Unit-V Waiting Lines (Queuing Theory) (1)

Introduction:-

→ Queue is formed if the server is busy. Then remaining customers are waiting in the queue. Queuing theory is based on probability. Most of the basic work on queue or waiting line theory was ~~called~~ carried by AK Erlang. A queuing system can be completely described by queuing ~~system~~ theory.

Model 1:- Single Channel :-

Assumptions made in this model:

(1) Poisson arrival process.

(2) Exponential Service time distribution

(3) Single Server.

(4) ∞ source capacity.

(5) ∞ queuing capacity.

(6) First Come First Serve.

To determine various parameters:-

1. Probability that exactly 'n' no. of customers in the system $= P_n$

$$\boxed{P_n = \left(\frac{\lambda}{\mu}\right)^n P_0}$$

2. To determine probability that no customers in the system or probability that server is idle:

$$\boxed{\sum_{n=0}^{\infty} P_n = 1}$$

Probability that no customer in the system = Probability that server is idle $= P_0$.

$$\boxed{P_0 = 1 - \frac{\lambda}{\mu}}$$

Probability that the server is busy $= \boxed{1 - P_0 = \frac{\lambda}{\mu}}$.

3. Probability that no. of customers in the system $\geq N$.

Probability that no. of customers in the system $\geq N = P_{(n \geq N)}$.

$$\boxed{P_{n \geq N} = \left(\frac{\lambda}{\mu}\right)^N}$$

4. Average or expected number of customers in the system (L_s):

$$\boxed{L_s = \frac{\lambda}{\mu - \lambda}}$$

5. Waiting time in the system (W_s): The average time spent by the customer in the system.

$$\boxed{W_s = \frac{L_s}{\lambda}} ; \boxed{W_s = \frac{1}{\mu - \lambda}}$$

6. To determine average no. of customers in the queue (L_q):

$$\boxed{L_q = \sum_{n=1}^{\infty} (n-1) P_n} ; \boxed{L_q = L_s - (1 - P_0)} ; \boxed{L_q = \frac{\lambda^2}{\mu(\mu - \lambda)}}$$

7. To determine waiting ~~times~~ time in the queue (W_q):

$$\boxed{W_q = \frac{L_q}{\lambda}} ; \boxed{W_q = \frac{\lambda}{\mu(\mu - \lambda)}}$$

8. Probability that waiting time in the queue $\geq t$

$$\boxed{= \frac{\lambda}{\mu} e^{-(\mu - \lambda)t}}$$

9. Probability that waiting time in the system $\geq t$

$$\boxed{= e^{-(\mu - \lambda)t}}$$

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Eg:- In a railway yard goods train arrive at a rate of 30 trains per day. Assuming that inter arrival time follows an exponential distribution and the service time distribution is also exponential with an average of 36 min. Calculate the following: (2)

(i) The average no. of trains in the queue.

(ii) Average number of trains in the system.

(iii) The probability that no. of trains in the system exceeds 10.

Soln:- (λ) Arrival rate = 30 trains / day.

Service time = 36 minutes $= \frac{36}{60 \times 24} = 0.025$ day.

Service rate $= \mu = \frac{1}{\text{Service time}} = \frac{1}{0.025} = 40$ trains / day.

(i) Average no. of trains in the queue = L_q.

$$L_q = \frac{\lambda^2}{\mu(\mu-\lambda)} = \frac{30^2}{40(40-30)} = 2.25 \text{ trains.}$$

(ii) Average no. of trains in the system = L_s.

$$L_s = \frac{\lambda}{(\mu-\lambda)} = \frac{30}{(40-30)} = 3 \text{ trains.}$$

(iii) The probability that the no. of trains in the system exceeds 10.

$$p(n>10) = \left(\frac{\lambda}{\mu}\right)^{10} = \left(\frac{30}{40}\right)^{10} = (0.75)^{10} = 0.0563.$$

Q:- Customers arrive at a one-window drive in a bank according to Poisson's distribution with mean 10 per hour. Service time per customer is exponential with mean 5 minutes. The space in front of the window including that for serviced car can accommodate maximum of three cars. Other cars can wait outside the space.

(i) What is the probability than an arriving car can drive directly to the space in front of the window?

(ii) How long is an arriving customer expected to wait before starting the service?

Solⁿ:- Arrival rate $= \lambda = 10$ cars / hour.

$$\text{Service time} = 5 \text{ minutes} = \frac{5}{60} = 0.083 \text{ hr.}$$

$$\text{Service rate} = \mu = \frac{1}{\text{Service time}} = \frac{60}{5} = 12 \text{ Cars/hr.}$$

The probability that an arriving car can drive directly to the space in front of the window

$$= P_0 + P_1 + P_2.$$

$$\frac{\lambda}{\mu} = 0.834,$$

$$P_0 = 1 - \frac{\lambda}{\mu} = 1 - 0.834 = 0.166.$$

$$P_n = \left(\frac{\lambda}{\mu}\right)^n P_0.$$

$$P_1 = \left(\frac{\lambda}{\mu}\right)^1 P_0 = 0.834 \times 0.166 = 0.138$$

$$P_2 = \left(\frac{\lambda}{\mu}\right)^2 P_0 = 0.834^2 \times 0.166 = 0.115.$$

$$P_0 + P_1 + P_2 = 0.166 + 0.138 + 0.115 = 0.419.$$

(ii) Waiting time in the queue = W_q.

$$W_q = \frac{\lambda}{\mu(\mu-\lambda)} = \frac{10}{12(12-10)} = 0.416 \text{ hrs}.$$

eg:- Arrival rate of a telephone booth is according to Poisson's distribution with an average time of 9 minutes between two consecutive arrivals. The length of a Telephone call is assumed to be exponentially distributed with mean of 3minutes.

(i) Determine the probability that a person arriving at the booth will have to wait.

(ii) Average waiting length in the queue.

(iii) What is the probability that an arrival will have to wait for more than 10 minutes before the phone is free.

(iv) What is the probability that the customer will have to wait more than 10 minutes before the phone is available & to complete the call. (waiting time in system)

(v) Find the fraction of a day that the phone will be in use.

(vi) The telephone company will install a second booth when convinced that an arrival would expect to have to wait at least 4 minutes for the phone. Find the increase in flow of arrivals, which will justify a second booth.

Sol^n:- Time between two consecutive arrivals = 9 minutes.

Arrival rate $= \lambda = \frac{1}{9} = 0.11$ customer / minute.

Service time = 3 minutes

Service rate $= \mu = \frac{1}{\text{Service time}} = \frac{1}{3} = 0.33$ Customers / minute

$$\frac{\lambda}{\mu} = \frac{0.11}{0.33} = 0.33.$$

i) Probability that the service is busy or probability that a person will have to wait $= 1 - P_0$.

$$1 - P_0 = \frac{\lambda}{\mu} = \frac{0.11}{0.33} = 0.33$$

ii) Average no. of Customers in the queue $= L_q$.

$$L_q = \frac{\lambda^2}{\mu(\mu-\lambda)} = \frac{0.11^2}{0.33(0.33-0.11)} = 0.166 \text{ customer.}$$

iii) Probability that waiting time in the queue $\geq t =$

$$\frac{\lambda}{\mu} e^{-(\mu-\lambda)t}.$$

Probability that waiting time in the queue ≥ 10 minutes

$$= 0.33 e^{-(0.33-0.11)10} = 0.036.$$

iv) Probability that waiting time in the system $\geq t =$

$$\boxed{e^{-(\mu-\lambda)t}}$$

Probability that waiting time in the system ≥ 10 minutes

$$= e^{-(0.33-0.11)10} = 0.11.$$

v) Fraction of a day that the phone will be in use = probability that the server is busy $= 1 - P_0$.

fraction of a day that the phone will be in use (4)

$$= \frac{\lambda}{\mu} \cancel{0.33 \; 0.67} = \frac{0.11}{0.33} = 0.33.$$

(vi) Required waiting time in the queue = W_q = 4 minutes

$$W_q = \frac{\lambda}{\mu(\mu-\lambda)} = \frac{\lambda}{0.33(0.33-\lambda)} = 4.$$

Hence, new arrival rate = λ = 0.19 arrivals/minute.

Increase in arrival rate to justify second booth ~~= 0.19~~

$$= 0.19 - 0.11 = 0.08 \text{ customer/minute.}$$

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eg:- A self-servicing store employs one cashier at its counter. Nine customers arrive at an average of 5 minutes while the cashier can service 10 customers in 5 minutes. Assuming Poisson's distribution for arrival rate. Find:

(i) Average number of customers in system.
(ii) Average number of customers in queue.
(iii) Average time a customer spends in the system.
(iv) Average time a customer waits before being serviced.

Soln:- Arrival Rate = $\lambda = \frac{9}{5}$ = 1.8 Customers/minute.

Service rate = $\mu = \frac{10}{5}$ = 2 Customers/minute.

(i) Average no. of Customers in the system = L_s.

$$L_s = \frac{\lambda}{(\mu-\lambda)} = \frac{1.8}{(2-1.8)} = 9 \text{ customers.}$$

(ii) Average no. of Customers in the queue = L_q.

$$L_q = \frac{\lambda^2}{\mu(\mu-\lambda)} = \frac{1.8^2}{2(2-1.8)} = 8.1 \text{ Customers}$$

(iii) Average time a customer spends in the system

$$= W_s = \frac{1}{(\mu-\lambda)} = \frac{1}{(2-1.8)} = 5 \text{ minutes.}$$

(iv) Waiting time in the queue $= W_q$.

$$W_q = \frac{\lambda}{\mu(\mu-\lambda)} = \frac{1.8}{2(2-1.8)} = 4.5 \text{ minutes.}$$

Q:- A branch of Punjab National Bank has only one typist. Since the typing work varies in length, the mean service rate becomes 8 letters per hour. The letters arrive at a rate of 5 per hour during the entire 8 hrs workday. If the typewriter is valued at ₹. 1.50 per hour, determine:

(i) Equipment utilization.

(ii) The percent time that an arriving letter has to wait.

(iii) Average system time.

(iv) Average cost due to waiting on the part of typewriter (Idle time Cost).

Soln:- Arrival Rate $= \lambda = 5$ letters / hour.

Service Rate $= \mu = 8$ letters / hour.

(i) Probability that the server is busy or Equipment utilization $= 1 - P_0 = \frac{\lambda}{\mu} = \frac{5}{8} = 0.625$.

(ii) The percentage time that an arriving letter has to wait = probability that the server is busy $= 1 - P_0$.

$$1 - P_0 = \frac{\lambda}{\mu} = \frac{5}{8} = 0.625 = 62.5\%$$

(5)

(iii) Average time a customer spends in the system $= W_s$

$$W_s = \frac{1}{(\mu-\lambda)} = \frac{1}{(8-5)} = \frac{1}{3} = 0.33\text{ hrs} = 20\text{ minutes}$$

(iv) Idle time cost of type writer per day $= 8(1-0.625)1.50$

$= ₹.\ 4.50/\text{day}.$

eg:- On average 96 patients per 24 hrs day requires the service of an emergency clinic. Also, on an average a patient requires 10 minutes of active attention. Assume that the facility can handle one emergency at a time. Suppose that it costs the clinic ₹100 per patient treated to obtain an average service time of 10 minutes and that each minute of decrease in the average time would cost ₹10 per patient to be treated. Determine present average size of queue. How much would have to be budgeted by the clinic to decrease the average size of the queue from 1.33 patients to 0.5 patients?.

Soln:- Arrival rate $= \lambda = \frac{96}{24} = 4$ patients/hour.

Service ~~rate~~ time $=$ 10 minutes $= 0.116$ hour.

Service rate $= \mu = \frac{1}{\text{Service time}} = 6$ patients/hour.

Average number of patients in the queue $= L_q$.

$$L_q = \frac{\lambda^2}{\mu(\mu-\lambda)} = \frac{4^2}{6(6-4)} = 1.33 \text{ patients}.$$

If the service time is 10 minutes then average length of queue is 1.33 patients/hour. To reduce average length of queue from 1.33 patients to 0.5 patient then service time has to be reduced.

$$L_q = \frac{\lambda^2}{\mu(\mu-\lambda)}$$

$$0.5 = \frac{4^2}{\mu(\mu-4)}$$

$$0.5\mu^2 - 2\mu = 16.$$

$$0.5\mu^2 - 2\mu - 16 = 0.$$

$$\mu = \frac{-b \pm \sqrt{b^2-4ac}}{2a}.$$

$$\mu = \frac{2 \pm \sqrt{2^2 - 4 \times 0.5 * (-16)}}{2 \times 0.5}$$

$$\mu = 2 \pm \sqrt{36} = 8/\text{hours}.$$

Service time $= \frac{1}{8} \times 60 = 7.5$ minutes.

To reduce average length of queue from 1.33 patients to 0.5 patient ~~that~~ then service time has to be reduced from 10 minutes to 7.5 minutes.

Then cost per patient $= 100 + (10-7.5) \times 10 = 100 * 25 = \cancel{₹125}$/pat.

$= ₹125$ / patient.

Model 2:-

In this model the capacity of the system is N.

To determine various parameters:

(i) Probability that there are n customers in the system =

$$P_n = \left(\frac{\lambda}{\mu}\right)^n P_0.$$

(2) To determine probability that server is idle ~~or~~ probability that '0' customer in the system (P_0).

$$P_n = \left(\frac{\lambda}{\mu}\right)^n P_0.$$

$$P_0 = \frac{1-\frac{\lambda}{\mu}}{1-\left(\frac{\lambda}{\mu}\right)^{N+1}} = \frac{1-\rho}{1-\rho^{N+1}}.$$

(3) To determine average (or) expected number of customers in the system (L_s):

$$L_s = \sum_{n=0}^{N} nP_n.$$

(4) Waiting time in the system = W_s.

$$W_s = \frac{L_s}{\lambda}.$$

(5) Average number of customers in the queue (L_q):

$$L_q = L_s - (1-P_0).$$

(6) Waiting time in the queue (W_q): Average time customer spends in the queue.

$$W_q = \frac{L_q}{\lambda}.$$

Eg:- At a railway station only one train is handled at a time. The railway yard is sufficient only for two trains to wait while the other is given signal to leave the station. Trains arrive at the station at an average rate of 6 per hour and the railway station can handle them at an average of 12 per hour. Assuming Poisson's arrivals and exponential service distribution, find the steady state probabilities for the various number of trains in the system. Find also the average waiting time of a train coming into the yard.

Sol:- Maximum number of trains allowed into system = N = 3.

Arrival rate = λ = 6 trains / hour.

service rate = μ = 12 trains / hour.

$$\frac{\lambda}{\mu} = \frac{6}{12} = 0.5$$

To find steady state probabilities:

probability that 0 train in the system = P_0.

probability that 1 train in the system = P_1.

probability that 2 trains in the system = P_2.

probability that 3 trains in the system = P_3.

$$P_0 = \frac{1-\frac{\lambda}{\mu}}{1-\left(\frac{\lambda}{\mu}\right)^{N+1}}$$

$$P_0 = \frac{1-0.5}{1-(0.5)^4} = 0.533.$$

$$P_1 = P_0\left(\frac{\lambda}{\mu}\right)^1 = 0.533 \times 0.5 = 0.266.$$

$$P_2 = P_0\left(\frac{\lambda}{\mu}\right)^2 = 0.533 \times 0.5^2 = 0.133.$$

$$P_3 = P_0\left(\frac{\lambda}{\mu}\right)^3 = 0.533 \times 0.5^3 = 0.066.$$

ii, Average (or) expected number of trains in the system (L_s):

$$L_s = \sum_{n=0}^{N} nP_n.$$

$$L_s = \sum_{n=0}^{3} nP_n.$$

$$= 0P_0 + 1P_1 + 2P_2 + 3P_3 = 0 + 1 \times 0.266 + 2 \times 0.133 + 3 \times 0.066 = 0.732$$ train

(iii) Average (or) expected no. of trains in the queue (L_q) 7

$$L_q = L_s - (1 - P_0) = 0.732 - (1 - 0.533)$$
$$= 0.265 \text{ train.}$$

(iv) Waiting time in the queue (W_q): Average time customer spends in the queue.

$$W_q = \frac{L_q}{\lambda} = \frac{0.265}{6} = 0.044 \text{ hour} = 2.65 \text{ minutes.}$$

Eg:- The capacity of yard is to admit 9 trains. Calculate the following on the assumption that 30 trains per hour on average are received in the year. Service rate is 40 trains/hour. Determine:

(i) The probability that the yard is empty.

(ii) Average queue length.

Sol:- Maximum no. of trains allowed into system = N = 9 trains

Arrival rate = λ = 30 trains/hour.

Service rate = μ = 40 trains/hour.

$$\lambda/\mu = 30/40 = 0.75$$

$$P_0 = \frac{1 - \frac{\lambda}{\mu}}{1 - \left(\frac{\lambda}{\mu}\right)^{N+1}}$$

$$= \frac{1 - 0.75}{1 - (0.75)^{10}} = 0.26.$$

(ii) Average (or) expected no. of trains in the system (L_s):

$$L_s = \sum_{n=0}^{N} n P_n$$

$$L_s = \sum_{n=0}^{9} n\left(\frac{\lambda}{\mu}\right)^n P_0$$

$$L_s = P_0 \sum_{n=0}^{9} n\left(\frac{\lambda}{\mu}\right)^n$$

$$L_s = 0.26 \sum_{n=0}^{9} n(0.75)^n = 0.26\left[1(0.75)^1 + 2(0.75)^2 + 3(0.75)^3 + 4(0.75)^4 + 5(0.75)^5 + 6(0.75)^6 + 7(0.75)^7 + 8(0.75)^8 + 9(0.75)^9\right]$$

$$L_s = 2.79 \text{ trains}$$

(iii) Average (or) expected no. of trains in the queue (L_q):

$$L_q = L_s - (1-P_0) = 2.79 - (1-0.26) = 2.02 \text{ trains}.$$

eg:- A barber shop has space to accommodate 10 Customers. He can serve only one person at a time. If a customer comes to his shop and finds it full, he goes to the next shop. Customers randomly arrive at an average rate of 10 per hour and the barber service time is negative exponential with an average of 5 minutes per customer.

~~i) Write recurrence relations for steady state queuing system.~~

(ii) Determine P_0 & P_n.

Soln:- Max^m no. of customers allowed into barber shop

$= N = 10$ persons,

Arrival rate $= \lambda = 10$ persons/hour.

Service rate $= \mu = 12$ persons/hour. $= \left(\frac{60}{5}\right)$

Probability that '0' number of customers in the system $= P_0$.

$$\frac{\lambda}{\mu} = \frac{10}{12} = 0.833.$$

$$P_0 = \frac{1-\frac{\lambda}{\mu}}{1-\left(\frac{\lambda}{\mu}\right)^{N+1}}$$

$$P_0 = \frac{1-0.833}{1-(0.833)^{11}} = 0.192.$$

Probability that 10 number customers in the system $= P_{10}$

$$P_{10} = P_0\left(\frac{\lambda}{\mu}\right)^{10} = 0.192\,(0.832)^{10} = 0.030.$$

Eg:- A stenographer ~~has~~ can have 5 persons for whom she performs stenographic work. Arrival rate is Poisson and service time is exponential. Average arrival rate is 4 per hour with an average service time of 10 minutes. Cost of waiting is ₹8 per hour, while the cost of servicing is ₹2.50 per customer. Calculate:

(i) The average waiting time of an arrival.

(ii) The average length of the waiting line.

(iii) The average time, which the arrival spends in the system.

(iv) The minimum cost service rate.

Soln:- Maximum no. of customers to stenographer $= N = 5$ persons

Arrival rate $= \lambda = 4$ persons / hour.

Service ~~time~~ rate $= \mu = 6$ persons / hours $= \left(\frac{60}{10}\right)$

probability that '0' no. of customer in the system $= P_0$.

$$\rho = \frac{\lambda}{\mu} = \frac{4}{6} = 0.666.$$

$$P_0 = \frac{1-\frac{\lambda}{\mu}}{1-\left(\frac{\lambda}{\mu}\right)^{N+1}}$$

$$P_0 = \frac{1-0.666}{1-(0.666)^6} = 0.365.$$

Average (or) expected no. of Customers in the system (L_s):

$$L_s = \sum_{n=0}^{N} n P_n.$$

$$L_s = \sum_{n=0}^{5} n P_0 \left(\frac{\lambda}{\mu}\right)^n. \qquad \left[\because P_n = P_0\left(\frac{\lambda}{\mu}\right)^n\right]$$

$$L_s = P_0 \sum_{n=0}^{5} n \left(\frac{\lambda}{\mu}\right)^n.$$

$$L_s = 0.365 \sum_{n=0}^{5} n(0.666)^n.$$

$$L_s = 0.365\left[(0.666)^1 + 2(0.666)^2 + 3(0.666)^3 + 4(0.666)^4 + 5(0.666)^5\right] = 1.416 \text{ Customers}.$$

Average (or) expected no. of customers in the queue (L_q):

$$L_q = L_s - (1 - P_0) = 1.416 - (1 - 0.365) = 0.781 \text{ Customers}.$$

Waiting time in the system $= W_s$.

$$W_s = \frac{L_s}{\lambda} = \frac{1.461}{4} = 0.354 \text{ hours} = 21.24 \text{ minutes}.$$

Waiting time in the queue $= W_q$.

$$W_q = \frac{L_q}{\lambda} = \frac{0.78}{4} = 0.195 \text{ hours} = 11.7 \text{ minutes}$$

Cost / customer = Waiting time × Cost of Customer + Server Cost/hr

Cost / customer $= (0.354 \times 8) + 2.50 = ₹.5.33$.

Model 3 :- Multi channel Model

The customers arrive in Poisson fashion with mean arrival rate λ. This model is same as model 1 except no. of servers in the system is equal to s. These 's' no's of servers are arranged in parallel and customers can go to any of the free counters, where the service time at each counter is identical and follows the same negative exponential distribution. The mean service rate is μ.

1. To determine steady state probabilities P_n in multiserver model.

Table : To determine steady state probabilities P_n in multiserver model.

Steady state probabilities	Condition	Overall service rate	Number of idle servers	Number of persons waiting in the queue
$P_n = \frac{1}{n!} P_0 \left(\frac{\lambda}{\mu}\right)^n$	$n < s$	$n\mu$	$s-n$	Nil
$P_n = \frac{1}{s!\, s^{n-s}} P_0 \left(\frac{\lambda}{\mu}\right)^n$	$n \geq s$	$s\mu$	Nil	$n-s$

1. To determine probability that zero no. of customer in the system.

$$P_0 = \left(\sum_{n=0}^{s-1} \frac{(s\rho)^n}{n!} + \frac{(s\rho)^s}{s!(1-\rho)} \right)^{-1} ; \quad \rho = \frac{\lambda}{s\mu}.$$

2. To determine average no. of customers in the queue

$$L_q = \frac{\rho (s\rho)^s}{s!(1-\rho)^2} P_0 .$$

$$\boxed{L_q = L_s - \frac{\lambda}{\mu}.}$$

③. To determine waiting time in the queue W_q.

$$W_q = \frac{L_q}{\lambda}.$$

④ To determine waiting time in the system (W_s).

$$W_s = \frac{L_s}{\lambda}.$$

⑤ To determine average no. of customers in the system (L_s).

$$L_s = L_q + \frac{\lambda}{\mu}.$$

Eg:- A super market has two ~~girls~~ persons at the counters. The ~~service time~~ service time for each customer is exponential with mean of 4 minutes and people arrive in a poisson fashion at the counter ~~of~~ at the rate of 10 per hour.

(i) Calculate the probability that an arrival will have to wait for service.

(ii) Find the expected percentage of idle time for each ~~girl~~ counter ~~person~~.

(iii) Find waiting time in the queue & system.

Sol:- Number of servers $= S = 2$.

~~Arrival time~~ $= \lambda$. Arrival rate $= \lambda = 10$ customers / hour.

Service time $= 4$ minutes $= 0.066$ hours.

Service rate $= \mu = 1$ / service time $= 15$ / hour.

$$\rho = \frac{\lambda}{S\mu} = \frac{10}{15\times 2} = 0.33$$

$$P_0 = \left[\sum_{n=0}^{S-1} \frac{(S\rho)^n}{n!} + \frac{(S\rho)^S}{S!(1-\rho)}\right]^{-1}.$$

$$P_0 = \left[\sum_{n=0}^{2-1} \frac{(2\times 0.33)^n}{n!} + \frac{(2\times 0.33)^2}{2!(1-0.33)}\right]^{-1}$$

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$$P_0 = (1 + 0.66 + 0.33)^{-1}$$

$$P_0 = 0.505$$

(i) Probability that an arrival has to wait for service

$$= 1 - (P_0 + P_1)$$

$$P_n = \frac{1}{n!} P_0 \left(\frac{\lambda}{\mu}\right)^n$$

$$P_1 = \frac{1}{1!} \times 0.5 \left(\frac{10}{15}\right)^1 = 0.33$$

Probability that an arrival has to wait for service =

$$= 1 - (P_0 + P_1) = 1 - (0.505 + 0.33) = 0.17$$

(ii) Expected percentage idle time of ~~girl~~ counter $= (2P_0 + 1P_1 + 0P_2)/2$

$$= (2 \times 0.5 + 1 \times 0.3)/2 = 0.67$$

Expected percentage idle time of ~~girl~~ counter = 67%.

(iii) Length of the queue at each counter $= L_q$.

$$L_q = \frac{\rho (S\rho)^S}{S!(1-\rho)^2} P_0$$

$$L_q = \frac{0.33 (2 \times 0.33)^2}{2!(1-0.33)^2} \times 0.505 = 0.08 \text{ customer.}$$

To determine waiting time in the queue (W_q).

$$W_q = \frac{L_q}{\lambda}$$

$$W_q = \frac{0.08}{10} = 0.008 \text{ hours} = 0.48 \text{ minute.}$$

$L_s = L_q + \frac{\lambda}{\mu} = 0.08 + \frac{10}{15} = 0.746$ cust (Length of system) (avg no of customer in system).

$W_s = \frac{L_s}{\lambda} = \frac{0.746}{} = 0.0746$ hr (Waiting time in the system) eg:- A bank has two tellers working on savings accounts. The first teller handles withdrawals only. The second teller handles deposits only. It has been found that the service time distribution for the deposits and withdrawals both are exponential with mean service time 3 minutes per customer. Depositors are found to arrive in a poisson fashion with mean arrival rate of 16/hr & withdrawals found arriving in a Poisson fashion with mean arrival rate of 14 per hour. What would be the effect on the average waiting time for deposits and withdrawal, if each teller could handle both withdrawals and deposits? What would be the effect, if this could be only be accomplished by increasing service time to 3.5 minutes.

Sol:-

Table: To determine waiting time in queue.

Queue at withdrawal's Counter	Queue at depositor's Counter	Handling of withdrawal's & depositors by both tellers.
$\lambda = 14$ Customers/hour	$\lambda = 16$ customers/hr.	No. of servers $s = 2$.
Service time = 3 minutes	Service time = 3 min	Arrival Rate = $\lambda = 30$ Customers/hrs
$\mu = \frac{60}{3} = 20$/hour	$\mu = \frac{60}{3} = 20$/hr	Service time = 3 min/Customer
$W_q = \frac{\lambda}{\mu(\mu-\lambda)}$	$W_q = \frac{\lambda}{\mu(\mu-\lambda)}$	Service rate = $\mu = \frac{60}{3} = 20$/hrs.
$= \frac{14}{20(20-14)}$	$= \frac{16}{20(20-16)}$	$\rho = (\lambda/s\mu) = 30/(2\times20) = 0.75$
$= 0.116$ hr	$= 0.2$ hrs	$P_0 = \left[\sum_{n=0}^{s-1} \frac{(s\rho)^n}{n!} + \frac{(s\rho)^s}{s!(1-\rho)}\right]^{-1}$
$= 7$ min	$= 12$ minutes	$P_0 = 0.142$
		$L_q = \frac{\rho(s\rho)^s}{s!(1-\rho)^2} P_0 = 1.917$ customers
		$W_q = \frac{L_q}{\lambda} = \frac{1.917}{30} = 0.0643$ hrs. $= 3.86$ minutes

If each teller handles both deposits and withdrawals then waiting time in the queues is reduced to 3.86 minutes, So, it is better to Combine the duties. (11)

Case 2:- No. of servers $= S = 2$.

Arrival rate $= \lambda = 30$ Customers / hr.

Service time $= 3.5$ minutes / Customer.

Service rate $= \mu = 60/3.5 = 17.14$ Customers / hr.

$$\rho = (\lambda / s\mu) = 30/(2 \times 17.14) = 0.875.$$

$$P_0 = \left[\sum_{n=0}^{s-1} \frac{(s\rho)^n}{n!} + \frac{(s\rho)^s}{s!(1-\rho)} \right]^{-1}$$

$$P_0 = \left[\sum_{n=0}^{2-1} \frac{(2 \times 0.875)^n}{n!} + \frac{(2 \times 0.875)^2}{2!(1-0.875)} \right]^{-1}$$

$$P_0 = 0.066.$$

$$L_q = \frac{\rho(s\rho)^s}{s!(1-\rho)^2} P_0 = \frac{0.875(2 \times 0.875)^2}{2!(1-0.875)^2} 0.066 = 5.7$$

To determine waiting time in the queue (W_q).

$$W_q = \frac{L_q}{\lambda}$$

$$W_q = \frac{5.7}{30} = 0.19 \text{ hour}$$

$$= 11.4 \text{ minutes}.$$

Q:- Four Counters are being run on the frontier of a Country to check the passports and necessary papers of the tourists. The tourists choose a counter at random. Arrival at the frontier is Poisson with parameter λ and the service rate is exponential with parameter $\lambda/2$. What is the steady state average queue at each counter?

Soln:-

No. of servers $= s = 4$.

Arrival rate $= \lambda$.

Service rate $= \mu = \lambda/2 = 0.5\lambda$.

$$\rho = [\lambda/s\mu] = (\lambda/(4\times 0.5\lambda)) = 0.5.$$

$$P_0 = \left[\sum_{n=0}^{s-1} \frac{(s\rho)^n}{n!} + \frac{(s\rho)^s}{s!(1-\rho)}\right]^{-1}$$

$$P_0 = \left[\sum_{n=0}^{4-1} \frac{(4\times 0.5)^n}{n!} + \frac{(4\times 0.5)^4}{4!(1-0.5)}\right]^{-1}$$

$$P_0 = \left[1 + \frac{2}{1!} + \frac{2^2}{2!} + \frac{2^3}{3!} + \frac{2^4}{4!\times 0.5}\right]^{-1} = 0.130$$

$$L_q = \frac{\rho(s\rho)^s}{s!(1-\rho)^2} P_0.$$

$$L_q = \frac{0.5(4\times 0.5)^4}{4!(1-0.5)^2} \times 0.130$$

$$L_q = 0.1739$$

Waiting time in queue $= W_q = \frac{L_q}{\lambda} = \frac{0.1739}{\lambda}$

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Unit - VII
Dynamic Programming

Definition :- Dynamic Programming is a method of decomposing the optimization problem into a number of subproblems each of the lower dimensionality. Any problem with optimizing function and constraints can be divided into a number of subproblems with the help of customers constraints, then that type of problem can be solved by dynamic programming approach. In dynamic programming problem, computations are carried out in stages by breaking the problem into the subproblems. Since sub-problems are interdependent, a procedure must be devised to link computations that generate a feasible solution for the entire problem.

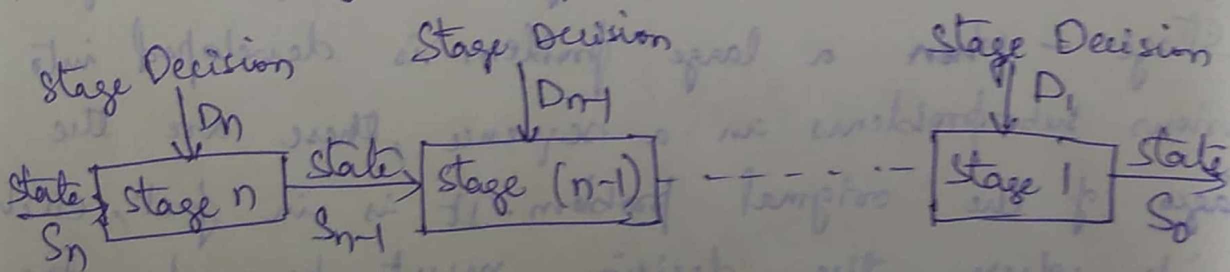
Important terms used in Dynamic Programming :-

① **Stage** - When a large problem is developed into various sub-problems in a sequence, these are the stages of the original problem. It is in fact, each point where the decision must be made. For eg, in salesman allocation problem, a stage may represent a group of cities, in the case of replacement problem, each year may represent a stage.

② **State** - Specific information describing the problem at different stages with the help of variables. The

Variables linking two stages are called the state variables. In the salesman allocation problem each stage has the variable of the number of remaining salesman and is called state variable, in the replacement problem the state is the state of beginning with a new machine.

(3) Principle of Optimality — Bellman's Principle of optimality states "An optimal policy (a sequence of decision) has the property that whatever the initial state and decisions are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision". According to this principle, a wrong decision (non-optimal) taken at one-stage does not mean that optimum decisions for the remaining stages cannot be taken. This can be shown diagrammatically as follows.



where n = stage number

S_n = Input to stage n from stage n+1

D_n = Decision Variable at stage n.

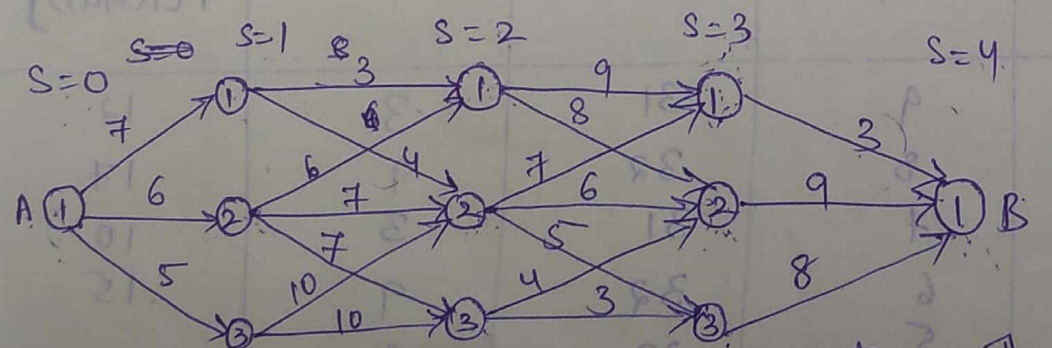
(4) Forward and Backward Recursive Approach :- It is the type of computation forward or Backward depending upon whether we proceed from stage 1 to n ie; $S_1 \rightarrow S_2 \rightarrow S_3 \rightarrow S_n$ or from stages S_n to S_1 ie; $S_n \rightarrow S_{n-1} \rightarrow S_{n-2} \rightarrow S_1$.

Applications of Dynamic Programming:-

The dynamic programming is used to solve the following problems:

1. Shortest route problems.
2. Non-linear programming problems.
3. Integer non-linear programming problems.
4. Cargo loading problems.
5. Determining optimum number of scientists in a research project.
6. Reliability problems.
7. Capital budgeting problems.
8. Linear programming problems.

eg:- (To determine the shortest route by tabular method).
Find the shortest path from vertex A to vertex B along the arcs joining the various vertices lying between A & B as shown in fig.



soln:- Divide the problem into five stages. These stages are 0, 1, 2, 3 & 4.
The objective is to determine the minimum path or the shortest route from A to B.
The problem can be solved by using the backward recursive approach, ie, from stage 4 to stage 0 using the tabular method.

Stage 4: Reached final stage (destination is reached).

Stage 3: Returns at stage 3 are shown in table below.

Table: Returns at stage 3.

Stage	Decision alternatives available (d)	Immediate returns due to the decision $r(d)$	Resulting stage due to the decision $T(S,d)$	Returns due to the resulting stage $F(T(S,d))$	Returns at stage $S=3$ $\min\{r(d)+F(T(S,d))\}$
$S=3$	31-41	3	41	0	3
	32-41	9	41	0	9
	33-41	8	41	0	8

Stage 2: Returns at stage 2 are shown in table below.

Table: Returns at stage 2.

Stage	Decision alternatives available (d)	Immediate returns due to the decision $r(d)$	Resulting stage due to the decision $T(S,d)$	Returns due to the resulting stage $F(T(S,d))$	Returns at stage $S=2$ $\min\{r(d)+F(T(S,d))\}$
$S=2$	21-31	9	31	3	12
	21-32	8	32	9	17
	22-31	7	31	3	10
	22-32	6	32	9	15
	22-33	5	33	8	13
	23-32	4	32	9	13
	23-33	3	33	8	11

Stage 1: Returns at stage 1 are shown in table below

Table: Returns at stage 1. (3)

Stage	Decision alternatives available (d)	Immediate returns due to the decision r(d)	Resulting stage due to the decision T(S,d)	Returns due to the resulting stage F{T(S,d)}.	Returns at stage S=1 min{r(d)+F[T(S,d)]}.
S=1	11-21	3	21	12	15
	11-22	4	22	10	14
	12-21	6	21	12	18
	12-22	7	22	10	17
	12-23	7	23	11	18
	13-22	10	22	10	20
	13-23	10	23	11	21

Stage 0: Returns at stage 0 are shown in table below.

Table: Returns at stage 0.

Stage	Decision alternatives available (d)	Immediate returns due to the decision r(d)	Resulting stage due to the decision T(S,d)	Returns due to the resulting stage F{T(S,d)}.	Returns at stage S=0. min{r(d)+F[T(S,d)]}.
S=0	01-11	7	11	14	21
	01-12	6	12	17	23
	01-13	5	13	20	25.

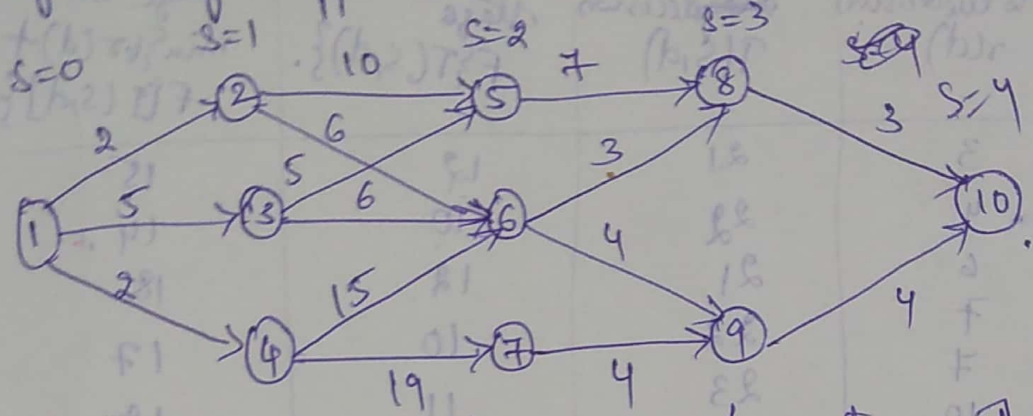
From above table (s=0);

The shortest distance from A to B = 21.

The minimum path (optimal path) is determined by moving from all tables (stage S=0 to S=3).

01-11-22-31-41 = 21.

Eg:- (To determine the shortest path by tabular method). Find the shortest path from 1 to 10 (below fig). using the dynamic programming approach.



Soln:- Divide the problem into five stages. These stages are 0, 1, 2, 3 and 4.

The objective is to determine the minimum path or the shortest route from 1 to 10.

The problem can be solved by using the backward recursive approach.

Stage 4: Reached final stage (destination is reached).

Stage 3: Returns at stage 3 are shown in table below.

Table: Returns at stage 3:

Stage	Decision alternatives available (d)	Immediate returns due to the decision $r(d)$	Resulting stage due to the decision $T(S,d)$	Returns due to the resulting stage $F[T(S,d)]$	Returns at stage S=3 $\min\{r(d)+F[T(S,d)]\}$.
S=3	38-410	3	410	0	3
	39-410	4	410	0	4

Stage 2: Returns at stage 2 are shown in below table

Table: Returns at stage 2

stage	Decision alternatives available (d)	Immediate returns due to the decision r(d)	Resulting stage due to the decision T(S,d)	Returns due to the resulting stage F[T(S,d)]	Returns at stage S=2 min {r(d)+F[T(S,d)]}
S=2	25-38	7	38	3	10
	26-38	3	38	3	6
	26-39	4	39	4	8
	27-39	4	39	4	8

stage 1: Returns at stage 1 are shown in below table.

Table: Returns at stage 1.

stage	Decision alternatives available (d)	Immediate returns due to the decision r(d)	Resulting stage due to the decision T(S,d)	Returns due to the resulting stage F[T(S,d)]	Returns at stage S=1 min {r(d)+F[T(S,d)]}
S=1	12-25	10	25	10	20
	12-26	6	26	6	12
	13-25	5	25	10	15
	13-26	6	26	6	12
	13-27	7	27	8	15
	14-26	15	26	6	21
	14-27	19	27	8	27

stage 0: Returns at stage 0 are shown in below table.

Table: Returns at stage 0.

stage	Decision alternatives available (d)	Immediate returns due to the decision r(d)	Resulting stage due to the decision T(S,d)	Returns due to the resulting stage F[T(S,d)]	Returns at stage S=0 min {r(d)+F[T(S,d)]}
S=0	01-12	2	12	12	14
	01-13	5	13	12	17
	01-14	2	14	27	29

From last table :- The shortest distance = 14.
The minimum path is determined by moving from last to first table. 01-12-26-38-410=14
1-2-6-8-10=14.

eg :- (To solve the linear programming problem):

Maximize $Z = x_1 + 9x_2$.

subject to $2x_1 + x_2 \leq 25$

$x_2 \leq 11$

$x_1, x_2 \geq 0$.

sol^n :- The number of variables = 2. So, the problem is divided into two stages.

There are two resource constraints:

B_1 = maximum of resource 1 = 25

B_2 = maximum of resource 2 = 11.

This problem can be solved by using the backward recursive approach, ie, from the last stage to the first stage.

$F(S_2)$ = returns at stage 2 = maximize $9x_2$.

x_2 = minimum of $(25 - 2x_1, 11)$

$F(S_1)$ = return at stage 1 = $\underset{0 \leq x_1 \leq 12.5}{\text{maximize}} \left[x_1 + 9 \min \text{ of } (25 - 2x_1, 11)\right]$.

Vary x_1 from 0 to 12.5 by small increment. The solution at which $F(S_1)$ is maximum is an optimal solution. The optimal solution is:

$x_1 = 7$, $x_2 = 11$ and $Z = 106$.

eg:- (To solve the linear programming problem). (5)

Maximize $Z = 3x_1 + x_2$.

subject to $2x_1 + x_2 \leq 6$.

$x_1 \leq 2$.

$x_2 \leq 4$

$x_1, x_2 \geq 0$.

Sol^n:- The no. of variables $= 2$. So, the problem is divided into two stages.

There are three resource Constraints.

$B_1 =$ maximum of resource $1 = 6$.

$B_2 =$ maximum of resource $2 = 2$.

$B_3 =$ maximum of resource $3 = 4$.

This problem can be solved by using the backward recursive approach, i.e., from the last stage to the first stage.

$F(S_2) =$ returns at stage $2 =$ maximize x_2.

$x_2 =$ minimum of $(6 - 2x_1, 4)$.

$F(S_1) =$ returns at stage 1 $=$

$= \underset{0 \leq x_1 \leq 2}{\text{maximize}} \left[3x_1 + \text{minimum of } (6 - 2x_1, 4) \right]$.

[$\because$ $x_1 \leq 2$ in the constraint $x_1 \leq 2$]

Vary x_1 from 0 to 2 by small increment.

The solution at which $F(S_1)$ is maximum is an optimal solution.

The optimal solution is:

$x_1 = 2$, $x_2 = 2$ and $Z = 8$.

eg:- (To solve the linear programming problem).

$$\text{Max } Z = 3x_1 + 4x_2$$

subject to $2x_1 + x_2 \leq 40$

$2x_1 + 5x_2 \leq 180$

$x_1, x_2 \geq 0.$

Soln:- The no. of variables = 2.

So, the problem is divided into two stages.

There are two resource constraints:

B_1 = Maximum of resource 1 = 40.

B_2 = Maximum of resource 2 = 180

This problem can be solved by using the backward recursive approach, ie, from the last stage to first stage.

$f(s_2)$ = returns at stage 2 = maximize $4x_2$

$$x_2 = \text{minimum of} \left[40 - 2x_1, \frac{180 - 2x_1}{5}\right].$$

$f(s_1)$ = returns at stage 1

$$= \underset{0 \leq x_1 \leq 20}{\text{maximize}} \left[3x_1 + 4 \text{ minimum of} \left[40 - 2x_1, \frac{180 - 2x_1}{5}\right)\right].$$

Vary x_1 from 0 to 20 by small increment.

The solution at which $f(s_1)$ is maximum is an optimal solution.

The optimal solution is:

$$x_1 = 2.5, \; x_2 = 35 \text{ and } Z = 147.5$$

eg:- (To solve the linear programming problem). ⑥

Maximize $Z = 50x_1 + 100x_2$.

subject to
$$2x_1 + 3x_2 \le 48.$$
$$x_1 + 3x_2 \le 42.$$
$$x_1 + x_2 \le 21$$
$$x_1, x_2 \geq 0.$$

Sol :- The no of variables $= 2$.

So, the problem is divided into two stages.

There are three resource constraints:

B_1 = maximum of resource $1 = 48$.

B_2 = maximum of resource $2 = 42$.

B_3 = maximum of resource $3 = 21$.

This problem can be solved by using the backward recursive approach, i.e., from the last stage to first stage.

$F(s_2)$ = returns at stage 2 = maximize $100x_2$

$$x_2 = \text{minimum of } \left[\frac{48-2x_1}{3}, \frac{42-x_1}{3}, 21-x_1\right].$$

$F(s_1)$ = returns at stage 1

$$= \underset{0 \le x_1 \le 21}{\text{maximize}} \left[50x_1 + 100 \text{ minimum of } \left(\frac{48-2x_1}{3}, \frac{42-x_1}{3}, 21-x_1\right)\right]$$

Vary x_1 from 0 to 21 by small increment. The solution at which $F(s_1)$ is maximum is an optimal solution. The optimal solution is:

$$x_1 = 21, \ x_2 = 0 \text{ and } Z = 1050.$$