Review: Time – Varying Fields

In the dynamics case, we can distinguish between two regimes:

Low Frequency (Slowly-Varying Fields) – The displacement current is negligible in the Maxwell's equations, since

$$\left| \frac{\partial \vec{D}(t)}{\partial t} \right| << \left| \vec{J}(t) \right|$$

High Frequency (Fast-Varying Fields) – The general set of Maxwell's equations must be considered, with no approximations.

In the low frequency regime we use the complete set of Maxwell's equations, but the displacement current is omitted

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{dt}$$

$$\nabla \times \vec{H} = \vec{J}$$

$$\nabla \cdot \vec{D} = \rho$$

$$\nabla \cdot \vec{B} = 0$$

$$\vec{D} = \varepsilon \, \vec{E}$$

$$\vec{B} = \mu \, \vec{H}$$

$$\vec{F} = q\left(\vec{E} + \vec{v} \times \vec{B}\right)$$

The concept of low frequency and slowly-varying phenomena is relative to the situation at hand. Any disturbance (time-variation) of the electromagnetic field propagates at the speed of light. If a length L is the maximum dimension of the system under study, the maximum propagation time for a disturbance is

Maximum Propagation Time ⇒ $t_d = \dfrac{\mathrm{L}}{v_p}$ ⇐ **Maximum Length** (L), **Phase velocity of light** (v_p)

We can assume slow-varying fields if the currents are practically constant during this time period.

For sinusoidal currents, with a period of oscillation T, we have

Period ⇒ $T = \dfrac{1}{f} = \dfrac{\lambda}{v_p} \gg t_d$ ⇐ **Wavelength** (λ), **Frequency** ⇒ f

and $\mathrm{L} \ll \lambda$

The electric potential is now by itself insufficient to completely describe the time-varying electric field, because there is also a direct dependence on the magnetic field variations. By recalling the definition of magnetic vector potential, we can derive a relationship between electric field and electric potential

Time-Varying Fields
$$\nabla \times \vec{E}(t) = -\frac{\partial \vec{B}(t)}{\partial t} = -\frac{\partial}{\partial t} \nabla \times \vec{A}(t)$$ $$\Rightarrow \nabla \times \left(\vec{E}(t) + \frac{\partial \vec{A}(t)}{\partial t} \right) = 0$$ $$\vec{E}(t) + \frac{\partial \vec{A}(t)}{\partial t} = -\nabla \phi(t)$$

Statics
$$\nabla \times \vec{E} = 0$$ $$\vec{E} = -\nabla \phi$$

We can also obtain an integral relation between electric field and magnetic flux, by integrating the curl of the electric field over a surface

$$\int\!\!\int_S \nabla \times \vec{E}(t) \cdot d\vec{S} = \int\!\!\int_S -\frac{\partial \vec{B}(t)}{\partial t}\, dS = -\frac{\partial}{\partial t} \underbrace{\int\!\!\int_S \vec{B}(t) \cdot d\vec{S}}_{\text{Magnetic Flux } \Phi(t)}$$

Stoke's Theorem

$$\oint \vec{E} \cdot d\vec{l} = -\frac{\partial\, \Phi(t)}{\partial t}$$

In the electrostatic case, we do not need to distinguish between voltage and potential difference. The voltage between two points is always defined as

$$V_{ba} = -\int_a^b \vec{E} \cdot d\vec{l} = -e.m.f.$$

but in terms of potential ϕ we have

Time-Varying Fields	Statics
$V_{ba}(t) = \int_a^b \left(\nabla\phi + \dfrac{\partial \vec{A}}{\partial t} \right) \cdot d\vec{l}$ $= \phi_b - \phi_a + \dfrac{\partial}{\partial t} \int_a^b \vec{A}(t) \cdot d\vec{l}$	$V_{ba} = -\int_a^b \vec{E} \cdot d\vec{l}$ $= \phi_b - \phi_a$

Note that for time-varying fields the line integral of the magnetic vector potential between two given points depends on the actual path of integration. In general:

$$\int_a^b \vec{A}(t) \cdot d\vec{l} \neq \vec{A}(b,t) - \vec{A}(a,t)$$

Consider now the integral of the electric field along a closed path:

Time-varying fields	Statics
$\oint \vec{E}(t) \cdot d\vec{l} \neq 0$	$\oint \vec{E}(t) \cdot d\vec{l} = 0$

The closed path could be a metallic wire which confines the current due to moving electric charge.

The line integral of the electric field gives the work necessary to move a unit charge along the path of integration, under the influence of time-varying electric and magnetic fields.

For a closed wire loop at rest, the work necessary to move a unit charge once around the loop is

$$W = \oint \frac{\text{Force}}{\text{Charge}} \cdot d\vec{l} = \oint \vec{E}(t) \cdot d\vec{l} = \int \nabla \times \left(\vec{E}(t)\right) \cdot d\vec{S}$$

$$= \int_S -\frac{\partial \vec{B}(t)}{\partial t} \cdot d\vec{S} = -\frac{\partial}{\partial t} \int_S \vec{B}(t) \cdot d\vec{S}$$

$$= -\frac{\partial}{\partial t} \underbrace{\Phi(t)}_{\text{Magnetic Flux}}$$

As a more general case, consider a wire loop in motion. The complete Lorentz force must be considered:

$$W = e.m.f. = \oint \frac{\text{Force}}{\text{Charge}} \cdot d\vec{l} = \oint \left(\vec{E}(t) + \vec{v}(t) \times \vec{B}(t) \right) \cdot d\vec{l}$$

$$= \int \left(\nabla \times \left(\vec{E}(t) + \vec{v}(t) \times \vec{B}(t) \right) \right) \cdot d\vec{S}$$

$$= \int \left(\underbrace{-\frac{\partial \vec{B}(t)}{\partial t} + \nabla \times \left(\vec{v}(t) \times \vec{B}(t) \right)}_{\dfrac{d\vec{B}(t)}{dt}} \right) \cdot d\vec{S} = -\frac{d}{dt} \underbrace{\int \vec{B}(t) \cdot d\vec{S}}_{\text{Flux } \Phi(t)}$$

If the velocity of motion is constant, note that

$$\nabla \times \left(\vec{v} \times \vec{B}(t) \right) =$$

$$\vec{v} \underbrace{\nabla \cdot \vec{B}}_{0} - \vec{B} \underbrace{\nabla \cdot \vec{v}}_{0} + \underbrace{\left(\vec{B} \cdot \nabla \right) \vec{v}}_{0} - \left(\vec{v} \cdot \nabla \right) \vec{B} = -\left(\vec{v} \cdot \nabla \right) \vec{B}$$