Transportation Problem :-

~~Formulation~~ Definition :- The transportation problem is to transport various amounts of a single homogeneous commodity, that are initially stored at various origins, to different destinations in such a way that the total transportation cost is minimum.

Mathematical Formulation :-

Suppose that there are 'm' sources & 'n' destinations. Let 'a_i' be the no. of supply units available at source 'i' ($i = 1, 2, 3 \ldots m$) and let 'b_j' be the no. of demand units required at destination 'j' ($j = 1, 2, 3 \ldots n$).

Let c_{ij} be the unit transportation cost for transporting the units from source 'i' to destination 'j'. The objective is to determine the no. of units to be transported from source 'i' to destination 'j' so that total transportation cost is minimum. In addition the supply limit at the sources & demand requirements at the destinations must be satisfied exactly.

If x_{ij} ($x_{ij} \geq 0$) is the units shipped from source 'i' to destination 'j' then the equivalent LPP model will be find x_{ij} ($i = 1, 2, \ldots m$; $j = 1, 2, \ldots m$) inorder to minimize $Z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$ & subject to constraints

$\sum_{i=1}^{m} x_{ij} = a_i$, $i = 1, 2, \ldots m$. $\sum_{j=1}^{m} x_{ij} = b_j$, $j = 1, 2 \ldots n$.

where $x_{ij} \geq 0$.

The two sets of constraints will be consistent i.e., the system will be in balance if $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$. This Restriction causes one of the constraints to be redundant so that problem will have $(m+n-1)$ constraints & $m \times n$ unknowns the above information can be put in the form of general matrix shown below.

Jobs to be done

$$
\begin{array}{lcccccccc}
 & & j_1 & j_2 & \cdots & j_j & \cdots & j_n & \text{supply} \downarrow \\
\text{Sources} & S_1 = & C_{11} & C_{12} & \cdots & C_{1j} & \cdots & C_{1n} & a_1 \\
 & S_2 = & C_{21} & C_{22} & \cdots & C_{2j} & \cdots & C_{2n} & a_2 \\
 & \vdots & \vdots & \vdots & & \vdots & & \vdots & \vdots \\
 & S_i = & C_{i1} & C_{i2} & \cdots & C_{ij} & \cdots & C_{in} & a_i \\
 & \vdots & \vdots & \vdots & & \vdots & & \vdots & \vdots \\
 & S_m = & C_{m1} & C_{m2} & \cdots & C_{mj} & \cdots & C_{mn} & a_m \\
\text{Demand} & & b_1 & b_2 & \cdots & b_j & \cdots & b_n &
\end{array}
$$

Feasible Solution:- A set of non-negative values x_{ij} $(i=1,2\cdots m)$ $(j=1,2\cdots n)$ that satisfies the constraints is called a feasible soln to the transportation problem

(or)

A set of non-(+ve) individual allocations $(x_{ij} \geq 0)$ which simultaneously removes deficiencies is called a feasible solution.

Basic feasible Solution (BFS) :- A feasible solution that Contains no more than (m+n-1) non-negative allocations is called a BFS to the transportation problem

(or)

A feasible solⁿ to a m-origin n-destination problem is said to be basic if the no. of +ve allocations are (m+n-1) i.e., one less than the sum of rows & Columns.

Non-degenerate Basic feasible Solution :- If the no. of allocations in a basic feasible solution are less than (m+n-1), it is called degenerate BFS (or) non-degenerate BFS.

Optimal Solution :- A feasible solⁿ (not necessarily basic) is said to be optimal if it minimises the total transportation Cost.

Balanced & Unbalanced Transportation model :- when a transportation model is said to be balanced when total supply from all sources is equal to total demand in all destinations is called balanced otherwise unbalanced.

Thus, for a balanced problem $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$ & for unbalanced $\sum_{i=1}^{m} a_i \neq \sum_{j=1}^{n} b_j$.

Methods for Initial Basic feasible solution :-

North-West Corner Rule :-

Eg :- Determine the basic feasible solⁿ to following transportation table using N-W Corner Rule.

				availability
1	2	1	4	30
3	3	2	1	50
4	2	5	9	20
Requirement 20	40	30	10	100

Solⁿ :- $\Sigma a_i = \Sigma b_j = 100.$

$x_{11} = \min(30, 20)$
$= 20.$ (∴ Eliminating 1st Column & allocating $x_{11} = 20$)

1	2	1	4	30−20
3	3	2	1	50
4	2	5	9	20
20	40	30	10.	

2	1	4	10
3	2	1	50
2	5	9	20
40 (−10)	30	10	

$\Rightarrow x_{12} = \min(40, 10)$
$= 10.$
(∴ Eliminating 2nd row & allocating $x_{12} = 10$)

3	2	1	50
2	5	9	20
30	30	10	

$\Rightarrow x_{22} = \min(50, 30)$
$= 30$
(∴ Eliminating 2nd Column & allocating $x_{22} = 30$)

~~2~~	~~1~~	~~20~~
5	9	20
30	10	

$\Rightarrow x_{23} = \min(20, 30)$
$= 20.$
(∴ eleminating 2nd row & allocating $x_{23} = 20$).

5	9	20
~~30~~ 20	10	

$\Rightarrow x_{33} = \min(20, 10)$
$= 10$
(∴ eleminating 3rd column & allocating $x_{33} = 10$).

9	10
10	

Total allocation :-

1(20)	2(10)	1	4
3	3 (30)	2 (20)	1
4	2	5 (10)	9 (10)

On multiplying each individual allocation by its corresponding unit in ( ) & adding, the ~~total~~ cost becomes = ~~5(10)~~ $1 \times 20 + 2(10) + 3(30) + 2(30) + 5(10) + 9(10)$

$= 20 + 20 + 90 + 60 + 50 + 90$

$= 40 + \cancel{200 + 90} = 110 + 200$

$= \cancel{310} = 310$

Q:-

10	20	5	7	10
13	9	12	8	20
4	5	7	9	30
14	7	1	0	40
3	12	5	19	50
60	60	20	10	150

Sol^n:- $\sum a_i = \sum b_j = 150$.

$x_{11} = \min(60, 10) = 10$

~~10~~	~~20~~	~~5~~	~~7~~	~~10~~
13	9	12	8	20
4	5	7	9	30
14	7	1	0	40
3	12	5	19	50
60 (−10)	60	20	10	

(eliminating 1st row & allocating $x_{11} = 10$);

~~13~~	~~9~~	~~12~~	~~8~~	~~20~~
4	5	7	9	30
14	7	1	0	40
3	12	5	19	50
50	60	20	10	

$\left(x_{21} = \min(20, 50) \right.$
$\left. = 20. \right)$

(eliminating 2nd row & allocating $x_{21} = 20$)

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~~14~~	~~5~~	~~7~~	~~9~~	~~30~~
14	7	1	0	40
3	12	5	19	50
30	60	20	10	

$x_{31} = \min(30, 30)$

$= 30.$

(eliminating 3rd row & 1st column & allocating $x_{31} = 30$).

~~7~~	~~1~~	~~0~~	~~40~~
12	5	19	50
60	20	10	

$x_{42} = \min(40, 60)$

$= 40.$

(allocating $x_{42} = 40$ by eliminating 4th row).

12	5	19	50
20	20	10	

$x_{52} = \min(50, 20)$

$= 20.$

(allocating $x_{52} = 20$ by eliminating 2nd column).

5	19	30
20	10	

$x_{53} = \min(20, 30)$

$= 20.$

(allocating $x_{53} = 20$ by eliminating 3rd column).

19	10
10	

$m+n-1 = 5+4-1 = 8$

(m = no. of rows)

n = no. of columns).

10(10)	20	5	7	10
13(20)	9	12	18	20
4(30)	5	7	9	30
14	7(40)	1	0	40
3	12(20)	5(20)	19(10)	50
60	60	20	10	

$m+n-1 = 8$

$7 \neq 8$.

(7 = no. of allocations)

$\therefore$ It is degenerate.

Now allocate 'ε' with x_{51} as '3' is not forming a cycle.

$= 10\times10 + 13\times20 + 4\times30 + 3\times\varepsilon + 7\times40 + 12\times20 + 5\times20$
$+ 19\times10$

$= 1290 + 3\varepsilon \quad (\because \varepsilon = 0)$.

$= 1290/-$

Least Cost Method (or) Lowest Cost Entry Method (LCM) (or) Matrix minimum method.

Step 1:- Identify the cell with smallest cost & allocate $x_{ij} = \min(a_i, b_j)$.

Case (i) :- If $\min(a_i, b_j) = a_i$ then put $x_{ij} = a_i$ cross out the i^{th} row & decrease b_j by a_i & go to step 2.

Case ii) If min $(a_i, b_j) = b_j$ then put $x_{ij} = b_j$ crossout the j^{th} column & decrease a_i by b_j, go to step 2.

Case iii) If min $(a_i, b_j) = a_i = b_j$ then put $x_{ij} = a_i = b_j$ cross out either i^{th} row or j^{th} column but not both, go to step 2.

Step 2:- Repeat step 1 for resulting transporting until all the requirements are satisfied.

Eg:- For the following transportation problem by using least method.

				Availability
1	2	1	4	30
3	3	2	10	50
4	2	5	9	20
20	40	30	10	

Requirement 20 40 30 10

Sol^n:- Least cost is '1' which is occuring at x_{11} & x_{13}

we are considering x_{11}.

$x_{11} = \min(30, 20) = 20$ (allocate $x_{11} = 20$ & eliminate 1^{st} column)

1	2	1	4	30
3	3	2	10	50
4	2	5	9	20
20	40	30	10	

2	1	4	10
3	2	10	50
2	5	9	20
40	30	10	

least is '1' at x_{13}.

$x_{13} = \min(10, 30)$
$= 10$

(allocate $x_{13} = 10$ & eliminate ~~2nd~~ 1st row)

3	2	10	50
2	5	9	20
40	20	10	

least is '2' at x_{23})

$x_{23} = \min(20, 50)$
$= 20$.

(allocate $x_{23} = 20$ & eliminate 3rd column).

3	10	30
2	9	20
40	10	

least is '2' at $x_{32} = \min(20, 40)$
$= 20$.

allocate $x_{32} = 20$ & eliminate ~~col~~ 3rd row)

3	10	30.
20	10	

least is '3' at x_{22}.

$x_{22} = \min(20, 30)$
$= (20)$

allocate $x_{22} = 20$ & eliminate 2nd ~~row~~ column).

10	10
10	

Total allocation

20	40	30	10	
1 (20)	2	1 (10)	4	30
3	3 (20)	2 (20)	10 (10)	50
4	2 (20)	5	9	20

$m+n-1 = 6$ (m = no. of rows
n = no. of columns).

$= 1\times20 + 1\times10 + 3\times20 + 2\times20 + 1\times10 + 2\times20.$

Total Cost = 270/-

no. of allocations = 6.
$m+n-1 = 6.$
$6 = 6$ (solⁿ is ~~degenerate~~ feasible)

Eg:-

10	20	5	7	10
13	9	12	18	20
4	5	7	19	30
14	7	1	0	40
3	12	5	19	50
60	60	20	10.	

least value is 0 at x_{44}.
$x_{44} = \min(10, 40) = 10.$
allocate $x_{44} = 10$ & eliminate 4^{th} column.

10	20	5	~~10~~ 9
13	9	12	20
4	5	7	30
14	7	1	30
3	12	5	50
60	60	20	

least is '1' at $x_{43} = \min(30, 20) = 20.$

allocate x_{43} ~~d~~ = 20 & eliminate 3rd column.

10	20	10
13	9	20
4	5	30
14	7	10
3	12	50
60	60	

least is '3' at $x_{51} = \min(60, 50)$
$= 50$.

allocate $x_{51} = 50$ & eliminate 5th row.

10	20	10
13	9	20
4	5	30
14	7	10
10	60	

least is '4' at $x_{31} = \min(30, 10)$
$= 10$.

Allocate $x_{31} = 10$ & eliminate 1st column.

20	10
9	20
5	20
7	10
60	

least is '5' at x_{32}

$x_{32} = \min(20, 60)$
$= 20$.

allocate $x_{32} = 20$ & eliminate 3rd row.

Final allocation :-

10	20 (10)	5	7
13	9 (20)	12	8
(10) 4	5 (20)	7	9
14	7 (10)	1 (20)	0 (10)
3 (50)	12	5	19

Total Cost $= 20\times10 + 9\times20 + 4\times10 + 5\times20 + 7\times10 + 1\times20 + 3\times50$

$= 760$

Vogel's Approximation method (VAMP) (or) Unit Cost Penalty Method :-

Step 1:- Find the difference (penalty) b/w the smallest cost & next smallest cost in each row (column) & write them in brackets against the corresponding row (or) column.

Step 2 :- Identify the row (or) Column with largest penalty if a tie occurs break the tie arbitrarily choose the cell with smallest cost in that selected row (or) Column & allocate as much as possible to the cell & cross out the satisfied row (or) Column & go to Step 3.

Step 3 :- Again compute the row & column penalities for reduced transportation table & go to step 2, repeat the procedure until all the requirements are satisfied.

(Q) Find the IBFS for the following transportation table.

	D_1	D_2	D_3	D_4	
S_1	11	13	17	14	250
S_2	16	18	14	10	300
S_3	21	24	13	10	400
	200	225	275	250	$\boxed{950}$

Since $\Sigma a_i = \Sigma b_j = 950$ the given problem is balanced.

$\therefore$ There exists a feasible solⁿ to the problem.

					Penalty
11 (200)	13	17	14	250/50	[2]
16	18	14	10	300	[4]
21	24	13	10	400	[3]
200	225	275	250		
penalty → [5]	[5]	[1]	[0]		

First let us find the diff b/w smallest & next smallest cost in each row & column & write them in brackets against respective rows & columns.

The largest of these differences is '5' & is associated with 1st 2 columns of the T. Table, we choose the 1st column arbitrarily.

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(8)

In this selected column the cell (1,1) has the minimum unit transportation cost $C_{11} = 11$.
$\therefore$ allocate $x_{11} = \min[250, 200] = 200$ & decrease the 200 by 200 & crossover satisfied column, the resulting table is

13 (50)	17	14	50	[1]
18	14	10	300	[4]
24	13	10	400	[3]
225	275	250.		
[5]	[1]	[0].		

The row & column differences are now computed for the reduced T.T. ~~The~~ The largest of these differences is '5' which is associated with the second column.
$\therefore C_{12} = 13$. is the min. cost & we allocate $x_{12} = \min[225, 50] = 50$, for cell (1,2) & decrease 225 by 50 & crossover satisfied row. The resulting transportation table is

18 (175)	14	10	300	[4]
24	13	10	400	[3]
175	275	250		
[6]	[1]	[0]		

Continuing in this manner we get remaining

14	10 (125)	125 [4]
13	10	400 [3]
275 [1]	250 [0]	

13	10 (125)	400 [3]
275	125	

(275) 13	275
275	

11 (200)	13 (50)	17	14
16	18 (175)	14	10 (125)
21	24	13 (275)	10 (125)

$m+n-1 = 6.$

$6 = 6$ (no. of allocations).

Total Cost $= 11 \times 200 + 13 \times 50 + 18 \times 175 + 10 \times 125 +$
$13 \times 275 + 10 \times 125$

$=$ Rs. 12075/-

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Stepping Stone Method for Optimality Test :-

4	6	8	8	40
6	8	6	7	60
5	7	6	8	50
20	30	50	50	$\boxed{150}$

find optimal solⁿ for given transportational problem.

Solⁿ:-

$\Sigma a_i = \Sigma b_j = 150$

4	6	8	8	40	[2] ←
6	8	6	7	60	[0]
5	7	6	8	50	[1]
20	30	50	50		
[1]	[1]	[0]	[1]		

$x_{11} = \min[40, 20]$
$= 20.$

6	8	8	20	[2]
8	6	7	60	[1]
7	6	8	50	[1]
30	50	50		
[1]	[0]	[1]		

$x_{12} = \min[20, 30] = 20.$

8	6	7	60	[1]
7	6	8	50	[1]
10	50	50		
[1]	[0]	[1]		

$x_{23} = \min[60, 50]$
$= 50.$

8	7	10	[1]
7	8	50	[1]
10	50		

$x_{24} = \min[10, 50] = 10$

7	8	50	[1]
10	40		

$x_{32} = \min[50, 10].$
$= 10.$

8	40
40.	

Total allocation :-

4 (20)	6 (20)	8	8	40
6	8	6 (50)	7 (10)	60
5	7 (10)	6	8 (40)	50
20	30	50	50	

$m+n-1 = 3+4-1 = 6.$

$\Rightarrow 4\times20 + 6\times20 + 6\times50$
$+ 7\times10 + 7\times10 + 8\times4$
$= 960/-$

To check whether the solⁿ is optimal or not; we need to go for the optimality test, here the allocated elements are basic. [P.T.O] variables, unallocated are non-basic variables. To verify the optimality we need to allocate a variable to unallocated cost variable by taking +1 & we need to create a loop which ends at the starting point.

x_{13}

4 (20)	6(-1) (20)	8(+1)	8
6	8 (-1)	6(-1) (50)	7(+1) (10)
5	7 (10)	6	8(-1) (40)

$= 8(1)+6(-1)+7(1)$
$+8(-1)+7(1)+6(-1)$

$= 8-6+7-8+7-6$

$= 2.$

x_{14}

4 (20)	6(-1) (20)	8	8 +1
6	8	6 (50)	7 (10)
5	7(+1) (10)	6	8(-1) (40)

$= 8(1)+8(-1)+7(1)+6(-1)$
$= 8-8+7-6$
$= 1.$

x_{21}

4(-1) (20)	6(+1) (20)	8	8
6 (+1)	8	6 (50)	7(-1) (10)
5	7(-1) (10)	6	8+1 (40)

$= 6(1)+4(-1)+6(1)+7(-1)+8(1)+7(-1)$
$= 2.$

x_{22}

4 (20)	6 (20)	8	8
6	8(+1)	6(50)	7(-1) (10)
5	7(-1) (10)	6	8(40) (+1)

$= 8(1)+7(-1)+8(1)+7(-1)$
$= 2.$

x_{31}

4 (-1) (20)	6 (+1) (20)	8	8
6	8	6 (50)	7 (10)
5 (+1)	7 (-1) (10)	6	8 (40)

$= 5(1)+4(-1)+6(1)+7(-1)$
$= 0.$

x_{33}.

4 (20)	6 (20)	8	8
6	8	6 (-1) (50)	7 (+1) (10)
5	7 (10)	6 (+1)	8(40) (-1)

$= 6(1)+8(-1)+7(1)+6(-1)$

$= 6-8+7-6$

$= -1$

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$x_{13} \rightarrow$ increase of 2.

$x_{14} \rightarrow$ " " 1

$x_{21} \rightarrow$ " " 2

$x_{22} \rightarrow$ " " 2

$x_{31} \rightarrow$ " " 0.

$x_{33} \rightarrow$ Decrease of 1.

There are 6 Non- variables ∴ There is only one position where we can decrease the value.

4 (20)	6 (20)	8	8
6	8	6 $50-\theta$	7 $10+\theta$
5	7	6 θ	8 $40-\theta$

$\theta = 40.$

4 (20)	6 (20)	8	8
6	8	6 (10)	7 (50)
5	7 10	6 40	8

$\Rightarrow$

4 20	6 20	8	8
6	8	6 10	7 50
5	7 10	6 40	8

$\Rightarrow m+n-1 = 3+4-1 = 6.$

$= 4\times20 + 6\times20 + 6\times10 + 7\times50 + 7\times10 + 6\times40 = 920/-$

x_{13}

4 (20)	6 (−1) (20)	8 (+1)	8
6	8	6 (10)	7 (10)
5	7 (+1) (10)	(−1) 6 (40)	8

$\Rightarrow 8(1) + 6(-1) + 7(1) + 6(-1)$

$= 3.$

x_{14}

4 (20)	$6^{(-1)}$ (20)	8	8 (+1)
6	8	6 (10) (+1)	7 (50) (-1)
5	7 (+1) (10)	$6^{(-1)}$ (40)	8

$= 8(1) + 7(-1) + 6(1)$
$+ 6(-1) + 7(1) + 6(-1)$
$= 2.$

x_{21}

$4^{(-1)}$ (20)	$6^{(+1)}$ (20)	8	8
6 (+1)	8	$6^{(-1)}$ (10)	7 (50)
5	$7^{(-1)}$ (10)	$6^{(+1)}$ (40)	8

$= 6(1) + 4(-1) + 6(1) + 7(-1)$
$+ 6(1) + 6(-1)$
$= 1.$

x_{22}

4 (20)	6 (20)	8	8
6	$8^{(+1)}$	$6^{(10)}_{(-1)}$	7 (50)
5	7 (-1) (10)	6 (40) (+1)	8

$= 8(1) + 6(-1) + 6(1) + 7(-1).$
$= 1.$

x_{31}

4 (-1) (20)	6 (+1) (20)	8	8
6	8	$6^{(10)}$	7 (50)
5 (+1)	7 (10) (-1)	$6^{(40)}$	8

$= 5(1) + 7(-1) + 6(1) + 4(-1)$
$= 0.$

x_{34}

4 (20)	6 (20)	8	8
6	8	$6^{(+1)}$ (10)	$7^{(-1)}$ (50)
5	7 (10)	6 (-1) (40)	8 (+1)

$= 8(1) + 7(-1) + 6(1) + 6(-1)$
$= 1.$

$x_{13} \rightarrow$ increase of 3.
$x_{14} \rightarrow$ " " 2
$x_{21} \rightarrow$ " " 1
$x_{22} \rightarrow$ " " 1
$x_{31} \rightarrow$ " " 0
$x_{34} \rightarrow$ " " 1

All the non-basic variables (or) unallocated cells, we have got the '+ve' values but we have got a '0' at x_{31} which indicates that there is no change in the optimal cost but there is an alternate optimal solution.

4 (20)−θ	6 20+θ	8	8
6	8	6	7
5 θ	7 10−θ	6	8

θ = 10

4 20−10	6 20+10
5 10	7 10−10

4 (10)	6 (30)	8	8
6	8	6 (10)	7 (50)
5 (10)	7	6 (40)	8

Total cost = $4\times10+6\times30+6\times10+7\times50+5\times10+6\times40 = 920/-$

UV Method (or) MODI Method (Modified Distribution)

4	6	8	8	40
6	8	6	7	60
5	7	6	8	50
20	30	50	50	

Solⁿ:-

					Penalty
4	6	8	8	40	[2] ←
6	8	6	7	60	[0]
5	7	6	8	50	[1]
20	30	50	50		
Penalty → [1]	[1]	[0]	[1]		

$x_{11} = \min\{40, 20\}$
$= 20$.

Allocate $x_{11} = 20$ & eliminate 1st column.

6	8	8	20	[2]
8	6	7	60	(1)
7	6	8	50	(1)
30	50	50		
[1]	[0]	[1]		

$\Rightarrow x_{12} = \min\{20, 30\} = 20.$

allocate $x_{12} = 20$ & eliminate 2nd row.

8	6	7	60	(1)
7	6	8	50	[1]
10	50	50		
[1]	[0]	[1]		

$x_{23} = \min\{60, 50\}$

$= 50$

allocate $x_{23} = 50$ & eliminate 3rd Column.

8	7	10	[1]
7	8	50	[1]
10	50/40		
[1]	[1]		

$x_{24} = \min\{10, 50\}$

$= 10.$

allocate $x_{24} = 10$ & eliminate 2nd row.

7	8	50(40)
10	40	

$x_{32} = \min\{50, 10\}.$

$= 10$

allocate $x_{32} = 10$ & eliminate 3rd column

8 (40)	40
40	

4 (20)	6 (20)	8	8	40
6	8	6 (50)	7 (10)	60
5	7 (10)	6	8 (40)	50
20	30	50	50	

$= 4(20) + 6(20) + 6(50) + 7(10) + 7(10) + 8(40)$

$=$ Rs. 960/-

for allocated cell $\boxed{C_{ij} = u_i + v_j}$

for unallocated cell $d_{ij} = \boxed{C_{ij} - (u_i + v_j)}$

Initially Assuming $\boxed{u_1 = 0}$.

	$v_1=4$	$v_2=6$	$v_3=6$	$v_4=7$
$u_1=0$	4 (20)	6 (20)	8	8
$u_2=0$	6	8	6 (50)	7 (10)
$u_3=1$	5	7 (10)	6	8 (40)

$$\boxed{C_{ij} = u_i + v_j}$$

$C_{11} = u_1 + v_1$

$4 = 0 + v_1$

$\boxed{v_1 = 4}$

$C_{12} = u_1 + v_2$

$6 = 0 + v_2$

$\boxed{v_2 = 6}$

$C_{32} = u_3 + v_2$

$7 = u_3 + 6$

$\boxed{u_3 = 1}$

$C_{34} = u_3 + v_4$.

$8 = 1 + v_4$

$\boxed{v_4 = 7.}$

$C_{24} = u_2 + v_4$

$7 = u_2 + 7$

$\boxed{u_2 = 0.}$

$C_{23} = u_2 + v_3$

$6 = 0 + v_3$

$\boxed{v_3 = 6.}$

for unallocated

	$v_1=4$	$v_2=6$	$v_3=6$	$v_4=7$
$u_1=0$	4 (20)	6 (20)	8 2	8
$u_2=0$	6	8	6 (50)	7 (10)
$u_3=1$	5	7 (10)	6	8 (40)

$d_{ij} = c_{ij} - (u_i + v_j)$.

$d_{13} = c_{13} - (u_1 + v_3)$

$= 8 - (0+6)$

$= 2$

$d_{14} = c_{14} - (u_1 + v_4)$

$= 8 - (0+7)$

$= 1$.

$d_{21} = c_{21} - (u_2 + v_1)$

$= 6 - (0+4)$

$= 2$.

$d_{22} = c_{22} - (u_2 + v_2)$

$= 8 - (0+6)$

$= 2$

$d_{31} = c_{31} - (u_3 + v_1)$

$= 5 - (1+4)$

$= 0$

$d_{33} = c_{33} - (u_3 + v_3)$

$= 6 - (1+6)$

$= -1$

$d_{13} \rightarrow$ increase of 2.

$d_{14} \rightarrow$ " " 1.

$d_{21} \rightarrow$ " " 2.

$d_{22} \rightarrow$ " " 2

$d_{31} \rightarrow$ " " 0

$d_{33} \rightarrow$ decrease of 1.

4 (20)	6 (20)	8 2	8 1
6 2	8 2	6 (50)	7 (10)
5 0	7 (10)	6 (-1)	8 (40)

$\theta = 40$.

6 $50-\theta$	7 $10+\theta$
6 θ	8 $40-\theta$

4 (20)	6 (20)	8	8
6	8	6 (10)	7 (50)
5	7 (10)	6 (40)	8

$m+n-1 = 3+4-1 = 6$.

$= 4\times20 + 6\times20 + 6\times10 + 7\times50 + 7\times10 + 6\times40$

$=$ Rs. 920/-

	$v_1=4$	$v_2=6$	$v_3=5$	$v_4=6$
$u_1=0$	4 (20)	6 (20)	8	8
$u_2=1$	6	8	6 (10)	7 (50)
$u_3=1$	5	7 (10)	6 (40)	8

for allocated cells :-

$c_{11} = u_1 + v_1$

$4 = 0 + v_1$

$\boxed{v_1 = 4}$

$c_{12} = u_1 + v_2$

$6 = 0 + v_2$

$\boxed{v_2 = 6}$

$c_{32} = u_3 + v_2$

$7 = u_3 + 6$

$\boxed{1 = u_3}$.

$c_{33} = u_3 + v_3$

$6 = 1 + v_3$

$\boxed{v_3 = 5}$

$c_{23} = u_2 + v_3$

$6 = u_2 + 5$

$\boxed{u_2 = 1}$.

$c_{24} = u_2 + v_4$

$7 = 1 + v_4$

$\boxed{v_4 = 6}$

For unallocated :-

$$\boxed{d_{ij} = c_{ij} - (u_i + v_j)}$$

$d_{13} = c_{13} - (u_1 + v_3)$

$= 8 - (0 + 5)$

$\boxed{d_{13} = 3}$

$d_{14} = c_{14} - (u_1 + v_4)$

$= 8 - (0 + 6)$

$\boxed{d_{14} = 2}$

$d_{21} = c_{21} - (u_2 + v_1)$

$= 6 - (1 + 4)$

$\boxed{d_{21} = 1}$

$d_{22} = c_{22} - (u_2 + v_2)$

$= 8 - (1 + 6)$

$\boxed{d_{22} = 1}$

$d_{31} = c_{31} - (u_3 + v_1)$

$= 5 - (1 + 4)$

$\boxed{d_{31} = 0}$

$d_{34} = c_{34} - (u_3 + v_4)$

$= 8 - (1 + 6)$

$\boxed{d_{34} = 1}$

$d_{13} \rightarrow$ increased by 3.

$d_{14} \rightarrow$ " " 2.

$d_{21} \rightarrow$ " " 1

$d_{22} \rightarrow$ " " 1

$d_{31} \rightarrow$ " " 0

$d_{34} \rightarrow$ " " 1 Optimal sol^n.

$\therefore$ Alternative sol^n.

4 $(20-\theta)$	6 $(20+\theta)$	8	8
6	8	6	7
5 (θ)	7 $(10-\theta)$	6	8

$\theta = 10.$

4 $(20-10)$	6 $(20+10)$
5 (10)	7 $(10-10)$

4 (10)	6 (30)	8	8
6	8	6 (10)	7 (50)
5 (10)	7	6 (40)	8

$m+n-1 = 6.$

$= 920/-$

Eg:- A Company has three plants A, B and C and three warehouses X, Y and Z. Number of units available at the plants is 60, 70 and 80 respectively. Demands at X, Y and Z are 50, 80 and 80, respectively. Unit costs of transportation are as follows:

	X	Y	Z
A	8	7	3
B	3	8	9
C	11	3	5

What would be your transportation plan? Give minimum distribution costs.

Sol:- Step 1:- First, write the given cost-requirement table in the following manner.

	X	Y	Z	Available
A	8	7	3	60
B	3	8	9	70
C	11	3	5	80.
	50	80	80	

Using either the "lowest Cost Entry Method" or "Vogel's Approximation Method" obtain the initial solution.

Since the no. of occupied cells in the solⁿ table is not equal to m+n-1, the problem is degenerate at the very beginning and so the attempt to assign u_i & v_j values to the table will not succeed. However, it is possible to resolve the degeneracy by addition of Δ to some suitable cell.

8	7	60(3)	60
50(3)	8	20(9)	70
11	80(3)	5(Δ)	80
50	80	80	

So proceed to resolve the degeneracy by allocating 'Δ' to least cost independent empty cell (3,3) as shown in above table.

Now, proceed to test this solⁿ for optimality. Values of u_i & v_j will be obtained in below table.

for allocated cells $c_{ij} = u_i + v_j$.

for unallocated cells $c_{ij} - (u_i + v_j)$

$C_{33} = u_3 + v_3$.

$5 = 0 + v_3$.

$\boxed{v_3 = 5}$

$C_{23} = u_2 + v_3$

$9 = u_2 + 5$

$\boxed{u_2 = 4}$

$C_{13} = u_1 + v_3$.

$3 = u_1 + 5$

$\boxed{u_1 = -2.}$

$C_{21} = u_2 + v_1$.

$3 = 4 + v_1$

$\boxed{v_1 = -1}$

$C_{32} = u_3 + v_2$

$3 = 0 + v_2$

$\boxed{v_2 = 3}$

8	7	1
.	8	.
11	.	.

				$u_i \downarrow$
			•(3)	−2
	•(3)		•(9)	4
		•(3)	•(5)	0
$v_j \rightarrow$	−1	3	5	

Matrix c_{ij} for empty cells only.

(8)	(7)	.
.	(8)	.
(11)	.	.

Matrix (u_i+v_j) for empty cells.

−3	1	.
.	7	.
−1	.	.

Matrix $[c_{ij}-(u_i+v_j)]$ Since all $[c_{ij}-(u_i+v_j)]$ are +ve the sol^n is: ↓

11	6	.
.	1	.
12	.	.

		60(3)
50(3)		20(9)
	80(3)	

Thus minimum cost is 180+150+180+240 = Rs. 750/-

Unbalanced transportation problem

Eg:-

					a_i (Supply) ↓
4	2	3	2	6	8
5	4	5	2	1	12
6	5	4	7	3	14
demand b_j → 4	4	6	8	8	

$\Sigma a_i \neq \Sigma b_j$

∴ The problem is unbalanced.

Step 1:- Modifying the problem to balanced type;

Since the 'a_i' (supply) is 'four' quantities more than the total demand (b_j); consider a fictitious requirement by four quantities. Thus the modified transportation cost matrix becomes:

	M_1	M_2	M_3	M_4	M_5	M_f	Supply ↓
	4	2	3	2	6	0	8
	5	4	5	2	1	0	12
	6	5	4	7	3	0	14
demand →	4	4	6	8	8	4	$\boxed{34}$

To find the initial sol^n; applying Vogel's (penalty) method in usual manner, the initial sol^n obtained is;

4	2(4)	3	2(4)	6	0
5	4	5	2(4)	1 (8)	0
6(4)	5	4(6)	7	3	0(4)

This gives the transportation cost $= 4(2) + \ldots = 80/-$

To test the initial solⁿ for optimality.

Since the total no. of allocation is 7 (instead of $m+n-1 = 8$)

∴ This is a degenerate basic feasible solⁿ.

∴ allocate an infinitesimal quantity 'Δ' to empty cell (1, 1). Then proceeding in the usual manner; following tables for testing the optimality of the solⁿ are obtained

u_i & v_j

						u_i ↓
(Δ) 4	2(4)	3	2(4)	6	0	0
5	4	5	2(4)	1 (8)	0	0
6 (4)	5	4(6)	7	3	0 (4)	2
$v_j \rightarrow$ 4	2	2	2	1	-2	

$u_i + v_j$ for empty cell.

.	.	2	.	1	-2	0
4	2	2	.	.	-2	0
.	4	.	4	3	.	2
4	2	2	2	1	-2	

.	.	1	.	5	8
1	2	3	.	.	2
.	1	.	3	0	.

Since all $d_{ij} = c_{ij} - (u_i + v_j)$ for empty cells are non-negative, the solⁿ under test is optimal. Further '0' in the cell (3, 5) indicates that alternative solⁿ's will also exist.

Assignment Problems :-

→ The Assignment problem is a particular case of transportation problem in which the number of jobs (origins (or) source) are equal to the no. of facilities (destinations (or) machines (or) persons).

→ The objective is to maximise total profit of allocation or to minimise the total cost.

→ An assignment problem is a completely degenerate form of a transportation problem.

→ The units available at each origin & the units demanded at each destination are equal to one.

Mathematical formulation of an assignment problem :-

Given 'n' jobs (or activities) and n persons (facilities) and effectiveness (in terms of cost, profit, time & others) of each person for each job, the problem lies in assigning each person to one and one job so that the given measure of effectiveness is optimised.

Let C_{ij} be the cost of assigning i^{th} person to j^{th} job. The assignment problem can be stated in the form of $n \times n$ cost matrix. C_{ij} has shown in the following table.

Jobs

	1	2	3	...	j	...	n	Supply
1	C_{11}	C_{12}	C_{13}	...	C_{1j}	...	C_{1n}	1
2	C_{21}	C_{22}	C_{23}	...	C_{2j}	...	C_{2n}	1
3	C_{31}	C_{32}	C_{33}	...	C_{3j}	...	C_{3n}	1
persons ⋮	⋮	⋮	⋮		⋮			
i	C_{i1}	C_{i2}	C_{i3}	...	C_{ij}	...	C_{in}	1
⋮	⋮							
n	C_{n1}	C_{n2}	C_{n3}	...	C_{nj}	...	C_{nn}	1
Demand	1	1	1	...	1	...	1	

Let x_{ij} denote the assignment of a person 'i' to job 'j' such that $x_{ij} = \left\{ \begin{matrix} 1 \\ 0 \end{matrix} \right\}$ for '1' if person 'i' is assigned to job 'j', otherwise zero.

The assignment problem is simply written as LPP in following form.

$$\boxed{\min Z = \sum_{i=1}^{n} \sum_{j=1}^{n} C_{ij} x_{ij}}$$

Subject to Constraints $\sum_{j=1}^{n} x_{ij} = 1$, for $i = 1, 2 \ldots n$.
(one job is done by ith person)

$$\sum_{i=1}^{n} x_{ij} = 1, \quad \text{for } j = 1, 2 \ldots n.$$

and (only one person should be assigned jth job)

~~$\sum_{i=1}^{n} x_{ij}=1$ for $j=1,2 \ldots n$ with~~

~~$x_{ij}=0$ or 1 $\forall$ i and j.~~

NOTE: ~~As assignment problem is an $n \times n$ transportation problem with each $a_i = b_j = 1$.~~

Hungarian Method or Floats Technique or Reduced Cost Matrix:-

Various steps of computational process (or) procedure for obtaining an optimal solution namely summarised as follows:-

Step 1:- If the no. of rows are not equal to no. of columns or vice-versa then a dummy row (or) column must be added with zero cost elements.

Step 2:- Find the smallest cost in each row of cost matrix, subtract this smallest cost from each element in that row therefore there will exist atleast one zero in each row of this new matrix which called the first reduced cost matrix.

Step 3:- In the reduced cost matrix find the smallest element in each column element subtract the smallest cost element from each element in that column as a result there could be atleast one zero in each row element & column of the second reduced cost matrix.

Step 4:- Determine an optimum assignment as follows

i) Examining the rows successively until a row with exactly one zero found. Box [□] around the zero element as an assigned cell and cross out [x] all other zero's in its column. Proceed until all the rows have been examined if there are more than one zero in any row then donot considered that row & pass on to the next row.

ii) Repeat the procedure of columns of reduced cost matrix if there is no single zero in any row (or) column of the reduced matrix then arbitrarily choose a row (or) column] having a min. no. of 0's arbitrarily choose zero in row (or) column & cross the remaining 0's in the row (or) column repeat this 2 steps until all zero's are assigned (or) crossed out.

Step 5:- An optimal assignment is formed if the no of assigned cells equals to no of rows (columns) if zero cell is arbitrarily choosen there may be an alternate optimum if no optimum sol^n is found (some rows (or) columns without any assignments]. Then go to next step.

Step 6:- Draw the minimum no. of horizontally/vertical lines through all the zero's as follows:-

i) Mark (√) to those rows where no assignment has been made.

ii) Mark to those columns which have 0's in the marked rows.

(iii) Mark rows (not already marked) which have assignment in marked columns.

(iv) The process may be repeated until no more rows (or) columns can be checked.

(v) Draw straight lines through all unmarked rows & marked columns.

Step 7:- If the minimum no. of lines passing through all zero's is equal to the no. of rows (or) columns the optimal solution is attained by an arbitrarily allocation in position of 0's not crossed in step 3 otherwise go to next step.

Step 8:- Revise the cost matrix as follows

(1) find the elements that are uncovered by a line choose the smallest of these elements and subtract this element from all the uncrossed elements & add the same at the point of intersection of two elements.

(2) Other elements crossed by the lines remaining unchanged.

Step 9:- Go to step 4 & repeat the procedure till the optimum solution is obtained.

Eg:- The assignment cost of assigning any one operator to an even m/c is given in the following table. find optimal assignment.

$$\begin{array}{c|cccc} & I & II & III & IV \\ A & 10 & 5 & 13 & 15 \\ B & 3 & 9 & 18 & 3 \\ C & 10 & 7 & 3 & 2 \\ D & 5 & 11 & 9 & 7 \end{array}$$

The given problem is a balanced problem we can find optimal assignment.

Sol^n:-

$$\begin{bmatrix} 10 & 5 & 13 & 15 \\ 3 & 9 & 18 & 3 \\ 10 & 7 & 3 & 2 \\ 5 & 11 & 9 & 7 \end{bmatrix}$$

In every row check least element & subtract.

$$\begin{bmatrix} 5 & 0 & 8 & 10 \\ 0 & 6 & 15 & 0 \\ 8 & 5 & 1 & 0 \\ 0 & 6 & 4 & 2 \end{bmatrix}$$

The third column of the table is not having zero so find smallest element in 3rd column & subtract it

$\therefore$ 1 is the smallest element.

$$\begin{bmatrix} 5 & 0 & 7 & 10 \\ 0 & 6 & 14 & 0 \\ 8 & 5 & 0 & 0 \\ 0 & 6 & 3 & 2 \end{bmatrix}$$

Check for single zero in all rows if it occurs in that column (or) mark & cross out any zero in assign.

$$\begin{array}{c|cccc} M/cs \rightarrow & I & II & III & IV \\ \hline A & 5 & \boxed{0} & 7 & 10 \\ B & \cancel{0} & 6 & 14 & \boxed{0} \\ C & 8 & 5 & \boxed{0} & \cancel{0} \\ D & \boxed{0} & 6 & 3 & 2 \end{array}$$

operators $\rightarrow$

A $\rightarrow$ II, B $\rightarrow$ IV, C $\rightarrow$ III, D $\rightarrow$ I.

$\therefore$ Sum up the original costs $= 5+3+3+5 = 16$

Eg.:- A department has 5 employees with 5 jobs to be performed and the time is in hours. Each man will take to perform each job in hours is given in the cost matrix. ~~How should~~ jobs to be allocated 1 per employee so as to minimise total man hours.

Jobs

$$\begin{array}{cc|ccccc} & & I & II & III & IV & V \\ \text{Employees} & A & 10 & 5 & 13 & 15 & 16 \\ & B & 3 & 9 & 18 & 13 & 6 \\ & C & 10 & 7 & 2 & 2 & 2 \\ & D & 7 & 11 & 9 & 7 & 12 \\ & E & 7 & 9 & 10 & 4 & 12 \end{array}$$

Soln:-

$$\begin{bmatrix} 5 & \boxed{0} & 8 & 10 & 11 \\ \boxed{0} & 6 & 15 & 10 & 3 \\ 8 & 5 & \boxed{0} & \cancel{0} & \cancel{0} \\ \cancel{0} & 4 & 2 & \boxed{0} & 5 \\ 3 & 5 & 6 & \cancel{0} & 8 \end{bmatrix}$$

we have got only 4 assignment which is not optimal.

$$\begin{bmatrix} 5 & \boxed{0} & 8 & 10 & 11 \\ \boxed{0} & 6 & 15 & 10 & 3 \\ 8 & 5 & \boxed{0} & \cancel{0} & \cancel{0} \\ \cancel{0} & 4 & 2 & \boxed{0} & 5 \\ 3 & 5 & 6 & \cancel{0} & 8 \end{bmatrix}$$

The smallest element in the uncovered elements is 2 subtract this 2 from all uncovered elements & add this at intersected element, there is no change in the covered elements

$$\begin{bmatrix} 7 & \boxed{0} & 8 & 12 & 11 \\ \boxed{0} & 4 & 13 & 10 & 1 \\ 10 & 5 & \cancel{0} & 2 & \boxed{0} \\ \cancel{0} & 2 & \boxed{0} & \cancel{0} & 3 \\ 3 & 3 & 4 & \boxed{0} & 6 \end{bmatrix}$$

$5+3+2+9+4$

$= 23$ hrs.

$A \to II$, $B \to I$, $C \to V$, $D \to III$, $E \to IV$.

Eg:- A department has 5 employees with 5 jobs to be performed and the time is in hours. Each man will take to perform each job in hours is given in the cost matrix. How should jobs to be allocated 1 per employee so as to minimise total man hours.

$$
\begin{array}{c|ccccc}
 & I & II & III & IV & V \\
\hline
A & 10 & 5 & 13 & 15 & 16 \\
B & 3 & 9 & 18 & 13 & 6 \\
C & 10 & 7 & 2 & 2 & 2 \\
D & 7 & 11 & 9 & 7 & 12 \\
E & 7 & 9 & 10 & 4 & 12
\end{array}
$$

Soln:-

$$
\begin{bmatrix}
5 & \boxed{0} & 8 & 10 & 11 \\
\boxed{0} & 6 & 15 & 10 & 3 \\
8 & 5 & \boxed{0} & \cancel{0} & \cancel{0} \\
\cancel{0} & 4 & 2 & \boxed{0} & 5 \\
3 & 5 & 6 & \cancel{0} & 8
\end{bmatrix}
$$

We have got only 4 assignment which is not optimal.

$$
\begin{bmatrix}
5 & \boxed{0} & 8 & 10 & 11 \\
\boxed{0} & 6 & 15 & 10 & 3 \\
8 & 5 & \boxed{0} & \cancel{0} & \cancel{0} \\
\cancel{0} & 4 & 2 & \boxed{0} & 5 \\
3 & 5 & 6 & \cancel{0} & 8
\end{bmatrix}
\begin{array}{c} \\ \checkmark \\ \\ \checkmark \\ \checkmark \end{array}
$$
$$\checkmark \qquad\qquad\qquad \checkmark$$

The smallest element in the uncovered elements is 2, subtract this 2 from all uncovered elements & add this at intersected lines there is no change in the covered elements

$$
\begin{bmatrix}
7 & \boxed{0} & 8 & 12 & 11 \\
\boxed{0} & 4 & 13 & 10 & 1 \\
10 & 5 & \cancel{0} & 2 & \boxed{0} \\
\cancel{0} & 2 & \boxed{0} & \cancel{0} & 3 \\
3 & 3 & 4 & \boxed{0} & 6
\end{bmatrix}
$$

$5+3+2+9+4$
$= 23$ hrs.

A → II
B → I
C → V
D → III
E → IV

Eg:-

$$\begin{array}{c|cccc} & M_1 & M_2 & M_3 & M_4 \\ \hline J_1 & 0 & 2 & 6 & 1 \\ J_2 & 3 & 0 & 4 & 1 \\ J_3 & 0 & 3 & 6 & 3 \\ J_4 & 7 & 1 & 5 & 0 \end{array}$$

Sol:- As '0' is missing in 3rd column subtract least element from it.

$$\begin{bmatrix} \boxed{0} & 2 & 2 & 1 \\ 3 & \boxed{0} & \cancel{0} & 1 \\ \cancel{0} & 3 & 2 & 3 \\ 7 & 1 & 1 & \boxed{0} \end{bmatrix}$$

draw lines for untick marked rows & tick marked columns.

$$\begin{bmatrix} \boxed{0} & 2 & 2 & 1 \\ 3 & \boxed{0} & \cancel{0} & 1 \\ \cancel{0} & 3 & 2 & 3 \\ 7 & 1 & 1 & \boxed{0} \end{bmatrix}$$

least value is '1' &, subtract it & add it at intersection elements.

$$\begin{bmatrix} \cancel{0} & 1 & 1 & \boxed{0} \\ 4 & \boxed{0} & \cancel{0} & 1 \\ \boxed{0} & 2 & 1 & 2 \\ 8 & 1 & 1 & \cancel{0} \end{bmatrix}$$

∴ '1' is the smallest element

$$\begin{bmatrix} \boxed{0} & \cancel{0} & \cancel{0} & \cancel{0} \\ 5 & \boxed{0} & \cancel{0} & 2 \\ 0 & 1 & \boxed{0} & 2 \\ 8 & \cancel{0} & \cancel{0} & \boxed{0} \end{bmatrix}$$

$= 0 + 0 + 6 + 0.$

$= 6.$

$J_1 \to M_1$

$J_2 \to M_2$

$J_3 \to M_3$

$J_4 \to M_4.$

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Non-square Matrix (or) Solving unbalanced Assignment problem :-

*) A ~~Company~~ Company has one surplus truck in each of cities A, B, C, D, E & deficit truck in each of the cities is 1, 2, 3, 4, 5, 6 the distance b/w cities in Km's is shown in matrix below. Find the assignment of trucks in cities in surplus in cities in deficit so that total distance covered by vehicles is minimum.

Cities \ Trucks	1	2	3	4	5	6
A	12	10	15	22	18	8
B	10	18	25	15	16	12
C	11	10	3	8	5	9
D	6	14	10	13	13	12
E	8	12	11	7	13	10

The given problem is an unbalanced problem to find the optimal assignment we need to add a dummy row with all '0' costs for the given table.

$$\begin{array}{c} A \\ B \\ C \\ D \\ E \\ F \end{array}\begin{bmatrix} 12 & 10 & 15 & 22 & 18 & 8 \\ 10 & 18 & 25 & 15 & 16 & 12 \\ 11 & 10 & 3 & 8 & 5 & 9 \\ 6 & 14 & 10 & 13 & 13 & 12 \\ 8 & 12 & 11 & 7 & 13 & 10 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 2 & 7 & 14 & 10 & \boxed{0} \\ \boxed{0} & 8 & 15 & 5 & 6 & 2 \\ 8 & 7 & \boxed{0} & 5 & 2 & 6 \\ \cancel{0} & 8 & 4 & 7 & 7 & 6 \\ 1 & 5 & 4 & \boxed{0} & 6 & 3 \\ \cancel{0} & \boxed{0} & \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \end{bmatrix}$$

'2' is the smallest element.

$$\begin{bmatrix} 6 & 2 & 7 & 14 & 10 & \boxed{0} \\ \boxed{0} & 6 & 13 & 3 & 4 & 0 \\ 10 & 7 & \boxed{0} & 5 & 2 & 6 \\ \cancel{0} & 6 & 2 & 5 & 5 & 4 \\ 3 & 5 & 4 & \boxed{0} & 6 & 3 \\ 2 & \boxed{0} & 0 & 0 & 0 & \cancel{0} \end{bmatrix}$$

2 is least; add & subtract '2'.

$$\begin{bmatrix} 6 & \boxed{0} & 5 & 12 & 8 & 0 \\ \cancel{0} & 4 & 11 & 1 & 2 & \boxed{0} \\ 12 & 7 & \boxed{0} & 5 & 2 & 8 \\ \boxed{0} & 4 & \cancel{0} & 3 & 3 & 4 \\ 5 & 5 & 4 & \boxed{0} & 6 & 5 \\ 4 & \cancel{0} & \cancel{0} & \cancel{0} & \boxed{0} & 2 \end{bmatrix}$$

$10 + 12 + 3 + 6 + 7 + 0 = 38$ kms.

A → 2, B → 6, C → 3, D → 1, E → 4, Dummy.

Maximisation problem (Assignment Model)

Eg:- A Company has a team of 4 salesman & there are 4 districts where the company wants to start its business after taking into account the capabilities of salesman & the nature of districts, the Company estimates that profit perday in rupees for each salesman in each district is as below. find the assignment of salesman to various districts which will yield max. profit.

	1	2	3	4
A	16	10	14	11
B	14	11	15	15
C	15	15	13	12
D	13	12	14	15

Soln:- As the given problem is maximisation type to find the assignment, we need to convert the given problem into minimization case. Therefore the highest element is 16, subtract all the elements of the table from the highest element the resulting table is the loss matrix..

$$\begin{bmatrix} 0 & 6 & 2 & 5 \\ 2 & 5 & 1 & 1 \\ 1 & 1 & 3 & 4 \\ 3 & 4 & 2 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} \boxed{0} & 6 & 2 & 5 \\ 1 & 4 & \boxed{0} & \cancel{0} \\ \cancel{0} & \boxed{0} & 2 & 3 \\ 2 & 3 & 1 & \boxed{0} \end{bmatrix}$$

$= 16 + 15 + 15 + 15 = $ Rs. 61/-.

$A \to 1,\ B \to 3,\ C \to 2,\ D \to 4.$

Q2) A department has 4 subordinates & 4 tasks to be performed, the subordinates differ in efficiency & tasks differ in their intrinsic difficulty, the estimates of the profit in rupees each man would earn, is given in the effectiveness matrix. How should the tasks be allocated one to each man, so as to maximise total earnings.

	5	6	7	8
1	5	40	20	5
2	25	35	30	25
3	15	25	20	10
4	15	5	30	15

Highest is 40.

35	0	20	35
15	5	10	15
25	15	20	30
25	35	10	25

$\Rightarrow$

35	0	20	35
10	0	5	10
10	0	5	15
15	25	0	15

25	$\boxed{0}$	20	25
$\cancel{0}$	$\cancel{0}$	5	$\boxed{0}$
$\boxed{0}$	$\cancel{0}$	5	5
5	25	$\boxed{0}$	5

$\Rightarrow 40+25+15+30.$

$\boxed{= 110/-}$

$1 \to 2$; $2 \to 4$, $3 \to 1$, $4 \to 3$.

Restrictions on Assignments :-

Eg:- 4 New machines m_1, m_2, m_3, m_4 are to be installed in a machine shop there are five vacant places A, B, C, D, E available because of limited space, machine m_2 cannot be placed at C & m_3 cannot be placed at A. C_{ij} the assignment cost of m/c 'i' to place 'j' in rupees is shown below.

	A	B	C	D	E
m_1	4	6	10	5	6
m_2	7	4	—	5	4
m_3	—	6	9	6	2
m_4	9	3	7	2	3

As the given matrix is non square matrix ; we add a dummy machine & associate '0' cost with the corresponding cells. As machine m_2 cannot be placed at 'C' & m_3 cannot be placed at 'A'. We assign infinite cost (∞) in cells (m_2, C) & (m_3, A), resulting in the following matrix.

M_1	4	6	10	5	6
M_2	7	4	∞	5	4
M_3	∞	6	9	6	2
M_4	9	3	7	2	3
Dummy	0	0	0	0	0

$\Rightarrow$

$\boxed{0}$	2	6	1	2
3	$\boxed{0}$	∞	1	0
∞	4	7	4	$\boxed{0}$
7	1	5	$\boxed{0}$	1
$\cancel{0}$	$\cancel{0}$	$\boxed{0}$	$\cancel{0}$	$\cancel{0}$

$= 4 + 4 + 2 + 2 + 0 = 12/-$

Eg:- ~~The~~ Five jobs ~~rafts~~ are available to work on ~~5~~ 5 m/c's & the respective cost in rupees for each m/c, combo is given in the matrix below. A 6th m/c is available to replace one of the existing machines & the associated costs are given in the following table. Determine (1) whether the new m/c can be accepted (2) The optimal assignments & the associated costs.

Jobs ↓ M/Cs →	A	B	C	D	E	F
1	19	15	–	16	13	22
2	13	–	15	–	21	14
3	15	17	19	20	12	18
4	20	22	16	18	17	–
5	–	16	14	19	18	15
6	0	0	0	0	0	0

as 1st job is cannot be assigned to 'C' &
2nd " " " " " to 'B' & 'D'.
4th " " " " " 'F'
5th " " " " " 'A', ; we assign infinite cost (∞) in cells (1, C), (2, B) (2, D), (4, F), (5, A) resulting in the following matrix.

Jobs ↓ M/cs →	A	B	C	D	E	F
1	19	15	∞	16	13	22
2	13	∞	15	∞	21	14
3	15	17	19	20	12	18
4	20	22	16	18	17	∞
5	∞	16	14	19	18	15
6	0	0	0	0	0	0

Choosing least value in each row & subtracting that value from all the ~~other~~ elements in that row.

6	2	∞	3	$\boxed{0}$	9
$\boxed{0}$	∞	2	∞	8	1
3	5	7	8	~~0~~	6
4	6	$\boxed{0}$	~~2~~	1	~~∞~~
∞	2	~~0~~	5	4	1
~~0~~	$\boxed{0}$	~~0~~	~~0~~	~~0~~	~~0~~

after ~~subtracting each~~ making every row containing 'zero' we get the below matrix.

~~6~~	~~1~~	∞	2	$\boxed{0}$	8
$\boxed{0}$	∞	2	∞	8	~~0~~
3	4	7	7	~~0~~	5
~~4~~	5	$\boxed{0}$	~~1~~	~~1~~	∞
∞	1	~~0~~	4	4	$\boxed{0}$
1	$\boxed{0}$	1	~~0~~	1	~~0~~

13 + 13 + 19 + 15 + 0.

Total Cost = 60/—

Travelling Salesman Problems (Routing problems):-

→ Travelling salesman problem is one of the problems considered as puzzles by the mathematicians.

→ Suppose a salesman wants to visit a certain no. of cities alloted to him. He knows the distance of b/w every pair of cities, usually denoted by C_{ij} i.e., from city i to city j. His problem is to select such a route that starts from his home city, passes through each city once and only once & returns to his home city in the shortest possible distance.

The problem may be classified in 2 forms:-

(1) Symmetrical :- The problem is said to be symmetrical if the distance (or cost or time) b/w every pair of cities is independent of the direction of his journey.

(2) Assymmetrical :- The problem is said to be assymetrical if for one or more pair of cities, the distance changes with the direction. For eg, flying from East to west usually takes longer time than from West to east on account of prevailing winds. Similar is the case while going up-hill from city A to B instead of coming down-hill from B to A.

Furthermore, if no. of cities is only two, obviously there is no choice. If no. of cities become three, say A, B and C one of them [say A) is the home base, then there are two possible routes:

$A \to B \to C$ and $A \to C \to B$.

for four cities, A, B, C and D, there are $3! = 6$ possible routes.

$A \to B \to C \to D$, $A \to B \to D \to C$. $A \to C \to B \to D$.

$A \to C \to D \to B$, $A \to D \to B \to C$ $A \to D \to C \to B$.

But if no. of cities is increased to 21, there are $20! = 2402902008,176,640,000$. different routes. Even a fast electronic computer testing one route per micro second and working 8 hrs a day 365 days a year, would take almost a quarter of a million years to find the best solution. Thus, it is practically impossible to find the best route by ~~~~trying each ~~other~~ one.

→ In general if there are n cities, there are $(n-1)!$ possible routes. At present, the best procedure is to solve the problem as if it were an assignment problem.

→ It becomes necessary to formulate this type of sequencing problem as if it were an assignment problem, with the additional restriction on his choice of route.

Formulation of Travelling-Salesman problem as assignment problem:-

Suppose c_{ij} is the distance (or cost or time) from city i to city j & $x_{ij}=1$, if the salesman goes directly from city i to city j; & $x_{ij}=0$ otherwise.
Then minimize $\sum_{i}\sum_{j} x_{ij}c_{ij}$ with the additional restriction that the x_{ij} must be so chosen that no city is visited twice before the tour of all cities is completed. In particular, he cannot go directly from city i to i itself. This possibility may be avoided in the minimization process by adopting the convention $c_{ii}=\infty$ which ensures that x_{ii} can never be unity.

→ Alternatively, omit the variable x_{ii} from the problem specification. It is also important to note that only single $x_{ij}=1$ for each value of i & j.

→ The distance (or cost or time) matrix for this problem is given in table below.

		To			
		A_1	A_2	...	A_n
From	A_1	∞	c_{12}	...	c_{1n}
	A_2	c_{21}	∞	...	c_{2n}
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	A_n	c_{n1}	c_{n2}	...	∞

Solution Procedure

→ Detailed procedure is explained with reference to an equivalent example involving the order in which five products A_1, A_2, A_3, A_4 & A_5 are processed over a production facility. To find an order in which products are produced so that the setup costs are minimized, is an asymmetrical travelling-salesman problem.

→ The change-over / set-up costs b/w products must be produced once & only once & production must return to the first product. The following eg's will make the procedure clear.

Eg:- Given the matrix of set-up costs, show how to sequence the production so as to minimize the set-up cost per cycle.

	A_1	A_2	A_3	A_4	A_5
A_1	∞	2	5	7	1
A_2	6	∞	3	8	2
A_3	8	7	∞	4	7
A_4	12	4	6	∞	5
A_5	1	3	2	8	∞

Solⁿ:- Consider the problem as an assignment problem

Step 1:- First, subtract the smallest element from each row & each column to get the reduced matrix in above table.

∞	1	4	6	0
4	∞	1	6	0
4	3	∞	0	3
8	0	2	∞	1
0	2	1	7	∞

In 3^{rd} Column, zero is not exiting so find smallest element in 3^{rd} column & then reduce the 3^{rd} column by that least element

	A_1	A_2	A_3	A_4	A_5
A_1	∞	1	3	6	$\boxed{0}$
A_2	4	∞	$\boxed{0}$	6	∞
A_3	4	3	∞	$\boxed{0}$	3
A_4	8	$\boxed{0}$	1	∞	1
A_5	$\boxed{0}$	2	∞	7	∞

Although zero's of this matrix gives a solⁿ to the assignment problem, but this is not a solⁿ of the 'travelling salesman' problem. This solⁿ ($A_1 \to A_5$, $A_5 \to A_1$, $A_2 \to A_3$, $A_3 \to A_4$ & $A_4 \to A_2$) indicates to produce the products A_1, then A_5 & then again A_1, without producing the products A_2, A_3 & A_4, thereby violating the additional restriction of producing each product once & only once before returning to the first product.

Step 3 :- Again, examine the matrix for some of the 'next best' solution to the assignment problem, and try to find out one solⁿ which satisfies the additional restriction. The smallest element other than zero is 1, so try the effect of putting such an element in the solⁿ.

	A_1	A_2	A_3	A_4	A_5
A_1	α	$\boxed{1}$	3	6	x
A_2	4	α	$\boxed{0}$	6	x
A_3	4	3	α	$\boxed{0}$	3
A_4	8	x	1	α	$\boxed{1}$
A_5	$\boxed{0}$	2	x	7	α

optimal travelling solⁿ :-

$A_1 \to A_2$, $A_2 \to A_3$, $A_3 \to A_4$, $A_4 \to A_5$, $A_5 \to A_1$

Total cost $= 2+3+4+5+1 = 15$ /- .

Eg :- Solve the following "Travelling Salesman problem" given by the following data :-

$c_{12} = 4$; $c_{13} = 7$, $c_{14} = 3$, $c_{23} = 6$, $c_{24} = 3$ & $c_{34} = 7$ where $c_{ij} = c_{ji}$.

$c_{12} = 3 \Rightarrow c_{21} = 3$

$c_{13} = 7 \Rightarrow c_{31} = 7$

$c_{14} = 3 \Rightarrow c_{41} = 3$

$c_{23} = 6 \Rightarrow c_{32} = 6$.

$c_{24} = 4 \Rightarrow c_{42} \Rightarrow 4$

$c_{34} = 7 \Rightarrow c_{43} \Rightarrow 7$

	1	2	3	4
1	α	3	7	3
2	3	α	6	4
3	7	6	α	7
4	3	4	7	α