Operations Research.

Unit - I

Development :-

→ In order to understand 'what Operations Research is today', we must know something of its history & evolution.

→ The main origin of Operation Research was during the Second World-war. At that time, the military management in England called upon a team of scientists to study the strategic & tactical problems related to air & land defence of the Country. Since they were having very limited military resources, it was necessary to decide upon the most effective utilization of them, e.g. the efficient ocean transport, effective bombing, etc.

→ During World-War II the Military Commands of U.K. & U.S.A. engaged several inter-disciplinary teams of scientists to undertake scientific research into strategic & tactical military operations.

→ Their mission was to formulate specific proposals and plans for aiding the Military commands to arrive at the decisions on optimal utilization of scarce military resources & efforts, & also to implement the decisions effectively. The OR teams were not actually engaged in military operations & in fighting the war. But, they were only advisors & significantly instrumental in winning the war to the extent that the scientific & systematic approaches involved in OR provided a good intellectual support to the strategic initiatives of the military commands. Hence OR can be associated with "an art of winning the war without actually fighting it".

→ As the name implies, 'Operations Research' was apparently invented because the team was dealing with research on (military) operations. The work of this team of scientists was named as Operations Research in England.

→ The encouraging results obtained by the British OR teams quickly motivated the United states military management to start with similar activities.

→ Successful applications of the U.S teams included the ~~United States military management to start with similar activities~~: invention of new flight patterns, planning sea mining & effective utilization of electronic equipment.

→ The work of ~~the~~ OR team was given various names in the United states: Operational Analysis, Operations Evaluation, Operations Research, Systems Analysis, Systems Evaluation, Systems Research, ~~Systems Analysis~~ & Management Science. The name Operations Research was and is the most widely used.

Definition:-

→ OR is a scientific method of providing executive department with a quantitative basis for decision regarding the operations under their control. —Morse & Kimbal.

→ OR is a scientific method of providing executive depts with an analytical & objective basis for decisions.

→ OR is the application of scientific methods, techniques & tools to problems involving the operations of systems so as to provide these in control of the operations with optimum solutions to the problem.

→ OR is the art of giving bad answers to problems to which otherwise worse answers are given.

→ OR is the art of winning the war without actually fighting it.

→ OR is a scientific approach to problem solving for executive management.

→ OR is the systematic method of oriented study of the basic structure, characteristics, functions & relationships of an organization to provide the executive with a sound, scientific & quantitative basis for decision making.

Main Characteristics (Features) of Operations Research :-

he main characteristics of OR are as follows:

1. Inter-disciplinary Team approach:- In OR the optimum solution is found by a team of scientists selected from various disciplines such as mathematics, statistics, economics, engineering, physics, etc.

For eg; while investigating the inventory management in a factory, perhaps we may require an engineer who knows the functions of various iteams of stores. We also require a cost accountant & a mathematician-cum-statistician. Each member of such OR team is benefitted by the view points of others, so that the workable solution obtained through such collaborative study has a greater chance of acceptance by management.

2. Wholistic approach to the system:- The most of the problem tackled by OR have the characteristic that OR tries to find the best (optimum) decisions relative to largest possible portion of the total organization. The nature of organization is essentially immaterial.

for eg; in attempting to solve a maintanance problem in a factory, OR tries to ~~find~~ consider ~~the best topti~~ how this affects ~~on~~ the production department as a whole. If possible, it also tries to consider how this affects the production department ~~as a whole~~ in turn affects other departments & the business as a whole. It may even try to go further and investigate how the effect on this particular business organization in turn affects ~~other department & the business as a whole. It may even try to go further & investigate how the effect on this particular business organization in turn~~ affects the industry as a whole, etc.

3. Imperfectness of solutions:- By OR techniques we cannot obtain perfect answers to our problems But, only the quality of the solution is improved from worse to bad answers.

4. Use of Scientific research:- OR uses techniques of scientific research to reach the optimum solution.

5. To optimize the total output:- OR tries to optimize total return by maximizing the profit & minimizing the cost or loss.

Main Phases of Operations Research :-

→ The procedure for an OR study generally involves the following major phases:-

Phase I: Formulating the problem:- Before proceeding to find the solution of a problem, first of all one must be able to formulate the problem in the form of an appropriate model. To do so, the following information will be required.

(i) Who has to take the decision?
(ii) What are the objectives?
(iii) What are the ranges of controlled variables?
(iv) What are the uncontrolled variables that may affect the possible solutions?
(v) What are the restrictions or constraints on the variables?

Since wrong formulation cannot yield a right decision (solⁿ), one must be considerably careful while execution of this phase.

Phase II: Constructing a mathematical model:- The second phase of the investigations is concerned with the reformulation of the problem in an appropriate form which is convenient for analysis. The most suitable form of this purpose is to construct a mathematical model representing the system under study. It requires the identification of both static & dynamic structural elements. A mathematical model should include the following three important basic factors:

(i) Decision variables & parameters
(ii) Constraints or Restrictions
(iii) Objective function.

Phase III: Deriving the solutions from the model: This phase is devoted to the Computation of those values of decision variables that maximise (or minimise) the objective function. Such solution is called an optimal solution which is always in the best interest of the problem under consideration. ~~The general techniques for deriving the solution of OR model are discussed in the following sections & further details are given in the~~

Phase IV: Testing the model & its Solution (updating the model):— After Completing the model, it is once again tested as a whole for the errors if any. A model may be said to be valid if it can provide a reliable prediction of the system's performance. A good practitioner of Operations Research realises that his model be applicable for a longer time & thus he updates the model time to time by taking into account the past, present & future specifications of the problem.

Phase V:— Controlling the Solution:— This phase establishes control over the solution with any degree of satisfaction. The model requires immediate modification as soon as the controlled variables (one or more) change significantly, otherwise the model goes out of control. As the Conditions are constantly changing in the world, the model & the solution may not remain valid for a long time.

Phase VI: Implementing the Solution:— Finally, the tested results of the model are implemented to work. This phase is primarily executed with the cooperation of Operations Research experts & those who are responsible for managing & operating the systems.

Modelling in Operations Research

Definition:- A model used in OR is defined as a representation of an actual object or situation. It shows the relationships & inter-relationships ~~&~~ of action & reaction in terms of cause & effect.

① Distribution Model ③ Waiting line or queing model ⑤ Network Model
② Production & Inventory Model ④ Game theory (Competitive strategy model)

→ Since, a model is an abstraction of reality, it thus appears to be less complete than reality itself. For a model to be complete, it must be a representative of those aspects of ~~ref~~reality that are being investigated.

→ The main objective of ~~the~~ a model is to provide means for analyzing the behaviour of the system for the purpose of improving its performance.

Models can be classified according to following characteristics:-

① **Classification by Structure**

⑥ Job sequencing model ⑧ Simulation model
⑦ Replacement model

(i) **Iconic models:** Iconic models represent the system as it is by scaling it up or down (i.e., by enlarging or reducing the size).

For eg:- a toy airplane is an iconic model of a real one.

(ii) **Analogue models:-** The models in which one set of properties is used to represent another set of properties, are called analogue models. After the problem is solved, the solution is reinterpreted in terms of the original system.

For eg:- graphs, are very simple analogues because distance is used to represent the properties such as: time, number, percent, age, weight and many other properties.

(iii) **Symbolic (Mathematical) models:-** The symbolic or mathematical model is one which employs a set of mathematical symbols (i.e., letters, numbers, etc) to represent the decision variables of the system. These variables are related together by means of a mathematical equation or a set of equations to describe the behaviour of the system. The solⁿ of the problem is then obtained by applying well-developed mathematical techniques to the model.

→ The symbolic model is usually the easiest to manipulate experimentally & it is most general & abstract.

② **Classification by Purpose** :-

Models can also be classified by purpose of its utility. The purpose of a model may be descriptive, predictive or prescriptive.

(i) **Descriptive models** :- A descriptive model simply describe some aspects of a situation based on observations, survey, questionnaire results or other available data. The result of an opinion poll represents a descriptive model.

(ii) **Predictive models** :- Such models can answer 'what if' type of questions, i.e, they can make predictions regarding certain events. For example, based on the survey results, television networks such models attempt to explain and predict the election results before all the votes are actually counted.

(iii) **Prescriptive models** :- Finally, when a predictive model has been repeatedly successful, it can be used to prescribe a source of action. For example, linear programming is a prescriptive model because it prescribes what the managers ought to do.

③ **Classification by Nature of Environment** :-

These are mainly of two types :-

(i) **Deterministic models** :- Such models assume conditions of complete certainty & perfect knowledge. For example, linear programming, transportation & assignment models are deterministic type of models.

(ii) Probablistic (or Stochastic) models :- These types of models usually handle such situations in which the Consequences or payoff of managerial actions cannot be predicted with Certainity. However, it is possible to forecast a pattern of events, based on which managerial decisions can be made. For example, insurance companies are willing to insure against risk of fire, accidents, sickness and soon, because of the pattern of events have been compiled in the form of probability distributi

④ Classification by Behaviour :-

(i) Static models :- These models donot consider the impact of changes that takes place during the planning horizon ie, they are independent of time. Also, in a static model only one decision is needed for the duration of a given time period.

(ii) Dynamic models :- In these models, time is considered as one of the important variables & admit the impact of changes generated by time. Also, in dynamic models, not only one but a series of interdependent decisions is required during the planning horizon.

⑤ Classification by Method of Solution :-

(i) Analytical models :- These models have a specific mathematica structure & thus can be solved by known analytical (or) mathematical techniques. For example, a general linear programming model as well as the specially structured transportation & assignment models are analytical models.

(ii) Simulation models :- They also have a mathematical structure but they cannot be solved by purely using the 'tools' & 'techniques' of mathematics. A simulation model is essentially computer assisted experimentation on a mathematical structure of a real time structure in order to study the system under a variety of assumptions.

⑥ Classification by Use of Digital Computers:-

The development of the digital computer has led to the introduction of the following types of modelling in OR.

(i) **Analogue and Mathematical models combined**:- Sometimes analogue models are also expressed in terms of mathematical symbols. Such models may belong to both the types (ii) & (iii) in classification '1' above.

~~(i) Analogue and Mathematical models combined:- Sometimes analogue models are also expressed in terms of mathematical symbols. Such models may belong to both the types.~~

For eg; Simulation model is of analogue type but mathematical formulae are also used in it Managers very frequently use this model to 'simulate' their decisions by summarizing the activities of industry in a scale-down period.

(ii) **Function models**:- Such models are grouped on the basis of the function being performed.

For eg:- ~~such~~ a function may serve to acquaint to scientist with such things as-tables carrying data, a blue-print of layouts, a program representing a sequence of operations.

(iii) **Quantitative models**:- Such models are used to measure the observations.

For example, degree of temperature, yardstick, a unit of measurement of length value, etc.

Other examples of quantitative models are:- (i) transformation models which are useful in converting a measurement of one ~~such~~ scale ~~of~~ to another & (ii) the test ~~method~~ models that act as 'standards' against which measurements are compared.

(iv) **Heuristic models**:- These models are mainly used to explore alternative strategies that were overlooked previously, whereas mathematical models are used to represent systems possessing well-defined strategies. Heuristic models donot claim to find the best solⁿ to the problem.

Applications of Operations Research :-

Some of the areas of management decision making, where the 'tools' & 'techniques' of OR are applied, can be outlined as follows:

① Finance - Budgeting & Investments.

(i) cash-flow analysis, long range capital requirements, dividend policies, investment portfolios,

(ii) Credit policies, credit risks and delinquent account procedures.

(iii) Claim and complaint procedure.

② Purchasing, Procurement and Exploration

(i) Rules for buying, supplies and stable or varying prices.

(ii) Determination of quantities and timining of purchases.

(iii) Bidding policies.

(iv) Strategies for exploration and exploitation of raw material sources.

(v) Replacement policies.

③ Production Management :-

(i) Physical Distribution :-

ⓐ Location and size of warehouses, distribution centres & retail outlets.

ⓑ Distribution policy.

(ii) Facilities Planning :-

ⓐ Numbers and location of factories, warehouses, hospitals, etc.

ⓑ Loading and unloading facilities for railroads and trucks determining the transport schedule.

(iii) **Manufacturing** :-

ⓐ Production scheduling and sequencing.

ⓑ Stabilization of production and employment training, layoffs & optimum product mix.

(iv) **Maintenance and Project Scheduling** :-

ⓐ Maintenance policies and preventive maintenance.

ⓑ Maintenance crew sizes.

ⓒ Project scheduling and allocation of resources.

④ **Marketing** :-

(i) Product selection, timing, competitive actions.

(ii) Number of salesman, frequency of calling on accounts percent of time spent on prospects.

(iii) Advertising media with respect to cost & time.

⑤ **Personnel Management** :-

(i) Selection of suitable personnel on minimum salary.

(ii) Mixes of age & skills.

(iii) Recruitment policies and assignment of jobs.

⑥ **Research and Development** :-

(i) Determination of the areas of concentration of research & development.

(ii) Project selection

(iii) Determination of time cost trade-off and control of development projects.

(iv) Reliability and alternative designs.

From all above areas of applications we may Conclude that OR Can be widely used in taking timely management decisions and also used as a corrective measure. The application of this tool involves certain data & not merely a personality of decision maker, & hence we can say: OR has replaced management by personality.

Linear Programming :- The General LP Problems calls for optimizing (maximizing / minimizing) a linear function of variables called the 'OBJECTIVE FUNCTION' subject to a set of linear equations and/or inequalities called the CONSTRAINTS or RESTRICTIONS.

Formulation of LP Problems :-

Ex:1 (Production Allocation problem) A firm manufactures two type of production A & B & sells them at a profit of Rs. 2 on type A & Rs. 3 on type B. Each product is processed on two machines G & H. Type A requires one minute of processing time on G & two minutes on H; type B requires one minute on G & one minute on H. The machine G is available for not more than 6 hour 40 minutes while machine H is available for 10 hours during any working day. (Formulate the problem as a LPP).

Formulation :- Let x_1 be the number of products of type A & x_2 the no. of products of type B.

The given information can be systematically arranged ~~After~~ ~~to~~ in the form of the following table.

Machine	~~Type~~ Time of products (minutes)		Available Time (minutes)
	Type A (x_1 units)	Type B (x_2 units)	
G	1	1	400
H	2	1	600
Profit per unit	Rs. 2	Rs. 3	

Since the profit on type 'A' is Rs. 2 per product, $2x_1$ will be the profit on selling x_1 units of type A. Similarly, $3x_2$ will be the profit on selling x_2 units of type B. Therefore, total profit on selling 'x_1' units of A & x_2 units of B is given by

$$\boxed{P = 2x_1 + 3x_2}$$ (objective function).

Since machine G takes 1 minute time on type A & 1 minute time on type B, the total number of minutes required on machine G is given by: $\boxed{x_1 + x_2.}$

Similarly, the total number of minutes required on machine H is given by $\boxed{2x_1 + x_2.}$

But machine 'G' is not available for more than ~~6 hours &~~ i.e. 6hrs 40minutes (=400 mins). Therefore,

$$\boxed{x_1 + x_2 \leq 400}$$ (first Constraint).

Also, the machine H is available for 10 hours only, therefore,

$$\boxed{2x_1 + x_2 \leq 600}$$ (second Constraint)

Since it is not possible to produce negative quantities,

$$\boxed{x_1 \geq 0 \quad \& \quad x_2 \geq 0}$$ (non-negativity restrictions).

Hence, the allocation problem of the firm can be finally put in the form:

Find x_1 & x_2 such that the profit $P = 2x_1 + 3x_2$ is max^m, subject to the constraints:

$x_1 + x_2 \leq 400$;

$2x_1 + x_2 \leq 600$

$x_1 \geq 0$; $x_2 \geq 0$.

Q2 :- A company produces two types of Hats. Each hat of the first type requires twice as much labour times as the second type. If all types hats are of the second type only, the company can produce a total of 500 hats a day. The market limits daily sales of the first & second type of 150 & 250 hats. Assuming that the profits per hat are Rs. 8 for type A & Rs. 5 for type B, formulate the problem as a linear programming model in order to determine the no. of hats to be produced of each type so as to maximize the profit.

Formulation :- Let the company produce x_1 hats of type A & x_2 hats of type B each day. So the profit P after selling these two products is given by the linear function:

$$\boxed{P = 8x_1 + 5x_2} \text{ (objective function)}$$

Since the company can produce at the most 500 hats in a day and A type of hats require twice as much time as that of type B, production restriction is given by $2tx_1 + tx_2 \leq 500t$, where t is the labour time per unit of second type; i.e; But there are limitations on the sale of hats, therefore further restrictions are :

$$\boxed{x_1 \leq 150} \; ; \boxed{x_2 \leq 250}$$

Also, since the company cannot produce negative quantities,

$x_1 \geq 0$ & $x_2 \geq 0$.

Hence, the problem can be finally put in the form:

Find x_1 & x_2 such that the profit $P = 8x_1 + 5x_2$ is maximum,
Subject to restrictions : $2x_1 + x_2 \leq 500$; $x_1 \leq 150$; $x_2 \leq 250$; $x_1 \geq 0$; $x_2 \geq 0$

Graphical Solution :-

Eg 1:- Find a geometrical interpretation and solution as well for the following LP problem:

Maximize $z = 3x_1 + 5x_2$, subject to restrictions;

$x_1 + 2x_2 \leq 2000$; $x_1 + x_2 \leq 1500$; $x_2 \leq 600$ & $x_1 \geq 0$, $x_2 \geq 0$.

Graphical Solutions:-

Step 1:- (To graph the inequality-constraints). Consider two mutually $\perp^r$ lines OX_1 & OX_2 as axes of Coordinates. Obviously any pt (x_1, x_2) in the positive quadrant will certainly satisfy non-negativity restrictions : $x_1 \geq 0$, $x_2 \geq 0$. To plot the line $x_1 + 2x_2 = 2000$, put $x_2 = 0$, find $x_1 = 2000$ from this equation.

→ Then mark a point 'L' such that OL=2000 by assuming a suitable scale, say ~~500~~ 500 units = 2cm. Similarly, again put $x_1 = 0$; to find $x_2 = 1000$ & mark another point M such that OM=1000.

→ Now join the points L & M. This line will represent the equation $x_1 + 2x_2 = 2000$ as shown in the above figure.

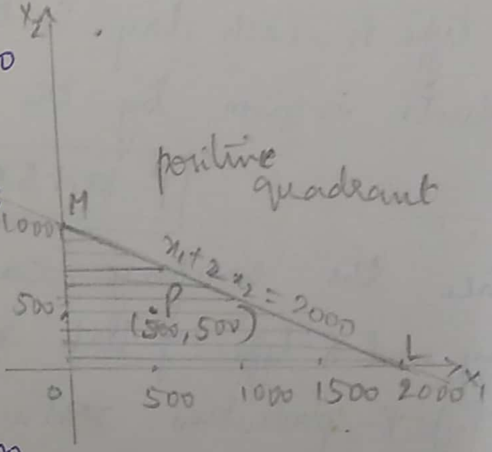
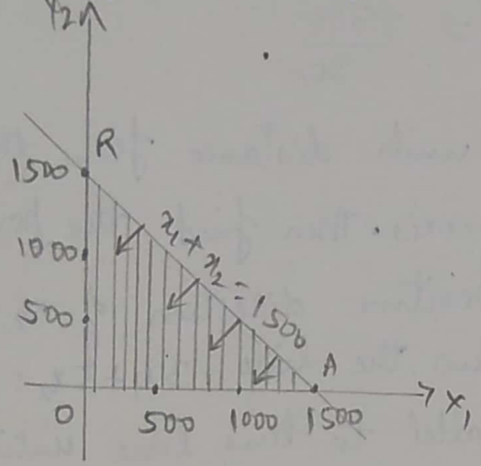


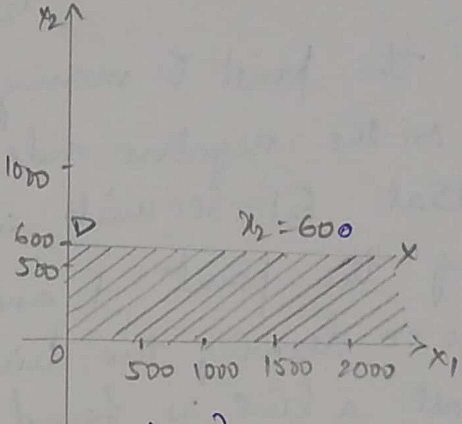
figure: 1

→ Clearly, any point P lying on or below the line $x_1 + 2x_2 = 2000$ will satisfy the inequality $x_1 + 2x_2 \leq 2000$. (If we take a point (500, 500), i.e., $x_1 = 500$; $x_2 = 500$, then we have $500 + 2 \times 500 < 2000$, which is true).

Similar procedure is now adopted to plot the other two lines : $x_1 + x_2 = 1500$ and $x_2 = 600$. Any pt on or below the lines $x_1 + x_2 = 1500$ & $x_2 = 600$ will also satisfy other two inequalities $x_1 + x_2 \leq 1500$ & $x_2 \leq 600$ respectively.

Step 2:- Find the feasible region or solution of space by combining three figures (fig 1, fig 2, fig 3) together.





A Common shaded area OABCD is obtained (fig 4) which is a set of points satisfying the inequality constraints'.

$$\boxed{x_1+2x_2\le 2000\ ;\ x_1+x_2\le 1500\ ;\ x_2\le 600}.$$

& non-negativity restrictions as $x_1\geq 0$, $x_2\geq 0$. Hence any point in the shaded area (including its boundary) is feasible solution to the given LPP.

Step 3:- Find the coordinates of the corner points of feasible region O, A, B, C and D.

Step 4:- Locate the corner points of optimal solution either by calculating the value of Z for each corner point O, A, B, C & D (or by adopting the following procedure).

→ Here, the problem is to find the point or points in the feasible region which maximize (1) the objective function. For some fixed value of Z, $Z=3x_1+5x_2$ is a straight line and any point on it gives the same value of Z.

→ Also, it should be noted that the lines corresponding to different values of Z are parallel, because the gradient $(-3/5)$ of the line $Z=3x_1+5x_2$ remains the same throughout.

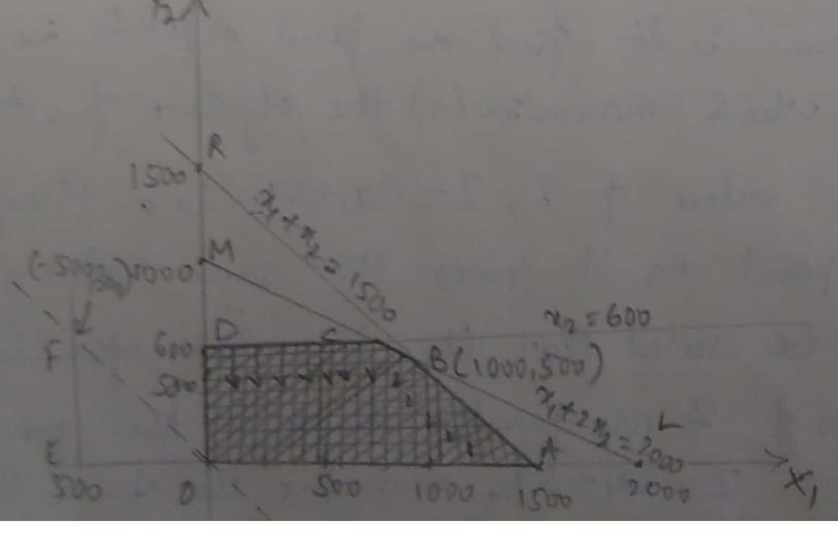
For $z=0$, i.e., $0=3x_1+5x_2$, means a line which passes through the origin. To draw the line $3x_1+5x_2=0$; determine the ratio $\frac{x_1}{x_2}=\frac{-5}{3}=\frac{-500}{300}$.

→ Mark the point E moving 500 units distance from the origin on the negative side of x_1-axis. Then find the points F such that EF = 300 units in the positive direction of x_2-axis.

→ Joining the point F and O draw the line $3x_1+5x_2=0$.. Now go on drawing the lines parallel to this line until at least a line is found which is farthest from the origin but passes through at least one corner of the feasible region at which the maximum value of Z is attained.

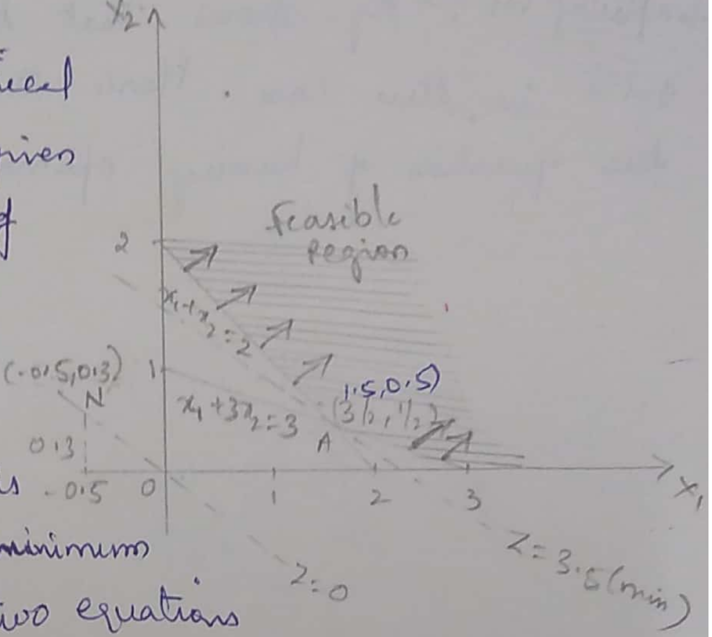
→ It is also possible that such a line may coincide with one of the edge of feasible region. In this case, every point on that edge gives the maximum value of z.

→ In this example, maximum value of z is attained at the corner point B(1000, 500), which is the point of intersection of lines $x_1+2x_2=2000$ & $x_1+x_2=1500$. Hence, the required solution is $x_1=1000$; $x_2=500$ & max. value $z=$ Rs. 5500.



Eg :- (Minimization Problem).

Consider the problem :- Min $Z = 1.5x_1 + 2.5x_2$ subject to $x_1 + 3x_2 \geq 3$, $x_1 + x_2 \geq 2$, $x_1, x_2 \geq 0$.



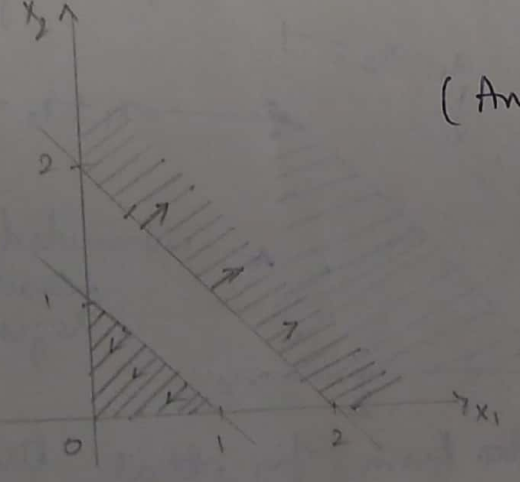
Graphical solution :- The geometrical interpretation of the problem is given in fig. The minimum value of Z is $Z_A = 3.5$. This minimum is attained at the point of intersection A of the lines $x_1 + 3x_2 = 3$ and $x_1 + x_2 = 2$. This is the unique point to give the minimum value of Z. Now, Solving these two equations simultaneously, the optimum solⁿ is; $x_1 = 3/2$, $x_2 = 1/2$ and minimum $Z = 3.5$.

Eg :- (problem with inconsistent system of constraints)

Maximize. $Z = 3x_1 - 2x_2$ subject to $x_1 + x_2 \leq 1$, $2x_1 + 2x_2 \geq 4$ & $x_1, x_2 \geq 0$.

Graphical Solution :- The problem is represented graphically in fig.

→ The fig. shows that there is no point (x_1, x_2) which satisfies both the constraints simultaneously. Hence the problem has no solⁿ because the Constraints are inconsistent.

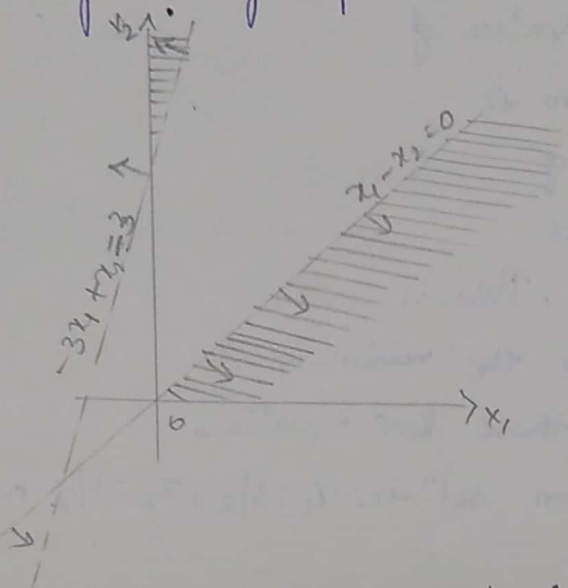


(An Infeasible solⁿ)

Eg (Constraints can be consistent and yet there may be no solution).

Max. $z = x_1 + x_2$ subject to $x_1 - x_2 \geq 0$, $-3x_1 + x_2 \geq 3$ & $x_1, x_2 \geq 0$.

Graphical solⁿ :- Fig. shows that there is no region of feasible solⁿ's in this case. Hence there is no feasible solⁿ. So the question of having optimal solⁿ does not arise.

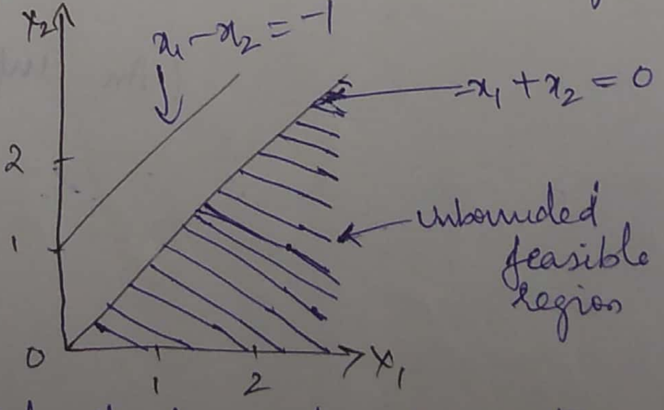


(Infeasible solⁿ)

Eg :- Use the graphical method to solve the following LPP.

Max ; $Z = 3x_1 + 4x_2$. subject to Constraints $x_1 - x_2 \leq -1$
$-x_1 + x_2 \leq 0$.
$x_1, x_2 \geq 0$.

Solⁿ :- Graph each constraint by first treating it as a linear equation in the same way. ~~as discuss~~ Then use the inequality condⁿ of each constraint to mark the feasible region as shown in following fig.



It may be noted ~~the~~ from fig, that there exist infinite ~~infi~~ no. of pts in the Convex region for which the value the objective funcⁿ increases as we move from $O = (0,0)$ the right. That is, both the variables x_1 & x_2 can be made arbitrarily large & the value of objective funcn Z is also increased. Thus, the problem has an unbounded soln.

~~Eg~~ **Simplex Method :-**

Computational procedure :-

Eg:- Consider the following LPP:

Max $Z = 3x_1 + 2x_2$, subject to the constraints:

$x_1 + x_2 \le 4$, $x_1 - x_2 \le 2$ & $x_1, x_2 \ge 0$.

Soln:- **Step 1:** First, observe whether all the right side constants of the constraints are non-negative. If not, it can be changed into positive value on multiplying both sides of the constraints by '-1'. In this example, all the b_i's (right side constants) are already '+ve'.

Step 2:- Next convert the inequality eqn constraints to eqn's by introducing the non-negative slack or surplus variables. The coefficients of slack or surplus variables are always taken zero in the objective function. In this example, all inequality constraints being '$\le$', only slack variables S_1 & S_2 are needed. Therefore the ~~pro~~ given problem now becomes:

Maximize $Z = 3x_1 + 2x_2 + 0S_1 + 0S_2$, subject to the constraints:

$$x_1 + x_2 + S_1 = 4$$
$$x_1 - x_2 + S_2 = 2$$
$$x_1, x_2, S_1, S_2 \ge 0.$$

Step 3:- Now present the constraint eqn's in matrix form:

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ S_1 \\ S_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}.$$

Step 4:- Construct the starting simplex table using the notations ~~already explained in~~ . It ~~should~~ be remembered that the values of non-basic variables are always zero at each iteration. So $x_1 = x_2 = 0$ here.

Column X_B gives the values of basic variables as indicated in the first column. So $s_1 = 4$ & $s_2 = 2$ here. The complete starting basic feasible solⁿ can be immediately read from table 1 as $s_1 = 4, s_2 = 2, x_1 = 0, x_2 = 0$ & the value of the objective functⁿ is zero.

Table 1 Starting Simplex Table.

		$c_j \rightarrow$	3	2	0	0	
Basic variables	C_B	X_B	x_1	x_2	$X_1 (S_1)$ (B_1)	$X_2 (S_2)$ (B_2)	Min Ratio,
S_1	0	4	1	1	1	0	$4/1 = 4$
$\leftarrow S_2$	0	2	1	-1	0	1	$2/1 = 2$
	$Z = C_B X_B$ $= 0$		$\Delta_1 = -3$ $\uparrow$	$\Delta_2 = -2$	$\Delta_3 = 0$	$\Delta_4 = 0$	$\Delta_j = z_j - c_j = C_B x_j - c_j$

Step 5:- Now, proceed to test the basic feasible solⁿ for optimality by the rules given below. This is done by computing the 'net evaluation' Δ_j for each variable x_j (column vector x_j) by the formula.

Thus, we get

$$\Delta_1 = C_B x_1 - c_1 = (0,0)(1,1) - 3 = (0 \times 1 + 0 \times 1) - 3 = -3$$

$$\Delta_2 = C_B x_2 - c_2 = (0,0)(1,-1) - 2 = (0 \times 1 - 0 \times 1) - 2 = -2$$

$$\Delta_3 = C_B x_3 - c_3 = (0,0)(1,0) - 0 = (0 \times 1 + 0 \times 0) - 0 = 0$$

$$\Delta_4 = 0$$

Optimality Test :-

(i) If all $\Delta_j (= z_j - c_j) \geq 0$, the solⁿ under test will be optimal. Alternative optimal solⁿ's will exist if any non-basic Δ_j is also zero.

(ii) If atleast one Δ_j is '-ve', the solⁿ under test is not optimal, then proceed to improve the solⁿ in next step.

(iii) If corresponding to any '-ve' Δ_j, all elements of the column x_j

are '-ve' or zero (≤ 0), then the solⁿ under test will be unbounded.

Apply these rules for testing the optimality of starting basic feasible solⁿ, it is observed that Δ_1 & Δ_2 both are '-ve'. Hence, we have to proceed to improve the solⁿ in ~~set~~ step-6.

Step 6:- In order to improve the basic feasible solⁿ, the vector entering the basis matrix & the vector to be removed from the basis matrix are determined by the following rules. Such vectors are usually named as 'incoming vector' & 'outgoing vector' respecti

Incoming vector :- The incoming vector x_k is always selected corresponding to the most negative value of Δ_j (say Δ_k). Here $\Delta_k = \min[\Delta_1, \Delta_2) = \min\{-3, -2\} = -3 = \Delta_1$. Therefore, $k = 1$ & hence column vector x_1 must enter the basis matrix. The column x_1 is marked by an upward arrow (↑).

Outgoing Vector :- The outgoing vector β_r is selected corresponding to the minimum ratio of elements of 'x_B' by the corresponding positive elements of predetermined incoming vector x_k. This rule is called the Minimum Ratio Rule. In mathematical form; this ~~ca~~ rule can be written as;

For $k = 1$, $$\frac{x_{Br}}{x_{rk}} = \min_i \left[\frac{x_{Br}}{x_{ik}}, \ x_{ik} > 0\right]$$

$$\frac{x_{Br}}{x_{r1}} = \min\left(\frac{x_{B1}}{x_{11}}, \frac{x_{B2}}{x_{21}}\right) = \min\left\{\frac{4}{1}, \frac{2}{1}\right\}$$

or $$\frac{x_{Br}}{x_{r1}} = \frac{2}{1} = \frac{x_{B2}}{x_{21}}.$$

Comparing both sides of this eqⁿ, we get $r = 2$. So the vector β_2, i.e., x_4 marked with downward arrow (↓) should be removed from the basis matrix. The starting table is now modified as given below.

→ Key element / Pivot element

		$c_j \rightarrow$	3	2	0	0	
Basic variables	C_a	x_a	x_1	x_2	$x_3(S_1)$ (β_1)	$x_4(S_2)$ (β_2)	Min Ratio (x_B/x_1).
S_1	0	4	1	1	1	0	$4/1$
← S_2	0	2	$\boxed{1}$	-1	0	1	$2/1$ ← Min. Ratio
	$z = C_B x_B = 0$		-3 (min Δ_j) ↑	-2	0	0 ↓	← $\Delta_j = z_j - c_j = C_B\beta_j -$

entering vector

leaving vector.

Step 7:- In order to bring $\beta_2 = \begin{bmatrix}0\\1\end{bmatrix}$ in place of incoming vector $x_1 = \begin{bmatrix}1\\1\end{bmatrix}$, unity must occupy in the marked '□' position & zero at all other places of x_1. If the no. in the marked '□' position is other than unity, divide ~~the~~ all elements of that row by the 'key element'. The element at the intersection of minimum ratio arrow (←) & incoming vector arrow (↑) is called the key element or pivot element.

Then, subtract appropriate multiples of this new row from the other (remaining) rows, so as to obtain zeros in the remaining positions of the column x_1. Thus, the process can be fortified by the simple matrix transformation as follows:

The intermediate coefficient matrix is:

$$\begin{array}{c|c|c|c|c|c|l}
 & x_B & x_1 & x_2 & x_3 & x_4 & \\
R_1 & 4 & 1 & 1 & 1 & 0 & \\
R_2 & 2 & 1 & -1 & 0 & 1 & \\
R_3 & z=0 & -3 & -2 & 0 & 0 & \leftarrow \Delta_j
\end{array}$$

Apply $R_1 \rightarrow R_1 - R_2$, $R_3 \rightarrow R_3 + 3R_2$ to obtain

X_B	x_1	x_2	x_3	x_4	
2	0	2	1	-1	
2	1	-1	0	1	
Z=6	0	-5	0	3	$\leftarrow \Delta_j$

Now, Construct the improved simplex table as follows:

$C_j \rightarrow$ 3 2 0 0

Basic Variables	C_B	X_B	x_1	x_2	$x_3(S_1)$	$x_4(S_2)$	Min Ratio
S_1	0	2	0	$\leftarrow\boxed{2}\uparrow$	-1	-1	2/2 ← Key row
[illegible]	3	2	1	-1	0	1	2/-1 (-ve) ratio is not counted
	$Z = C_B X_B = 6$		0	-5 ↑	0 ↓	3	$\leftarrow \Delta_j$

Key column

From this table the improved basic feasible solⁿ is read as: $x_1 = 2$, $x_2 = 0$, $S_1 = 2$, $S_2 = 0$. The improved value of $Z = 6$.

It is of particular interest the note here that Δ_j's are also computed while transforming the table by matrix method. However, the correctness of Δ_j's can be verified by computing them independently by using the formula $\Delta_j = C_B X_j - C_j$.

Step 8:- Now repeat steps 5 through 7 as & when needed until an optimum solⁿ is obtained.

Δ_k = most '-ve' $\Delta_j = -5 = \Delta_2$.

Therefore, K=2 & hence x_2 should be the entering vector (Key Column). By minimum ratio rule.

Minimum Ratio $\left(\frac{X_B}{X_2}, X_2 > 0\right) = \text{Min}\left[\frac{2}{2}, -\right]$ [∵ '-ve' is not Counted so the 2nd is not Considered]

Since first ratio is minimum, remove the first vector β_1 from the basis matrix. Hence the key element is 2. Dividing the first row by key element 2, the intermediate coefficient matrix is obtained as:

	X_B	x_1	x_2	x_3	x_4	
R_1	1	0	1	1/2	-1/2	
R_2	2	1	-1	0	1	
R_3	Z=6	0	-5	0	3	← Δ_j

Applying $R_2 \rightarrow R_2 + R_1$, $R_3 \rightarrow R_3 + 5R_1$

1	0	1	1/2	-1/2	
3	1	0	1/2	1/2	
Z=11	0	0	5/2	1/2	← Δ_j

Now construct the next improved simplex table as follows: <u>Final Simplex Table</u>

$C_j \rightarrow$ 3 2 0 0

Basic variables	C_B	X_B	$x_1(\beta_1)$	$x_2(\beta_2)$	S_1	S_2	
→ x_2	2	1	0	1	1/2	-1/2	
x_1	3	3	1	0	1/2	1/2	
	$Z = C_B X_B = 11$		0	0	5/2	1/2	← Δ_j

The sol^n is read from the table is:

$x_1=3,\ x_2=1,\ S_1=0,\ S_2=0$ & max $Z=11$.

Also using the formula $\Delta_j = C_B X_j - C_j$ verify that all Δ_j's are non-negative. Hence the optimum sol^n is

$x_1=3,\ x_1=1$, max $Z=11$.

Artificial Variables Techniques :- (Big M Method & Two-phase Metho

~~Two Phase Method~~

Eg:- Solve the problems: Minimize $Z = x_1 + x_2$; subject to $2x_1 + x_2 \geq 4$,

$x_1 + 7x_2 \geq 7$ & $x_1, x_2 \geq 0$.

Sol^n:- First convert the problem of minimization to maximizatio by writing the objective function as:

$\text{Max}(-Z) = -x_1 - x_2$ or. Max. $Z' = -x_1 - x_2$, where $Z' = -Z$.

Since all b_i (4 & 7) are positive, the 'surplus variables' $x_3 \geq 0$ & $x_4 \geq 0$ are introduced, then constraints becomes:

$$2x_1 + x_2 - x_3 = 4$$

$$x_1 + 7x_2 - x_4 = 7.$$

But the basis matrix B would not be an identity matrix due to '-ve' coefficients of x_3 & x_4. Hence the starting basic feasible solution cannot be obtained.

On the other hand, if so-called 'artificial variables' $a_1 \geq 0$ and $a_2 \geq 0$ are introduced, the constraint equations can be written as:

$$Z = 0x_1 + 0x_2 + 0x_3 + 0x_4 - 1a_1 + a_2$$

$$2x_1 + x_2 - x_3 + a_1 = 4.$$

$$x_1 + 7x_2 - x_4 + a_2 = 7.$$

It should be noted that $a_1 < x_3$, $a_2 < x_4$, otherwise the constraints of the problem will not hold.

Phase I :- Construct the first table where A_5 & A_6 denote the artificial column-vectors corresponding to x_5 & x_6, respectively.

Table I

Basic variables	x_B	x_1	x_2	x_3	x_4	A_1	A_2
$x_5(A_1)$	4	2	1	-1	0	1	0
$x_6(A_2)$	7	1	7	0	-1	0	1
			↑	✗	✗	✗	↓

Now remove each artificial column vector A_1 & A_2 from the basis matrix. To remove vector A_2 first, Select the entering vector either A_1 or A_2, being careful to choose any one that will yield a non-negative (feasible) revised solution. Take the vector x_2 to enter the basis matrix. It can be easily verified that if the vector A_2 is entered in place of x_1, the resulting solⁿ will not be feasible. Thus, transformed table (Table II) is obtained.

Table II

Basic variables	x_B	x_1	x_2	x_3	x_4	A_1	A_2
a_1	3	13/7	0	-1	$\boxed{1/7}$	1	-1/7
x_2	1	1/7	1	0	-1/7	0	1/7
					↑	↓	

(Delete column A_2 forever at this stage)

→ This table gives the solution: $x_1=0$, $x_2=1$, $x_3=0$, $x_4=0$, $a_1=3$, $a_2=0$. when the artificial variable a_2 becomes zero (non-basic), we forget about it & never consider the corresponding vector A_2 again for re-entry into the basis matrix.

Similarly, remove A_1 from the basis matrix by introducing it in place of x_4 by the same method. Thus Table III is obtained.

Table III

Basic Variables	X_B	x_1	x_2	x_3	x_4	A_1
x_4	21	13	0	-7	1	7
x_2	4	4	1	1	0	1

This table gives the sol^n : $x_1=0$, $x_2=4$, $x_3=0$, $x_4=21$, $a_1=0$. Since the artificial variable a_1 becomes zero (non-basic), so drop the corresponding column A_1 from this table. Thus, the sol^n ($x_1=0$, $x_2=4$, $x_3=0$, $x_4=21$) is the basic feasible solution & now usual simplex routine can be started to obtain the required optimal sol^n.

Phase II Now in order to test the starting above sol^n for optimality, construct the starting simplex Table IV.

Table IV

		$C_j \rightarrow$	-1	-1	0	0	
Basic Variables	C_B	X_B	x_1	x_2	x_3	x_4	Min. Ratio (X_B/x_1).
← x_4	0	21	13	0	-7	1	21/13 ←
x_2	-1	4	2	1	-1	0	4/2
	$z' = C_B X_B = -4$		-1 ↑	0	1	0 ↓	← Δ_j

Compute $\Delta_1 = -1$ & $\Delta_3 = 1$.
Key element $\boxed{13}$ indicates that X_4 shall be removed from the basis matrix. Thus, by usual transformation method Table V is formed.

Table II

Basic Variables	C_B	X_B	x_1	x_2	X_3	X_4	Min Ratio column
$C_j \rightarrow$			-1	-1	0	0	
$\rightarrow x_1$	-1	$21/13$	1	0	$-7/13$	$1/13$	
x_2	-1	$10/13$	0	1	$1/13$	$-2/13$	
	$z' = -31/13$		0	0	$6/13$	$1/13$	$\leftarrow \Delta_j \geq 0.$

Also, verify that

$$\Delta_3 = C_B X_3 - C_3 = (-1,-1)\left(-7/13, 1/13\right) = 6/13.$$

$$\Delta_4 = C_B X_4 - C_4 = (-1,-1)\left(1/13, -2/13\right) = 1/13.$$

Since all $\Delta_j \geq 0$; the required optimal solⁿ is :

$x_1 = 21/13$, $x_2 = 10/13$ and min $Z = 31/13$ (because $Z = -Z$).

Alternative Approach of Two-Phase Simplex Method:

Eg :- Use two-phase simplex method to solve the problem:
Minimize $Z = x_1 - 2x_2 - 3x_3$, subject to the constraints :

$-2x_1 + x_2 + 3x_3 = 2$, $2x_1 + 3x_2 + 4x_3 = 1$, and $x_1, x_2, x_3 \geq 0$.

Solⁿ :- first convert the objective funⁿ into maximization form;

Max $z' = -x_1 + 2x_2 + 3x_3$; where $z' = -z$.

Introducing the artificial variables $a_1 \geq 0$ and $a_2 \geq 0$, the constraints of the given problem become,

$$-2x_1 + x_2 + 3x_3 + a_1 = 2.$$

$$2x_1 + 3x_2 + 4x_3 + a_2 = 1.$$

$$x_1, x_2, x_3, a_1, a_2 \geq 0.$$

Phase I :- Auxillary L.P. problem is : Max $z' = 0x_1 + 0x_2 + 0x_3 - 1a_1 - 1$

Subject to above given constraints.

The following solⁿ table is obtained for auxillary problem

Table I

	$C_j \rightarrow$		0	0	0	-1	-1	
Basic Variables	C_B	x_B	x_1	x_2	x_3	A_1	A_2	Min. Ratio (x_B/x_k)
a_1	-1	2	-2	1	3	1	0	2/3
$\leftarrow a_2$	1	1	2	3	$\boxed{4}$	0	1	$1/4 \leftarrow$
	$z'^* = 3$		0	-4	-7^* ↑	0	0 ↓	$\leftarrow \Delta_j$
a_1	-1	5/4	-7/2	-5/4	0	1	-3/4	
$\Rightarrow x_3$	0	1/4	1/2	3/4	1	0	1/4	
	$z'^* = -5/4$		7/4	5/4	0	0	3/4	$\leftarrow \Delta_j \geq 0$

Since all $\Delta_j \geq 0$, an optimum basic feasible solⁿ to the auxillary L.P.P. has been attained. But at the same time max z'^* is negative and the artificial variable a_1 appears in the basic solⁿ at a positive level. Hence the original problem does not ~~ap~~ ~~possess~~ possess any feasible solⁿ. Here there is no need to enter Phase II.

Big M-Method (Charne's Penalty Method):-

Eg 1:- Solve using big M Method the following LPP.

Max $Z = -2x_1 - x_2$, subject to $3x_1 + x_2 = 3$, $4x_1 + 3x_2 \geq 6$,

$x_1 + 2x_2 \leq 4$, & $x_1, x_2 \geq 0$.

Solⁿ:- Step 1:- Introducing slack, surplus and artificial variables the system of constraint eqⁿ becomes:

$3x_1 + x_2 + a_1 = 3.$

$4x_1 + 3x_2 - x_3 + a_2 = 6.$

$x_1 + 2x_2 + x_4 = 4$ (∵ it is '≤' no ~~[illegible]~~ ~~[illegible]~~ surplus & artificial variables will be added).

which can be written in matrix form as:

$$\left[\begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & A_1 & A_2 \\ 3 & 1 & 0 & 0 & 1 & 0 \\ 4 & 3 & -1 & 0 & 0 & 1 \\ 1 & 2 & 0 & 1 & 0 & 0 \end{array}\right] \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ a_1 \\ a_2 \end{pmatrix} = \begin{bmatrix} 3 \\ 6 \\ 4 \end{bmatrix}$$

Step 2:- Assigning the large negative price '$-M$' to the artificial variables a_1 & a_2, the objective funcⁿ becomes:

$$\text{Max. } Z = -2x_1 - x_2 + 0x_3 + 0x_4 - Ma_1 - Ma_2 .$$

Step 3:- Construct starting simplex table 'I'.

Table I

$c_j \rightarrow$			-2	-1	0	0	$-M$	$-M$	
Basic variables	C_B	X_B	x_1	x_2	x_3	x_4	A_1	A_2	Min Ratio (X_B/X_1)
← a_1	$-M$	3	$\boxed{3}$ key element	1	0	0	1	0	$3/3 = 1$ ←
a_2	$-M$	6	$\boxed{4}$	3	-1	0	0	1	$6/4 = 1.5$
x_4	0	4	1	2	0	1	0	0	$4/1 = 4$
	$Z = -9M$		$(2-7M)$ ↑	$(1-4M)$	M	0	0	0	← Δ_j

To apply optimality test, compute

$$\Delta_1 = C_B X_1 - C_1 = (-M, -M, 0)(3, 4, 1) - (-2) = 2 + (-3M - 4M + 0) = 2 - 7M$$

$$\Delta_2 = C_B X_2 - C_2 = 1 - 4M$$

$$\Delta_3 = C_B X_3 - C_3 = M.$$

$\therefore \Delta_K = \min(\Delta_1, \Delta_2, \Delta_3) = \min[2-7M, 1-4M, M] = \Delta_1$ Therefore, x_1 will be entered.

Using minimum ratio rule, find the key element 3 which indicates that A_1 should be removed. Now the transform table (II) is obtained.

Table II

Basic variables	C_B	X_B	x_1	x_2	x_3	x_4	A_1	A_2	Min Ratio (X_B/X_K)
$C_j \rightarrow$			-2	-1	0	0	$-M$	$-M$	
$\rightarrow x_1$	-2	1	1	$1/3$	0	0	$1/3$	0	$1/(1/3)$
$\leftarrow a_2$	$-M$	2	0	$\boxed{5/3}$	-1	0	$4/3$	1	$2/(4/3) \leftarrow$
x_4	0	3	0	$5/3$	0	1	$-4/3$	0	$3/(5/3)$
	$Z=-2-2M$		0	$(1-5M)/3 \uparrow$	M	0	$(-2+7M)/3$	$0 \downarrow$	$\leftarrow \Delta_j$

Again Compute, $\Delta_2 = C_B X_2 - C_2 = (-2,-M,0)(1/3, 5/3, 5/3)+1 = (1-5M)/3$

and similarly, $\Delta_3 = M$, $\Delta_5 = (-2+7M)/3$.

Since minimum Δ_j rule & minimum ratio rule decide the key element $5/3$, so enter x_2 and remove a_2.

Therefore, the second improved table (III) is formed.

Table III

Basic variables	C_B	X_B	x_1	x_2	x_3	x_4	A_1	A_2	Min Ratio
$C_j \rightarrow$			-2	-1	0	0	$-M$	$-M$	
x_1	-2	$3/5$	1	0	$1/5$	0	$3/5$	$-1/5$	
x_2	-1	$6/5$	0	1	$-3/5$	0	$-4/5$	$3/5$	
x_4	0	1	0	0	1	1	1	-1	
	$Z = C_B X_B = -12/5$		0	0	$1/5$	0	$M-2/5$	$M-1/5$	$\leftarrow \Delta_j \geq 0$.

To test the soln for optimality compute,

$\Delta_3 = C_B X_3 - C_3 = (-2,-1,0)(1/5, -3/5, 1) - 0 = 1/5$

$\Delta_5 = C_B A_2 - C_5 = (-2,-1,0)(3/5,-4/5,1) + M = M - 2/5.$

$\Delta_6 = C_B A_2 - C_6 = (-2,-1,0)(-1/5,-3/5,-1) + M = M - 1/5.$

Since M is as large as possible, $\Delta_3, \Delta_5, \Delta_6$ are all '+ve'. Consequently, the optimal solⁿ is $x_1 = 3/5$, $x_2 = 6/5$, $\max z =$

Eg:- Use penalty (Big-M) method to maximize :
$Z = 3x_1 - x_2$ subject to the constraints :

$2x_1 + x_2 \geq 2$, $x_1 + 3x_2 \leq 3$, $x_2 \leq 4$ and $x_1, x_2 \geq 0$.

Solⁿ:- By introducing the surplus variable $x_3 \geq 0$, artificial variable $a_1 \geq 0$, and slack variables $x_4 \geq 0$, $x_5 \geq 0$, the problem becomes : Max $Z = 3x_1 - x_2 + 0x_3 + 0x_4 + 0x_5 - Ma_1$, subject to constraints :

$$2x_1 + x_2 - x_3 + a_1 = 2$$
$$x_1 + 3x_2 + x_4 = 3$$
$$x_2 + x_5 = 4$$
$$x_1, x_2, x_3, x_4, x_5, a_1 \geq 0.$$

In matrix form,
$$\begin{bmatrix} 2 & 1 & -1 & 0 & 0 & 1 \\ 1 & 3 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ a_1 \end{pmatrix} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}.$$

Now the solⁿ is obtained in Table I.

		$c_j \rightarrow$	3	-1	0	0	0	-M	
Basic variable	C_B	X_B	x_1 ↑	x_2	x_3	x_4	x_5	A_1	Min Ratio (X_B/x_k)
← a_1	-M	2	$\boxed{2}$	1	1	0	0	1	2/2
x_4	0	3	1	3	0	1	0	0	3/1
x_5	0	4	0	1	0	0	1	0	—
	Z = -2M		(-2M-3) ↑	(-M+1)	M	0	0	0 ↓	

Basic variables	C_B	x_B	x_1	x_2	x_3	x_4	x_5	A_1	Min Ratio (x_B/x_2)
→ x_1	3	1	1	$1/2$	$-1/2$	0	0	x	—
← x_4	0	2	0	$5/2$	$1/2$	1	0	x	$2/(1/2)$ ←
x_5	0	4	0	1	0	0	1	x	—
	$Z = 3$		0	$5/2$	$(-3/2)$ ↑	0	0	x	← Δ_j

Basic variables	C_B	x_B	x_1	x_2	x_3	x_4	x_5	A_1	Min Ratio (x_B/x_2)
x_1	3	3	1	3	0	1	0	x	
→ x_3	0	4	0	5	1	2	0	x	
x_5	0	4	0	1	0	0	1	x	
	$Z = 9$		0	10	0	3	0	x	← $\Delta_j \geq 0$

Thus, the optimum solution is obtained as: $x_1 = 3$, $x_2 = 0$
max $Z = 9$.

Big-M Method

Minimization Problem.

Minimize $Z = 5x_1 + 4x_2$

Subject to Constraints $4x_1 + x_2 \geq 40$

$2x_1 + 3x_2 \geq 90$

Converting Minimization into maximization problem and solving using Big-M Method.

$\text{Min } Z = \text{Max } (-Z) = -5x_1 - 4x_2.$

subject to $4x_1 + x_2 \geq 40$

$2x_1 + 3x_2 \geqq 90$

Converting the given inequality Constraints into equation form by introducing surplus and artificial variables.

$4x_1 + x_2 - S_1 + a_1 = 40.$

$2x_1 + 3x_2 - S_2 + a_2 = 90.$

Modified objective function.

$\text{Max } Z = -5x_1 - 4x_2 + 0S_1 + 0S_2 - Ma_1 - Ma_2.$

'M' is assumed as a very big value.

Simplex Table I :-

			-5	-4	0	0	-M	-M	
Basic variables	C_B	X_B	X_1	X_2	S_1	S_2	A_1	A_2	Min. Ratio
← A_1	-M	40	$\boxed{4}$	1	-1	0	1	0	$40/4 = 10$
A_2	-M	90	2	3	0	-1	0	1	$90/2 = 45$
	$Z = \Sigma C_B X_B$ $= -130M$		$\Delta_1 =$ $-6M+5$ ↑	$\Delta_2 =$ $-4M+4$	$\Delta_3 =$ M	$\Delta_4 = M$	$\Delta_5 = 0$	$\Delta_6 = 0$	

∵ 'Δ_1' & 'Δ_2' having '-ve' Coefficient of 'M' and 'M' is a very big value; least negative 'Δ' is Δ_1.

∴ '[illegible]' is inComing vector

key Min Ratio = $\frac{X_B}{\text{Incoming vector}} = X_B / X_1$ ∴

min Ratio is for 'A_1' vector

∴ A_1 is leaving vector.

key Element = 4.

$R_1 \to R_1/4$; $R_2 \to R_2 - 2R_1$

Simplex Table II :

			-5	-4	0	0	-M	-M	
B.V	C_B	X_B	X_1	X_2	S_1	S_2	A_1	A_2	Min Ratio $\frac{X_B}{X_2}$
X_1	-5	10	1	1/4	-1/4	0	1/4	0	$\frac{10}{1/4} = 40$
← A_2	-M	70	0	5/2	5/2	-1	-1/2	1	$70 \times \frac{2}{5} = 28$
	Z =	-50 - 70M	$\Delta_1 = 0$	$\Delta_2 = \frac{1}{4} - M$ ↑	$\Delta_3 = \frac{5}{4} - \frac{M}{2}$	$\Delta_4 = M$	$\Delta_5 = \frac{3}{2}M - \frac{5}{4}$	$\Delta_6 = 0$	

∴ 'Δ_2' is having least '-ve' Coefficient of 'M' ;

X_2 is incoming vector

Simplex Table III : $R_2 \to R_2 \times 2/5$; $R_1 \to R_1 - \frac{R_2}{4}$.

			-5	-4	0	0	-M	-M
B.V	C_B	X_B	X_1	X_2	S_1	S_2	A_1	A_2
X_1	-5	3	1	0	-3/10	-1/10	1/8	-1/10
X_2	-4	28	0	1	4/5	-2/5	-1/5	2/5
	Z = -127		$\Delta_1 = 0$	$\Delta_2 = 0$	$\Delta_3 = 7/10$	$\Delta_4 = 21/10$	$\Delta_5 = M + \frac{7}{40}$	$\Delta_6 = M - \frac{11}{10}$

∵ 'M' is very large value '$\Delta_6 = M - \frac{11}{10}$' can never be '-ve'

∴ all the 'Δ' values are positive.

∴ Max Z = -127 for $x_1 = 3$; $x_2 = 28$. is optimal solution.

∴ Min Z = -(-127)

Min Z = 127 for $x_1 = 3$; $x_2 = 28$.