Let $x[n]$ be nonzero on the interval $n1 \leq n \leq n2$. The z-transform is

$$X(z) = \sum_{n=n_1}^{n_2} x[n]z^{-n}$$

The ROC for a finite duration signal includes entire z-plane except $z = 0$ or/and $z = \infty$

·If a signal is causal ($n2 > 0$) then $X(z)$ will have a term containing $z-1$, hence ROC can not include $z = 0$

·If a signal is non-causal ($n1 < 0$) then $X(z)$ will have a term containing powers of z, hence ROC can not include $z = \infty$The ROC for a finite duration signal includes entire z-plane except $z = 0$ or/and $z = \infty$

·If $n2 \leq 0$ then the ROC will include $z = 0$If $n1 \geq 0$ then the ROC will include $z = \infty$

·This shows the only signal whose ROC is entire z-plane is $x[n] = c\delta[n]$,where c is a constant

Finite duration signals

·The condition for convergence is $|X(z)| < \infty$

$$|X(z)| = |\sum_{n=-\infty}^{\infty} x[n]z^{-n}|$$

$$\leq \sum_{n=-\infty}^{\infty} |x[n]z^{-n}|$$

magnitude of sum of complex numbers $\leq$sum of individual magnitudes

· Magnitude of the product is equal to product of the magnitudes

$$\sum_{n=-\infty}^{\infty} |x[n]z^{-n}| = \sum_{n=-\infty}^{\infty} |x[n]||z^{-n}|$$

· split the sum into negative and positive time parts

· Let

$$I_-(z) = \sum_{n=-\infty}^{-1} |x[n]||z|^{-n}$$

$$I_+(z) = \sum_{n=0}^{\infty} |x[n]||z|^{-n}$$

- Note that $X(z) = I_-(z)+I_+(z)$. If both $I_-(z)$ and $I_+(z)$ are finite, then $X(z)$ if finite
- If $x[n]$ is bounded for smallest $+ve$ constants A_-, A_+, r_- and r_+ such That

$|x[n]| \leq A_-(r_-)^n,\ n < 0$

$|x[n]| \leq A_+(r_+)^n,\ n \geq 0$

· The signal that satisfies above two bounds grows no faster than $(r_+)^n$ for $+ve\ n$ and $(r_-)^n$ for $-ve\ n$

· If the $n < 0$ bound is satisfied then

$$I_-(z) \leq A_- \sum_{n=-\infty}^{-1} (r_-)^n |z|^{-n}$$

$$= A_- \sum_{n=-\infty}^{-1} \left(\frac{r_-}{|z|}\right)^n = A_- \sum_{k=1}^{\infty} \left(\frac{|z|}{r_-}\right)^k$$

- Sum converges if $|z| \leq r_-$
- If the $n \geq 0$ bound is satisfied then

$$I_+(z) = A_+ \sum_{n=0}^{\infty} (r_+)^n |z|^{-n}$$

$$= A_+ \sum_{n=0}^{\infty} \left(\frac{r_+}{|z|}\right)^n$$

• Sum converges if $|z| > r_+$

• If $r_+ < |z| < r_-$, then both $I_+(z)$ and $I_-(z)$ converge and $X(z)$ converges

To summarize:

• If $r+ > r-$ then no value of z for which convergence is guaranteed

• Left handed signal is one for which $x[n] = 0$ for $n \geq 0$

• Right handed signal is one for which $x[n] = 0$ for $n < 0$

• Two sided signal that has infinite duration in both +ve and -ve directions

• The ROC of a right-sided signal is of the form $|z| > r+$

• The ROC of a left-sided signal is of the form $|z| < r-$

• The ROC of a two-sided signal is of the form $r+ < |z| > r-$

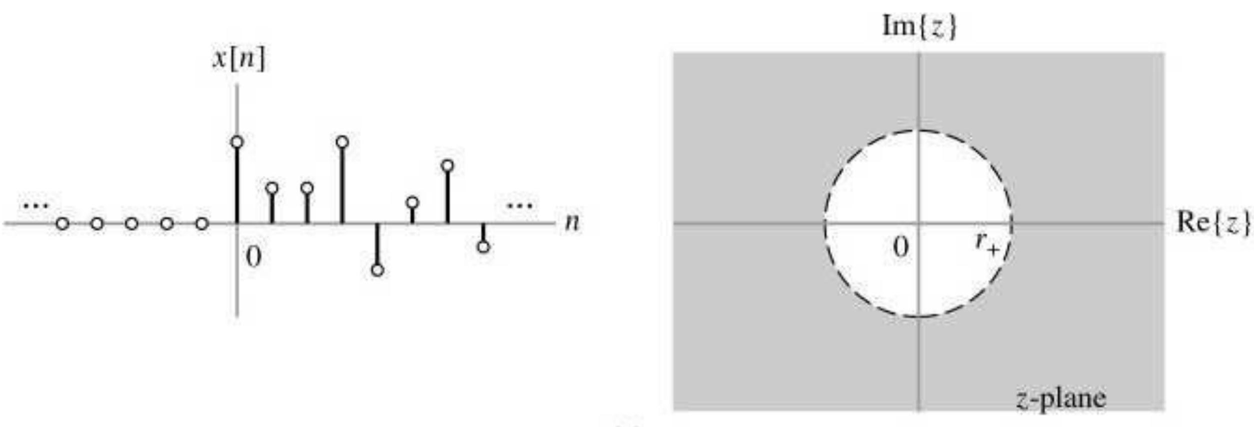


Figure : ROC of left sided sequence

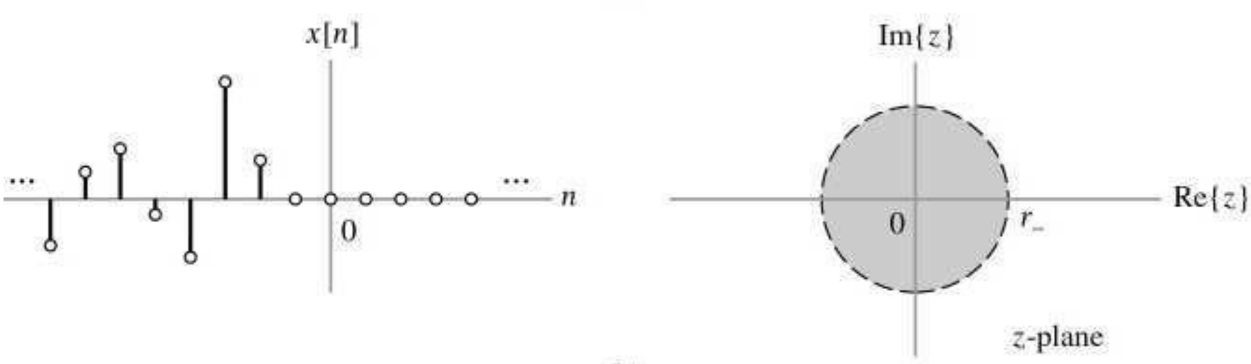


Figure: ROC of right sided sequence

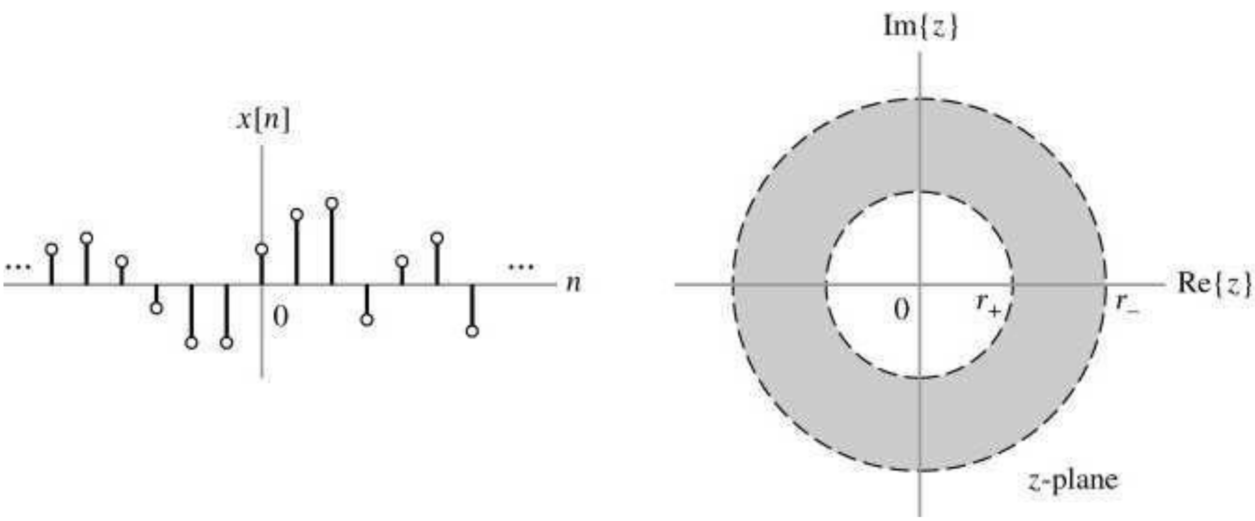


Figure: ROC of two sided sequence

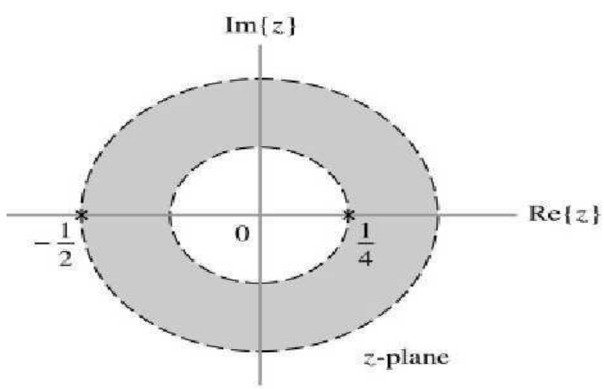


Figure : ROC of Example 1

Properties of *z*-transform

· We assume that

$$x[n] \overset{z}{\longleftrightarrow} X(z), \qquad \text{with ROC} \quad R_x$$

$$y[n] \overset{z}{\longleftrightarrow} Y(z), \qquad \text{with ROC} \quad R_y$$

• General form of the ROC is a ring in the *z*-plane, so the effect of an operation on the ROC is described by the a change in the radii of ROC

P1: Linearity

· The *z*-transformof a sum of signals is the sumof individual *z*-transforms

$$ax[n] + by[n] \overset{z}{\longleftrightarrow} aX(z) + bY(z),$$ with ROC at least $R_x \cap R_y$

P2: Time reversal

• Time reversal or reflection corresponds to replacing z by z^{-1}. Hence, if R_x is of the form $a < |z| < b$ then the ROC of the reflected signal is $a < 1/|z| < b$ or $1/b < |z| < 1/a$

If

$$x[n] \overset{z}{\longleftrightarrow} X(z),$$ with ROC R_x

Then

$$x[-n] \overset{z}{\longleftrightarrow} X\left(\frac{1}{z}\right),$$ with ROC $1/R_x$

Proof: Time reversal

· Let

$y[n] = x[$

$$Y(z) = \sum_{n=-\infty}^{\infty} x[-n] z^{-n}$$

Let $l = -n$, then

$$Y(z) = \sum_{l=-\infty}^{\infty} x[l] z^{l}$$

$$Y(z) = \sum_{l=-\infty}^{\infty} x[l] \left(\frac{1}{z}\right)^{-l}$$

$$Y(z) = X\left(\frac{1}{z}\right)$$

P3: Time shift

· Time shift of *no* in the time domain corresponds to multiplication of z^{-no} in the z-domain

If

$$x[n] \stackrel{z}{\longleftrightarrow} X(z),$$ with ROC *Rx*

Then

$$x[n - n_o] \stackrel{z}{\longleftrightarrow} z^{-n_o} X(z),$$ with ROC *Rx* except $z = 0$ or $|z| = \infty$

Time shift, *no* >0

· Multiplication by z^{-no} introduces a pole of order *no* at $z = 0$

· The ROC can not include $z = 0$, even if *Rx* does include $z = 0$

· If $X(z)$ has a zero of at least order *no* at $z = 0$ that cancels all of the new poles then ROC can include $z = 0$

Time shift, *no* <0

· Multiplication by z^{-no} introduces *no* poles at infinity

· If these poles are not canceled by zeros at infinity in $X(z)$ then the ROC of $z^{-no}X(z)$ can not include $|z| = \infty$

Proof: Time shift

· Let

$$y[n] = x[n - n_o]$$

$$Y(z) = \sum_{n=-\infty}^{\infty} x[n - n_o] z^{-n}$$

Let $l = n - n_o$, then

$$Y(z) = \sum_{l=-\infty}^{\infty} x[l] z^{-(l+n_o)}$$

$$Y(z) = z^{-n_o} \sum_{l=-\infty}^{\infty} x[l] z^{-l}$$

$$Y(z) = z^{-n_o} X(z)$$

P4: Multiplication by α^n

· Let α be a complex number

If

$$x[n] \overset{z}{\longleftrightarrow} X(z),$$

with ROC R_x

Then

$$\alpha^n x[n] \xleftrightarrow{z} X(\frac{z}{\alpha}),$$ with ROC $|\alpha|Rx$

· $|\alpha|Rx$ indicates that the ROC boundaries are multiplied by $|\alpha|$.

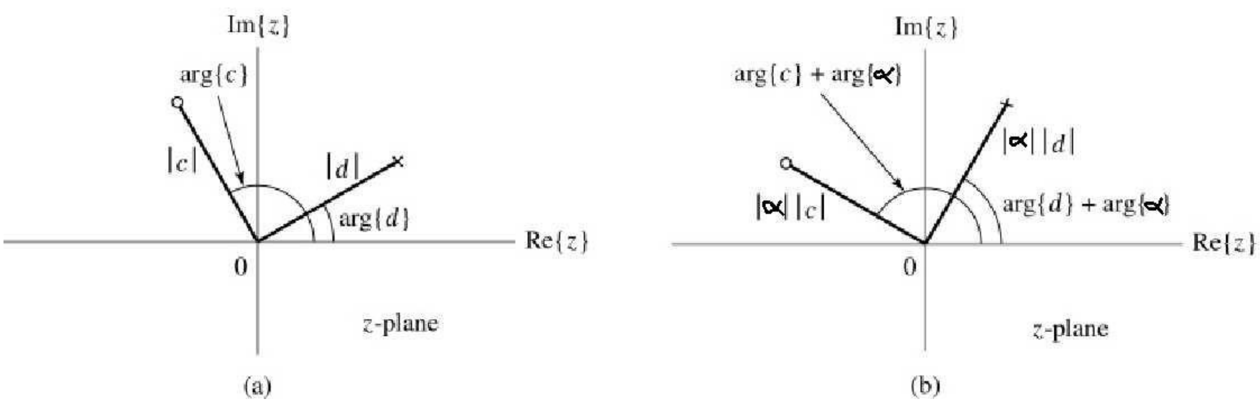
·If Rx is $a < |z| < b$ then the new ROC is $|\alpha|a < |z| < |\alpha|b$

·If $X(z)$ contains a pole d, ie. the factor $(z-d)$ is in the denominator then $X(z/\alpha)$ has a factor $(z-\alpha d)$ in the denominator and thus a pole atαd.

·If $X(z)$ contains a zero c, then $X(z/\alpha)$ has a zero at αc

·This indicates that the poles and zeros of $X(z)$ have their radii changed by $|\alpha|$

·Their angles are changed by $arg\{\alpha\}$



·If $|\alpha| = 1$ then the radius is unchanged and if α is +ve real number then the angle is unchanged

Proof: Multiplication by α^n

·Let $y[n] = \alpha^n x[n]$

$$Y(z) = \sum_{n=-\infty}^{\infty} \alpha^n x[n] z^{-n}$$

$$Y(z) = \sum_{l=-\infty}^{\infty} x[l] (\frac{z}{\alpha})^{-n}$$

$$Y(z) = X(\frac{z}{\alpha})$$

P5: Convolution

·Convolution in time domain corresponds to multiplication in the *z*domain

If

$x[n] \xleftrightarrow{z} X(z),$ with ROC *Rx*

$y[n] \xleftrightarrow{z} Y(z),$ with ROC *Ry*

$x[n] * y[n] \xleftrightarrow{z} X(z)Y(z),$ with ROC at least $Rx \cap Ry$

·Similar to linearity the ROC may be larger than the intersection of *Rx* and*Ry*

Proof: Convolution

·Let

$$c[n] = x[n] * y[n]$$

$$C(z) = \sum_{n=-\infty}^{\infty} (x[n] * y[n]) z^{-n}$$

$$C(z) = \sum_{n=-\infty}^{\infty} (\sum_{k=-\infty}^{\infty} x[k] * y[n-k]) z^{-n}$$

$$C(z) = \sum_{k=-\infty}^{\infty} x[k] (\underbrace{(\sum_{n=-\infty}^{\infty} y[n-k]) z^{-(n-k)})}_{Y(z)} z^{-k}$$

$$C(z) = (\underbrace{\sum_{k=-\infty}^{\infty} x[k] z^{-k}}_{X(z)}) Y(z)$$

$$C(z) = X(z) Y(z)$$

P6: Differentiation in the *z* domain

·Multiplication by *n* in the time domain corresponds to differentiation with respect to *z* and multiplication of the result by $-z$ in the *z*-domain.

If

$x[n] \overset{z}{\longleftrightarrow} X(z),$ with ROC *Rx* Then

$nx[n] \overset{z}{\longleftrightarrow} -z \frac{d}{dz} X(z)$ with ROC *Rx*

·ROC remains unchanged

Proof: Differentiation in the *z* domain

·We know

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

Differentiate with respect to *z*

$$\frac{d}{dz}X(z) = \sum_{n=-\infty}^{\infty} (-n)x[n]z^{-n}z^{-1}$$

• Multiply with −*z*

$$-z\frac{d}{dz}X(z) = \sum_{n=-\infty}^{\infty} -(-n)x[n]z^{-n}z^{-1}z$$

$$-z\frac{d}{dz}X(z) = \sum_{n=-\infty}^{\infty} nx[n]z^{-n}$$

Then

$$nx[n] \xleftrightarrow{z} -z\frac{d}{dz}X(z)$$ with ROC *Rx*

Inverse *z*-transform

Partial fraction method

· In case of LTI systems, commonly encountered form of *z*-transform is

$$X(z) = \frac{B(z)}{A(z)}$$

$$X(z) = \frac{b_0 + b_1 z^{-1} + \ldots + b_M z^{-M}}{a_0 + a_1 z^{-1} + \ldots + a_N z^{-N}}$$ Usually $M < N$

· If $M > N$ then use long division method and express $X(z)$ in the form

$$X(z) = \sum_{k=0}^{M-N} f_k z^{-k} + \frac{\tilde{B}(z)}{A(z)}$$

where $\tilde{B}(z)$ now has the order one less than the denominator polynomial and use partial fraction method to find z-transform

· The inverse z-transform of the terms in the summation are obtained from the transform pair and time shift property

$$1 \xleftrightarrow{z} \delta[n]$$

$$z^{-n_o} \xleftrightarrow{z} \delta[n-n_o]$$

· If $X(z)$ is expressed as ratio of polynomials in z instead of $z-1$ then convert into the polynomial of z^{-1}

· Convert the denominator into product of first-order terms

$$X(z) = \frac{b_0 + b_1 z^{-1} + \ldots + b_M z^{-M}}{a_0 \prod_{k=1}^{N} (1 - d_k z^{-1})}$$

where d_k are the poles of $X(z)$

For distinct poles

· For all distinct poles, the $X(z)$ can be written as

$$X(z) = \sum_{k=1}^{N} \frac{A_k}{(1 - d_k z^{-1})}$$

·Depending on ROC, the inverse z-transform associated with each term is then determined by using the appropriate transform pair

·We get

$$A_k(d_k)^n u[n] \stackrel{z}{\longleftrightarrow} \frac{A_k}{1-d_k z^{-1}},$$ with ROC $z > dk$OR

$$-A_k(d_k)^n u[-n-1] \stackrel{z}{\longleftrightarrow} \frac{A_k}{1-d_k z^{-1}},$$ with ROC $z < dk$

·For each term the relationship between the ROC associated with $X(z)$ and each pole determines whether the right-sided or left sided inversetransform is selected

For Repeated poles

·If pole di is repeated r times, then there are r terms in the partial fraction expansion associated with that pole

$$\frac{A_{i_1}}{1-d_i z^{-1}}, \frac{A_{i_2}}{(1-d_i z^{-1})^2}, \ldots, \frac{A_{i_r}}{(1-d_i z^{-1})^r}$$

·Here also, the ROC of $X(z)$ determines whether the right or left sided inverse transform is chosen

$$A\frac{(n+1)\ldots(n+m-1)}{(m-1)!}(d_i)^n u[n] \stackrel{z}{\longleftrightarrow} \frac{A}{(1-d_i z^{-1})^m},$$ withROC$|z|>di$

·If the ROC is of the form $|z| < di$, the left-sided inverse z-transform is chosen, ie

$$-A\frac{(n+1)\ldots(n+m-1)}{(m-1)!}(d_i)^n u[-n-1] \stackrel{z}{\longleftrightarrow} \frac{A}{(1-d_i z^{-1})^m},$$

Deciding ROC

· The ROC of $X(z)$ is the intersection of the ROCs associated with the individual terms in the partial fraction expansion.

· In order to chose the correct inverse z-transform, we must infer the ROC of each term from the ROC of $X(z)$.

· By comparing the location of each pole with the ROC of $X(z)$.

· Chose the right sided inverse transform: if the ROC of $X(z)$ has the radius greater than that of the pole associated with the given term

· Chose the left sided inverse transform: if the ROC of $X(z)$ has the radius less than that of the pole associated with the given term

Power series expansion

· Express $X(z)$ as a power series in z^{-1} or z as given in z-transform equation

· The values of the signal $x[n]$ are then given by coefficient associated with z^{-n}

· Main disadvantage: limited to one sided signals

· Signals with ROCs of the form $|z| > a$ or $|z| < a$

· If the ROC is $|z| > a$, then express $X(z)$ as a power series in z^{-1} and we get right sided signal

· If the ROC is $|z| < a$, then express $X(z)$ as a power series in z and we get left sided signal.

UNIT 8

Z TRANSFORMS-2

Learning Objectives: Transform analysis of LTI Systems, unilateral Z Transform and its application to solve difference equations

The Transfer Function

- We have defined the transfer function as the *z*-transform of the impulse response of an LTI system

$$H(z) = \sum_{k=-\infty}^{\infty} h[k]z^{-k}$$

- Then we have $y[n] = x[n] * h[n]$ and $Y(z) = X(z)H(z)$
- This is another method of representing the system
- The transfer function can be written as

$$H(z) = \frac{Y(z)}{X(z)}$$

- This is true for all z in the ROCs of $X(z)$ and $Y(z)$ for which $X(z)$ in nonzero
- The impulse response is the *z*-transform of the transfer function
- We need to know ROC in order to uniquely find the impulse response
- If ROC is unknown, then we must know other characteristics such as stability or causality in order to uniquely find the impulse response

Relation between transfer function and difference equation

• The transfer can be obtained directly from the difference-equation description of an LTI system

• We know that

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

• We know that the transfer function $H(z)$ is an eigen value of the system associated with the eigen function z^n, ie. if $x[n] = z^n$ then the output of an LTI system $y[n] = z^n H(z)$

• Put $x[n-k] = z^{n-k}$ and $y[n-k] = z^{n-k} H(z)$ in the difference equation,

we get

$$z^n \sum_{k=0}^{N} a_k z^{-k} H(z) = z^n \sum_{k=0}^{M} b_k z^{-k}$$

• We can solve for $H(z)$

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

• The transfer function described by a difference equation is a ratio of polynomials in z^{-1} and is termed as a rational transfer function.

• The coefficient of z^{-k} in the numerator polynomial is the coefficient associated with $x[n-k]$ in the difference equation

• The coefficient of z^{-k} in the denominator polynomial is the coefficient associated with $y[n-k]$ in the difference equation

• This relation allows us to find the transfer function and also find thedifference equation description for a system, given a rational function

Transfer function

• The poles and zeros of a rational function offer much insight into LTI system characteristics

• The transfer function can be expressed in pole-zero form by factoring the numerator and denominator polynomial

• If c_k and d_k are zeros and poles of the system respectively and $\tilde{b} = b_0/a_0$ is the gain factor, then

$$H(z) = \frac{\tilde{b} \prod_{k=1}^{M} (1 - c_k z^{-1})}{\prod_{k=1}^{N} (1 - d_k z^{-1})}$$

• This form assumes there are no poles and zeros at $z = 0$

• The pth order pole at $z = 0$ occurs when $b_0 = b_1 = \ldots = b_{p-1} = 0$

• The lth order zero at $z = 0$ occurs when $a_0 = a_1 = \ldots = a_{l-1} = 0$

• Then we can write $H(z)$ as

$$H(z) = \frac{\tilde{b} z^{-p} \prod_{k=1}^{M-p} (1 - c_k z^{-1})}{z^{-l} \prod_{k=1}^{N-l} (1 - d_k z^{-1})}$$

where $\tilde{b} = b_p/a_l$

• In the example we had first order pole at $z = 0$

• The poles, zeros and gain factor $\tilde{b}$ uniquely determine the transfer function

• This is another description for input-output behavior of the system

• The poles are the roots of characteristic equation

Causality, stability and Inverse systems

Causality

• The impulse response of an LTI system is zero for $n < 0$

• The impulse response of a causal LTI system is determined from the transfer function by using right sided inverse transforms

• The pole inside the unit circle in the z-plane contributes an exponentially decaying term

• The pole outside the unit circle in the z-plane contributes an exponentially increasing term

Stability

• The system is stable: if impulse response is absolutely summable and DTFT of impulse response exists

• The ROC must include the unit circle: the pole and unit circle together define the behavior of the system

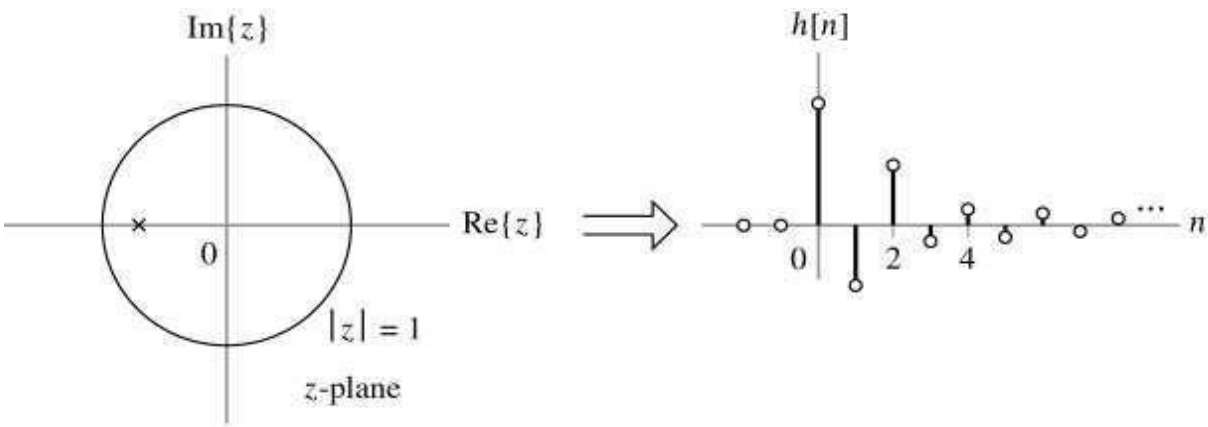


Figure: When the pole is inside the unit circle

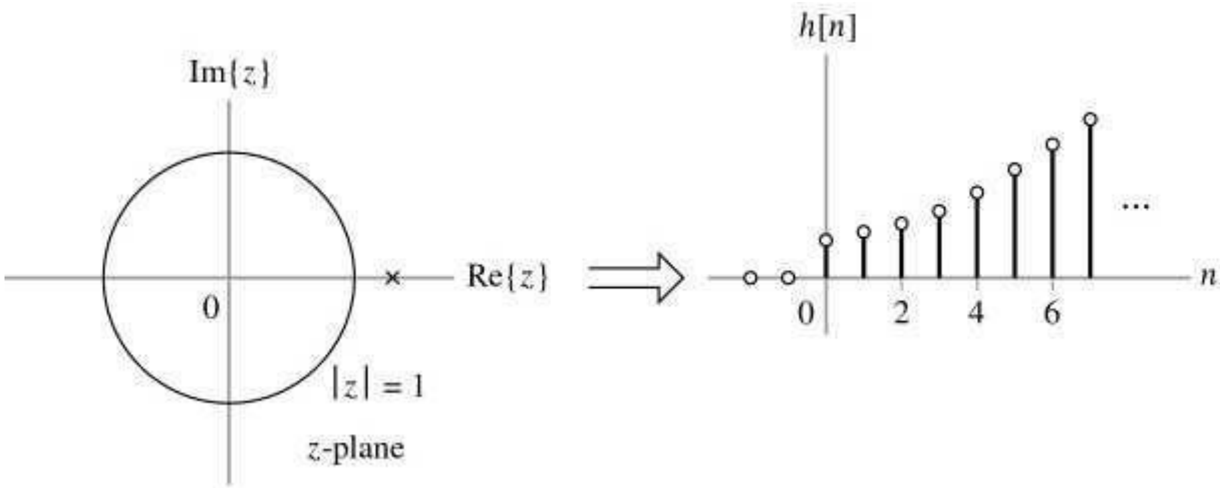


Figure: When the pole is outside the unit circle

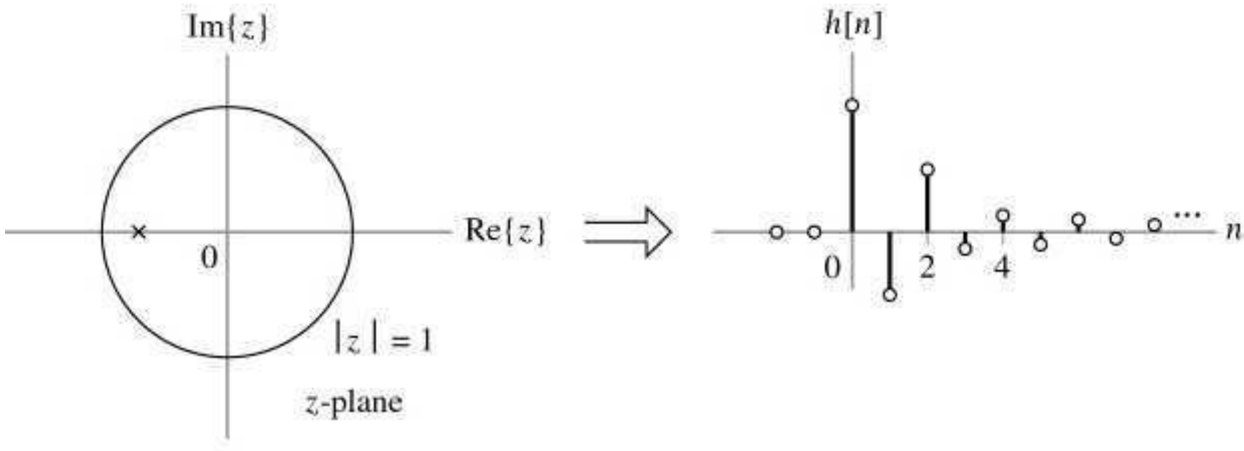


Figure: Stability: When the pole is inside the unit circle

• A stable impulse response can not contain any increasing exponential term

• The pole inside the unit circle in the *z*-plane contributes right-sided exponentially decaying term

• The pole outside the unit circle in the *z*-plane contributes left-sided exponentially decaying term

Causal and stable system

• Stable and causal LTI system: all the poles must be inside the unit circle

• A inside pole contributes right sided or causal exponentially decaying system

• A outside pole contributes either left sided decaying term which is not causal or right-sided exponentially increasing term which is not stable

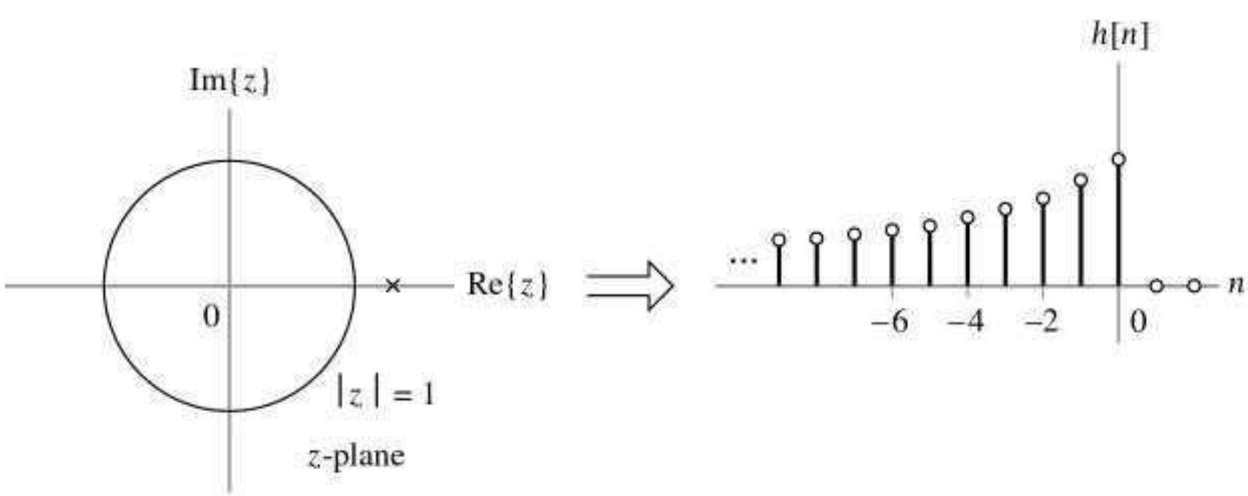


Figure : Stability: When the pole is outside the unit circle

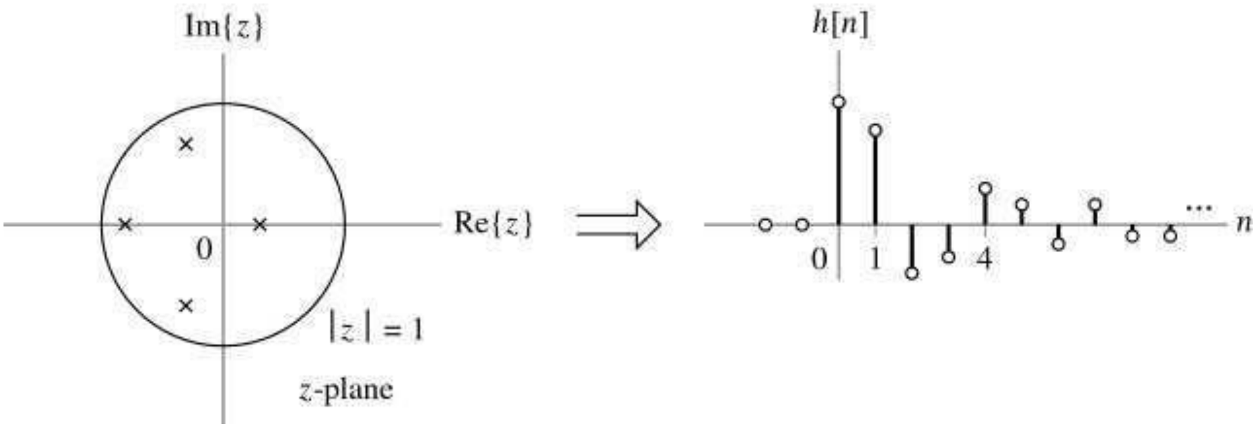


Figure : Location of poles for the causal and stable system

• Example of stable and causal system: all the poles are inside the unit Circle

Inverse system

• Impulse response (h^{inv}) of an inverse system satisfies $h^{inv}[n] * h[n] = \delta[n]$ where $h[n]$ is the impulse response of a system to be inverted

• Take inverse z-transform on both sides gives

$H^{inv}(z)H(z) = 1$

$Hinv(z) = 1/H(z)$

• The transfer function of an LTI inverse system is the inverse of thetransfer function of the system that we desire to invert

• If we write the pole-zero form of $H(z)$ as

$$H(z) = \frac{\tilde{b} z^{-p} \prod_{k=1}^{M-p}(1 - c_k z^{-1})}{z^{-l} \prod_{k=1}^{N-l}(1 - d_k z^{-1})}$$

where $\tilde{b} = bp/al$

• Then we can write H^{inv}as

$$H^{inv}(z) = \frac{z^{-l}\prod_{k=1}^{N-l}(1-d_k z^{-1})}{\tilde{b}z^{-p}\prod_{k=1}^{M-p}(1-c_k z^{-1})}$$

• The zeros of $H(z)$ are the poles of $H^{inv}(z)$

• The poles of $H(z)$ are the zeros of $H^{inv}(z)$

• System defined by a rational transfer function has an inverse system

• We need inverse systems which are both stable and causal to invert the distortions introduced by the system

• The inverse system $H^{inv}(z)$ is stable and causal if all poles are inside the unit circle

• Poles of $H^{inv}(z)$ are zeros of (z)

• Inverse system $H^{inv}(z)$: stable and causal inverse of an LTI system $H(z)$ exists *if and only if* all the zeros of $H(z)$ are inside the unit circle

• The system with all its poles and zeros inside the unit circle is called as *minimum-phase* system

• The magnitude response is uniquely determined by the phase response and vice-Vera

• For a *minimum-phase* system the magnitude response is uniquely determined by the phase response and vice-versa

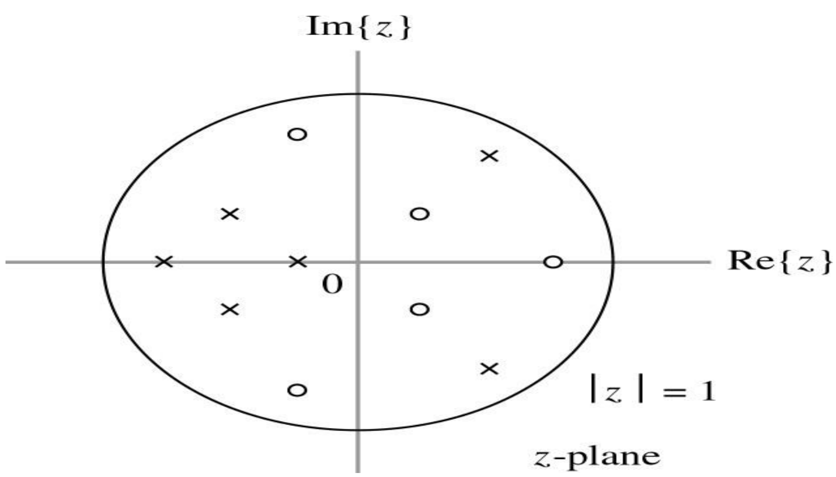


Figure: Location of poles in a *minimum-phase* system

Unilateral z-transform

· Useful in case of causal signals and LTI systems

· The choice of time origin is arbitrary, so we may choose $n = 0$ as the time at which the input is applied and then study the response for times $n \geq 0$

Advantages

· We do not need to use ROCs

· It allows the study of LTI systems described by the difference equation with initial conditions

Unilateral z-transform

· The unilateral z-transform of a signal $x[n]$ is defined as

$$X(z) = \sum_{n=0}^{\infty} x[n] z^{-n}$$

which depends only on $x[n]$ for $n \geq 0$

· The unilateral and bilateral z-transforms are equivalent for causal signals

$$\alpha^n u[n] \xleftrightarrow{z_u} \frac{1}{1 - \alpha z^{-1}}$$

$$a^n \cos(\Omega_o n) u[n] \xleftrightarrow{z_u} \frac{1 - a\cos(\Omega_o) z^{-1}}{1 - 2a\cos(\Omega_o) z^{-1} + a^2 z^{-2}}$$

Properties

· The same properties are satisfied by both unilateral and bilateral ztransformswith one exception: the time shift property

·The time shift property for unilateral z-transform: Let $w[n] = x[n-1]$

·The unilateral z-transform of $w[n]$ is

$$W(z) = \sum_{n=0}^{\infty} w[n]z^{-n} = \sum_{n=0}^{\infty} x[n-1]z^{-n}$$

$$W(z) = x[-1] + \sum_{n=1}^{\infty} x[n-1]z^{-n}$$

$$W(z) = x[-1] + \sum_{m=0}^{\infty} x[m]z^{-(m+1)}$$

·The unilateral z-transform of $w[n]$ is

$$W(z) = x[-1] + z^{-1}\sum_{m=0}^{\infty} x[m]z^{-m}$$

$$W(z) = x[-1] + z^{-1}X(z)$$

·A one-unit time shift results in multiplication by $z-1$ and addition of

the constant $x[-1]$

·In a similar way, the time-shift property for delays greater than unity is

$$x[n-k] \xleftrightarrow{z_u} x[-k] + x[-k+1]z^{-1} +$$

$$\ldots + x[-1]z^{-k+1} + z^{-k}X(z) \quad \text{for } k > 0$$

- In the case of time advance, the time-shift property changes to

$$x[n+k] \xleftrightarrow{z_u} -x[0]z^k - x[-1]z^{k-1} +$$

$$\ldots - x[k-1]z + z^k X(z) \ \text{ for } \ k > 0$$