TORSION OF SHAFTS AND SPRINGS

TORSION

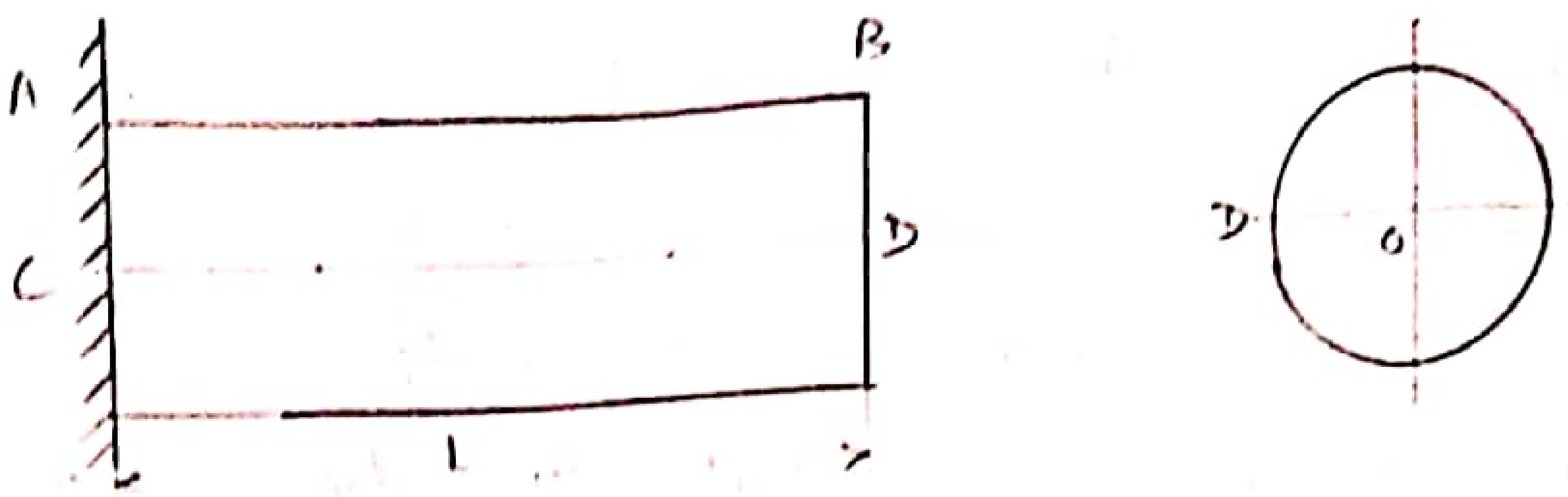
A shaft is said to be in torsion, when equal and opposite torques are applied at the two ends of the shaft.

The torque is equal to the product of Force applied (Tangentially to the ends of a shaft) and radius of the shaft.

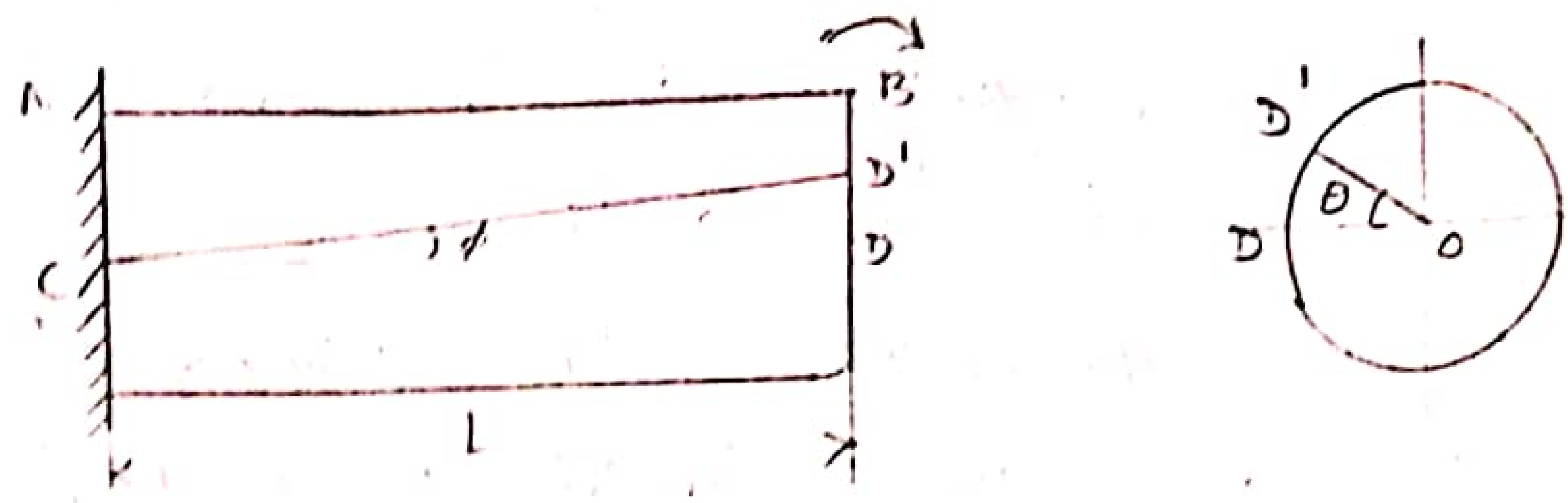
Due to application of the torques at the two ends the shaft is subjected to a twisting moment.

DERIVATION OF TORSION EQUATION :-

Consider a shaft fixed at one end AA and free at the end BB as shown in below. Let CD is any line on the outer surface of the shaft. Now let the shaft is subjected to a torque T at the end BB as shown in fig (2). As a result of this torque T, the shaft at the end BB will rotate clockwise and every cross-section of the shaft will be subjected to shear stresses. The point D will shift to D' and hence line CD will be deflected to CD' as shown in fig (2) The line OD will be shifted to OD' as shown in fig 2b.



Before Applying.



After applying.

we know $\tan\phi = \frac{DD'}{CD}$ $\boxed{\because \tan\phi = \phi}$

$$\boxed{\phi = \frac{DD'}{CD}}$$

we also know

shear strain = $\frac{\text{Distorsion at outer surface}}{\text{Length of shaft}}$

$$\phi = \frac{DD'}{L}$$

Here $DD' = R\theta$

$$\phi = \frac{R\theta}{L}$$

Now the modulus of Rigidity(G) is given by

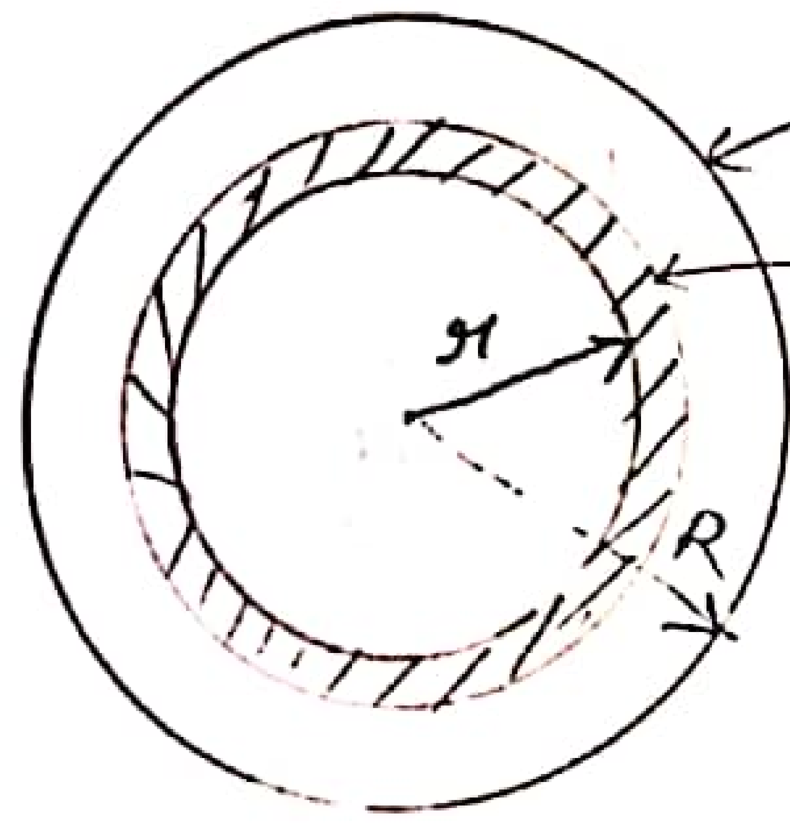
$$G = \frac{\text{shear stress}}{\text{shear strain}} = \frac{\tau}{\phi}$$

$$G = \frac{\tau}{\frac{R\theta}{L}} \Rightarrow G = \frac{L\tau}{R\theta}$$

$$\frac{\tau}{R} = \frac{G\theta}{L}$$

For outer Area

$$\tau = \frac{F}{A}$$



For <u>small</u> <u>elemental</u> <u>strip</u>

Force $(dF) = \tau \cdot dA$

$$\boxed{\text{Torque } (T) = \text{Force } (F) \times \text{Radius}}$$

$$dT = dF \times r$$

$$dT = \tau \, dA \cdot r$$

To get Total Torque we have to integrate above eqn

$$\int dT = \int \frac{G\theta}{L} \times r \times dA \cdot r$$

At radius r shear stress

$$\tau = \frac{G\theta}{L} \times r$$

$$T = \frac{G\theta}{L} \int r^2 dA$$

$$\boxed{\therefore \int r^2 dA = J}$$

$$= \frac{G\theta}{L} \times J$$

$$\frac{T}{J} = \frac{G\theta}{L} = \frac{\tau}{R}$$

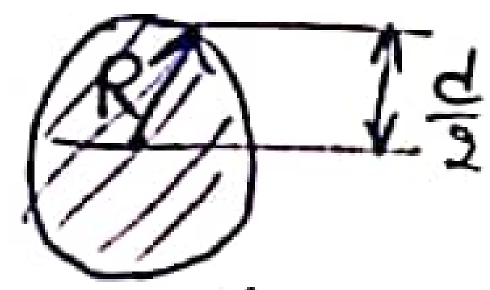
T = Torque
J = Polar MOI
G = modulus of Rigidity
θ = Angle of twist in radians
L = length of shaft

τ = Shear stress
R = Distance from the centre of shaft.

maximum torque transmitted by solid circular shaft:-

we know

$$\frac{T}{J} = \frac{\tau}{R}$$



For solid 0^{lar} shaft polar moment of inertia =

$$= I_{xx} + I_{yy}$$

$$= \frac{\pi d^4}{64} + \frac{\pi d^4}{64} \Rightarrow \frac{\cancel{2}^{1}\pi d^4}{\cancel{64}_{32}}$$

$$= \frac{\pi d^4}{32}$$

$$\frac{T}{\frac{\pi d^4}{32}} = \frac{\tau}{\frac{d}{2}}$$

$$\frac{32T}{\pi d^4} = \frac{2\tau}{d}$$

$$T = \frac{\cancel{2}^{1}\pi \cancel{d^4}^{3} \cdot \tau}{\cancel{32}_{16} \cdot \cancel{d}}$$

$$\boxed{T = \frac{\pi d^3}{16} \cdot \tau}$$

maximum torque transmitted by Hollow circular shaft

$$\frac{T}{J} = \frac{\tau}{R}$$

$$T = \frac{\tau}{\frac{D}{\cancel{2}_{1}}} \times \frac{\pi(D^4 - d^4)}{\cancel{32}_{16}}$$

$$\boxed{T = \frac{\pi(D^4 - d^4)}{16D} \cdot \tau}$$

$$\boxed{J = \frac{\pi(D^4 - d^4)}{32}}$$

power Transmitted by shafts:-

$$P = \omega \times T$$

$$\omega = \frac{2\pi N}{60}$$

N = speed of the shaft in r.p.m

ω = Angular speed of the shaft.

T = Average (or) mean torque. (N-m)

P = power (watt)

$$\boxed{P = \frac{2\pi NT}{60}}$$

Q1) A solid shaft of 150mm diameter is used to transmit torque. Find the maximum torque transmitted by the shaft if the max shear stress induced to the shaft is $45\ N/mm^2$.

sol Given data:-

Diameter of shaft (d) = 150mm

max. shear stress (τ) = $45\ N/mm^2$

we know

$$T_{max} = \frac{\pi d^3}{16} \cdot \tau$$

$$= \frac{\pi (150)^3}{16} \times 45 = \frac{477129384.3}{16}$$

$$\boxed{T = 29820.58\ N\text{-}m}$$

The shearing stress of a solid shaft is not to exceed 40N/mm² when the torque transmitted is 20000 N-m. Determine the minimum diameter of the shaft.

Given data :-

Shear stress $(\tau) = 40\ N/mm^2$.

Torque $(T) = 20000\ N\text{-}m$.

$= 20000 \times 10^3\ N\text{-}mm$

we know

$$T = \frac{\pi d^3}{16} \times \tau$$

$$d^3 = \frac{16T}{\pi \tau} = \frac{16 \times 2 \times 10^7}{\pi \times 40} = \frac{32 \times 10^7}{125.66}$$

$$d = \sqrt[3]{2546479.089}$$

$$\boxed{d = 136.55\ mm}$$

A hollow shaft of external diameter 120mm transmits 300kW power at 200rpm. Determine the maximum internal diameter if the maximum stress in the shaft is not to exceed 60 N/mm².

Given data :-

External diameter $(D) = 120mm$

Power $(P) = 200r.p.m$

maximum shear stress (τ) = $60\ N/mm^2$

we know for hollow circular shaft

$$T = \frac{\pi (D^4 - d^4)}{16D} \cdot \tau$$

But we also know

$$P = \frac{2\pi N T}{60}$$

$$\frac{300 \times 10^3 \times 60}{2\pi \times 200} = T$$

$$T = \frac{18 \times 10^6}{400\pi} = \cancel{141371.66\ N\text{-}mm}\ 14323.94\ N\text{-}m$$

$$= 14323.94 \times 10^3\ N\text{-}mm$$

$\boxed{\cancel{P = 141.371 N\text{-}m}}$

$$\overset{14323.94 \times 10^3}{\cancel{141.371 \times 10^3}} = \frac{\pi (120^4 - d^4)}{16 \times 120} \times 60$$

$$\frac{\overset{14323.94 \times 10^3}{\cancel{141.371 \times 10^3}} \times 16 \times 120}{60\pi} = 120^4 - d^4$$

$$\frac{27501974.17 \times 10^3\ \cancel{271433605.3}}{188.49} = 207360000 - d^4$$

$$d^4 = 207360000 - \overset{145906.8096 \times 10^3}{\cancel{1440042.471}}$$

$d^4 = (61453190.4)$ $\cancel{= \sqrt[4]{205919957.5}}$ $\cancel{d = \sqrt[4]{20721.4093}}$

$d = \sqrt[4]{61453190.4}$ $\cancel{d = 119.79\ mm}$ $\cancel{d =}$

$$\boxed{d = 88.53\ mm}$$

A hollow shaft is to transmit 300 KW power at 80 r.p.m. of the shear stress is not to exceed 60 N/mm² and the internal diameter is 0.6 of the external diameter. find the external and internal diameters. Assuming that the maximum torque is 1.4 times the mean.

Given data:

power $(P) = 300\,KW = 300 \times 10^3\,W$

speed $(N) = 80$ r.p.m.

shear stress $(\tau) = 60\,N/mm^2$

Internal diameter $(d) = 0.6\,D$

maximum Torque $(T_{max}) = 1.4\,T_{mean}$

we know mean Torque $(T_{mean}) = \frac{60P}{2\pi N}$ $\boxed{\therefore P = \frac{2\pi NT}{60}}$

$$T_{mean} = \frac{60 \times 300 \times 10^3}{2 \times \pi \times 80} = \frac{18 \times 10^6}{502.65}$$

$$T_{mean} = 35809.86\ N\text{-}m.$$

$$T_{mean} = 35809.86 \times 10^3\ N\text{-}mm$$

$$T_{max} = 1.4\,T_{mean}$$

$$= 1.4 \times 35809.86 \times 10^3$$

$$\boxed{T_{max} = 50133.80 \times 10^3\ N\text{-}mm}$$

we Also know for hollow shaft

$$T = \frac{\pi(D^4 - d^4)}{16D} \times \tau$$

$$50133.80 \times 10^3 = \frac{\pi\left[D^4 - (0.6D)^4\right]}{16D} \times 60$$

$$\frac{802140 \times 10^3 D}{60\pi} = D^4 - 0.1296D^4$$

$$= D^4(1 - 0.1296)$$

$$4889114.049 \cancel{D} = D^{\cancel{4}\,3}(0.8704)$$

$$D^3 = \frac{4255484.868}{0.8704}$$

$$D = \sqrt[3]{4889114.049}$$

$$\boxed{D = 169.72\,mm}$$

$$d = 0.6 \times D$$

$$d = 0.6 \times 169.72$$

$$\boxed{d = 102\ mm}$$

Determine the diameter of a solid steel shaft which will transmit 90kW at 160r.p.m. Also determine the length of the shaft if the twist must not exceed 1° over the entire length. The maximum shear stress is limited to $60\,N/mm^2$. Take the value of modulus of Rigidity $= 8 \times 10^4\,N/mm^2$:

Given data :-

Power $(P) = 90 KW = 90 \times 10^3 W$

speed $(N) = 160 r.p.m$

Angle of twist $(\theta) = 1^\circ \Rightarrow 1 \times \frac{\pi}{180} = 0.0174$ radian.

shear stress $(\tau) = 60 N/mm^2$.

modulus of Rigidity $(G) = 8 \times 10^4 N/mm^2$.

we know

$$T = \frac{60P}{2\pi N} = \frac{60 \times 90000}{2 \times \pi \times 160}$$ $$\boxed{\therefore P = \frac{2\pi NT}{60}}$$

$= 14323.94 N\text{-}m$

$= 14323.94 \times 10^3 N\text{-}mm$.

we Also know for solid circular shaft

$$T = \frac{\pi d^3}{16} \times \tau$$

from the above eqn we can calculate the diameter

$$d^3 = \frac{16T}{\pi \times \tau} = \frac{16 \times 14323.94 \times 10^3}{\pi \times 60} = \frac{229183118.1}{60\pi}$$

$$d = \sqrt[3]{1215854.204}$$

$$\boxed{d = 106.73 \text{ mm}}$$

For calculating the length of the shaft we have

$$\frac{T}{J} = \frac{G\theta}{L}$$

For solid circular shaft polar m.o.I (J) is equal to

$$J = \frac{\pi d^4}{32}$$

$$\frac{14323.94 \times 10^3}{\frac{\pi (106.73)^4}{32}} = \frac{8 \times 10^4 \times 0.0174}{L} \Rightarrow \frac{32 \times 14323.94 \times 10^3}{\pi (106.73)^4} = \frac{1392}{L}$$

$$L = \frac{1392 \times 407657944.6}{458366080}$$

$$= \frac{5.67 \times 10^{11}}{458366080}$$

$$\boxed{L = 1238 \text{ mm}}$$

Q) A hollow shaft of diameter ratio 3:8 (internal dia to external dia) is to transmit 375 KW power at 100 r.p.m. The maximum torque being 20% greater than the mean. The shear stress not to exceed 60 N/mm² and twist in a length of 4m not to exceed 2°. Calculate its external and internal diameters which would satisfy both the above conditions. Assume modulus of rigidity, $G = 0.85 \times 10^5$ N/mm².

Sol Given data :-

Diameter ratio $\left(\frac{d}{D}\right) = \frac{3}{8} \Rightarrow d = \frac{3}{8}D = 0.375D$

Power (P) $= 375$ KW $= 375 \times 10^3$ W

Speed (N) $= 100$ r.p.m

maximum Torque $(T_{max}) = 1.2\, T_{mean}$

Shear stress $(\tau) = 60$ N/mm²

Angle of twist $(\theta) = 2^\circ \times \frac{\pi}{180} = 0.0349$

length $(L) = 4m = 4000mm$

modulus of Rigidity $(G) = 0.85 \times 10^5\ N/mm^2$.

we know for ~~sol~~. hollow circular shaft

$$T_{max} = \frac{\pi(D^4 - d^4)}{16D} \times \tau$$

For Calculating T_{mean} (mean Torque)

$$T_{mean} = \frac{60P}{2\pi N} = \frac{60 \times 375 \times 10^3}{2 \times \pi \times 100} = \frac{225 \times 10^5}{628.31}$$

$$T_{mean} = 35810.34\ N\text{-}m$$
$$= 35810.34 \times 10^3\ N\text{-}mm$$

$$T_{max} = 1.2\ T_{mean}$$
$$= 1.2 \times 35810.34 \times 10^3$$
$$= 42972.41 \times 10^3\ N\text{-}mm$$

$$T_{max} = \frac{\pi(D^4 - d^4)}{16D} \times \tau$$

$$42972.41 \times 10^3 = \frac{\pi\left[D^4 - (0.375D)^4\right]}{16D} \times 60$$

$$\frac{42972.41 \times 10^3 \times 16}{60 \times \pi} = \frac{D^4 - 0.01977D^4}{D}$$

$$\frac{687558.68 \times 10^3}{60\pi} = \frac{\cancel{D^4}^{3}(1 - 0.01977)}{\cancel{D}}$$

$$\cancel{3600004883} = D^3(0.9802) = 3647611.45$$

$$\boxed{D = 154.96\ mm}$$

Scanned with CamScanner

from given data

$$d = \frac{3}{8} D \Rightarrow \frac{3}{8} \times 154.96$$

$$\boxed{d = 58.11 \text{ mm}}$$

Condition 2:-

$$\frac{T}{J} = \frac{G\theta}{L}$$

$$\frac{42972.41 \times 10^3}{\frac{\pi \left(D^4 - (0.375 D)^4\right)}{32}} = \frac{0.85 \times 10^5 \times 0.0349}{4000}$$

$$\frac{32 \times 42972.41 \times 10^3 \times 4000}{\pi \times 0.85 \times 10^5 \times 0.0349} = D^4 - 0.01977 D^4$$

$$\frac{5.50 \times 10^{12}}{9319.53} = D^4 (1 - 0.01977)$$

$$D^4 = \frac{590158224.8}{0.9802}$$

$$D = \sqrt[4]{602079336.9}$$

$$\boxed{D = 156.64 \text{ mm}}$$

$$d = \frac{3}{8} \times D \Rightarrow \frac{3}{8} \times 156.64$$

$$\boxed{d = 58.74 \text{ mm}}$$

The diameter of the shaft greater of the two values

COMBINED BENDING AND TORSION :-

When a shaft is transmitting Torque or Power, it is subjected to shear stresses. At the same time the shaft is also subjected to bending moments due to gravity or inertia loads. Due to the bending moment, bending stresses are setup in the shaft. Hence each particle in a shaft is subjected to shear stress and bending stress.

For design purpose it is necessary to find the principal stresses and maximum shear stress.

major principal stress $= \frac{\sigma}{2} + \sqrt{\left(\frac{\sigma}{2}\right)^2 + \tau^2}$

$$= \frac{\frac{\cancel{32}^{16} M}{\pi d^3}}{\cancel{2}_1} + \sqrt{\left(\frac{\frac{\cancel{32}^{16} M}{\pi d^3}}{\cancel{2}_1}\right)^2 + \left(\frac{16T}{\pi d^3}\right)^2}$$

$$= \frac{16M}{\pi d^3} + \sqrt{\left(\frac{16M}{\pi d^3}\right)^2 + \left(\frac{16T}{\pi d^3}\right)^2}$$

$$= \frac{16}{\pi d^3}\left(M + \sqrt{M^2 + T^2}\right)$$

minor principal stress

$$= \frac{\sigma}{2} - \sqrt{\left(\frac{\sigma}{2}\right)^2 + \tau^2}$$

$$\frac{M}{I} = \frac{\sigma}{y}$$

$$\sigma = \frac{M}{I} \times y$$

$$= \frac{M}{\frac{\pi d^4}{64}} \times \frac{d}{2}$$

$$= \frac{\cancel{64}^{32} M}{\pi \cancel{d^4}_3} \times \frac{\cancel{d}^1}{\cancel{2}}$$

$$\boxed{\sigma = \frac{32M}{\pi d^3}}$$

similarly

$$\frac{T}{J} = \frac{\tau}{R}$$

$$\tau = \frac{T}{\frac{\pi \cancel{d^4}^3}{\cancel{32}_{16}}} \times \frac{\cancel{d}^1}{\cancel{2}_1}$$

$$\boxed{\tau = \frac{16T}{\pi d^3}}$$

minor pricipal stress

$$= \frac{\frac{\cancel{32}^{16} M}{\pi d^3}}{\cancel{2}} - \sqrt{\left(\frac{\frac{\cancel{32}^{16} M}{\pi d^3}}{\cancel{2}}\right)^2 + \left(\frac{16T}{\pi d^3}\right)^2}$$

$$= \frac{16M}{\pi d^3} - \sqrt{\left(\frac{16M}{\pi d^3}\right)^2 + \left(\frac{16T}{\pi d^3}\right)^2}$$

$$= \frac{16}{\pi d^3}\left(M - \sqrt{M^2 + T^2}\right).$$

maximum shear stress $=$ $\frac{\text{major principal stress} - \text{minor principal stress}}{2}$

$$= \frac{\frac{16}{\pi d^3}\left(M + \sqrt{M^2+T^2}\right) - \frac{16}{\pi d^3}\left(M - \sqrt{M^2+T^2}\right)}{2}$$

$$= \frac{\cancel{\frac{16}{\pi d^3} M} + \frac{16}{\pi d^3}\sqrt{M^2+T^2} - \cancel{\frac{16M}{\pi d^3}} + \frac{16}{\pi d^3}\sqrt{M^2+T^2}}{2}$$

$$= \frac{\frac{16}{\pi d^3}\sqrt{M^2+T^2}\,(1+1)}{2} = \frac{16}{\pi d^3}\sqrt{M^2+T^2} \times \frac{\cancel{2}}{\cancel{2}}$$

$$= \frac{16}{\pi d^3}\sqrt{M^2+T^2}.$$

position of max. shear with normal cross-section

$$\boxed{\tan 2\theta = \frac{T}{M}}$$

Similarly for Hollow shaft

major principal stress $= \frac{16D}{\pi(D^4-d^4)}\left(M + \sqrt{M^2+T^2}\right)$

minor principal stress $= \frac{16D}{\pi(D^4-d^4)}\left(M - \sqrt{M^2+T^2}\right)$

max. shear stress $= \frac{16D}{\pi(D^4-d^4)}\left(\sqrt{M^2+T^2}\right)$

Q) A solid shaft of diameter 80mm is subjected to a twisting moment of 8MN-mm and bending moment of 5MN-mm at a point. Determine
(i) principal stresses (ii) position of the plane on which they act.

Sol Given data :-

Diameter of shaft (d) = 80mm

Twisting moment (T) = 8 MN-mm
$= 8 \times 10^6$ N-mm

Bending moment (M) = 5 MN-mm
$= 5 \times 10^6$ N-mm

we know for solid shaft

major principal stress $= \frac{16}{\pi D^3}\left(M + \sqrt{M^2 + T^2}\right)$

$= \frac{16}{\pi (80)^3}\left(5 \times 10^6 + \sqrt{(5\times10^6)^2 + (8\times10^6)^2}\right)$

$\frac{16}{(512000)\pi}\left(5 \times 10^6 + \sqrt{(2.5\times10^{13}) + (6.4\times10^{13})}\right)$

$= 9.9471 \times 10^{-6}\left[5 \times 10^6 + \sqrt{8.9 \times 10^{13}}\right)$

~~$= 9.9471 \times 10^{-6} \times (4996394.45 \cdot (5003605.55))$~~

~~major principal stress $= 49.69$ N/mm²~~

$= 9.9471 \times 10^{-6}\left[5 \times 10^6 + 9433981.132\right]$

$= 9.9471 \times 10^{-6}\left[14433981.13\right]$

$= 143.57$ N/mm²

Minor principal stress $= \frac{16}{\pi D^3}\left[M - \sqrt{M^2+T^2}\right]$

$$= \frac{16}{\pi(80)^3}\left[5\times10^6 - \sqrt{(5\times10^6)^2 + (8\times10^6)^2}\right]$$

$$= 9.9471\times10^{-6}\left[5\times10^6 - \sqrt{8.9\times10^{13}}\right]$$

$$= 9.9471\times10^{-6}\left[5\times10^6 - 9433981.13\right]$$

$$= 9.9471\times10^{-6}\left[-4433981.13\right]$$

$$= -44.10\ N/mm^2.\ \text{(Compressive)}$$

ii) position of plane on which they act

$$\tan 2\theta = \frac{T}{M} = \frac{8\times10^6}{5\times10^6}$$

$$\tan 2\theta = \frac{8}{5}$$

$$2\theta = \tan^{-1}(1.6)$$

$$\theta = \frac{57.99}{2} = 28.99^\circ$$

9) The maximum allowable shear stress in a hollow shaft of external diameter equal to twice the internal diameter is $80\ N/mm^2$. Determine the diameter of the shaft if it is subjected to a torque of 4×10^6 N-mm and bending moment of 3×10^6 N-mm.

Sol Given data :-

External diameter (D) = 2 X Internal diameter.

$$D = 2d$$

shear stress (τ) = $80\ N/mm^2$.

Torque (T) = 4×10^6 N-mm

Bending moment (M) = 3×10^6 N-mm.

For calculating maximum shear stress we know

$$\text{max. shear stress} = \frac{16D}{\pi(D^4 - d^4)} \left[\sqrt{M^2 + T^2}\right]$$

$$80 = \frac{16 \times 2d}{\pi\left[(2d)^4 - d^4\right]} \times \sqrt{(3\times10^6)^2 + (4\times10^6)^2}$$

$$80 = \frac{32d}{\pi\left[16d^4 - d^4\right]} \times \sqrt{9\times10^{12} + 1.6\times10^{13}}$$

$$= \frac{32d}{\pi(15d^4)} \times \sqrt{2.5\times10^{13}}$$

$$\frac{d^{\cancel{4}\,3}}{\cancel{d}} = \frac{32\times(5\times10^6)}{\pi\times15\times80} = \frac{16\times10^7}{3769.911}$$

$$d^3 = 42441.32$$

$$d = \sqrt[3]{42441.32}$$

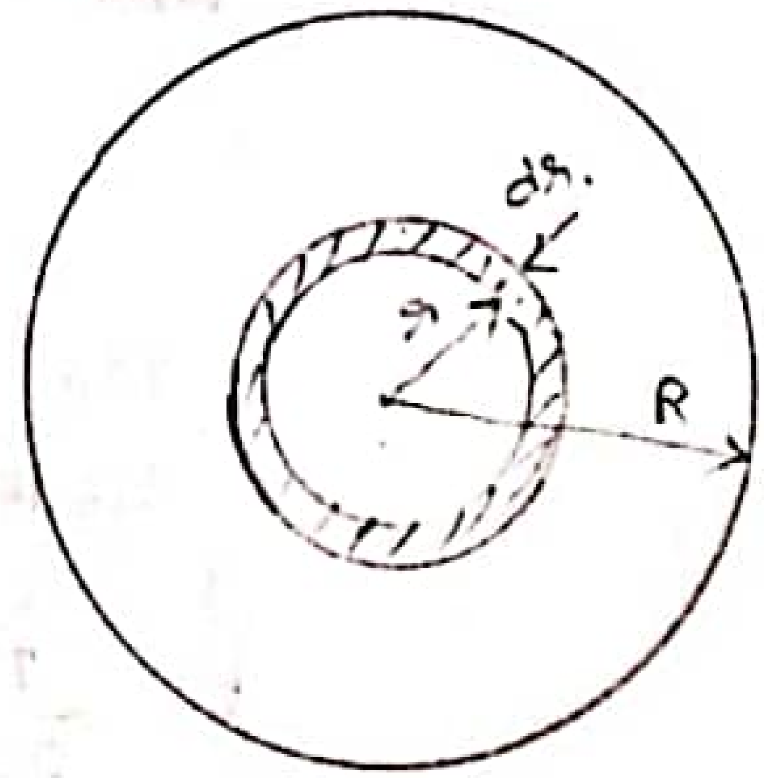
$$\boxed{d = 34.88\ mm}$$

$$D = 2d \Rightarrow 2\times34.88$$

$$\boxed{D = 69.76\ mm}$$

EXPRESSION FOR STRAIN ENERGY STORED IN A BODY DUE TO TORSION :-

Consider a solid shaft which is subjected to torsion. Take an elementary ring of width dr at a radius r as shown in fig.



we know shear stress due to torsion at a radius r is equal to

$$\frac{\tau_r}{r} = \frac{\tau}{R}$$

shear strain energy is given by

$$= \frac{(\text{Shear stress})^2}{2G} \times \text{volume.}$$

$$\boxed{U = \frac{\tau^2}{2G} \times \text{Volume}}$$

volume of ring = Area × length

$$= 2\pi r \times dr \times l$$

shear strain energy in the ring at a radius r

$$= \frac{\left(\frac{\tau}{R} \times r\right)^2}{2G} \times 2\pi r\, dr \times l \Rightarrow \frac{\tau^2}{R^2} \times \frac{r^2}{2G} \times 2\pi r\, dr . l$$

$$= \frac{\tau^2 \cdot l}{2GR^2}\left[r^2\, 2\pi r \cdot dr\right] \Rightarrow \frac{\tau^2 \cdot l}{2GR^2}\left[r^2 \cdot dA\right]$$

Total strain energy $= \dfrac{\tau^2 \cdot l}{2GR^2}\int r^2 dA = \dfrac{\tau^2 l}{2GR^2} \times J$

$$= \frac{\tau^2 \cdot l}{2GR^2} \times \frac{\pi d^4}{32} \Rightarrow \cancel{\frac{\tau^2 \cdot l \times 4}{2 \cdot G \times d^4} \times \frac{\pi d^4}{32}}$$

$$= \frac{\tau^2 \cdot l}{2GR^2} \times \frac{\pi (2R)^4}{32} \Rightarrow \frac{\tau^2 \cdot l}{2GR^2} \times \frac{\pi}{32} \times 16R^4$$

$$= \frac{\tau^2 \cdot l}{2GR^{\cancel{2}}} \times \frac{\pi}{\cancel{32}_{2}} \times \cancel{16}R^2 \cdot \cancel{R^2} \Rightarrow \frac{\tau^2 \times}{4G} \times \pi R^2 \cdot l$$

$$\boxed{U = \frac{\tau^2}{4G} \times V}$$

Total strain energy in a hollow shaft due to torsion

we know

$$\cancel{U = \frac{\tau^2}{4G} \times l} \Rightarrow \cancel{\frac{\tau^2}{4G} \times \pi}$$

$$= \frac{\tau^2 \times l}{2G\,R^2} \times J \Rightarrow \frac{\tau^2 \times l}{2GR^2} \times \frac{\pi}{32}\left[D^4 - d^4\right]$$

$$= \frac{\tau^2 \times l}{2G\left(\frac{D}{2}\right)^2} \times \frac{\pi}{32}\left[D^2 + d^2\right]\left[D^2 - d^2\right]$$

$$= \frac{\cancel{4}\,\tau^2 \times l}{2G \times (D^2)} \times \frac{\pi}{\cancel{32}_{8}}(D^2 + d^2)(D^2 - d^2)$$

$$= \frac{\tau^2}{4G \cdot D^2} \times \frac{\pi}{4}(D^2 - d^2) \times l \times (D^2 + d^2)$$

$$\boxed{U = \frac{\tau^2}{4G \cdot D^2} \times V(D^2 + d^2)}$$

Q) Determine the maximum strain energy stored in a solid shaft of diameter 10cm and length of 1.25m, if the maximum allowable shear stress is 50 N/mm². Take $G = 8 \times 10^4$ N/mm².

Sol Given data

Dia. of shaft (d) = 10 cm

Area of shaf $= \frac{\pi}{4} d^2 = \frac{\pi}{4}(10 \times 10)^2$

$= 7853.98 mm^2$

Length $(L) = 1.25 m = 1250 mm$

Shear stress $(\tau) = 50 N/mm^2$

modulus of Rigidity $(G) = 8 \times 10^4 N/mm^2$.

$$U = \frac{\tau^2}{4G} \times V$$

$$= \frac{(50)^2}{4 \times 8 \times 10^4} \times 7853.98 \times 1250$$

$$= \frac{2.45 \times 10^{10}}{32 \times 10^4}$$

$$\boxed{U = 76699.02 \ N\text{-}mm.}$$

9) The external and internal diameters of a hollow shaft are 40cm and 20cm. Determine the maximum strain energy stored in the hollow shaft if the maximum allowable shear stress is $50 N/mm^2$ and length of the shaft is 5m. Take $G = 8 \times 10^4 N/mm^2$.

Sol Given data:-

External diameter $(D) = 40 cm = 400 mm$

Internal diameter $(d) = 20 cm = 200 mm$

Shear stress $(\tau) = 50 N/mm^2$.

Length of the shaft $(L) = 5 m = 5000 mm$

modulus of Rigidity (G) = 8×10^4 N/mm².

we know strain energy for hollow shaft

$$U = \frac{\tau^2}{4GD^2}(D^2+d^2) \times V$$

$$= \frac{(50)^2}{4 \times 8 \times 10^4 \times (400)^2}(400^2+200^2) \times \frac{\pi}{4}(400^2-200^2)$$

$$= \frac{2500 \times (200000) \times \pi(120000)}{2.048 \times 10^{11}} \times (5000)$$

$$= \frac{1.8849 \times 10^{14}}{2.048 \times 10^{11}} \times 5000 \Rightarrow \frac{9.4245 \times 10^{17}}{2.048 \times 10^{11}}$$

$$= 4601806.64 \text{ N-mm}$$

$$U = 4601.806 \text{ N-m}$$

9) A solid circular shaft of 10 cm diameter of length 4 m is transmitting 112.5 KW power at 150 r.p.m. Determine (i) maximum shear stress induced in the shaft and (ii) strain energy stored in the shaft. Take $G = 8 \times 10^4$ N/mm².

Sol Given data:-

Diameter of shaft (d) = 10 cm = 100 mm.

length of shaft (L) = 4 m

Power (P) = 112.5 KW ⇒ 112500 W

Speed (N) = 150 r.p.m

modulus of Rigidity (G) = 8×10^4 N/mm².

we know

$$power = \frac{2\pi N T}{60}$$

$$T = \frac{112500 \times 60}{2 \times \pi \times 150} = \frac{6750000}{942.47} = 7161.97 \text{ N-m}$$

$$T = 7161972.43 \text{ N-mm.}$$

we also know for solid shaft

$$T = \frac{\pi d^3}{16} \times \tau$$

$$7161972.43 = \frac{\pi \times (100)^3}{16} \times \tau$$

$$\tau = \frac{114591559}{3141592.654} = 36.47 \text{ N/mm}^2.$$

$$\text{strain energy } (U) = \frac{\tau^2}{4G} \times V$$

$$= \frac{(36.47)^2}{4 \times 8 \times 10^4} \times \frac{\pi}{4} (100)^2 \times 4000$$

$$= \frac{1330.47 \times \pi \times 10000 \times 400}{4 \times 8 \times 10^4 \times 4}$$

$$= \frac{1.67191 \times 10^{10}}{1280000} =$$

$$\boxed{U = 13061.7968 \text{ N-mm}}$$

19

Columns and Struts

19.1. INTRODUCTION

Column or strut is defined as a member of a structure, which is subjected to axial compressive load. If the member of the structure is vertical and both of its ends are fixed rigidly while subjected to axial compressive load, the member is known as *column*, for example a vertical pillar between the roof and floor. If the member of the structure is not vertical and one or both of its ends are hinged or pin joined, the bar is known as *strut*. Examples of struts are : connecting rods, piston rods etc.

19.2. FAILURE OF A COLUMN

The failure of a column takes place due to the anyone of the following stresses set up in the columns :

(*i*) Direct compressive stresses,

(*ii*) Buckling stresses, and

(*iii*) Combined of direct compressive and buckling stresses.

19.2.1. Failure of a Short Column. A short column of uniform cross-sectional area A, subjected to an axial compressive load P, is shown in Fig. 19.1. The compressive stress induced is given by

$$p = \frac{P}{A}$$

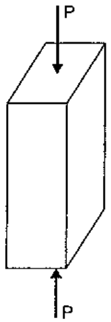


Fig. 19.1

If the compressive load on the short column is gradually increased, a stage will reach when the column will be on the point of failure by crushing. The stress induced in the column corresponding to this load is known as crushing stress and the load is called crushing load.

Let P_c = Crushing load,

σ_c = Crushing stress, and

A = Area of cross-section.

Then $$\sigma_c = \frac{P_c}{A}.$$

All short columns fail due to crushing.

19.2.2. Failure of a Long Column. A long column of uniform cross-sectional area A and of length l, subjected to an axial compressive load P, is shown in Fig. 19.2. A column is known as long column if the length of the column in comparison to its lateral dimensions, is very large. Such columns do not fail by crushing alone, but also by bending (also known buckling) as shown

in Fig. 19.2. The load at which the column just buckles, is known as *buckling load or critical just or crippling load*. The buckling load is less than the crushing load for a long column. Actually the value of buckling load for long columns is low whereas for short columns the value of buckling load is relatively high.

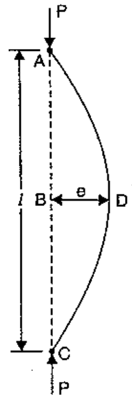


Fig. 19.2

Refer to Fig. 19.2.

Let l = Length of a long column

P = Load (compressive) at which the column has just buckled

A = Cross-sectional area of the column

e = Maximum bending of the column at the centre

σ_0 = Stress due to direct load $= \dfrac{P}{A}$

σ_b = Stress due to bending at the centre of the column $= \dfrac{P \times e}{Z}$

where Z = Section modulus about the axis of bending.

The extreme stresses on the mid-section are given by

Maximum stress $= \sigma_0 + \sigma_b$

and Minimum stress $= \sigma_0 - \sigma_b$.

The column will fail when maximum stress (*i.e.*, $\sigma_0 + \sigma_b$) is more than the crushing stress σ_c. But in case of long columns, the direct compressive stresses are negligible as compared to buckling stresses. Hence very long columns are subjected to buckling stresses only.

19.3. ASSUMPTIONS MADE IN THE EULER'S COLUMN THEORY

The following assumptions are made in the Euler's column theory :

1. The column is initially perfectly straight and the load is applied axially.
2. The cross-section of the column is uniform throughout its length.
3. The column material is perfectly elastic, homogeneous and isotropic and obeys Hooke's law.
4. The length of the column is very large as compared to its lateral dimensions.
5. The direct stress is very small as compared to the bending stress.
6. The column will fail by buckling alone.
7. The self-weight of column is negligible.

19.4. END CONDITIONS FOR LONG COLUMNS

In case of long columns, the stress due to direct load is very small in comparison with the stress due to buckling. Hence the failure of long columns take place entirely due to buckling (or bending). The following four types of end conditions of the columns are important :

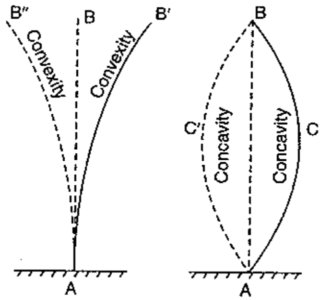
1. Both the ends of the column are hinged (or pinned).
2. One end is fixed and the other end is free.
3. Both the ends of the column are fixed.
4. One end is fixed and the other is pinned.

For a hinged end, the deflection is zero. For a fixed end the deflection and slope are zero. For a free end the deflection is not zero.

19.4.1. Sign Conventions. The following sign conventions for the bending of the columns will be used :

1. A moment which will bend the column with its *convexity* towards its initial central line as shown in Fig. 19.3 (a) is taken as positive. In Fig. 19.3 (a), AB represents the initial centre line of a column. Whether the column bends taking the shape AB' or AB'', the moment producing this type of curvature is positive.

2. A moment which will tend to bend the column with its *concavity* towards its initial centre line as shown in Fig. 19.3 (b) is taken as negative.



(a) Positive (b) Negative

Fig. 19.3

19.5. EXPRESSION FOR CRIPPLING LOAD WHEN BOTH THE ENDS OF THE COLUMN ARE HINGED

The load at which the column just buckles (or bends) is called crippling load. Consider a column AB of length l and uniform cross-sectional area, hinged at both of its ends A and B. Let P be the crippling load at which the column has just buckled. Due to the crippling load, the column will deflect into a curved form ACB as shown in Fig. 19.4.

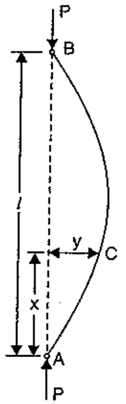


Fig. 19.4

Consider any section at a distance x from the end A.

Let y = Deflection (lateral displacement) at the section.

The moment due to the crippling load at the section $= -P \, . \, y$

(– ve sign is taken due to sign convention given in Art. 19.4.1)

But moment $$= EI \frac{d^2y}{dx^2}.$$

Equating the two moments, we have

$$EI \frac{d^2y}{dx^2} = -P \, . \, y \quad \text{or} \quad EI \frac{d^2y}{dx^2} + P \, . \, y = 0$$

or $$\frac{d^2y}{dx^2} + \frac{P}{EI} \, . \, y = 0$$

The solution* of the above differential equation is

$$y = C_1 \, . \cos \left(x \sqrt{\frac{P}{EI}} \right) + C_2 \, . \sin \left(x \sqrt{\frac{P}{EI}} \right) \quad \ldots(i)$$

where C_1 and C_2 are the constants of integration. The values of C_1 and C_2 are obtained as given below :

---

*The equation $\frac{d^2y}{dx^2} + \frac{P}{EI} \times y = 0$ can be written as $\frac{d^2y}{dx^2} + \alpha^2 y = 0$ where $\alpha^2 = \frac{P}{EI}$ or $\alpha = \sqrt{\frac{P}{EI}}$

The solution of the equation is $y = C_1 \cos (\alpha x) + C_2 \sin (\alpha x)$

$$= C_1 \cos \left(\sqrt{\frac{P}{EI}} \times x \right) + C_2 \sin \left(\sqrt{\frac{P}{EI}} \times x \right) \text{ as } \alpha = \sqrt{\frac{P}{EI}}$$

(i) At A, $x = 0$ and $y = 0$ (See Fig. 19.4)

Substituting these values in equation (i), we get

$$0 = C_1 \, . \cos 0° + C_2 \sin 0$$
$$= C_1 \times 1 + C_2 \times 0 \qquad (\because \cos 0 = 1 \text{ and } \sin 0 = 0)$$
$$= C_1$$
$$\therefore \quad C_1 = 0. \qquad \ldots(ii)$$

(ii) At B, $x = l$ and $y = 0$ (See Fig. 19.4).

Substituting these values in equation (i), we get

$$0 = C_1 \, . \cos \left(l \times \sqrt{\frac{P}{EI}} \right) + C_2 \, . \sin \left(l \times \sqrt{\frac{P}{EI}} \right)$$
$$= 0 + C_2 \, . \sin \left(l \times \sqrt{\frac{P}{EI}} \right) \qquad [\because C_1 = 0 \text{ from equation } (ii)]$$
$$= C_2 \sin \left(l \sqrt{\frac{P}{EI}} \right) \qquad \ldots(iii)$$

From equation (iii), it is clear that either $C_2 = 0$

or $$\sin \left(l \sqrt{\frac{P}{EI}} \right) = 0.$$

As $C_1 = 0$, then if C_2 is also equal to zero, then from equation (i) we will get $y = 0$. This means that the bending of the column will be zero or the column will not bend at all. Which is not true.

$$\therefore \quad \sin \left(l \sqrt{\frac{P}{EI}} \right) = 0$$
$$= \sin 0 \text{ or } \sin \pi \text{ or } \sin 2\pi \text{ or } \sin 3\pi \text{ or } \ldots$$

or $$l \sqrt{\frac{P}{EI}} = 0 \text{ or } \pi \text{ or } 2\pi \text{ or } 3\pi \text{ or } \ldots$$

Taking the least practical value,

$$l \sqrt{\frac{P}{EI}} = \pi$$

or $$P = \frac{\pi^2 EI}{l^2}. \qquad \ldots(19.1)$$

19.6. EXPRESSION FOR CRIPPLING LOAD WHEN ONE END OF THE COLUMN IS FIXED AND THE OTHER END IS FREE

Consider a column AB, of length l and uniform cross-sectional area, fixed at the end A and free at the end B. The free end will sway sideways when load is applied at free end and curvature in the length l will be similar to that of upper half of the column whose both ends are hinged. Let P is the crippling load at which the column has just buckled. Due to the crippling load P, the column will deflect as shown in Fig. 19.5 in which AB is the original position of the column and AB', is the deflected position due to crippling load P.

Consider any section at a distance x from the fixed end A.

Let y = Deflection (or lateral displacement) at the section

a = Deflection at the free end B.

Then moment at the section due to the crippling load $= P\,(a - y)$

(+ve sign is taken due to sign convention given in Art. 19.4.1)

But moment is also $= EI \dfrac{d^2y}{dx^2}$

$\therefore$ Equating the two moments, we get

$$EI \frac{d^2y}{dx^2} = P\,(a - y) = P\,.\,a - P\,.\,y$$

or
$$EI \frac{d^2y}{dx^2} + P\,.\,y = P\,.\,a$$

or
$$\frac{d^2y}{dx^2} + \frac{P}{EI}\,.\,y = \frac{P}{EI}\,.\,a. \qquad \text{...(A)}$$

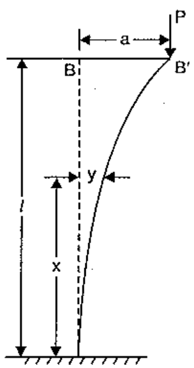


Fig. 19.5

The solution* of the differential equation is

$$y = C_1\,.\,\cos\left(x\sqrt{\frac{P}{EI}}\right) + C_2\,.\,\sin\left(x\sqrt{\frac{P}{EI}}\right) + a \qquad \text{...(i)}$$

where C_1 and C_2 are constant of integration. The values of C_1 and C_2 are obtained from boundary conditions. The boundary conditions are :

(*i*) For a fixed end, the deflection as well as slope is zero.

Hence at end A (which is fixed), the deflection $y = 0$ and also slope $\dfrac{dy}{dx} = 0$.

Hence at A, $x = 0$ and $y = 0$

Substituting these values in equation (*i*), we get

$$\begin{aligned} 0 &= C_1\,.\,\cos 0 + C_2 \sin 0 + a \\ &= C_1 \times 1 + C_2 \times 0 + a \qquad (\because \cos 0 = 1, \sin 0 = 0) \\ &= C_1 + a \end{aligned}$$

$$\therefore \qquad C_1 = -a \qquad \text{...(ii)}$$

At A, $x = 0$ and $\dfrac{dy}{dx} = 0$.

Differentiating the equation (*i*) w.r.t. x, we get

$$\frac{dy}{dx} = C_1\,.\,(-1)\sin\left(x\sqrt{\frac{P}{EI}}\right).\sqrt{\frac{P}{EI}} + C_2\cos\left(x\sqrt{\frac{P}{EI}}\right).\sqrt{\frac{P}{EI}} + 0$$

*The equation (*A*) can be written as

$$\frac{d^2y}{dx^2} + \alpha^2 \times y = \alpha^2 \times a \text{ where } \alpha^2 = \frac{P}{EI} \text{ or } \alpha = \sqrt{\frac{P}{EI}}$$

The complete solution of this equation is $y = C_1 \cos(\alpha\,.\,x) + C_2 \sin(\alpha\,.\,x) + \dfrac{\alpha^2 \times a}{\alpha^2}$

$$= C_1 \cos\left(\sqrt{\frac{P}{EI}} \times x\right) + C_2 \sin\left(\sqrt{\frac{P}{EI}} \times x\right) + a$$

$$= -C_1\,.\,\sqrt{\frac{P}{EI}}\sin\left(x\sqrt{\frac{P}{EI}}\right) + C_2\,.\,\sqrt{\frac{P}{EI}}\cos\left(x\sqrt{\frac{P}{EI}}\right)$$

But at A, $x = 0$ and $\dfrac{dy}{dx} = 0$.

$\therefore$ The above equation becomes as

$$\begin{aligned} 0 &= -C_1\,.\,\sqrt{\frac{P}{EI}} \sin 0 + C_2 \sqrt{\frac{P}{EI}} \cos 0 \\ &= -C_1 \sqrt{\frac{P}{EI}} \times 0 + C_2\,.\,\sqrt{\frac{P}{EI}} \times 1 = C_2 \sqrt{\frac{P}{EI}}\,. \end{aligned}$$

From the above equation it is clear that either $C_2 = 0$.

or
$$\sqrt{\frac{P}{EI}} = 0.$$

But for the crippling load P, the value of $\sqrt{\dfrac{P}{EI}}$ cannot be equal to zero.

$$\therefore \qquad C_2 = 0.$$

Substituting the values of $C_1 = -a$ and $C_2 = 0$ in equation (*i*), we get

$$y = -a\,.\,\cos\left(x\sqrt{\frac{P}{EI}}\right) + a. \qquad \text{...(iii)}$$

But at the free end of the column, $x = l$ and $y = a$.

Substituting these values in equation (*iii*), we get

$$a = -a\,.\,\cos\left(l\,.\sqrt{\frac{P}{EI}}\right) + a$$

or
$$0 = -a\,.\,\cos\left(l\,.\sqrt{\frac{P}{EI}}\right) \text{ or } a \cos\left(l\,.\sqrt{\frac{P}{EI}}\right) = 0$$

But 'a' cannot be equal to zero

$$\therefore \quad \cos\left(l\,.\sqrt{\frac{P}{EI}}\right) = 0 = \cos\frac{\pi}{2} \text{ or } \cos\frac{3\pi}{2} \text{ or } \cos\frac{5\pi}{2} \text{ or ......}$$

$$\therefore \quad l\,.\sqrt{\frac{P}{EI}} = \frac{\pi}{2} \text{ or } \frac{3\pi}{2} \text{ or } \frac{5\pi}{2} \text{ ......}$$

Taking the least practical value,

$$l\,.\sqrt{\frac{P}{EI}} = \frac{\pi}{2} \quad \text{or} \quad \sqrt{\frac{P}{EI}} = \frac{\pi}{2l}$$

or
$$P = \frac{\pi^2 EI}{4l^2} \qquad \text{...(19.2)}$$

19.7. EXPRESSION FOR CRIPPLING LOAD WHEN BOTH THE ENDS OF THE COLUMN ARE FIXED

Consider a column AB of length l and uniform cross-sectional area fixed at both of its ends A and B as shown in Fig. 19.6. Let P is the crippling load at which the column has buckled.

Due to the crippling load P, the column will deflect as shown in Fig. 19.6. Due to fixed ends, there will be fixed end moments (say M_0) at the ends A and B. The fixed end moments will be accting in such direction so that slope at the fixed ends becomes zero.

Consider a section at a distance x from the end A. Let the deflection of the column at the section is y. As both the ends of the column are fixed and the column carries a crippling load, there will be some fixed end moments at A and B.

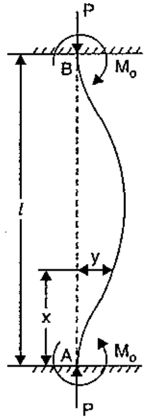


Fig. 19.6

Let M_0 = Fixed end moments at A and B.

Then moment at the section $= M_0 - P.y$

But moment at the section is also $= EI\,\dfrac{d^2y}{dx^2}$

$\therefore$ Equating the two moments, we get

$$EI\,\frac{d^2y}{dx^2} = M_0 - P.y$$

or
$$EI\,\frac{d^2y}{dx^2} + P\,.\,y = M_0$$

or
$$\frac{d^2y}{dx^2} + \frac{P}{EI}\,.\,y = \frac{M_0}{EI} \qquad ...(A)$$

$$= \frac{M_0}{EI} \times \frac{P}{P} = \frac{P}{EI}\,.\,\frac{M_0}{P}$$

The solution* of the above differential equation is

$$y = C_1\,.\,\cos\left(x\,.\sqrt{\frac{P}{EI}}\right) + C_2\,.\,\sin\left(x\,.\sqrt{\frac{P}{EI}}\right) + \frac{M_0}{P} \qquad ...(i)$$

where C_1 and C_2 are constant of integration and their values are obtained from boundary conditions. Boundary conditions are :

(*i*) At A, $x = 0$, $y = 0$ and also $\dfrac{dy}{dx} = 0$ as A is a fixed end.

(*ii*) At B, $x = l$, $y = 0$ and also $\dfrac{dy}{dx} = 0$ as B is also a fixed end.

---

*The equation (*A*) can be written as

$$\frac{d^2y}{dx^2} + \alpha^2 \times y = \frac{M_0}{EI} \quad \text{where } \alpha^2 = \frac{P}{EI} \quad \text{or} \quad \alpha = \sqrt{\frac{P}{EI}}$$

The complete solution of this equation is

$$y = C_1 \cos(\alpha\,.\,x) + C_2 \sin(\alpha\,.\,x) + \frac{M_0}{EI \times \alpha^2}$$

$$= C_1 \cos\left(\sqrt{\frac{P}{EI}} \times x\right) + C_2 \sin\left(\sqrt{\frac{P}{EI}} \times x\right) + \frac{M_0}{EI \times \dfrac{P}{EI}}$$

$$= C_1 \cos\left(x \times \sqrt{\frac{P}{EI}}\right) + C_2 \sin\left(x \times \frac{P}{\sqrt{EI}}\right) + \frac{M_0}{P}$$

Substituting the value $x = 0$ and $y = 0$ in equation (*i*), we get

$$0 = C_1 \times 1 + C_2 \times 0 + \frac{M_0}{P} \qquad (\because \cos 0 = 1)$$

$$= C_1 + \frac{M_0}{P}$$

$\therefore$
$$C_1 = -\frac{M_0}{P} \qquad ...(ii)$$

Differentiating the equation (*i*), with respect to x, we get

$$\frac{dy}{dx} = C_1\,.\,(-1) \sin\left(x\,.\sqrt{\frac{P}{EI}}\right).\sqrt{\frac{P}{EI}} + C_2 \cos\left(x\,.\sqrt{\frac{P}{EI}}\right).\sqrt{\frac{P}{EI}} + 0$$

$$= -C_1 \sin\left(x\,.\sqrt{\frac{P}{EI}}\right).\sqrt{\frac{P}{EI}} + C_2 \cos\left(x\,.\sqrt{\frac{P}{EI}}\right).\sqrt{\frac{P}{EI}}$$

Substituting the value $x = 0$ and $\dfrac{dy}{dx} = 0$, the above equation becomes

$$0 = -C_1 \times 0 + C_2 \times 1 \times \sqrt{\frac{P}{EI}} \qquad (\because \sin 0 = 0 \text{ and } \cos 0 = 1)$$

$$= C_2\sqrt{\frac{P}{EI}}\,.$$

From the above equation, it is clear that either $C_2 = 0$ or $\sqrt{\dfrac{P}{EI}} = 0$. But for a given crippling load P, the value of $\sqrt{\dfrac{P}{EI}}$ cannot be equal to zero.

$\therefore$
$$C_2 = 0.$$

Now substituting the values of $C_1 = -\dfrac{M_0}{P}$ and $C_2 = 0$ in equation (*i*), we get

$$y = -\frac{M_0}{P} \cos\left(x\sqrt{\frac{P}{EI}}\right) + 0 + \frac{M_0}{P}$$

$$= -\frac{M_0}{P} \cos\left(x\sqrt{\frac{P}{EI}}\right) + \frac{M_0}{P} \qquad ...(iii)$$

At the end B of the column, $x = l$ and $y = 0$.

Substituting these values in equation (*iii*), we get

$$0 = -\frac{M_0}{P} \cos\left(l\,.\sqrt{\frac{P}{EI}}\right) + \frac{M_0}{P}$$

or
$$\frac{M_0}{P} \cos\left(l\,.\sqrt{\frac{P}{EI}}\right) = \frac{M_0}{P}$$

or
$$\cos\left(l\,.\sqrt{\frac{P}{EI}}\right) = \frac{M_0}{P} \times \frac{P}{M_0} = 1 = \cos 0, \cos 2\pi, \cos 4\pi, \cos 6\pi, \ldots\ldots$$

$$\therefore \quad l \cdot \sqrt{\frac{P}{EI}} = 0, 2\pi, 4\pi, 6\pi, ...$$

Taking the least practical value,

$$l \cdot \sqrt{\frac{P}{EI}} = 2\pi \quad \text{or} \quad P = \frac{\pi^2 EI}{l^2} . \qquad ...(19.3)$$

19.8. EXPRESSION FOR CRIPPLING LOAD WHEN ONE END OF THE COLUMN IS FIXED AND THE OTHER END IS HINGED (OR PINNED)

Consider a column AB of length l and uniform cross-sectional area fixed at the end A and hinged at the end B as shown in Fig. 19.7. Let P is the crippling load at which the column has buckled. Due to the crippling load P, the column will deflect as shown in Fig. 19.7.

There will be fixed end moment (M_0) at the fixed end A. This will try to bring back the slope of deflected column zero at A. Hence it will be acting anticlock wise at A. The fixed end moment M_0 at A is to be balanced. This will be balanced by a horizontal reaction (H) at the top end B as shown in Fig. 19.7.

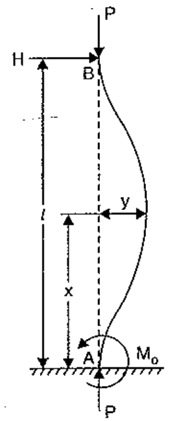


Fig. 19.7

Consider a section at a distance x from the end A.

Let y = Deflection of the column at the section,

M_0 = Fixed end moment at A, and

H = Horizontal reaction at B.

The moment at the section = Moment due to crippling load at B

+ Moment due to horizontal reaction at B

$$= -P \cdot y + H \cdot (l - x)$$

But the moment at the section is also

$$= EI \frac{d^2y}{dx^2}$$

Equating the two moments, we get

$$EI \frac{d^2y}{dx^2} = -P \cdot y + H(l - x)$$

or

$$EI \frac{d^2y}{dx^2} + P \cdot y = H(l - x)$$

or

$$\frac{d^2y}{dx^2} + \frac{P}{EI} \cdot y = \frac{H}{EI}(l - x) \qquad \text{(Dividing by } EI) \quad ...(A)$$

$$= \frac{H}{EI}(l - x) \times \frac{P}{P} = \frac{P}{EI} \cdot \frac{H(l - x)}{P}$$

The solution* of the above differential equation is

$$y = C_1 \cos\left(x\sqrt{\frac{P}{EI}}\right) + C_2 \sin\left(x\sqrt{\frac{P}{EI}}\right) + \frac{H}{P}(l - x) \qquad ...(i)$$

where C_1 and C_2 are constants of integration and their values are obtained from boundary conditions. Boundary conditions are :

(*i*) At the fixed end A, $x = 0$, $y = 0$ and also $\frac{dy}{dx} = 0$

(*ii*) At the hinged end B, $x = l$ and $y = 0$.

Substituting the value $x = 0$ and $y = 0$ in equation (*i*), we get

$$0 = C_1 \times 1 + C_2 \times 0 + \frac{H}{P}(l - 0) = C_1 + \frac{H \cdot l}{P}$$

$$\therefore \quad C_1 = -\frac{H}{P} \cdot l \qquad ...(ii)$$

Differentiating the equation (*i*) w.r.t. x, we get

$$\frac{dy}{dx} = C_1(-1)\sin\left(x \cdot \sqrt{\frac{P}{EI}}\right) \cdot \sqrt{\frac{P}{EI}} + C_2 \cos\left(x \cdot \sqrt{\frac{P}{EI}}\right) \cdot \sqrt{\frac{P}{EI}} - \frac{H}{P}$$

$$= -C_1 \sin\left(x \cdot \sqrt{\frac{P}{EI}}\right) \cdot \sqrt{\frac{P}{EI}} + C_2 \cos\left(x \cdot \sqrt{\frac{P}{EI}}\right) \cdot \sqrt{\frac{P}{EI}} - \frac{H}{P}$$

At A, $x = 0$ and $\frac{dy}{dx} = 0$.

$$\therefore \quad 0 = -C_1 \times 0 + C_2 \cdot 1 \cdot \sqrt{\frac{P}{EI}} - \frac{H}{P} \qquad (\because \sin 0 = 0, \cos 0 = 1)$$

$$= C_2\sqrt{\frac{P}{EI}} - \frac{H}{P} \quad \text{or} \quad C_2 = \frac{H}{P}\sqrt{\frac{EI}{P}} .$$

Substituting the values of $C_1 = -\frac{H}{P} \cdot l$ and $C_2 = \frac{H}{P}\sqrt{\frac{EI}{P}}$ in equation (*i*), we get

$$y = -\frac{H}{P} \cdot l \cos\left(x\sqrt{\frac{P}{EI}}\right) + \frac{H}{P}\sqrt{\frac{EI}{P}} \sin\left(x\sqrt{\frac{P}{EI}}\right) + \frac{H}{P}(l - x)$$

---

*The equation (*A*) can be written as

$$\frac{d^2y}{dx^2} + \alpha^2 \cdot y = \frac{H}{EI}(l - x) \text{ where } \alpha^2 = \frac{P}{EI} \quad \text{or} \quad \alpha = \sqrt{\frac{P}{EI}}$$

The complete solution of this equation is

$$y = C_1 \cos(\alpha \cdot x) + C_2 \sin(\alpha \cdot x) + \frac{H(l - x)}{EI \times \alpha^2}$$

$$= C_1 \cos\left(\sqrt{\frac{P}{EI}} \times x\right) + C_2 \sin\left(\sqrt{\frac{P}{EI}} \times x\right) + \frac{H}{EI} \times \frac{(l - x)}{\left(\frac{P}{EI}\right)}$$

$$= C_1 \cos\left(x \times \sqrt{\frac{P}{EI}}\right) + C_2 \sin\left(x \times \sqrt{\frac{P}{EI}}\right) + \frac{H}{P}(l - x)$$

At the end B, $x = l$ and $y = 0$.

Hence the above equation becomes as

$$0 = -\frac{H}{P}\, l \cos\left(l\sqrt{\frac{P}{EI}}\right) + \frac{H}{P}\sqrt{\frac{EI}{P}} \sin\left(l\sqrt{\frac{P}{EI}}\right) + \frac{H}{P}(l - l)$$

$$= -\frac{H}{P}\, l \cos\left(l\sqrt{\frac{P}{EI}}\right) + \frac{H}{P}\sqrt{\frac{EI}{P}} \sin\left(l\sqrt{\frac{P}{EI}}\right) + 0$$

or $$\frac{H}{P}\sqrt{\frac{EI}{P}} \sin\left(l\sqrt{\frac{P}{EI}}\right) = \frac{H}{P}\, l \cos\left(l\sqrt{\frac{P}{EI}}\right)$$

or $$\sin\left(l\sqrt{\frac{P}{EI}}\right) = \frac{H}{P} \cdot l \times \frac{P}{H} \times \sqrt{\frac{P}{EI}} \cdot \cos\left(l\sqrt{\frac{P}{EI}}\right)$$

$$= l \cdot \sqrt{\frac{P}{EI}} \cdot \cos\left(l \cdot \sqrt{\frac{P}{EI}}\right)$$

or $$\tan\left(l\sqrt{\frac{P}{EI}}\right) = l \cdot \sqrt{\frac{P}{EI}}.$$

The solution to the above equation is, $l \cdot \sqrt{\frac{P}{EI}} = 4.5$ radians

Squaring both sides, we get

$$l^2 \cdot \frac{P}{EI} = 4.5^2 = 20.25$$

$$\therefore \quad P = 20.25\,\frac{EI}{l^2}$$

But approximately $20.25 = 2\pi^2$

$$\therefore \quad P = \frac{2\pi^2 EI}{l^2} \qquad \text{...(19.4)}$$

19.9. EFFECTIVE LENGTH (OR EQUIVALENT LENGTH) OF A COLUMN

The effective length of a given column with given end conditions is the length of an equivalent column of the same material and cross-section with hinged ends, and having the value of the crippling load equal to that of the given column. Effective length is also called equivalent length.

Let L_e = Effective length of a column,

l = Actual length of the column, and

P = Crippling load for the column.

Then the crippling load for any type of end condition is given by

$$P = \frac{\pi^2 EI}{L_e^2}. \qquad \text{...(19.5)}$$

The crippling load (P) in terms of actual length and effective length and also the relation between effective length and actual length are given in Table 19.1.

TABLE 19.1

S.No.	*End conditions of column*	*Crippling load in terms of*		*Relation between effective length and actual length*
		Actual length	*Effective length*	
1.	Both ends hinged	$\frac{\pi^2 EI}{l^2}$	$\frac{\pi^2 EI}{L_e^2}$	$L_e = l$
2.	One end is fixed and other is free	$\frac{\pi^2 EI}{4l^2}$	$\frac{\pi^2 EI}{L_e^2}$	$L_e = 2l$
3.	Both ends fixed	$\frac{4\pi^2 EI}{l^2}$	$\frac{\pi^2 EI}{L_e^2}$	$L_e = \frac{l}{2}$
4.	One end fixed and other is hinged	$\frac{4\pi^2 EI}{l^2}$	$\frac{\pi^2 EI}{L_e^2}$	$L_e = \frac{l}{\sqrt{2}}$

There are two values of moment of inertia *i.e.*, I_{xx} and I_{yy}.

The value of I (moment of inertia) in the above expressions should be taken as the least value of the two moments of inertia as the column will tend to bend in the direction of least moment of inertia.

19.9.1. Crippling Stress in Terms of Effective Length and Radius of Gyration. The moment of inertia (I) can be expressed in terms of radius of gyration (k) as

$I = Ak^2$ where A = Area of cross-section.

As I is the least value of moment of inertia, then

k = Least radius of gyration of the column section.

Now crippling load P in terms of effective length is given by

$$P = \frac{\pi^2 EI}{L_e^2} = \frac{\pi^2 E \times Ak^2}{L_e^2} \qquad (\because \ I = Ak^2)$$

$$= \frac{\pi^2 E \times A}{\frac{L_e^2}{k^2}} = \frac{\pi^2 E \times A}{\left(\frac{L_e}{k}\right)^2} \qquad \text{...(19.6)}$$

And the stress corresponding to crippling load is given by

$$\text{Crippling stress} = \frac{\text{Crippling load}}{\text{Area}} = \frac{P}{A}$$

$$= \frac{\pi^2 E \times A}{A\left(\frac{L_e}{k}\right)^2} \qquad \text{(Substituting the value of } P\text{)}$$

$$= \frac{\pi^2 E}{\left(\frac{L_e}{k}\right)^2} \qquad \text{...(19.7)}$$

19.9.2. Slenderness Ratio. The ratio of the actual length of a column to the least radius of gyration of the column, is known as slenderness ratio.

Mathematically, slenderness ratio is given by

$$\text{Slenderness ratio} = \frac{\text{Actual length}}{\text{Least radius of gyration}} = \frac{l}{k} \quad \text{...(19.8)}$$

19.10. LIMITATION OF EULER'S FORMULA

From equation (19.6), we have

$$\text{Crippling stress} = \frac{\pi^2 E}{\left(\frac{L_e}{k}\right)^2}$$

For a column with both ends hinged, $L_e = l$. Hence Crippling stress becomes as $= \dfrac{\pi^2 E}{\left(\frac{l}{k}\right)^2}$,

where $\frac{l}{k}$ is slenderness ratio.

If the slenderness ratio $\left(i.e., \frac{l}{k}\right)$ is small, the crippling stress (or the stress at failure) will be high. But for the column material, the crippling stress cannot be greater than the crushing stress. Hence when the slenderness ratio is less than a certain limit, Euler's formula gives a value of crippling stress greater than the crushing stress. In the limiting case, we can find the value of l/k for which crippling stress is equal to crushing stress.

For example, for a mid steel column with both ends hinged,

Crushing stress = 330 N/mm^2

Young's modulus, $E = 2.1 \times 10^5$ N/mm^2.

Equating the crippling stress to the crushing stress corresponding to the minimum value of slenderness ratio, we get

Crippling stress = Crushing stress

or
$$\frac{\pi^2 E}{\left(\frac{l}{k}\right)^2} = 330 \quad \text{or} \quad \frac{\pi^2 \times 2.1 \times 10^5}{\left(\frac{l}{k}\right)^2} = 330$$

$$\therefore \quad \left(\frac{l}{k}\right)^2 = \frac{\pi^2 \times 2.1 \times 10^5}{330} = 6282$$

$$\therefore \quad \frac{l}{k} = \sqrt{6282} = 79.27, \text{ say } 80.$$

Hence, if the slenderness ratio is less than 80 for mild steel column with both ends hinged, then Euler's formula will not be valid.

Problem 19.1. *A solid round bar 3 m long and 5 cm in diameter is used as a strut with both ends hinged. Determine the crippling (or collapsing) load. Take $E = 2.0 \times 10^5$ N/mm^2.*

Sol. Given :

Length of bar, $l = 3 \text{ m} = 3000 \text{ mm}$

Diameter of bar, $d = 5 \text{ cm} = 50 \text{ mm}$

Young's modulus, $E = 2.0 \times 10^5 \text{ N/mm}^2$

Moment of inertia, $I = \frac{\pi}{64} \times 5^4 = 30.68 \text{ cm}^4 = 30.68 \times 10^4 \text{ mm}^4$

Let P = Crippling load.

As both the ends of the bar are hinged, hence the crippling load is given by equation (19.1).

$$\therefore \quad P = \frac{\pi^2 EI}{l^2} = \frac{\pi^2 \times 2 \times 10^5 \times 30.68 \times 10^4}{3000^2}$$

$$= 67288 \text{ N} = \mathbf{67.288 \text{ kN}}. \quad \textbf{Ans.}$$

Problem 19.2. *For the problem 19.1 determine the crippling load, when the given strut is used with the following conditions :*

(i) One end of the strut is fixed and the other end is free

(ii) Both the ends of strut are fixed

(iii) One end is fixed and other is hinged.

Sol. Given :

The data from Problem 19.1, is $l = 3000$ mm, diameter = 50 mm, $E = 2.0 \times 10^5$ N/mm^2 and $I = 30.68 \times 10^4$ mm^4.

Let P = Crippling load.

(i) Crippling load when one end is fixed and other is free

Using equation (19.2), $P = \dfrac{\pi^2 EI}{4l^2} = \dfrac{\pi^2 \times 2 \times 10^5 \times 30.68 \times 10^4}{4 \times 3000^2} = \mathbf{16822 \text{ N}}.$ **Ans.**

Alternate Method

The crippling load for any type of end condition is given by equation (19.5) as,

$$P = \frac{\pi^2 EI}{L_e^2} \quad \text{...}(i)$$

where L_e = Effective length.

The effective length (L_e) when one end is fixed and other end is free from Table 19.1 on page 819 is given as

$$L_e = 2l = 2 \times 3000 = 6000 \text{ mm}$$

Substituting the value of L in equation (*i*), we get

$$P = \frac{\pi^2 \times 2 \times 10^5 \times 30.68 \times 10^4}{6000^2} = \mathbf{16822 \text{ N}}. \quad \textbf{Ans.}$$

(ii) Crippling load when both the ends are fixed

Using equation (19.3), $P = \dfrac{4\pi^2 EI}{l^2} = \dfrac{4\pi^2 \times 2 \times 10^5 \times 30.68 \times 10^4}{3000^2}$

$$= 269152 \text{ N} = \mathbf{269.152 \text{ kN}}. \quad \textbf{Ans.}$$

Alternate Method

Using equation (19.5), $P = \dfrac{\pi^2 EI}{L_e^{\,2}}$

where L_e = Effective length

$= \dfrac{l}{2}$ (when both the ends are fixed)

$= \dfrac{3000}{2}$ ($\because$ $l = 3000$)

$= 1500$ mm

$$\therefore \quad P = \frac{\pi^2 \times 2.0 \times 10^5 \times 30.68 \times 10^4}{1500^2} = \textbf{269152 N.} \quad \textbf{Ans.}$$

(iii) Crippling load when one end is fixed and the other is hinged

Using equation (19.4), $P = \dfrac{2\pi^2 EI}{l^2} = \dfrac{2 \times \pi^2 \times 2.0 \times 10^5 \times 30.68 \times 10^4}{3000^2} = \textbf{134576 N.}$ **Ans.**

Alternate Method

Using equation (19.5), $P = \dfrac{\pi^2 EI}{L_e^{\,2}}$

where L_e = Effective length.

$= \dfrac{l}{\sqrt{2}}$ (when one end is fixed and the other is hinged)

$= \dfrac{3000}{\sqrt{2}}$

$$\therefore \quad P = \frac{\pi^2 \times 2.0 \times 10^5 \times 30.68 \times 10^4}{\left(\dfrac{3000}{\sqrt{2}}\right)^2} = \textbf{134576 N.} \quad \textbf{Ans.}$$

Problem 19.3. *A column of timber section 15 cm × 20 cm is 6 metre long both ends being fixed. If the Young's modulus for timber = 17.5 kN/mm², determine :*

(i) Crippling load and

(ii) Safe load for the column if factor of safety = 3.

Sol. Given :

Dimension of section = 15 cm × 20 cm

Actual length, $l = 6$ m $= 6000$ mm

Young's modulus, $E = 17.5$ kN/mm²

(i) Let P = Crippling load

Using equation (19.5), we get

$$P = \frac{\pi^2 EI}{L_e^{\,2}} \qquad \ldots(i)$$

where L_e = Effective length

$= \dfrac{l}{2}$ (when both the ends are fixed)

$= \dfrac{6000}{2} = 3000$ mm ($\because$ $l = 6000$ mm)

I = Least value of moment of inertia

Moment of inertia of the section about X-X axis,

$$I_{XX} = \frac{15 \times 20^3}{12} = 10000 \text{ cm}^4$$
$$= 10000 \times 10^4 \text{ mm}^4$$

And moment of inertia of the section about Y-Y axis,

$$I_{YY} = \frac{20 \times 15^3}{12} = 5625 \text{ cm}^4$$
$$= 5625 \times 10^4 \text{ mm}^4.$$

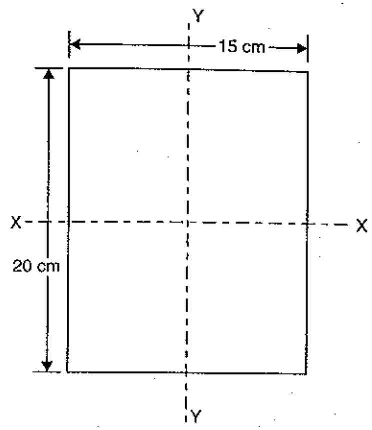


Fig. 19.8

Since I_{YY} is less than I_{XX}, therefore the column will tend to buckle in Y-Y direction.

And the value of I will be the least value of the two moment of inertia.

$\therefore$ $I = 5625$ cm⁴ $= 5625 \times 10^4$ mm⁴

Substituting the values of $I = 5625 \times 10^4$ mm⁴ and $L = 3000$ mm in equation (*i*), we get

$$P = \frac{\pi^2 \times 17.5 \times 5625 \times 10^4}{3000} = \textbf{1079.48 kN.} \quad \textbf{Ans.}$$

(ii) Safe load for the column

Factor of safety = 3.0 (given)

$\therefore$ Safe load $= \dfrac{\text{Crippling load}}{\text{Factor of safety}} = \dfrac{1079.48}{3} = \textbf{359.8 say 360 kN.}$ **Ans.**

Problem 19.4. *A hollow mild steel tube 6 m long 4 cm internal diameter and 6 mm thick is used as a strut with both ends hinged. Find the crippling load and safe load taking factor of safety as 3. Take E = 2 × 10⁵ N/mm².*

Sol. Given :

Length of tube, $l = 6$ m $= 600$ cm

Internal dia., $d = 4$ cm

Thickness, $t = 5$ mm $= 0.5$ cm

$\therefore$ External dia., $D = d + 2t = 4 + 2 \times 0.5 = 4 + 1 = 5$ cm

Young's modulus, $E = 2 \times 10^5$ N/mm²

Factor of safety = 3.0

Moment of inertia of section, $I = \dfrac{\pi}{64}(D^4 - d^4) = \dfrac{\pi}{64}[5^4 - 4^4]$ cm⁴

$= \dfrac{\pi}{64}(625 - 256) = 18.11$ cm⁴ $= 18.11 \times 10^4$ mm⁴

Since both ends of the strut are hinged,

$\therefore$ Effective length, $L_e = l = 600$ cm $= 6000$ mm

Let P = Crippling load

Using equation (19.5), we get

$$P = \frac{\pi^2 EI}{L_e^2}$$

$$= \frac{\pi^2 \times 2.0 \times 10^5 \times 18.11 \times 10^4}{6000^2} = \textbf{9929.9 say 9930 N.} \quad \textbf{Ans.}$$

And safe load $= \dfrac{\text{Crippling load}}{\text{Factor of safety}} = \dfrac{9930}{3.0} = \textbf{3310 N.}$ **Ans.**

Problem 19.4 (a). *A simply supported beam of length 4 metre is subjected to a uniformly distributed load of 30 kN/m over the whole span and deflects 15 mm at the centre. Determine the crippling loads when this beam is used as a column with the following conditions :*

(i) one end fixed and other end hinged

(ii) both the ends pin jointed. (Annamalai University, 1990)

Sol. Given :

Length, L = 4 m = 4000 mm

Uniformly distributed load, w = 30 kN/m = 30,000 N/m

$$= \frac{30{,}000}{1000} \text{ N/mm} = 30 \text{ N/mm}$$

Deflection at the centre, δ = 15 mm.

For a simply supported beam, carrying U.D.L. over the whole span, the deflection at the centre is given by,

$$\delta = \frac{5}{384} \times \frac{w \times L^4}{EI}$$

or $$15 = \frac{5}{384} \times \frac{30 \times 4000^4}{EI}$$

$\therefore$ $$EI = \frac{5}{384} \times \frac{30 \times 4000^4}{15}$$

$$= \frac{5}{384} \times \frac{3 \times 256}{15} \times 10^{13} = \frac{2}{3} \times 10^{13} \text{ N mm}^2.$$

(i) Crippling load when the beam is used as a column with one end fixed and other end hinged.

The crippling load P for this case in terms of actual length is given by equation (19.4) as

$$P = \frac{2\pi^2 \times EI}{L_e^2}, \text{ where } l = \text{actual length} = 4000 \text{ mm}$$

$$= \frac{2 \times \pi^2 \times \frac{2}{3} \times 10^{13}}{4000^2} = \textbf{8224.5 kN.} \quad \textbf{Ans.}$$

(ii) Crippling load when both the ends are pin-jointed

This is given by equation (19.1) in terms of actual length as

$$P = \frac{2\pi^2 \times EI}{l^2} \quad \text{where } l = \text{actual length} = 4000 \text{ mm}$$

$$= \frac{\pi^2 \times \frac{2}{3} \times 10^{13}}{4000^2} = \textbf{4112.25 kN.} \quad \textbf{Ans.}$$

Problem 19.5. *A solid round bar 4 m long and 5 cm in diameter was found to extend 4.6 mm under a tensile load of 50 kN. This bar is used as a strut with both ends hinged. Determine the buckling load for the bar and also the safe load taking factor of safety as 4.0.*

Sol. Given :

Actual length of bar, L = 4 m = 4000 mm

Dia. of bar, d = 5 cm

$\therefore$ Area of bar, $A = \dfrac{\pi}{4} \times 5^2 = 6.25\pi \text{ cm}^2 = 6.25\pi \times 10^2 \text{ mm}^2 = 625\pi \text{ mm}^2$

Extension of bar, δL = 4.6 mm

Tensile load, W = 50 kN = 50000 N.

In this problem, the values of Young's modulus (E) is not given. But it can be calculated from the given data.

$$\therefore \text{ Young's modulus, } E = \frac{\text{Tensile stress}}{\text{Tensile strain}} = \frac{\left(\dfrac{\text{Tensile load}}{\text{Area}}\right)}{\left(\dfrac{\text{Extension of bar}}{\text{Length of bar}}\right)}$$

$$\left(\because \text{ Stress} = \frac{\text{Load}}{\text{Area}} \text{ and strain} = \frac{\delta L}{L}\right)$$

$$= \frac{\left(\dfrac{W}{A}\right)}{\dfrac{\delta L}{L}} = \frac{W}{A} \times \frac{L}{\delta L} = \frac{50000}{625\,\pi} \times \frac{4000}{4.6} = 2.214 \times 10^4 \text{ N/mm}^2.$$

Since the strut is hinged at its both ends,

$\therefore$ Effective length, L_e = Actual length = 4000 mm

Let P = Crippling or buckling load.

Using equation (19.5), we get

$$P = \frac{\pi^2 EI}{L_e^2}$$

$$= \frac{\pi^2 \times 2.214 \times 10^4 \times \frac{\pi}{64} \times 5^4 \times 10^4}{4000 \times 4000} \qquad \left(\because I = \frac{\pi}{64} \times 5^4 \times 10^4 \text{ mm}^4\right)$$

$$= \textbf{4189.99 say 4190 N.} \quad \textbf{Ans.}$$

And safe load $= \dfrac{\text{Crippling load}}{\text{Factor of safety}} = \dfrac{4190}{4} = \textbf{1047.5 N.}$ **Ans.**

Problem 19.6. *A hollow alloy tube 5 m long with external and internal diameters 40 mm and 25 mm respectively was found to extend 6.4 mm under a tensile load of 60 kN. Find the buckling load for the tube when used as a column with both ends pinned. Also find the safe load for the tube, taking a factor of safety = 4.* (AMIE, Summer 1989)

Sol. Given :

Actual length, L = 5 m = 5000 mm

External dia., D = 40 mm

Internal dia., d = 25 mm

Extension, δL = 6.4 mm

Tensile load, W = 60 kN = 60,000 N

Safety factor = 4

Cross-section area, $A = \frac{\pi}{4}(D^2 - d^2) = \frac{\pi}{4}(40^2 - 25^2) = 766 \text{ mm}^2$

Moment of inertia, $I = \frac{\pi}{64}(D^4 - d^4) = \frac{\pi}{64}(40^4 - 25^4)$

$$= \frac{\pi}{64}(2560000 - 390625) = \frac{\pi}{64} \times 2169375 = 106500 \text{ mm}^4$$

The value of Young's modulus is obtained as given below

$$E = \frac{\text{Tensile stress}}{\text{Tensile strain}} = \frac{\left(\frac{W}{A}\right)}{\left(\frac{\delta L}{L}\right)}$$

$$= \frac{\left(\frac{60,000}{766}\right)}{\left(\frac{6.4}{5000}\right)} = \frac{60,000}{766} \times \frac{5000}{64} = 6.11945 \times 10^4 \text{ N/mm}^2$$

Since the column is pinned at both the ends.

$\therefore$ Effective length, L_e = Actual length = 5000 mm

Let P = Buckling load

Using equation (19.5), we get

$$P = \frac{\pi^2 EI}{L_e^2} = \frac{\pi^2 \times 6.11945 \times 10^4 \times 106500}{5000^2} = \textbf{2573 N.} \quad \textbf{Ans.}$$

And safe load $= \frac{\text{Buckling load}}{\text{Factor of safety}} = \frac{2573}{4} = \textbf{643.2 N.}$ **Ans.**

Problem 19.7. *Calculate the safe compressive load on a hollow cast iron column (one end rigidly fixed and other hinged) of 15 cm external diameter, 10 cm internal diameter and 10 m in length. Use Euler's formula with a factor of safety of 5 and E = 95 kN/mm².*

(AMIE, Winter 1981)

Sol. Given :

External dia., D = 15 cm

Internal dia., d = 10 cm

Actual length of column, l = 10 m = 10000 mm

Factor of safety = 5.0

Young's modulus, $E = 95 \text{ kN/mm}^2 = 95000 \text{ N/mm}^2$

Moment of inertia of hollow column,

$$I = \frac{\pi}{64}(D^4 - d^4) = \frac{\pi}{64}(15^4 - 10^4) = \frac{\pi}{64}(50625 - 10000)$$

$$= 1994.175 \text{ cm}^4 = 1994.175 \times 10^4 \text{ mm}^4$$

Since one end of the column is fixed and other end is hinged.

$\therefore$ Effective length, $L_e = \frac{l}{\sqrt{2}} = \frac{10000}{\sqrt{2}}$

Let P = Crippling load.

Using equation (19.5), we get

$$P = \frac{\pi^2 EI}{L_e^2}$$

$$= \frac{\pi^2 \times 95000 \times 1994.175 \times 10^4}{\left(\frac{10000}{\sqrt{2}}\right)^2} \qquad \left(\because L = \frac{10000}{\sqrt{2}}\right)$$

$$= \frac{\pi^2 \times 95000 \times 1994.175 \times 10^4 \times 2}{10000 \times 10000} = 373950 \text{ N} = 373.95 \text{ kN}$$

But safe load $= \frac{\text{Crippling load}}{\text{Factor of safety}} = \frac{373.95}{5} = \textbf{74.79 kN.}$ **Ans.**

Problem 19.8. *Determine Euler's crippling load for an I-section joist 40 cm × 20 cm × 1 cm and 5 m long which is used as a strut with both ends fixed. Take Young's modulus for the joist as 2.1 × 10⁵ N/mm².*

Sol. Given :

Dimensions of I-section = 40 cm × 20 cm × 1 cm

Length actual, l = 5 m = 5000 mm

Young's modulus, $E = 2.1 \times 10^5 \text{ N/mm}^2$.

Moment of inertia of the section about X-X axis,

I_{XX} = M.O.I. of rectangle of dimension 20 cm × 40 cm – M.O.I. of rectangle of dimension [(20 – 1), (40 – 1 – 1)]

$$= \frac{1}{12} bd^3 - \frac{1}{12} b_1 d_1^3$$

$$= \frac{1}{12} \times 20 \times 40^3 - \frac{1}{12} \times 19 \times 38^3 \qquad (\because b_1 = 19, d_1 = 38)$$

$$= 19786 \text{ cm}^4.$$

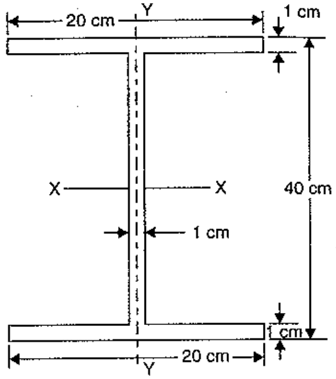


Fig. 19.9

Similarly the moment of inertia of the section about Y-Y axis

I_{YY} = M.O.I. of rectangle of dimension (38 × 1) + M.O.I. of two rectangles of dimension (1 × 20)

$$= \frac{1}{12} \times 38 \times 1^3 + 2 \times \frac{1}{12} \times 1 \times 20^3$$

$$= 3.166 \times 1333.33 = 1336.5 \text{ cm}^4.$$

$\therefore$ Least value of the moment of inertia is about Y-Y axis.

$\therefore$ $I = 1336.5 \text{ cm}^4 = 1336.5 \times 10^4 \text{ mm}^4$

As both the ends of the strut are fixed

$\therefore$ Effective length, $L_e = \frac{l}{2} = \frac{5000}{2} = 2500$ mm

Let P = Crippling load.

Using equation (19.5), we get

$$P = \frac{\pi^2 EI}{L_e^2} = \frac{\pi^2 \times 2.1 \times 10^5 \times 1336.5 \times 10^4}{2500^3}$$

$= 4432080$ N $=$ **4.432 MN. Ans.**

Problem 19.9. *Determine the crippling load for a T-section of dimensions 10 cm × 10 cm × 2 cm and of length 5 m when it is used as strut with both of its ends hinged. Take Young's modulus, $E = 2.0 \times 10^5$ N/mm².*

Sol. Given :

Dimensions of T-section $=$ 10 cm × 10 cm × 2 cm

Length actual, $l = 5$ m $= 5000$ mm

Young's modulus, $E = 2.0 \times 10^5$ N/mm².

First of all, calculate the C.G. of the section. The given section is symmetrical about the axis Y-Y, hence the C.G. of the section will lie on Y-Y axis.

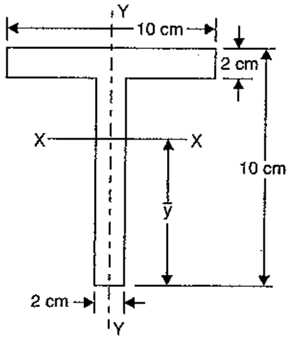


Fig. 19.10

Let $\bar{y}$ = Distance of C.G. of the section from bottom end.

For the flange, we have $a_1 = 10 \times 2 = 20$ cm²

y_1 = Distance of C.G. of area a_1 from the bottom end

$= 8 + 1 = 9$ cm

For the web, we have $a_2 = 8 \times 2 = 16$ cm²

y_2 = Distance of C.G. of area a_2 from bottom end $= \frac{8}{2} = 4$ cm

Using the relation, $\bar{y} = \dfrac{a_1 y_1 + a_2 y_2}{a_1 + a_2}$

$$= \frac{20 \times 9 + 16 \times 4}{20 + 16} = \frac{180 + 64}{36} = 6.777 \text{ cm}$$

Moment of inertia of the section about the axis X-X,

$$I_{XX} = \left(\frac{10 \times 8^3}{12} + 20 \times 2.223^2\right) + \left(\frac{2 \times 8^3}{12} + 16 \times 2.777^2\right)$$

$= (6.667 + 98.834) + (85.333 + 123.387) = 314.221$ cm⁴.

Moment of inertia of the section about the axis Y-Y,

$$I_{YY} = \frac{2 \times 10^3}{12} + \frac{8 \times 2^3}{12} = 166.67 + 5.33 = 172 \text{ cm}^4.$$

Least value of moment of inertia is about Y-Y axis

$\therefore$ $I = 172$ cm⁴ $= 172 \times 10^4$ mm⁴

Since the strut is hinged at both of its end

$\therefore$ Effect length, $L_e = l = 5000$ mm

Let P = Crippling load

Using equation (19.5), we get

$$P = \frac{\pi^2 EI}{L_e^2} = \frac{\pi^2 \times 2.0 \times 10^5 \times 172 \times 10^4}{5000^3} = \mathbf{135805.7\ N.\ Ans.}$$

Problem 19.10. *Calculate the Euler's critical load for a strut of T-section, the flange width being 10 cm, overall depth 8 cm and both flange and stem 1 cm thick. The strut is 3 m long and is built in at both ends. Take $E = 2 \times 10^5$ N/mm⁵.* (AMIE, Winter 1990)

Sol. Given :

Actual length, $l = 3$ m $= 3000$ mm

Value of $E = 2 \times 10^5$ N/mm²

The dimensions of T-section are shown in Fig. 19.10 (*a*). First of all, calculate the C.G. of the section. The given section is symmetrical about the y-y axis.

Hence the C.G. will lie on y-y axis.

Let $\bar{y}$ = Distance of C.G. of the section from the bottom end.

For the flange a_1 = Area of flange $= 10 \times 1 = 10$ cm²

y_1 = Distance of C.G. of area a_1 from the bottom end

$= 7 + \frac{1}{2} = 7.5$ cm.

For the stem or web, a_2 = Area of stem $= 7 \times 1 = 7$ cm²

y_2 = Distance of C.G. of area a_2 from the bottom end

$= \frac{7}{2} = 3.5$ cm

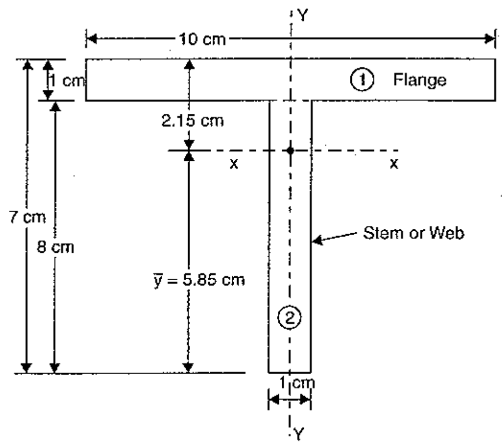


Fig. 19.10 (*a*)

Now using the relation, $\bar{y} = \dfrac{a_1 \times y_1 + a_2 \times y_2}{a_1 + a_2} = \dfrac{10 \times 7.5 + 7 \times 3.5}{(10 + 7)}$

$$= \frac{75 + 24.5}{17} = \frac{99.5}{17} = 5.85 \text{ cm.}$$

Now calculate the moment of inertia about x-x axis and y-y axis.

$$I_{XX} = \left[\frac{10 \times 1^3}{12} + a_1 \times (2.15 - 0.5)^2\right] + \left[\frac{1 \times 7^3}{12} + a_a \times (5.85 - 3.5)^2\right]$$

$$= \left[\frac{10}{12} + 10 \times 1.65^2\right] + \left[\frac{343}{12} + 7 \times 2.35^2\right] = 95.298 \text{ cm}^4$$

and $$I_{YY} = \frac{1 \times 10^3}{12} + \frac{7 \times 1^3}{12} = 83.33 + 0.583 = 83.916 \text{ cm}^4.$$

The least value of moment of inertia is about y-axis.

$\therefore$ $$I = I_{YY} = 83.916 \text{ cm}^4 = 839160 \text{ mm}^4.$$

As the strut is fixed at both ends, hence its effective length (L_e) will be half of its actual length.

$\therefore$ $$L_e = \frac{l}{2} = \frac{3000}{2} = 1500 \text{ mm}$$

Let P = Euler's critical load.

Using equation (19.5), $$P = \frac{\pi^2 EI}{L_e^2} = \frac{\pi^2 \times 2 \times 10^5 \times 839160}{1500^3}$$

$= 736190$ N $=$ **736.19 kN. Ans.**

Problem 19.11. *A strut length l, moment of inertia of cross-section = I uniform throughout and modulus of material = E, is fixed at its lower end, and its upper end is elastically supported laterally by a spring of stiffness k. Show from the first principles that the crippling load P is given by $\frac{\tan \alpha l}{\alpha l} = 1 - \frac{P}{kl}$, where $\alpha^2 = \frac{P}{EI}$.* (AMIE, Summer 1984)

Sol. Given :

Length of strut $= l$

Moment of inertia $= I$

Young's modulus $= E$

Stiffness of spring $= k$

Crippling load $= P$.

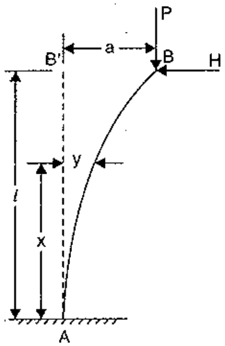


Fig. 19.11

The strut is shown in Fig. 19.11 in which

H = Lateral force due to spring of stiffness k

a = Deflection of the end B

Then H = Stiffness of spring × Deflection of end B

$= k \times a.$ ...(*i*)

Consider any section at a distance x from the fixed end A.

Let y = Deflection at the section.

Then moment at the section = Moment due to crippling load + Moment due to lateral force H

$$= P(a - y) - H(l - x) = P \,.\, a - P \,.\, y - H(l - x)$$

But moment is also $$= EI \frac{d^2y}{dx^2}$$

$\therefore$ Equating the two moments, we get

$$EI \frac{d^2y}{dx^2} = P.a - P.y - H(l - x)$$

$$EI \frac{d^2y}{dx^2} + P.y = P.a - H(l - x)$$

or $$\frac{d^2y}{dx^2} + \frac{P}{EI} \,.\, y = \frac{P}{EI} \,.\, a - \frac{H}{EI}(l - x) \qquad \text{(Dividing by } EI\text{)} \quad ...(A)$$

$$= \frac{P}{EI} \,.\, a - \frac{H}{P} \,.\, \frac{P}{EI}(l - x)$$

$$= \frac{P}{EI} \,.\, a - \frac{P}{EI} \,.\, \frac{H}{P}(l - x) = \frac{P}{EI}\left[a - \frac{H}{P}(l - x)\right]$$

The solution* of the above differential equation is

$$y = C_1 \,.\, \cos\left(x \,.\, \sqrt{\frac{P}{EI}}\right) + C_2 \,.\, \sin\left(x \,.\, \sqrt{\frac{P}{EI}}\right) + a - \frac{H}{P}(l - x)$$

$$= C_1 \,.\, \cos(x \,.\, \alpha) + C_2 \,.\, \sin(x \,.\, \alpha) + a - \frac{H}{P}(l - x)$$

$$\left(\because \alpha^2 = \frac{P}{EI} \quad \therefore \alpha = \sqrt{\frac{P}{EI}}\right) \quad ...(ii)$$

where C_1 and C_2 are constant of integration and their values are obtained from boundary conditions. The boundary conditions are :

(*i*) For fixed end A, $x = 0$ and $y = 0$ also $\frac{dy}{dx} = 0$

(*ii*) At $x = l$, $y = a$.

Substituting the values $x = 0$, $y = 0$ in equation (*ii*), we get

$$0 = C_1 \,.\, 1 + C_2 \,.\, 0 + a - \frac{H}{P}(l - 0) \qquad (\because \cos 0 = 1, \sin 0 = 0)$$

$$= C_1 + a - \frac{H}{P} l$$

$\therefore$ $$C_1 = -a + \frac{H}{P} l \quad ...(iii)$$

Differentiating equation (*ii*) w.r.t. x, we get

$$\frac{dy}{dx} = -C_1 \,.\, \alpha \,.\, \sin(x \,.\, \alpha) + C_2 \,.\, \alpha \,.\, \cos(x \,.\, \alpha) + 0 - \frac{H}{P}(-1)$$

$$= -C_1 \,.\, \alpha \,.\, \sin(x \,.\, \alpha) + C_2 \,.\, \alpha \,.\, \cos(x \,.\, \alpha) + \frac{H}{P}$$

---

*The equation (A) can be written as

$$\frac{d^2y}{dx^2} + \alpha^2 \times y = \alpha^2 \times a - \frac{H}{EI}(l - x) \quad \text{where } \alpha = \frac{P}{EI} \quad \text{or} \quad \alpha = \sqrt{\frac{P}{EI}}$$

The complete solution of this equation is

$$y = C_1 \cos(\alpha \,.\, x) + C_2 \sin(\alpha \times x) + \frac{\alpha^2 \times a - \frac{H}{EI}(l - x)}{\alpha^2}$$

$$= C_1 \cos\left(\sqrt{\frac{P}{EI}} \times x\right) + C_2 \sin\left(\sqrt{\frac{P}{EI}} \times x\right) + a - \frac{H(l - x)}{EI \times \frac{P}{EI}}$$

$$= C_1 \cos\left(x \times \sqrt{\frac{P}{EI}}\right) + C_2 \sin\left(x \times \sqrt{\frac{P}{EI}}\right) + a - \frac{H}{P}(l - x)$$

Substituting the values at $x = 0$, $\frac{dy}{dx} = 0$, in the above equation, we get

$$0 = -C_1 . \alpha . 0 + C_2.\alpha.1 + \frac{H}{P} \qquad (\because \sin 0 = 0, \cos 0 = 1)$$

$$= C_2 . \alpha + \frac{H}{P}$$

$$\therefore \quad C_2 = -\frac{H}{\alpha P} \qquad ...(iv)$$

Substituting the values of C_1 and C_2 in equation (*ii*), we get

$$y = -\left(a - \frac{H}{P}l\right) . \cos(x.\alpha) - \frac{H}{\alpha P}\sin(x.\alpha) + a - \frac{H}{P}(l - x) \qquad ...(v)$$

At the end B, $x = l$ and $y = a$.

Substituting these values in equation (*v*), we get

$$a = -\left(a - \frac{H}{P}l\right) . \cos(l.\alpha) - \frac{H}{\alpha P}\sin(l.\alpha) + a - \frac{H}{P}(l - l)$$

$$= -\left(a - \frac{H}{P}l\right) . \cos(l.\alpha) - \frac{H}{\alpha P}\sin(l.\alpha) + a$$

or $$a + \left(a - \frac{H}{P}l\right)\cos(l.\alpha) - a = -\frac{H}{\alpha P}\sin(l.\alpha)$$

or $$\left(a - \frac{H}{P}l\right)\cos(l.\alpha) = -\frac{H}{\alpha P}\sin(l.\alpha)$$

or $$\frac{\sin(l.\alpha)}{\cos(l.\alpha)} = \frac{\left(a - \frac{H}{P}l\right)}{-\frac{H}{\alpha P}}$$

or $$\tan(l.\alpha) = -\left(a - \frac{H}{P}l\right) . \frac{\alpha P}{H}$$

or $$\tan(l.\alpha) = -\frac{a\,\alpha . P}{H} + \frac{H}{P}l . \frac{\alpha.P}{H}$$

$$= -\frac{a\,\alpha . P}{H} + l.\alpha = -\frac{a\,\alpha.P}{k.a} + \alpha\, l \quad [\because H = k.a \text{ from equation } (i)]$$

$$= -\frac{\alpha . P}{k} + \alpha\, l$$

Dividing by $\alpha.l$, we get

$$\frac{\tan(\alpha.l)}{\alpha.l} = -\frac{\alpha.P}{k.\alpha.l} + 1 = -\frac{P}{k.l} + 1$$

or $$\frac{\tan(\alpha.l)}{\alpha.l} = \mathbf{1 - \frac{P}{kl}}. \quad \textbf{Ans.}$$

Problem 19.12. *Determine the ratio of buckling strengths of two columns one hollow and the other solid. Both are made of the same material and have the same length, cross-sectional area and end conditions. The internal diameter of hollow column is half of its external diameter.* (AMIE, Winter 1982, Summer 1985)

Sol. Given :

Let d = Dia. of solid column

D = External dia. of hollow column

$\therefore$ Internal dia. of hollow shaft $= \frac{D}{2}$

Both the shafts are made of the same material and have the same length, cross-sectional area and end conditions. As areas of solid and hollow shafts are equal.

$\therefore$ Area of solid shaft = Area of hollow shaft

or $$\frac{\pi}{2}d^2 = \frac{\pi}{2}\left[D^2 - \left(\frac{D}{2}\right)^2\right] \text{ or } d^2 = \left[D^2 - \frac{D^2}{4}\right] = \frac{3D^2}{4}$$

$$\therefore \quad d = \frac{\sqrt{3}D}{2} \qquad ...(i)$$

Buckling load of a column as given by Euler's formula is

$$P = \frac{\pi^2 EI}{L^2} \qquad \text{[see equation (19.5)]}$$

As the shafts are having the same length, same material, same cross-sectional areas and same end conditions. Hence the value of A, E and L for both shafts are same.

$$\therefore \quad P \propto I \qquad (\because A, E \text{ and } L \text{ are same}) \;...(ii)$$

Let P_1 = Buckling load of the hollow column

P_2 = Buckling load of solid column

I_1 = Least moment of inertia of the hollow column

$$= \frac{\pi}{64}\left[D^4 - \left(\frac{D}{2}\right)^2\right] = \frac{\pi}{64}\left[D^4 - \frac{D^4}{16}\right] = \frac{\pi}{64} \times \frac{15}{16}D^4$$

I_2 = Least moment of inertia of the solid column

$$= \frac{\pi}{64}d^4.$$

From equation (*ii*), we have

$$\frac{P_1}{I_1} = \frac{P_2}{I_2} \quad \text{or} \quad \frac{P_1}{P_2} = \frac{I_1}{I_2}$$

$$= \frac{\frac{\pi}{64} \times \frac{15}{16}D^4}{\frac{\pi}{64}d^4} \qquad \left(\because I_1 = \frac{\pi}{64} \times \frac{15}{16}D^4, I_2 = \frac{\pi}{64}d^4\right)$$

$$= \frac{15}{16}\frac{D^4}{d^4} = \frac{15}{16}\frac{D^4}{\left(\frac{\sqrt{3}D}{2}\right)^4} \qquad \left(\because d = \frac{\sqrt{3}D}{2}\right)$$

$$= \frac{15}{16} \times \frac{D^4}{\frac{9}{16}D^4} = \frac{15}{9} = \frac{5}{3} = 1.667$$

$$\therefore \quad \frac{\text{Buckling load of hollow column}}{\text{Buckling load of solid column}} = \mathbf{1.667.} \quad \textbf{Ans.}$$

Problem 19.12 (a). *Using Euler's formula, calculate the critical stresses for a series of struts having slenderness ratio of 40, 80, 120, 160 and 200 under the following conditions :*

(i) *both ends hinged and*

(ii) *both ends fixed.*

Take $E = 2.05 \times 10^5$ *N/mm*^{2}. (Annamalai University, 1990)

Sol. Given :

(i) *Critical stresses when both ends hinged*

Slenderness ratios, $\frac{l}{k} = 40, 80, 120, 160$ and 200

The critical stress or crippling stress is given by equation (19.7).

$$\therefore \quad \text{Critical stress} = \frac{\pi^2 \times E}{\left(\frac{L_e}{k}\right)^2}$$

where L_e = Effective length

But for both ends hinged, $L_e = l$

where l = actual length.

$$\therefore \quad \text{Critical stress} = \frac{\pi^2 E}{\left(\frac{l}{k}\right)^2} \qquad (\because \; L = l) \;...(ii)$$

When $\frac{l}{k} = 40$, the critical stress becomes as

$$= \frac{\pi^2 \times E}{40^2} = \frac{\pi^2 \times 2.05 \times 10^5}{1600} = \textbf{1264.54 N/mm}^2. \quad \textbf{Ans.}$$

When $\frac{l}{k} = 80$, the critical stress becomes as

$$= \frac{\pi^2 \times E}{80^2} = \frac{\pi^2 \times 2.05 \times 10^5}{6400} = \textbf{316.135 N/mm}^2. \quad \textbf{Ans.}$$

When $\frac{l}{k} = 120$, the critical stress becomes as

$$= \frac{\pi^2 \times E}{120^2} = \frac{\pi^2 \times 2.05 \times 10^5}{14400} = \textbf{140.5 N/mm}^2. \quad \textbf{Ans.}$$

When $\frac{l}{k} = 160$, the critical stress

$$= \frac{\pi^2 \times E}{160^2} = \frac{\pi^2 \times 2.05 \times 10^5}{25600} = \textbf{79.03 N/mm}^3. \quad \textbf{Ans.}$$

When $\frac{l}{k} = 200$, the critical stress

$$= \frac{\pi^2 \times E}{200^2} = \frac{\pi^2 \times 2.05 \times 10^5}{400{,}00} = \textbf{50.58 N/mm}^2. \quad \textbf{Ans.}$$

(ii) *Critical stresses when both ends fixed*

Using equation (19.7),

$$\text{Crippling or critical stress} = \frac{\pi^2 \times E}{\left(\frac{L_e}{k}\right)^2}$$

When both ends fixed, $L_e = \frac{l}{2}$

$$\therefore \quad \text{Critical stress} = \frac{\pi^2 \times E}{\left(\frac{l}{2k}\right)^2} = \frac{4\pi^2 \times E}{\left(\frac{l}{k}\right)^2}$$

When $\frac{l}{k} = 40$, the critical stress becomes as

$$= \frac{4\pi^2 \times E}{\left(\frac{l}{k}\right)^2} = \frac{4 \times \pi^2 \times 2.05 \times 10^5}{40^2}$$

$$= \frac{4 \times \pi^2 \times 2.05 \times 10^5}{1600} = \textbf{5058.16 N/mm}^2. \quad \textbf{Ans.}$$

When $\frac{l}{k} = 80$, the critical stress becomes as

$$= \frac{4\pi^2 \times E}{\left(\frac{l}{k}\right)^2} = \frac{4 \times \pi^2 \times 2.05 \times 10^5}{80^2} = \textbf{1264.54 N/mm}^2. \quad \textbf{Ans.}$$

When $\frac{l}{k} = 120$, the critical stress becomes as

$$= \frac{4\pi^2 \times E}{120^2} = \frac{4 \times \pi^2 \times 2.05 \times 10^5}{120^2} = \textbf{562.02 N/mm}^2. \textbf{ Ans.}$$

When $\frac{l}{k} = 160$, the critical stress becomes as

$$= \frac{4\pi^2 \times 2.05 \times 10^5}{160^2} = \textbf{316.135 N/mm}^2. \textbf{ Ans.}$$

When $\frac{l}{k} = 200$, the critical stress becomes as

$$= \frac{4\pi^2 \times 2.05 \times 10^5}{200^2} = \textbf{202.32 N/mm}^2. \textbf{ Ans.}$$

19.11. RANKINE'S FORMULA

In Art. 19.10, we have learnt that Euler's formula gives correct results only for very long columns. But what happens when the column is a short or the column is not a very long. On the basis of results of experiments performed by Rankine, he established an empirical formula which is applicable to all columns whether they are short or long. The empirical formula given by Rankine is known as Rankine's formula, which is given as

$$\frac{1}{P} = \frac{1}{P_C} + \frac{1}{P_E} \quad \text{...}(i)$$

where P = Crippling load by Rankine's formula

P_C = Crushing load = $\sigma_c \times A$

σ_c = Ultimate crushing stress

A = Area of cross-section

P_E = Crippling load by Euler's formula

$= \dfrac{\pi^2 EI}{L_e^2}$, in which L_e = Effective length

For a given column material the crushing stress σ_c is a constant. Hence the crushing load P_C (which is equal to $\sigma_c \times A$) will also be constant for a given cross-sectional area of the column. In equation (*i*), P_C is constant and hence value of P depends upon the value of P_E. But for a given column material and given cross-sectional area, the value of P_E depends upon the effective length of the column.

(*i*) *If the column is a short*, which means the value of L_e is small, then the value of P_E will be large. Hence the value of $\dfrac{1}{P_E}$ will be small enough and is negligible as compared to the value of $\dfrac{1}{P_C}$. Neglecting the value of $\dfrac{1}{P_E}$ in equation (*i*), we get

$$\frac{1}{P} \rightarrow \frac{1}{P_C} \quad \text{or} \quad P \rightarrow P_C.$$

Hence the crippling load by Rankine's formula for a short column is approximately equal to crushing load. In Art. 19.2.1 also we have seen that short columns fail due to crushing.

(*ii*) *If the column is long*, which means the value of L_e is large. Then the value of P_E will be small and the value of $\dfrac{1}{P_E}$ will be large enough compared with $\dfrac{1}{P_C}$. Hence the value of $\dfrac{1}{P_C}$ may be neglected in equation (*i*).

$$\therefore \quad \frac{1}{P} = \frac{1}{P_E} \quad \text{or} \quad P \rightarrow P_E.$$

Hence the crippling load by Rankine's formula for long columns is approximately equal to crippling load given by Euler's formula.

Hence the Rankine's formula $\dfrac{1}{P} = \dfrac{1}{P_C} + \dfrac{1}{P_E}$ gives satisfactory results for all lengths of columns, ranging from short to long columns.

Now the Rankine's formula is $\dfrac{1}{P} = \dfrac{1}{P_C} + \dfrac{1}{P_E} = \dfrac{P_E + P_C}{P_C \cdot P_E}$.

Taking reciprocal to both sides, we have

$$P = \frac{P_C \cdot P_E}{P_E + P_C} = \frac{P_C}{1 + \dfrac{P_C}{P_E}}$$

(Dividing the numerator and denominator by P_E)

$$= \frac{\sigma_c \times A}{1 + \dfrac{\sigma_c \cdot A}{\left(\dfrac{\pi^2 EI}{L_e^2}\right)}} \qquad \left(\because \; P_c = \sigma_c \cdot A \text{ and } P_E = \frac{\pi^2 EI}{L_e^2}\right)$$

But $I = Ak^2$, where k = least radius of gyration

$\therefore$ The above equation becomes as

$$P = \frac{\sigma_c \cdot A}{1 + \dfrac{\sigma_c \cdot A \cdot L_e^2}{\pi^2 E \cdot Ak^2}} = \frac{\sigma_c \cdot A}{1 + \dfrac{\sigma_c}{\pi^2 E} \cdot \left(\dfrac{L_e}{k}\right)^2}$$

$$= \frac{\sigma_c \cdot A}{1 + a \cdot \left(\dfrac{L_e}{k}\right)^2} \quad \text{...(19.9)}$$

where $a = \dfrac{\sigma_c}{\pi^2 E}$ and is known as Rankine's constant.

The equation (19.9) gives crippling load by Rankine's formula. As the Rankine formula is empirical formula, the value of 'a' is taken from the results of the experiments and is not calculated from the values of σ_c and E.

The values of σ_c and a for different columns material are given below in Table 19.2.

TABLE 19.2

S. No.	*Material*	σ_c *in N/mm²*	a
1.	Wrought Iron	250	$\frac{1}{9000}$
2.	Cast Iron	550	$\frac{1}{1600}$
3.	Mild Steel	320	$\frac{1}{7500}$
4.	Timber	50	$\frac{1}{750}$

Problem 19.13. *The external and internal diameter of a hollow cast iron column are 5 cm and 4 cm respectively. If the length of this column is 3 m and both of its ends are fixed, determine the crippling load using Rankine's formula. Take the values of* $\sigma_c = 550\ N/mm^2$ *and* $a = \dfrac{1}{1600}$ *in Rankine's formula.*

Sol. Given :

External dia., $D = 5$ cm

Internal dia., $d = 4$ cm

$\therefore$ Area, $A = \dfrac{\pi}{4}(5^2 - 4^2) = 2.25\pi \text{ cm}^2 = 2.25\pi \times 10^2 \text{ mm}^2 = 225\pi \text{ mm}^2$

Moment of Inertia, $I = \dfrac{\pi}{64}[5^4 - 4^4] = 5.7656\,\pi \text{ cm}^4$

$= 5.7656\pi \times 10^4 \text{ mm}^4 = 57656\pi \text{ mm}^4$

$\therefore$ Least radius of gyration,

$$k = \sqrt{\frac{I}{A}} = \sqrt{\frac{57656\pi}{225\pi}} = 25.625 \text{ mm}$$

Length of column, l = 3 m = 3000 mm

As both the ends are fixed,

$\therefore$ Effective length, $L_e = \frac{l}{2} = \frac{3000}{2} = 1500$ mm

Crushing stress, $\sigma_c = 550$ N/mm^2

Rankine's constant, $a = \frac{1}{1600}$

Let P = Crippling load by Rankine's formula

Using equation (19.9), we have

$$P = \frac{\sigma_c \cdot A}{1 + \left(\frac{L_e}{k}\right)^2} = \frac{550 \times 225\pi}{1 + \frac{1}{1600} \times \left(\frac{1500}{25.625}\right)^2}$$

$$= \frac{550 \times 225\pi}{3.1415} = \mathbf{123750\ N.}\quad \textbf{Ans.}$$

Problem 19.14. *A hollow cylindrical cast iron column is 4 m long with both ends fixed. Determine the minimum diameter of the column if it has to carry a safe load of 250 kN with a factor of safety of 5. Take the internal diameter as 0.8 times the external diameter. Take $\sigma_c = 550\ N/mm^2$ and $a = \frac{1}{1600}$ in Rankine's formula.* (AMIE, Winter 1983)

Sol. Given :

Length of column, l = 4 m = 4000 mm

End conditions = Both ends fixed

$\therefore$ Effective length, $L_e = \frac{l}{2} = \frac{4000}{2} = 2000$ mm

Safe load, = 250 kN

Factor of safety, = 5

Let External dia., = D

Internal dia. = $0.8 \times D$

Crushing stress, $\sigma_c = 550$ N/mm^2

Value of 'a' $= \frac{1}{1600}$ in Rankine's formula

Now factor of safety $= \frac{\text{Cirippling load}}{\text{Safe load}}$ or $5 = \frac{\text{Cirippling load}}{250}$

$\therefore$ Crippling load, $P = 5 \times 250 = 1250$ kN = 1250000 N

Area of column, $A = \frac{\pi}{4}[D^2 - (0.8D)^2]$

$$= \frac{\pi}{4}[D^2 - 0.64D^2] = \frac{\pi}{4} \times 0.36D^2 = \pi \times 0.09D^2$$

Moment of Inertia, $I = \frac{\pi}{64}[D^4 - (0.8D^4] = \frac{\pi}{64}[D^4 - 0.4096D^4]$

$$= \frac{\pi}{64} \times 0.5904 \times D^4 = 0.009225 \times \pi \times D^4$$

But $I = A \times k^2$, where k is radius of gyration

$$\therefore \quad k = \sqrt{\frac{I}{A}} = \sqrt{\frac{0.009225 \times \pi D^4}{\pi \times 0.09 \times D^2}} = 0.32D$$

Now using equation (19.9), $P = \dfrac{\sigma_c \cdot A}{1 + a\left(\frac{L_e}{k}\right)^2}$

or $$1250000 = \frac{550 \times \pi \times 0.09\ D^2}{1 + \frac{1}{1600} \times \left(\frac{2000}{0.32D}\right)^2} \quad (\because\ A = \pi \times 0.09D^2)$$

$$\frac{1250000}{550 \times \pi \times 0.09} = \frac{D^2}{1 + \frac{24414}{D^2}} \text{ or } 8038 = \frac{D^2 \times D^2}{D^2 + 24414}$$

or $8038D^2 + 8038 \times 24414 = D^4$ or $D^4 - 8038D^2 - 8038 \times 24414 = 0$

or $D^4 - 8038\ D^2 - 196239700 = 0.$

The above equations is a quadratic equation in D^2. The solution is

$$\therefore \quad D^2 = \frac{8038 \pm \sqrt{8038^2 + 4 \times 1 \times 196239700}}{2}$$

$$\left(\text{Roots} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\right)$$

$$= \frac{8038 \pm \sqrt{646094 + 784958800}}{2} = \frac{8038 \pm 29147}{2}$$

$$= \frac{8038 + 29147}{2} \text{ (The other root is not possible)}$$

$$= 18592.5 \text{ mm}^2$$

$$\therefore \quad D = \sqrt{18592.5} = 136.3 \text{ mm}$$

$\therefore$ External diameter = **136.3 mm. Ans.**

Internal diameter $= 0.8 \times 1363 =$ **109 mm. Ans.**

Problem 19.15. *A 1.5 m long column has a circular cross-section of 5 cm diameter. One of the ends of the column is fixed in direction and position and other end is free. Taking factor of safety as 3, calculate the safe load using :*

(a) Rankine's formula, take yield stress, $\sigma_c = 560\ N/mm^2$ and $a = \frac{1}{1600}$ for pinned ends.

(b) Euler's formula, Young's modulus for C.I. = $1.2 \times 10^5\ N/mm^2$.

(AMIE, Summer 1976)

Sol. Given :

Length, l = 1.5 m = 1500 mm

Diameter, d = 5 cm

$\therefore$ Area, $A = \frac{\pi}{4} \times 5^2 = 19.635 \text{ cm}^2 = 19.635 \times 10^2 \text{ mm}^2$

Moment of inertia, $I = \frac{\pi}{64} \times 5^4 = 30.7 \text{ cm}^4 = 30.7 \times 10^4 \text{ mm}^4$

and least radius of gyration, $k = \sqrt{\frac{I}{A}} = \sqrt{\frac{30.7 \times 10^4}{19.635 \times 10^2}} = 12.5$ mm.

End conditions = One end is fixed and other end is free.

$\therefore$ Effective length, $L_e = 2l = 2 \times 1500 = 3000$ mm

Factor of safety = 3.

(a) Safe load by Rankine's formula

Yield stress, $\sigma_c = 560 \text{ N/mm}^2$

Rankine's constant, $a = \frac{1}{1600}$

Let P = Crippling load by Rankine's formula

Using equation (19.9), we have

$$P = \frac{\sigma_c \times A}{1 + a\left(\frac{L_e}{k}\right)^2}$$

$$= \frac{560 \times 1963.5}{1 + \frac{1}{1600} \times \left(\frac{3000}{12.5}\right)^2} \qquad (\because \; L = 3000 \text{ mm and } k = 12.5)$$

$$= 29708.1 \text{ N}$$

$\therefore$ Safe load $= \frac{\text{Crippling load}}{\text{Factor of safety}}$

$$= \frac{29708.1}{3} = \mathbf{9902.7\ N.} \quad \textbf{Ans.}$$

(b) Safe load by Euler's formula

Young's Modulus, $E = 1.2 \times 10^5 \text{ N/mm}^2$

Let P = Crippling load by Euler's formula

Using equation (19.5), $P = \frac{\pi^2 EI}{L_e^2}$

$$= \frac{\pi^2 \times 1.2\ 10^5 \times 30.7 \times 10^4}{3000^2} \qquad (\because \; L = 3000 \text{ mm})$$

$$= 40200 \text{ N}$$

$\therefore$ Safe load $= \frac{\text{Crippling load}}{\text{Factor of safety}} = \frac{420200}{3} = \mathbf{1340\ N.}$ **Ans.**

Problem 19.16. *A short length of tube, 4 cm internal diameter and 5 cm external diameter, failed in compression at a load of 240 kN. When a 2 metre length of the same tube was tested as a strut with fixed ends, the load at failure was 158 kN. Assuming that σ_c in Rankine's formula is given by the first test, find the value of the constant a in the same formula. What will be the crippling load of this tube if it is used as a strut 3 m long with one end fixed and the other hinged ?* (AMIE, Summer 1983)

Sol. Given :

External diameter, D = 5 cm

Internal diameter, d = 4 cm

$\therefore$ Area, $A = \frac{\pi}{4}(5^2 - 4^2) = \frac{9\pi}{4} = 2.25\,\pi \text{ cm}^2 = 225\,\pi \text{ mm}^2$

Moment of inertia, $I = \frac{\pi}{64}[5^4 - 4^4] = \frac{\pi}{64}(625 - 256)$

$$= 5.7656 \times \pi \text{ cm}^4 = 57656\,\pi \text{ mm}^4$$

$\therefore$ Least radius of gyration, $k = \sqrt{\frac{I}{A}} = \sqrt{\frac{57656\,\pi}{225\,\pi}} = 16$ mm

Crushing load = 240 kN.

The value of σ_c in Rankine's formula is given by the crushing load of 240 kN.

$\therefore$ The value of $\sigma_c = \frac{\text{Crushing load of 240 kN}}{\text{Area}}$

$$= \frac{240}{225\,\pi} = 0.3395 \text{ kN/mm}^2$$

Length of the strut, l = 2 m = 2000 mm

End condition = Both the ends are fixed

$\therefore$ Effective length, $L_e = \frac{l}{2} = \frac{2000}{2} = 1000$ mm

Crushing load of strut, P = 158 kN.

(i) Value of constant 'a'

Let a = Value of Rankine's constant

Using equation (19.9), we have

$$P = \frac{\sigma_c \, . \, A}{1 + a \, . \left(\frac{L_e}{k}\right)^2}$$

or

$$158 = \frac{0.33953 \times 225\,\pi}{1 + a \, . \left(\frac{1000}{16}\right)^2} = \frac{239.99}{1 + 3906.25 \times a}$$

$$1 + 3906.25 \times a = \frac{239.99}{158} = 1.5189.$$

$$\therefore \quad a = \frac{1.5189 - 1.0}{3906.25} = 0.0001328 = \frac{\mathbf{1}}{\mathbf{7530}}. \quad \textbf{Ans.}$$

(ii) Crippling load for the strut of length 3 m when one end is fixed and other is hinged

Actual length, l = 3 m = 3000 mm

End conditions = One end fixed and other is hinged

$\therefore$ Effective length, $L_e = \frac{l}{\sqrt{2}} = \frac{3000}{\sqrt{2}}$

Let P = Crippling load.

Using equation (19.9),

$$P = \frac{\sigma_c \cdot A}{1 + a\cdot\left(\frac{L_e}{k}\right)^2}$$

$$= \frac{0.33953 \times 225\pi}{1 + \frac{1}{7530} \times \left(\frac{3000}{\sqrt{2} \times 16}\right)^2} \qquad \left(\because\ k = 16, a = \frac{1}{7530}\right)$$

$$= \frac{0.33953 \times 225 \times \pi}{1 + 2.3344} = \mathbf{71.97\ kN.}\quad \textbf{Ans.}$$

Problem 19.17. *Find the Euler crushing load for a hollow cylindrical cast iron column 20 cm external diameter and 25 mm thick if it is 6 m long and is hinged at both ends. Take $E = 1.2 \times 10^6\ N/mm^2$.*

Compare the load with the crushing load as given by the Rankine's formula, taking $\sigma_c = 550\ N/mm^2$ and $a = \frac{1}{1600}$; for what length of the column would these two formulae give the same crushing load ? (AMIE, Winter 1984)

Sol. Given :

External dia., $D = 20$ cm

Thickness, $t = 25$ mm $= 2.5$ cm

$\therefore$ Internal dia., $d = (D - 2 \times t) = 20 - 2 \times 2.5 = 15$ cm.

Area, $A = \frac{\pi}{4}(20^2 - 15^2) = \frac{175\pi}{4} = 137.44\ cm^2 = 13744\ mm^2$

Moment of inertia, $I = \frac{\pi}{64}[20^4 - 15^4] = \frac{\pi}{64}(160000 - 50625)$

$= 5368.93\ cm^4 = 53689300\ mm^4$

$\therefore$ Least radius of gyration,

$$k = \sqrt{\frac{I}{A}} = \sqrt{\frac{53689300}{15744}} = 62.5\ mm$$

Length of column, $l = 6$ m $= 6000$ mm

End conditions = Both ends are hinged

$\therefore$ Effective length, $L_e = l = 6000$ mm

Value of $E = 1.2 \times 10^5\ N/mm^2$.

Euler's crushing load is given by equation (19.5),

$$P = \frac{\pi^2 EI}{L_e^2}$$

$$= \frac{\pi^2 \times 1.2 \times 10^5 \times 53689300}{6000^2} = \mathbf{1766307\ N.}\quad \textbf{Ans.}$$

Crushing load by Rankine's formula

The value of $\sigma_c = 550\ N/mm^2$

Value of $a = \frac{1}{1600}$

Let P = Crushing load by Rankine's formula

Using equation (19.9),

$$P = \frac{\sigma_c \cdot A}{1 + a\cdot\left(\frac{L_e}{k}\right)^2} = \frac{550 \times 13744}{1 + \frac{1}{1600} \times \left(\frac{6000}{62.5}\right)^2} = \mathbf{1118224.8\ N.\ Ans.}$$

The length of the column for which the above two formulae gives the same crushing load

Let L = Length of the column

Crushing load by Euler's formula $= \frac{\pi^2 EI}{L^2}$

Crushing load by Rankine's formula $= \frac{\sigma_c \cdot A}{1 + a\left(\frac{L}{k}\right)^2}$

Equating the crushing loads given by the above two formulae, we get

$$\frac{\pi^2 EI}{L^2} = \frac{\sigma_c \cdot a}{1 + a\left(\frac{L}{k}\right)^2}$$

Substituting the all known values except 'L', we get

$$\frac{\pi^2 \times 1.2 \times 10^5 \times 53689300}{L^2} = \frac{550 \times 13744}{1 + \frac{1}{1600} \times \left(\frac{L}{62.5}\right)^2}$$

or $$\frac{\pi^2 \times 1.2 \times 10^5 \times 53689300}{550 \times 13744} = \frac{L^2}{1 + \frac{L^2}{6250000}}$$

or $$8411800 = \frac{L^2}{1 + \frac{L^2}{6250000}}$$

or $$8411800\left(1 + \frac{L^2}{6250000}\right) = L^2$$

or $$8411800 + \frac{8411800}{6250000} = L^2 = L^2$$

or $$8411800 + 1.346\,L^2 = L^2 \text{ or } 1.346\,L^2 - L^2 = -8411800$$

or $$0.346\,L^2 = -8411800 \quad\text{or}\quad L = \sqrt{\frac{-8411800}{0.346}}$$

The above equation gives the imaginary value of length L.

Hence it is not possible to have the same length of the columns, which have the same crushing load for the two given formulae. **Ans.**

Problem 19.18. *A hollow cast iron column 200 mm outside diameter and 150 mm inside diameter, 8 m long has both ends fixed. It is subjected to an axial compressive load. Taking a factor of safety as 6, $\sigma_c = 560$ N/mm², $a = \frac{1}{1600}$, determine the safe Rankine load.*

(AMIE, Summer 1990)

Sol. Given :

External dia., $D = 200$ mm

Internal dia., $d = 150$ mm

Length, $l = 8$ m $= 8000$ mm

End conditions = Both the ends are fixed

Crushing stress, $\sigma_c = 560$ N/mm²

Rankine's constant, $a = \frac{1}{1600}$

Safety factor = 6

Area of cross-section, $A = \frac{\pi}{4}(D^2 - d^2) = \frac{\pi}{4}(200^2 - 150^2)$

$$= \frac{\pi}{4}(40000 - 22500) = 13744 \text{ mm}^2$$

Moment of inertia, $I = \frac{\pi}{64}(D^4 - d^4) = \frac{\pi}{64}(200^4 - 150^4)$

$$= \frac{\pi}{64}(1600000000 - 506250000) = 53689000 \text{ mm}^4$$

Least radius of gyration, $k = \sqrt{\frac{I}{A}} = \sqrt{\frac{53689000}{13744}} = 62.5$ mm

Let P = Crippling load by Rankine formula.

Using equation (19.9), $P = \dfrac{\sigma_c \times A}{1 + a\left(\dfrac{L_e}{k}\right)^2}$

where L_e = Effective length $= \frac{l}{2} = \frac{8000}{2} = 4000$ mm

$\therefore$

$$P = \frac{560 \times 13744}{1 + \frac{1}{1600} \times \left(\frac{4000}{62.5}\right)^2}$$

$$= \frac{7696640}{1 + 2.56} = \frac{7696640}{3.56} = 2161977 \text{ N} = 2161.977 \text{ kN}$$

$\therefore$ Safe load $= \dfrac{\text{Crippling load}}{\text{Factor of safety}} = \dfrac{2161.977}{6} =$ **360.3295 kN. Ans.**

Problem 19.19. *A hollow C.I. column whose outside diameter is 200 mm has a thickness of 20 mm. It is 4.5 m long and is fixed at both ends. Calculate the safe load by Rankine's formula using a factor of safety of 4. Calculate the slenderness ratio and the ratio of Euler's and Rankine's critical loads. Take $\sigma_c = 550$ N/mm², $a = \frac{1}{1600}$ in Rankine's formula and $E = 9.4 \times 10^4$ N/mm².*

(AMIE, Winter 1979 ; Annamalai University, 1991)

Sol. Given :

Outside diameter, $D = 200$ mm

Thickness, $t = 20$ mm

$\therefore$ Inside diameter, $d = D - 2 \times t = 200 - 2 \times 20 = 160$ mm

Area, $A = \frac{\pi}{4}(D^2 - d^2) = \frac{\pi}{4}(200^2 - 160^2) = 11310$ mm²

Moment of inertia, $I = \frac{\pi}{64}(D^4 - d^4) = \frac{\pi}{64}(200^4 - 160^4) = 46370000$ mm⁴

And the least radius of gyration,

$$k = \sqrt{\frac{I}{A}} = \sqrt{\frac{46370000}{11310}} = 64 \text{ mm}$$

Length of column, $l = 4.5$ m $= 4500$ mm

End condition = Both the ends are fixed

$\therefore$ Effective length, $L_e = \frac{l}{2} = \frac{4500}{2} = 2250$ mm

Factor of safety = 4

Value of $\sigma_c = 550$ N/mm²

Value of $a = \frac{1}{1600}$

Value of $E = 9.4 \times 10^4$ N/mm².

(i) Slenderness ratio

Using equation (19.8), we get

Slenderness ratio $= \frac{l}{k} = \frac{4500}{64} =$ **70.30. Ans.**

(ii) Safe load by Rankine's formula

Let P = Crippling load by Rankine's formula

Using equation (19.9), $P = \dfrac{\sigma_c \times A}{1 + a\left(\dfrac{L_e}{k}\right)^2} = \dfrac{550 \times 11310}{1 + \dfrac{1}{1600}\left(\dfrac{2250}{64}\right)^2} = 351100$ N

$\therefore$ Safe load $= \dfrac{\text{Crippling load}}{\text{Factor of safety}} = \dfrac{3511000}{4} =$ **8777 N. Ans.**

(iii) Ratio of Euler's and Rankine's critical loads

Let P_E = Euler's critical load

Euler's critical load is given by equation (19.5)

$$P_E = \frac{\pi^2 EI}{L_e^2} = \frac{\pi^2 \times 9.4 \times 10^4 \times 46370000}{2250^2} = 849770 \text{ N}$$

$$\therefore \frac{\text{Euler's critical load}}{\text{Rankine's critical load}} = \frac{P_E}{P}$$

$$= \frac{8497700}{3511000} \qquad (\because P = 3511000 \text{ N})$$

$=$ **2.42. Ans.**

Problem 19.20. *A column is made up of two channel ISJC 200 and two 25 cm × 1 cm flange plate as shown in Fig. 19.12.*

Determine by Rankine's formula the safe load, the column of 6 m length, with both ends fixed, can carry with a factor of safety 4. The properties of one channel are $A = 17.77\ cm^2$, $I_{XX} = 1{,}161.2\ cm^4$ and $I_{YY} = 84.2\ cm^4$. Distance of centroid from back of web = 1.97 cm. Take $\sigma_c = 0.32\ kN/mm^2$ and Rankine's constant = 1/7,500.

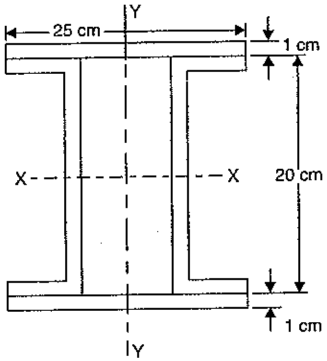


Fig. 19.12

Sol. Given :

Length of the column, $l = 6\ \text{m} = 600\ \text{mm}$

Factor of safety = 4

Yield stress, $\sigma_c = 0.32\ \text{kN/mm}^2$

Rankine's constant, $a = \dfrac{1}{7{,}500}$

Let P = Crippling load on the column.

From the geometry of figure, we find that area of column,

$$A = 2(17.77 + 25 \times 1)$$
$$= 85.54\ \text{cm}^2 = 8554\ \text{mm}^2$$

and moment of inertia of the column section about X-X axis,

$$I_{XX} = 2 \times 1{,}161.2 + 2\left(\frac{25 \times 1^3}{12} + 25 \times 1 \times 10.5^2\right)\ \text{cm}^4 = 7{,}839.0\ \text{cm}^4$$

Similarly, $$I_{YY} = 2\left\{\frac{1 \times 25^3}{12} + 8.42 + 17.77 \times (5 + 1.97)^2\right\}\ \text{cm}^4 = 4{,}499.0\ \text{cm}^4.$$

Since I_{YY} is less than I_{XX}, therefore the column will tend to buckle in y-y direction. Thus we shall make the value of I as $I_{YY} = 4{,}499\ \text{cm}^4$. Moreover as the column is fixed at its both ends, therefore equivalent length of the column,

$$L_e = \frac{l}{2} = \frac{6000}{2} = 3000\ \text{mm}$$

and $$I = 4499\ \text{cm}^4 = 4499 \times 10^4\ \text{mm}^4$$

We know that the least radius of gyration,

$$k = \sqrt{\frac{I}{A}} = \sqrt{\frac{4499 \times 10^4}{8554}} = 72.5\ \text{mm}$$

Using the relation, $$P = \frac{\sigma_c \,.\, A}{1 + a\left(\dfrac{L_e}{k}\right)^2} \text{ with usual notations}$$

$$= \frac{0.32 \times 8554}{1 + \dfrac{1}{7{,}500}\left(\dfrac{3000}{72.5}\right)^2} = 2228\ \text{kN}$$

We know that the safe load on the column

$$= \frac{P}{\text{Factor of safety}} = \frac{2228}{4} = \mathbf{557\ kN.}$$ **Ans.**

19.12. STRAIGHT LINE FORMULA

The Euler's formula and Rankine's formula give only the approximate values of crippling load due to the following reasons :

1. The pin joints are not practically frictionless.
2. The end fixation is never perfectly rigid.
3. In case of Euler's formula, the effect of direct compression has been neglected.
4. The load is not exactly applied as desired.
5. The members are never perfectly straight and uniform in section.
6. The material of the members is not homogeneous and isotropic.

On account of this, the empirical straight line formula are commonly used in practical designing. They are of the form

$$P = \sigma_c \times A - n\left(\frac{L_e}{k}\right) \times A \qquad \text{...(19.10)}$$

where P = Crippling load on the column,

σ_c = Compressive yield stress,

A = Area of cross-section of the column, $\dfrac{L_e}{k}$ = Slenderness ratio, and

n = A constant whose value depends upon the material of the column.

In equation (19.10), if P is plotted against $\left(\dfrac{L_e}{k}\right)$, we will get a straight line and hence the equation (19.10) represents a straight line formula. The equation (19.10) can also be written as

$$\frac{P}{A} = \sigma_c - n\,.\left(\frac{L_e}{k}\right)$$

where $\dfrac{P}{A}$ represents the stress corresponding to load A.

19.13. JOHNSON'S PARABOLIC FORMULA

The critical load according to Prof. Johnson, is given by

$$P = \sigma_c \,.\, A - r\left(\frac{L_e}{k}\right) \times A \qquad \text{...(19.11)}$$

where σ_c = Compressive yield stress,

r = A constant whose value depends upon the material of the column and is taken as

$= \dfrac{\sigma_c}{4\pi^2 E^2}$, where E = Young's modulus

$\dfrac{L_e}{k}$ = Slenderness ratio

A = Cross-sectional area of column.

In equation (19.11), if P is plotted against $\left(\dfrac{L_e}{k}\right)$, we will get a parabolic curve and hence the formula represented by equation (19.11) is known as a parabolic formula.

19.14. FACTOR OF SAFETY

The ratio of the critical load to the safe load is known as factor of safety. The values of the safety factors usually used for different materials are given below :

Column material	*Factor of safety*
Mild steel	3
Wrought iron	3
Cast iron	5
Timber	6

19.15. FORMULA BY INDIAN STANDARD CODE (I.S. CODE) FOR MILD STEEL

The direct stress in compression on the gross area of the section of an axially loaded compression member shall notexceed the values of σ_c, calculated as follows and given in Table 19.3.

$$\sigma_c = \sigma_c' = \frac{\dfrac{\sigma_y}{m}}{1 + 0.20 \sec\left[\dfrac{L_e}{k}\sqrt{\dfrac{mp_c'}{4E}}\right]} \qquad \left(\text{for } \frac{L_e}{k} = 0 \text{ to } 160\right)$$

and
$$\sigma_c = \sigma_c'\left(1.2 - \frac{L_e}{800\,k}\right) \qquad \left(\text{for } \frac{L_e}{k} = 160 \text{ and above}\right)$$

where σ_c = Allowable axial compression stress obtained from Table 19.3

σ_c' = A value obtained from the above secant formula

σ_y = The guaranted minimum yield stress taken as 260 N/mm^2 for mild steel

m = Factor of safety taken as 1.68

$\dfrac{L_e}{k}$ = Slenderness ratio

E = Modulus of elasticity = 2.047×10^5 N/mm^2.

Safe stresses (σ_c) in axial compression according to I.S. code for mild steel column is given in Table 19.3 for various values of slenderness ratio.

TABLE 19.3

Slenderness, ratio = $\frac{L_e}{k}$	*Safe stress,* σ_c *(N/mm²)*	*Slenderness ratio* = $\frac{L_e}{k}$	*Safe stress,* σ_c *(N/mm²)*
0	1.25	150	47.4
10	124.6	160	42.3
20	123.9	170	37.3
30	122.4	180	33.6
40	120.3	190	30.0
50	117.2	200	27.0
60	113.0	210	24.3
70	107.5	220	21.9
80	100.7	230	19.9
90	92.8	240	18.1
100	84.0	250	16.6
110	75.3	300	19.9
120	67.1	350	7.6
130	59.7		
140	53.1		

Problem 19.21. *Determine the safe load by I.S. code for a hollow cylindrical mild steel tube of 4.0 cm external diameter and 3 cm internal diameter when the tube is used as a column of length 2.5 m long with both ends hinged.*

Sol. Given :

External diameter, D = 4 cm

Internal diameter, d = 3 cm

$\therefore$ Area, $\quad A = \dfrac{\pi}{4}[4^2 - 3^2] = 1.75\pi \text{ cm}^2 = 175\pi \text{ mm}^2$

Moment of inertia, $\quad I = \dfrac{\pi}{64}[4^4 - 3^4] = 2.7343\pi \text{ cm}^4$

$= 27343\pi \text{ mm}^4$

Length of column, l = 2.5 m = 2500 mm

End conditions = Both ends hinged

$\therefore$ Effective length, $L_e = l$ = 2500 mm

To determine the safe load by I.S. code, first find the value of slenderness ratio. Then corresponding to the slenderness ratio, obtain the safe compressive stress (*i.e.*, value of σ_c) from the Table 19.3. Safe load will be equal to the product of σ_c and area A.

Now slenderness ratio $= \dfrac{L_e}{k}$

where k = Least radius of gyration $= \sqrt{\dfrac{I}{k}} = \sqrt{\dfrac{27343}{175\pi}} = 15.62$

$\therefore$ Slenderness ratio $= \dfrac{2500}{15.52} = 160.$

From Table 19.3 corresponding to slenderness ratio of 160, the allowable compressive stress is 42.3 N/mm^2.

$\therefore \quad \sigma_c = 42.3$ N/mm^2

$\therefore$ Safe load for the column $= \sigma_c \times A$

$= 42.3 \times 175\pi$ N = **23255.6 N. Ans.**

19.16. COLUMNS WITH ECCENTRIC LOAD

Fig. 19.13 (*a*) shows a column AB of length 'l' fixed at end A and free at end B. The column is subjected to a load P which is eccentric by an amount of 'e'. The free end will sway sideways by an amount 'a' and the column will deflect as shown in Fig. 19.13 (*b*).

Here a = deflection at free end B

e = Eccentricity

A = Area of cross-section of column

Consider any section at a distance x from the fixed end A.

Let y = deflection at the section then moment at the section = $P\,(a + e - y)$

(+ ve sign is taken due to sign convention given in Art. 19.4.1.)

But moment is also = $EI\dfrac{d^2y}{dx^2}$

$$\therefore \quad EI\frac{d^2y}{dx^2} = P\,(a + e - y) = P(a + e) - P \times y$$

or
$$EI\frac{d^2y}{dx^2} + P \times y = P\,(a + e)$$

or
$$\frac{d^2y}{dx^2} + \frac{P}{EI} \times y = \frac{P}{EI}(a + e)$$

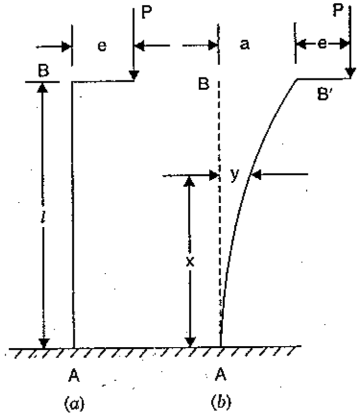


Fig. 19.13

The above equation can be written as

$$\frac{d^2y}{dx^2} + \alpha^2 y = \alpha^2\,(a + e)$$

where
$$\alpha^2 = \frac{P}{EI} \quad \text{or} \quad \alpha = \sqrt{\frac{P}{EI}}$$

The complete solution of the above equation is,

$$y = C_1 \cos(\alpha\,.\,x) + C_2 \sin(\alpha\,.\,x) + (a + e)$$

$$= C_1 \cos\left(\sqrt{\frac{P}{EI}} \times x\right) + C_2 \sin\left(\sqrt{\frac{P}{EI}} \times x\right) + (a + e)$$

$$= C_1 \cos\left(x \times \sqrt{\frac{P}{EI}}\right) + C_2 \sin\left(x \times \sqrt{\frac{P}{EI}}\right) + (a + e) \qquad ...(i)$$

and slope,
$$\frac{dy}{dx} = -C_1\left[\sin\left(x \times \sqrt{\frac{P}{EI}}\right)\right] \times \sqrt{\frac{P}{EI}} + C_2\left[\cos\left(x \times \sqrt{\frac{P}{EI}}\right)\right] \times \sqrt{\frac{P}{EI}} + 0 \qquad ...(ii)$$

($\because$ $a + e$ is constant, Hence differential is zero)

In equations (*i*) and (*ii*) C_1 and C_2 are constant of integration. Their values are obtained from boundary conditions.

(*i*) At A, $x = 0$, $y = 0$ and also $\dfrac{dy}{dx} = 0$ ($\because$ A is a fixed end)

From equation (*i*) where $x = 0$ and $y = 0$, we get

$$0 = C_1 + a + e \quad \therefore \quad C_1 = -(a + e)$$

From equation (*ii*) where $x = 0$ and $\dfrac{dy}{dx} = 0$, we get

$$0 = C_2 \times \sqrt{\frac{P}{EI}} \quad \therefore \quad C_2 = 0 \qquad \left(\because \frac{P}{EI} \text{ can not be zero}\right)$$

Substituting the values of C_1 and C_2 in equation (*i*), we get

$$y = -(a + e)\cos\left(x \times \sqrt{\frac{P}{EI}}\right) + (a + e) \qquad ...(iii)$$

(*ii*) at $x = l$, $y = a$, hence equation (*iii*) becomes as

$$a = -(a + e)\cos\left(l \times \sqrt{\frac{P}{EI}}\right) + (a + e)$$

or
$$(a + e)\cos\left(l \times \sqrt{\frac{P}{EI}}\right) = a + e - a = e$$

$$\therefore \quad a + e = \frac{e}{\cos\left(l \times \sqrt{\dfrac{P}{EI}}\right)} = e\sec\left(l \times \sqrt{\frac{P}{EI}}\right) \qquad ...(iv)$$

Maximum stress

Let us find the maximum compressive stress for the column section. Due to eccentricity, there will be bending stress and also direct stress.

$\therefore$ $\sigma_{max} = \sigma_0 + \sigma_b$ where σ_0 = direct stress = $\dfrac{P}{A}$

σ_b = Max. bending stress.

The maximum bending stress will be at the section where bending moment is maximum. Bending moment is maximum at the fixed end A.

$\therefore$ Max B.M. at A, $M = P \times (a + e)$

$$= P \times e\sec\left(l \times \sqrt{\frac{P}{EI}}\right) \qquad \left[\because a + e = e\sec\left(l \times \sqrt{\frac{P}{EI}}\right) \text{ from equation } (iv)\right]$$

Using
$$\frac{M}{I} = \frac{\sigma_b}{y} \quad \text{or} \quad \sigma_b = \frac{M}{I} \times y = \frac{M}{\left(\dfrac{I}{y}\right)}$$

$$= \frac{M}{Z} \quad \text{where } Z = \frac{I}{y} = \text{Section modulus}$$

$$= \frac{P \times e\sec\left(l \times \sqrt{\dfrac{P}{EI}}\right)}{Z} \qquad \left[\because M = P \times e\sec\left(l \times \sqrt{\frac{P}{EI}}\right)\right]$$

Hence maximum compressive stress becomes as

$$\therefore \quad \sigma_{max} = \sigma_0 + \sigma_b = \frac{P}{A} + \frac{P \times e\sec\left(l \times \sqrt{\dfrac{P}{EI}}\right)}{Z} \qquad ...(19.12)$$

The equation (19.12) is used for a column whose one end is fixed, other end is free and load is eccentric to the column. In this equation, l is the actual length of the column. The relation between actual length and effective length for a column whose one end is fixed and other end is free is given by

$$L_e = 2l \quad \text{or} \quad l = \frac{L_e}{2}$$

Substituting the value of l is equation (19.12), we get a general formula which can be used for any end condition. Hence general formula is

$$\sigma_{max} = \frac{P}{A} + \frac{P \times e \times \sec\left(\dfrac{L_e}{2} \times \sqrt{\dfrac{P}{EI}}\right)}{Z} \qquad ...(19.13)$$

Problem 19.22. *A column of circular section is subjected to a load of 120 kN. The load is parallel to the axis but eccentric by an amount of 2.5 mm. The external and internal diameters of columns are 60 mm and 50 mm respectively. If both the ends of the column are hinged and column is 2.1 m long, then determine the maximum stress in the column. Take E = 200 GN/m².*

Sol. Given :

Load, $P = 120$ kN $= 120 \times 10^3$ N

Eccentricity, $e = 2.5$ mm $= 2.5 \times 10^{-3}$ m

$D = 60$ mm $= 0.06$ m, $d = 50$ mm $= 0.05$ mm, $l = 2.1$ m

Both ends are hinged, $L_e = l = 2.1$ m

Value of $E = 200$ GN/m² $= 200 \times 10^9$ N/m²

The maximum stress is given by equation (19.13) as

$$\sigma_{max} = \frac{P}{A} + \frac{P \times e \times \sec\left(\frac{L_e}{2} \times \sqrt{\frac{P}{EI}}\right)}{Z} \quad \ldots(i)$$

where A = Area of section

$$= \frac{\pi}{4}[D^2 - d^2] = \frac{\pi}{4}[0.06^2 - 0.05^2]$$

$$= \frac{\pi}{4} \times 0.0011 = 8.639 \times 10^{-4} \text{ m}^2$$

$$I = \text{Moment of inertia} = \frac{\pi}{64}(D^4 - d^4)$$

$$= \frac{\pi}{64}(0.06^4 - 0.05^4) \text{ mm}^4$$

$$= \frac{\pi}{64}(1.296 \times 10^{-5} - 0.625 \times 10^{-5}) = 0.0329 \times 10^{-5}$$

Z = Section modulus

$$= \frac{I}{y} = \frac{\frac{\pi}{64}[D^4 - d^4]}{\left(\frac{D}{2}\right)} = \frac{\frac{\pi}{64}[0.06^4 - 0.05^4]}{0.03}$$

$$= \frac{\pi(1.296 \times 10^{-5} - 0.625 \times 10^{-5})}{64 \times 0.03}$$

$$= 1.0975 \times 10^{-5} \text{ m}^3$$

$$\sec\left(\frac{L_e}{2} \times \sqrt{\frac{P}{EI}}\right) = \sec\left(\frac{2.1}{2} \times \sqrt{\frac{120 \times 10^3}{200 \times 10^9 \times 0.329 \times 10^{-5}}}\right)$$

$= \sec(1.4179$ radians) (Here 1.4179 is in radians $= 1.4179 \times \frac{180}{\pi} = 81.239$)

$= \sec(81.239) = 6.566$

Substituting these values in equation (*i*) above, we get

$$\sigma_{max} = \frac{120 \times 10^3}{8.639 \times 10^{-4}} + \frac{(120 \times 10^3) \times (2.5 \times 10^{-3}) \times 6.566}{1.0975 \times 10^{-5}}$$

$$= 138.9 \times 10^6 + 179.48 \times 10^6 \text{ N/m}^2$$

$$= \mathbf{318.38 \times 10^6 \text{ N/m}^2 \text{ or } 318.38 \text{ N/mm}^2} \quad \textbf{Ans.}$$

Problem 19.23. *If the given column of problem 19.22 is subjected to an eccentric load of 100 kN and maximum permissible stress is limited to 320 MN/m², then determine the maximum eccentricity of the load.*

Sol. Given :

Data from problem 19.23

$D = 60$ mm $= 0.06$ m, $d = 50$ mm $= 0.05$ m, $l = 2.1$ m, $L_e = l = 2.1$ m,

$E = 200$ GN/m² $= 200 \times 10^9$ N/m²., $I = 0.0329 \times 10^{-5}$ m⁴,

$Z = 1.0975 \times 10^{-5}$ m³, $A = 8.639 \times 10^{-4}$ m²

Eccentric load, $P = 100$ kN $= 100 \times 10^3$ N

Max. stress, $\sigma_{max} = 320$ MN/m² $= 320 \times 10^6$ N/m²

Let e = Maximum eccentricity

Using equation (19.13), we get

$$\sigma_{max} = \frac{P}{A} + \frac{P \times e \times \sec\left(\frac{L_e}{2} \times \sqrt{\frac{P}{EI}}\right)}{Z} \quad \ldots(i)$$

Let us first find the value of $\sec\left(\frac{L_e}{2} \times \sqrt{\frac{P}{EI}}\right)$.

$$\sec\left(\frac{L_e}{2} \times \sqrt{\frac{P}{EI}}\right) = \sec\left[\frac{2.1}{2} \times \sqrt{\frac{100 \times 10^3}{200 \times 10^9 \times 0.0329 \times 10^{-5}}}\right]$$

$$= \sec[1.294 \text{ rads}] = \sec\left(1.294 \times \frac{180^0}{\pi}\right)$$

$$= \sec(74.16^0) = 3.665$$

Substituting the known values in equation (*i*), we get

$$320 \times 10^6 = \frac{100 \times 10^3}{8.639 \times 10^{-4}} + \frac{(100 \times 10^3) \times e \times 3.665}{1.0975 \times 10^{-5}}$$

$$= 115.754 \times 10^6 + 33394\, e \times 10^6$$

or $$320 = 115.754 + 33394 \times e$$

or $$e = \frac{320 - 115.754}{33394} \text{ m} = 6.116 \times 10^{-3} \text{ m} = \mathbf{6.116 \text{ mm.}} \quad \textbf{Ans.}$$

19.17. COLUMNS WITH INITIAL CURVATURE

Fig. 19.14 shows a column AB of length 'l' hinged at both of its ends. The column is having initial curvature and this position is shown by $AC'B$. Let P be the crippling load at which the column has just buckled (*i.e.*, has just started bending). This position is shown by $AC''B$. The initial shape of the column may be assumed circular, parabolic or sinusoidal without making much difference to the final result. But the most common form is

$$y' = C \times \sin\left(\frac{\pi x}{l}\right)$$

where C = Maximum initial deflection.

y' = Initial deflection at a distance x from end A.

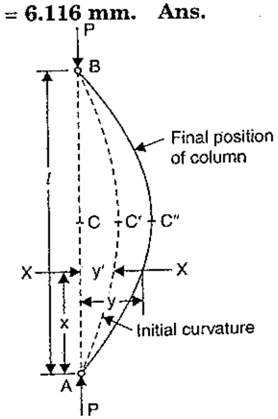


Fig. 19.14

$$\therefore \quad \frac{dy'}{dx} = \frac{C\pi}{l}\cos\left(\frac{\pi x}{l}\right) \qquad ...(i)$$

and
$$\frac{d^2y'}{dx^2} = \frac{C\pi}{l}\left(\frac{-\pi}{l}\right)\sin\left(\frac{\pi x}{l}\right)$$
$$= -\frac{C\pi^2}{l^2}\sin\left(\frac{\pi x}{l}\right). \qquad ...(ii)$$

At the crippling load 'P', the final shape of column is shown by $AC''B$. Let y is the final deflection at a distance x from end A.

Change of deflection = $(y - y')$.

Let change of deflection $= y^*$

Then $y^* = (y - y')$

This change of deflection is due to B.M equal to $P \times y$

$\therefore$ B.M. at the section due to crippling load $= -(P \times y)$.

(– ve sign is due to sign convention given in Art. 19.4.1.)

But bending moment is also $= EI\,\dfrac{d^2y^*}{dx^2}$ or $EI\,\dfrac{d^2(y-y')}{dx^2}$ $\qquad (\because \ y^* = y - y')$

Equating the two bending moments, we get

$$EI\,\frac{d^2(y-y')}{dx^2} = -P \times y \quad \text{or} \quad \frac{d^2y}{dx^2} - \frac{d^2y'}{dx^2} = -\frac{P \times y}{EI}$$

or
$$\frac{d^2y}{dx^2} + \frac{Py}{EI} = \frac{d^2y'}{dx^2}$$
$$= -\frac{C\pi^2}{l^2}\sin\left(\frac{\pi x}{l}\right)$$
$$\left[\because \ \frac{d^2y'}{dx^2} = -\frac{C\pi^2}{l^2}\sin\left(\frac{\pi x}{l}\right)\text{from equation } (ii)\right] \qquad ...(19.14)$$

Let the solution of the above differential equation is $y = mC\sin\left(\dfrac{\pi x}{l}\right)$ $\qquad ...(iii)$

where m is a constant of integration.

The value of m will be obtained by finding the value of $\dfrac{d^2y}{dx^2}$ and substituting this value and value of y in equation (19.14)

Let us find the value of $\dfrac{d^2y}{dx^2}$ and substitute this value in equation (19.14)

Differentiating equation (*iii*) w.r.t. x, we get

$$\therefore \quad \frac{dy}{dx} = mC \times \frac{\pi}{l}\cos\frac{(\pi x)}{l}$$

Again differentiating the above equation, we get

$$\frac{d^2y}{dx^2} = -mC \times \frac{\pi^2}{l^2}\sin\left(\frac{\pi x}{l}\right)$$

Substituting the values of $\dfrac{d^2y}{dx^2}$ and y in equation (19.14), we get

$$-mC \times \frac{\pi^2}{l^2}\sin\left(\frac{\pi x}{l}\right) + \frac{P}{EI} \times mC\sin\left(\frac{\pi x}{l}\right) = -\frac{C\pi^2}{l^2}\sin\left(\frac{\pi x}{l}\right)$$

Concelling $\sin\left(\dfrac{\pi x}{l}\right)$ to both sides, we get

$$-mC \times \frac{\pi^2}{l^2} + \frac{P}{EI} \times mC = -\frac{C\pi^2}{l^2}$$

Also concelling C to both sides, we get

$$-m \times \frac{\pi^2}{l^2} + \frac{P}{EI} \times m = -\frac{\pi^2}{l^2}$$

or
$$\frac{m\pi^2}{l^2} - \frac{P}{EI}m = \frac{\pi^2}{l^2} \quad \text{or} \quad m\left(\frac{\pi^2}{l^2} - \frac{P}{EI}\right) = \frac{\pi^2}{l^2}$$

or
$$m = \frac{\left(\dfrac{\pi^2}{l^2}\right)}{\left(\dfrac{\pi^2}{l^2} - \dfrac{P}{EI}\right)} = \left(\frac{1}{1 - \dfrac{P \times l^2}{EI \times \pi^2}}\right) \qquad ...(iv)$$

(Dividing numerator and denominator by π^2/l^2)

We know that P_E = Euler's load is given by

$$P_E = \frac{\pi^2 \times EI}{l^2}$$

Hence substituting the value of $\dfrac{\pi^2 \times EI}{l^2}$ in equation (*iv*), we get

$$m = \frac{1}{1 - \dfrac{P}{P_E}} = \frac{P_E}{P_E - P}$$

The above equation gives the value of m in terms of Euler's load and axial load Substituting the value of m in equation (*iii*), we get

$$y = \frac{P_E}{(P_E - P)} \times C \times \sin\frac{(\pi x)}{l} \qquad ...(19.15)$$

Equation (19.15) gives the final deflection at any distance x from end A.

Maximum deflection

The deflection will be maximum at the mid-point where $x = \dfrac{l}{2}$. Let this maximum deflection is y_{max}

$\therefore$ At $x = \dfrac{l}{2}, y = y_{max}$

Now the equation (19.15) becomes as

$$y_{max} = \frac{P_E}{(P_E - P)} \times C \times \sin\left(\frac{\pi}{l} \times \frac{l}{2}\right) \qquad \left(\because \ y = y_{max}, x = \frac{l}{2}\right)$$

$$= \left(\frac{P_E}{P_E - P}\right) \times C \times \sin\frac{\pi}{2}$$

$$= \left(\frac{P_E}{P_E - P}\right) \times C \qquad \left(\because \sin\frac{\pi}{2} = 1\right)$$

Maximum stress

The maximum stress (σ_{max}) is due to direct stress (σ_0) and maximum bending stress

$$\therefore \qquad \sigma_{max} = \sigma_0 + \sigma_b. \qquad ...(v)$$

The bending stress should be compressive. It will be maximum, where bending moment is maximum.

But $B.M. = P \times y$. Hence maximum $B.M. = P \times y_{max}$ where $y_{max} = \frac{(P_E)}{P_E - P} \times C$

$$\therefore \text{ Maximum } B.M., \quad M = P \times \frac{P_E}{(P_E - P)} \times C$$

$$\therefore \qquad \sigma_b = \frac{M}{Z} = \frac{P \times \left(\frac{P_E}{P_E - P}\right) \times C}{\left(\frac{Ak^2}{y_c}\right)} \quad \text{where} \quad Z = \frac{I}{y_c} = \frac{Ak^2}{y_c}$$

where y_c = Distance from the neutral axis of the extreme layer in compression

Substituting the value of σ_b in equation (v), we get

$$\sigma_{max} = \sigma_0 + \frac{P \times \frac{P_E}{(P_E - P)} \times C}{\frac{Ak^2}{y_c}} = \sigma_0 + \left[P \times \frac{P_E}{(P_E - P)} \times C\right] \times \frac{y_c}{Ak^2}$$

$$= \sigma_0 + \frac{P}{A} \times \left(\frac{P_E}{P_E - P}\right) \times \frac{C \times y_c}{k^2}$$

$$= \sigma_0 + \sigma_0 \left[\left(\frac{P_E}{P_E - P}\right) \times \frac{C \times y_c}{k^2}\right] \qquad \left[\because \frac{P}{A} = \sigma_0\right]$$

$$= \sigma_0 \left[1 + \left(\frac{P_E}{P_E - P}\right) \times \frac{C \times y_c}{k^2}\right]$$

or
$$\frac{\sigma_{max}}{\sigma_0} = 1 + \frac{P_E}{(P_E - P)} \times \frac{C \times y_c}{k^2}$$

or
$$\left(\frac{\sigma_{max}}{\sigma_0} - 1\right) = \frac{P_E}{P_E - P} \times \frac{C \times y_c}{k^2}$$

$$= \left(\frac{\sigma_E}{\sigma_E - \sigma_0}\right) \times \frac{C \times y_c}{k^2} = \frac{1}{\left(1 - \frac{\sigma_0}{\sigma_E}\right)} \times \frac{C \times y_c}{k^2}$$

or
$$\left(\frac{\sigma_{max}}{\sigma_0} - 1\right)\left(1 - \frac{\sigma_0}{\sigma_E}\right) = \frac{C \times y_c}{k^2} \qquad ...(19.16)$$

The term $\frac{P_E}{P_E - P}$ has been written in turms of stresses as

$$\frac{P_E}{P_E - P} = \frac{\frac{P_E}{A}}{\frac{P_E}{A} - \frac{P}{A}} = \frac{\sigma_E}{(\sigma_E - \sigma_0)} = \frac{1}{(1 - \sigma_0/\sigma_E)}.$$

In equation (19.16), C is the initial maximum deflection and y_c is the distance of extreme layer in compression from neutral axis.

Problem 19.24. *Determine the maximum stress developed in a circular steel strut which is subjected to an axial load of 140 kN. The outside and inside diameters of the strut are 200 mm and 140 mm respectively. It is 5 m long and has both of its end hinged. The strut is having initial curvature of sinusoidal form with initial maximum deflection of 8 mm. Take $E = 205\ GN/m^2$.*

Sol. Given :

Axial load, $P = 140$ kN $= 140 \times 10^3$ N, $D = 200$ mm $= 0.2$ m, $d = 140$ mm $= 0.14$ m, $l = 5$ m, both ends are hinged, initial maximum deflection, $C = 8$ mm $= 0.008$ m, $E = 205$ GN/m^2 $= 205 \times 10^9$ N/m^2.

Let σ_{max} = Maximum stress developed

Now,
$$A = \frac{\pi}{4}(D^2 - d^2) = \frac{\pi}{4}(0.2^2 - 0.14^2)\ \text{m}^2$$

$$= \frac{\pi}{4}(0.04 - 0.0196) = 0.016\ \text{m}^2$$

$$I = \frac{\pi}{64}(D^4 - d^4) = \frac{\pi}{64}(0.2^4 - 0.14^4)\ \text{m}^4$$

$$= \frac{\pi}{64}(0.0016 - 0.000384) = 5.968 \times 10^{-5}\ \text{m}^4.$$

$$k^2 = \frac{I}{A} = \frac{5.968 \times 10^{-5}}{0.016} = 0.00373$$

Let us now find the values of σ_0 (stress due to direct load), σ_E (stress due to Euler's load) and y_c

$$\sigma_0 = \frac{P}{A} = \frac{140 \times 10^3}{0.016} = 8.75 \times 10^6\ \text{N/m}^2$$

$$\sigma_E = \frac{P_E}{A} \text{ where } P_E = \text{Euler's load}$$

$$= \frac{\pi^2 EI}{l^2} = \frac{\pi^2 \times 205 \times 10^9 \times 5.968 \times 10^{-5}}{5^2} = 4.83 \times 10^6\ \text{N}$$

$$= \frac{4.83 \times 10^6}{0.016} = 301.87 \times 10^6\ \text{N/m}^2$$

y_c = distance of extreme layer in compression from neutral axis

$$= \frac{D}{2} = \frac{0.2}{2} = 0.1\ \text{m}.$$

These values are substituted in equation (19.16)

Now using equation (19.16), we get

$$\left(\frac{\sigma_{max}}{\sigma_0}-1\right)\left(1-\frac{\sigma_0}{\sigma_E}\right)=\frac{C\times y_c}{k^2}$$

or $$\left(\frac{\sigma_{max}}{8.75\times10^6}-1\right)\left(1-\frac{8.75\times10^6}{301.87\times10^6}\right)=\frac{0.008\times0.1}{0.00373}$$

or $$\left(\frac{\sigma_{max}}{8.75\times10^6}-1\right)(1-0.02898)=0.21447$$

or $$\left(\frac{\sigma_{max}}{8.75\times10^6}-1\right)=\frac{21447}{(1-0.02898)}=0.22088$$

$\therefore$ $$\sigma_{max}=(1+0.22088)\times8.75\times10^6$$
$$=\mathbf{10.683\times10^6\ N/m^2.}\quad \textbf{Ans.}$$

19.18. STRUT WITH LATERAL LOAD (OR BEAM COLUMNS)

Columns carry axial compressive loads. If the columns are also subjected to transverse loads, then they are known as beam columns. The transverse load is generally uniformly distributed. But let us consider two cases when

(*i*) Transverse load is a point load and acts at the centre

(*ii*) Transverse load is uniformly distributed.

19.18.1. Strut Subjected to Compressive Axial Load or Axial Thrust and a Transverse Point Load at the Centre. Both ends are Pinned. Fig. 19.15 shows a strut *AB* of length '*l*' subjected to compressive axial load *P* and a transverse point load *W* at the centre. The strut is pinned at both of its ends. Consider any section at a distance *x* from the end *A*. Let '*y*' is the deflection at this section. The bending moment at the section is given by,

$$M=-Py-\frac{W}{2}x \qquad \text{...(i)}$$

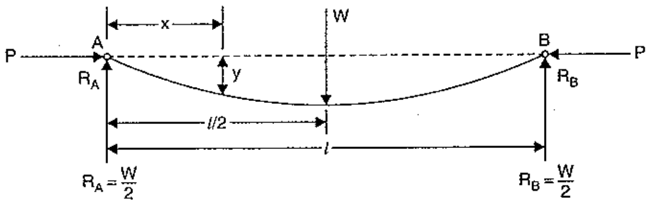


Fig. 19.15

B.M. is also given by, $M=EI\dfrac{d^2y}{dx^2}$ ...(*ii*)

Equating the two *B.M.* given by equations (*i*) and (*ii*),

$\therefore$ $$EI\frac{d^2y}{dx^2}=-Py-\frac{W}{2}x \quad \text{or} \quad \frac{d^2y}{dx^2}=-\frac{P}{EI}\times y-\frac{W}{2EI}\times x$$

or $$\frac{d^2y}{dx^2}+\frac{P}{EI}\times y=-\frac{W}{2EI}\times x.$$

The solution of the above differential equation is

$$y=C_1\cos\left(x\times\sqrt{\frac{P}{EI}}\right)+C_2\sin\left(x\times\sqrt{\frac{P}{EI}}\right)-\frac{W\times x}{2EI\times\frac{P}{EI}}$$

$$=C_1\cos\left(x\times\sqrt{\frac{P}{EI}}\right)+C_2\sin\left(x\times\sqrt{\frac{P}{EI}}\right)-\frac{W\times x}{2P} \qquad \text{...(iii)}$$

The slope at any section is given by

$$\frac{dy}{dx}=-C_1\times\sqrt{\frac{P}{EI}}\sin\left(x\times\sqrt{\frac{P}{EI}}\right)+C_2\times\sqrt{\frac{P}{EI}}\cos\left(x\times\sqrt{\frac{P}{EI}}\right)-\frac{W}{2P} \qquad \text{...(iv)}$$

The values of C_1 and C_2 are obtained from boundary conditions.

At $x=0$, $y=0$. Hence from equation (*iii*) we get

$0=C_1$

At $x=\dfrac{l}{2}$, $\dfrac{dy}{dx}=0$. Hence from equation (*iv*), we get

$$0=-C_1\times\sqrt{\frac{P}{EI}}\times\sin\left(\frac{l}{2}\times\sqrt{\frac{P}{EI}}\right)+C_2\times\sqrt{\frac{P}{EI}}\cos\left(\frac{l}{2}\times\sqrt{\frac{P}{EI}}\right)-\frac{W}{2P}$$

$$=0+C_2\times\sqrt{\frac{P}{EI}}\cos\left(\frac{l}{2}\times\sqrt{\frac{P}{EI}}\right)-\frac{W}{2P} \qquad (\because\ C_1=0)$$

or $$C_2=\frac{W}{2P}\times\sqrt{\frac{EI}{P}}\times\frac{1}{\cos\left(\frac{l}{2}\times\sqrt{\frac{P}{EI}}\right)}=\frac{W}{2P}\times\sqrt{\frac{EI}{P}}\times\sec\left(\frac{l}{2}\times\sqrt{\frac{P}{EI}}\right)$$

Substituting the value of C_1 and C_2 is equation (*iii*), we get

$$y=0+\frac{W}{2P}+\sqrt{\frac{EI}{P}}\times\sec\left(\frac{l}{2}\times\sqrt{\frac{P}{EI}}\right)\sin\left(x\times\sqrt{\frac{P}{EI}}\right)-\frac{W\times x}{2P}$$

$$=\frac{W}{2P}\times\sqrt{\frac{EI}{P}}\times\sec\left(\frac{l}{2}\times\sqrt{\frac{P}{EI}}\right)\sin\left(x\times\sqrt{\frac{P}{EI}}\right)-\frac{W\times x}{2P} \qquad \text{...(v)}$$

The above equation (*v*), gives the deflection at any section.

Let us now find maximum deflection, maximum bending moment and maximum stress induced.

Maximum deflection (y_{max}). The deflection is maximum at the centre, where $x=\dfrac{l}{2}$

Substituting $x=\dfrac{l}{2}$ in equation (*v*), we get maximum deflection.

$\therefore$ $$y_{max}=\frac{W}{2P}\times\sqrt{\frac{EI}{P}}\times\sec\left(\frac{l}{2}\times\sqrt{\frac{P}{EI}}\right)\sin\left(\frac{l}{2}\times\sqrt{\frac{P}{EI}}\right)-\frac{W}{2P}\times\frac{l}{2}$$

$$=\frac{W}{2P}\times\sqrt{\frac{EI}{P}}\times\tan\left(\frac{l}{2}\times\sqrt{\frac{P}{EI}}\right)-\frac{W\times l}{4P} \qquad \text{...(vi)}$$

Maximum bending moment. The *B.M.* is given by equation (*i*) as

$$M = -\left(P \times y + \frac{W}{2} \times x\right)$$

The bending *B.M.* will be maximum at the centre where $y = y_{max}$ and $x = \frac{l}{2}$. Substituting these values in the above equation, we get

$$M_{max} = -\left(P \times y_{max} + \frac{W}{2} \times \frac{l}{2}\right)$$

$$= -\left[P \times \left\{\frac{W}{2p} \times \sqrt{\frac{EI}{p}} \times \tan\left(\frac{l}{2} \times \sqrt{\frac{P}{EI}}\right) - \frac{W \times l}{4P}\right\} + \frac{W}{2} \times \frac{l}{2}\right]$$

[Substitute y_{max} here from equation (*vi*)]

$$= -\left[\frac{W}{2} \times \sqrt{\frac{EI}{P}} \times \tan\left(\frac{l}{2} \times \sqrt{\frac{P}{EI}}\right) - \frac{W \times l}{4} + \frac{W \times l}{4}\right]$$

$$= -\left[\frac{W}{2} \times \sqrt{\frac{EI}{P}} \times \tan\left(\frac{l}{2} \times \sqrt{\frac{P}{EI}}\right)\right]$$

The – ve sign is due to sign convention. Hence the magnitude of maximum *B.M.* is given by

$$M_{max} \text{ (magnitude)} = \frac{W}{2} \times \sqrt{\frac{EI}{P}} \times \tan\left(\frac{l}{2} \times \sqrt{\frac{P}{EI}}\right) \quad ...(19.17)$$

Maximum stress (σ_{max}). Maximum stress induced is due to direct axial compressive load and due to maximum bending stress

$\therefore$ $\sigma_{max} = \sigma_0 + \sigma_b$ where σ_0 = Stress due to direct axial compressive load

$= \frac{P}{A} + \sigma_b$ $\quad\sigma_b$ = Stress due to bending.

The stress due to bending of strut is given by,

$$\frac{M}{I} = \frac{\sigma_b}{y}$$

or

$$\sigma_b = \frac{M \times y}{I} = \frac{M \times y_c}{Ak^2}$$

where y_c = distance of the extreme layer in compression from neutral axis

$I = Ak^2$

k = Radius of gyration

$M = M_{max}$

Max. bending stress $= \frac{M_{max} \times y_c}{A\,k^2}$

$$= \left[\frac{\frac{W}{2} \times \sqrt{\frac{EI}{P}} \times \tan\left(\frac{l}{2} \times \sqrt{\frac{P}{EI}}\right)}{AK^2}\right] \times y_c$$

$$\left[\because \text{ From equation 19.17, } M_{max} = \frac{W}{2} \times \sqrt{\frac{EI}{P}} \times \tan\left(\frac{l}{2} \times \sqrt{\frac{P}{EI}}\right)\right]$$

Hence maximum stress induced becomes as

$$\therefore \quad \sigma_{max} = \frac{P}{A} + \left[\frac{W}{2} \times \sqrt{\frac{EI}{P}} \times \tan\left(\frac{l}{2} \times \sqrt{\frac{P}{EI}}\right)\right] \times \frac{y_c}{Ak^2} \quad ...(19.18)$$

Problem 19.25. *Determine the maximum stress induced in a cylindrical steel strut of length 1.2 m and diameter 30 mm. The strut is hinged at both its ends and subjected to an axial thrust of 20 kN at its ends and a transverse point load of 1.8 kN at the centre. Take E = 208 GN/m².*

Sol. Given :

l = 1.2 m ; d = 30 mm = 0.03 m ; axial thrust, P = 20 kN = 20 × 10^3 N ; transverse point load, W = 1.8 kN = 1.8 × 10^3 N; E = 208 GN/m² = 208 × 10^9 N/m²

Area, $\quad A = \frac{\pi}{4} d^2 = \frac{\pi}{4} (0.03^2) = 7.068 \times 10^{-4}$ m²

M.O.I, $\quad I = \frac{\pi}{64} d^4 = \frac{\pi}{64} (0.03)^4 = 3.976 \times 10^{-8}$ m⁴

Direct stress is due to axial thrust.

$$\therefore \quad \sigma_0 = \frac{P}{A} = \frac{20 \times 10^3}{7.068 \times 10^{-4}} = 28.29 \times 10^6 \text{ N/m}^2 = 28.29 \text{ MN/m}^2$$

Maximum bending stress is given by

$$\sigma_b = \frac{M_{max} \times y_c}{I} \quad ...(i)$$

Max. bending moment is given by equation (19.17), as

$$M_{max} = \frac{W}{2} \times \sqrt{\frac{EI}{P}} \times \tan\left(\frac{l}{2} \times \sqrt{\frac{P}{EI}}\right) \quad ...(ii)$$

Let us find $\sqrt{\frac{P}{EI}}$.

Now, $$\sqrt{\frac{P}{EI}} = \sqrt{\frac{20 \times 10^3}{(208 \times 10^9) \times (3.976 \times 10^{-8}}} = \sqrt{2.4183} = 1.555$$

and $$\sqrt{\frac{EI}{P}} = \frac{1}{1.555} = 0.643$$

Also $\frac{l}{2} \times \sqrt{\frac{P}{EI}} = \frac{1.2}{2} \times 1.555 = 0.933 \text{ rad} = 0.933 \times \frac{180°}{\pi} = 53.45°$

$$\therefore \quad \tan\left(\frac{l}{2} \times \sqrt{\frac{P}{EI}}\right) = \tan(0.933 \text{ rad}) = \tan(53.45°) = 1.349$$

Substituting known values in equation (*ii*), we get

$$M_{max} = \frac{1.8 \times 10^3}{2} \times 0.643 \times 1.349 = 780.66 \text{ Nm.}$$

Substituting the above value in equation (*i*), we get

$$\sigma_b = \frac{780.66 \times y_c}{I}$$

$$= \frac{780.66 \times 0.015}{3.976 \times 10^{-8}}$$

$$\left(\because \; y_c = \frac{d}{2} = \frac{30}{2} = 15 \text{ mm} = 0.015 \text{ m and } I = 3.976 \times 10^{-8} \text{ m}^4\right)$$

$= 294.51 \times 10^6$ N/m² = 294.51 MN/m²

$\therefore$ Maximum stress induced is given by

$$\sigma_{max} = \sigma_0 + \sigma_b$$
$$= 28.29 \text{ MN/m}^2 + 294.51 \text{ MN/m}^2 = \mathbf{322.8 \ MN/m^2}. \quad \textbf{Ans.}$$

19.18.2 Strut Subjected to Compressive Axial load or Axial thrust and a Transverse Uniformly Distributed load of Intensity w per unit length. Both ends are Pinned. Fig. 19.16 shows a strut AB of length 'l' subjected to axial thrust P at its ends and also a transverse uniformly distributed load of intensity w/unit length. The strut is pinned at both of its ends.

Consider any section at a distance 'x' from the end A. Let 'y' is deflection at this section. The bending moment at the section is given by,

$$M = -P \times y + (w \times x) \times \frac{x}{2} - \frac{w \times l}{2} \times x$$
$$= -P \times y + \frac{w \times x^2}{2} - \frac{w \times l \times x}{2} \quad \ldots(i)$$

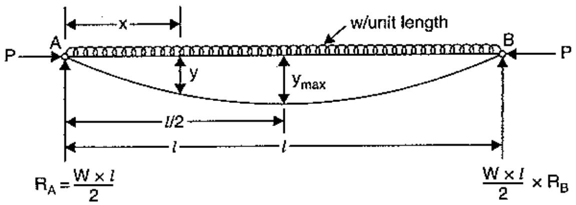


Fig. 19.16

Bending moment is also given by,

$$M = EI \frac{d^2y}{dx^2} \quad \ldots(ii)$$

Differentiating the equation (i), w.r.t.x., we get

$$\frac{dM}{dx} = -P\frac{dy}{dx} + \frac{w \times 2x}{2} - \frac{w \times l}{2}$$
$$= -P\frac{dy}{dx} + w \times x - \frac{w \times l}{2}$$

Differentiating the above equation again, we get

$$\frac{d^2M}{dx^2} = -P\frac{d^2y}{dx^2} + w \quad \ldots(iii)$$

From equation (ii), $\frac{d^2y}{dx^2} = \frac{M}{EI}$. Substituting this value in equation (iii), we get

$$\frac{d^2M}{dx^2} = -P \times \frac{M}{EI} + w \quad \text{or} \quad \frac{d^2M}{dx^2} + \frac{P}{EI} \times M = w$$

The above equation can be written as

$$\frac{d^2M}{dx^2} + \alpha^2 M = w \quad \text{where } \alpha^2 = \frac{P}{EI} \quad \text{or} \quad \alpha = \sqrt{\frac{P}{EI}}$$

This equation is a differential equation in M and is more useful as the maximum bending moment can be obtained directly from this.

The solution of the above equation is

$$M = C_1 \cos(\alpha \times x) + C_2 \sin(\alpha \, . \, x) + \frac{w}{\alpha^2}$$
$$= C_1 \cos\left(\sqrt{\frac{P}{EI}} \times x\right) + C_2 \sin\left(\sqrt{\frac{P}{EI}} \times x\right) + \frac{w \times EI}{P}$$
$$= C_1 \cos\left(x \times \sqrt{\frac{P}{EI}}\right) + C_2 \sin\left(x \times \sqrt{\frac{P}{EI}}\right) + \frac{w \times EI}{P} \quad \ldots(iv)$$

Let us find $\frac{dM}{dx}$ also from the above equation

$$\therefore \quad \frac{dM}{dx} = -C_1 \times \sqrt{\frac{P}{EI}} \sin\left(x \times \sqrt{\frac{P}{EI}}\right) + C_2 \times \sqrt{\frac{P}{EI}} \cos\left(x \times \sqrt{\frac{P}{EI}}\right) + 0$$
$$= -C_1 \sqrt{\frac{P}{EI}} \sin\left(x \times \sqrt{\frac{P}{EI}}\right) + C_2 \times \sqrt{\frac{P}{EI}} \cos\left(x \times \sqrt{\frac{P}{EI}}\right) \quad \ldots(v)$$

The values of C_1 and C_2 are obtained from boundary conditions. At $x = 0$, $M = 0$, hence from equation (iv), we have

$$0 = C_1 + \frac{w \times EI}{P} \quad \therefore \quad C_1 = -\frac{w \times EI}{P}$$

At $x = \frac{l}{2}$, $\frac{dM}{dx} = 0$ (as at the centre shear force $= R_A - w \times \frac{l}{2} = \frac{w \times L}{2} - \frac{w \times L}{2} = 0$) hence from equation (v), we get

$$0 = -C_1 \times \sqrt{\frac{P}{EI}} \sin\left(\frac{l}{2} \times \sqrt{\frac{P}{EI}}\right) + C_2 \times \sqrt{\frac{P}{EI}} \times \cos\left(\frac{l}{2} \times \sqrt{\frac{P}{EI}}\right)$$
$$= -\left(-\frac{w \times EI}{P}\right) \times \sqrt{\frac{P}{EI}} \sin\left(\frac{l}{2} \times \sqrt{\frac{P}{EI}}\right) + C_2 \sqrt{\frac{P}{EI}} \cos\left(\frac{l}{2} \times \sqrt{\frac{P}{EI}}\right)$$
$$\left(\because \ c_1 = -\frac{w \times EI}{P}\right)$$

or
$$0 = \frac{w \times EI}{P} \times \sin\left(\frac{l}{2} \times \sqrt{\frac{P}{EI}}\right) + C_2 \cos\left(\frac{l}{2} \times \sqrt{\frac{P}{EI}}\right)$$

(Concelling $\sqrt{\frac{P}{EI}}$ to both sides)

or
$$C_2 = -\frac{w \times EI}{P} \times \frac{\sin \frac{l}{2} \times \sqrt{\frac{P}{EI}}}{\cos\left(\frac{l}{2} \times \sqrt{\frac{P}{EI}}\right)} = -\frac{w \times EI}{P} \tan\left(\frac{l}{2} \times \sqrt{\frac{P}{EI}}\right)$$

Substituting the value of C_1 and C_2 in equation (iv), we get

$$M = \left(-\frac{w \times EI}{P}\right) \times \cos\left(x \times \sqrt{\frac{P}{EI}}\right) + \left[-\frac{wEI}{P} \tan\left(\frac{l}{2} \times \sqrt{\frac{P}{EI}}\right)\right] \sin\left(x \times \sqrt{\frac{P}{EI}}\right) + \frac{w \times EI}{P}$$
$$= \left(-\frac{w \times EI}{P}\right)\left[\cos\left(x \times \sqrt{\frac{P}{EI}}\right) + \tan\left(\frac{l}{2} \times \sqrt{\frac{P}{EI}}\right) \sin\left(x \times \sqrt{\frac{P}{EI}}\right) - 1\right].$$

Expression for maximum bending moment, maximum deflection and maximum stress

Maximum bending moment :

The bending moment is maximum at $x = \frac{l}{2}$. Hence above equation becomes as

$$M_{max} = \left(\frac{-w \times EI}{P}\right)\left[\cos\left(\frac{l}{2} \times \sqrt{\frac{P}{EI}}\right) + \tan\left(\frac{l}{2} \times \sqrt{\frac{P}{EI}}\right) \times \sin\left(\frac{l}{2} \times \sqrt{\frac{P}{EI}}\right) - 1\right]$$

$$= -\frac{w \times EI}{P}\left[\cos\left(\frac{l}{2} \times \sqrt{\frac{P}{EI}}\right) + \frac{\sin\left(\frac{l}{2} \times \sqrt{\frac{P}{EI}}\right)}{\cos\left(\frac{l}{2} \times \sqrt{\frac{P}{EI}}\right)} \times \sin\left(\frac{l}{2} \times \sqrt{\frac{P}{EI}}\right) - 1\right]$$

$$= -\frac{w \times EI}{P}\left[\frac{\cos^2\left(\frac{l}{2} \times \sqrt{\frac{P}{EI}}\right) + \sin^2\left(\frac{l}{2} \times \sqrt{\frac{P}{EI}}\right) - 1}{\cos\left(\frac{L}{2} \times \sqrt{\frac{P}{EI}}\right)}\right]$$

$$= -\frac{w \times EI}{P}\left[\frac{1}{\cos\left(\frac{L}{2} \times \sqrt{\frac{P}{EI}}\right)} - 1\right]$$

$$= -\frac{w \times EI}{P}\left[\sec\left(\frac{l}{2} \times \sqrt{\frac{P}{EI}}\right) - 1\right] \qquad \text{...(19.19)}$$

(– ve sign is due to sign convention)

Maximum deflection : The corresponding maximum deflection (y_{max}) is obtained from equation (*i*), in which M = Max, $y = y_{max}$ and $x = \frac{l}{2}$. Hence equation (*i*) becomes as

$$M_{max} = -P \times y_{max} + \frac{w}{2} \times \left(\frac{l}{2}\right)^2 - \frac{wl}{2} \times \frac{l}{2}$$

$$= -P \times y_{max} + \frac{w \times l^2}{8} - \frac{w \times l^2}{4}$$

$$= -P \times y_{max} - \frac{w \times l^2}{8}$$

$$= -\left(P \times y_{max} + \frac{wl^2}{8}\right)$$

But from equation (19.19),

$$M_{max} = -\frac{w \times EI}{P}\left[\sec\left(\frac{l}{2} \times \sqrt{\frac{P}{EI}}\right) - 1\right]$$

Equating the two values of M_{max}, we get

$$-\left(P \times y_{max} + \frac{wl^2}{8}\right) = -\frac{w \times EI}{P}\left[\sec\left(\frac{l}{2} \times \sqrt{\frac{P}{EI}}\right) - 1\right]$$

or
$$P \times y_{max} = \frac{w \times EI}{P}\left[\sec\left(\frac{l}{2} \times \sqrt{\frac{P}{EI}}\right) - 1\right] - \frac{wl^2}{8}$$

or
$$y_{max} = \frac{w \times EI}{P^2}\left[\sec\left(\frac{l}{2} \times \sqrt{\frac{P}{EI}}\right) - 1\right] - \frac{wl^2}{8P} \qquad \text{...(19.20)}$$

Maximum stress :

The maximum stress is then given by,

$$\sigma_{max} = \sigma_0 + \sigma_b$$

$$= \frac{P}{A} + \frac{M_{max}}{z} \quad \text{where } \frac{M_{max}}{z} = \frac{M_{max}}{\left(\frac{I}{y_c}\right)} = \frac{M_{max} \times y_c}{I}$$

where y_c = distance of extreme layer in compression from *N.A.*

Problem 19.26. *Determine the maximum stress induced in a horizontal strut of length 2.5 m and of rectangular cross-section 40 mm wide and 80 mm deep when it carries an axial thrust of 100 kN and a vertical load of 6 kN/m length. The strut is having pin joints at its ends. Take $E = 208$ GN/m².*

Sol. Given :

l = 2.5 m ; b = 40 mm = 0.04 m ; d = 80 mm = 0.08 m ; axial thrust, P = 100 kN = 100 × 1000 N ; uniformly distributed load, w = 6 kN/m = 6×10^3 N/m ; E = 208 GN/m² = 208×10^9 N/m² Area, $A = b \times d = 0.04 \times 0.08 = 0.0032$ m²

$$I = \frac{bd^3}{12} = \frac{0.04 \times 0.08^3}{12} = 1.7066 \times 10^{-6} \text{ m}^4.$$

Let us now find the value of maximum bending moment (*i.e.*, M_{max}). Using equation (19.19), we get

$$M_{max} = \frac{w \times EI}{P}\left[\sec\left(\frac{l}{2} \times \sqrt{\frac{P}{EI}}\right) - 1\right] \text{ (magnitude) (neglect – ve sign)}$$

where $\frac{P}{EI} = \frac{100 \times 1000}{208 \times 10^9 \times 1.7066 \times 10^{-6}} = 0.2817$ and $\frac{EI}{P} = \frac{1}{0.2817} = 3.55$

or
$$\sqrt{\frac{P}{EI}} = \sqrt{0.2817} = 0.5307$$

Also,
$$\frac{l}{2} \times \sqrt{\frac{P}{EI}} = \frac{2.5}{2} \times 0.5307 = 0.6634 \text{ radians}$$

$$= \frac{0.6634 \times 180^\circ}{\pi} = 38^\circ$$

$$\therefore \quad \sec\left[\frac{l}{2} \times \sqrt{\frac{P}{EI}}\right] = \sec 38^\circ = 1.269$$

$$\therefore \quad M_{max} = w \times \frac{EI}{P}\left[\sec\left(\frac{l}{2} \times \sqrt{\frac{P}{EI}}\right) - 1\right] = (6 \times 10^3) \times 3.55\ [1.269 - 1]$$

$$= 5729.70 \text{ Nm}$$

Stress due to direct load, $\sigma_0 = \dfrac{P}{A} = \dfrac{100 \times 1000}{0.0032}$ N/m^2 = 31.25×10^6 N/m^2

Max. bending stress, $\sigma_b = \dfrac{M_{max} \times y_c}{I}$

where y_c = distance of extreme layer in compression from neutral axis

$$= \frac{d}{2} = \frac{80}{2} = 40 \text{ mm} = 0.04 \text{ m}$$

$$\therefore \quad \sigma_b = \frac{5729.7 \times 0.04}{1.7066 \times 10^{-6}} = 134.3 \times 10^6 \text{ N/m}^2$$

$\therefore$ Maximum stress induced is given by

$$\sigma_{max} = \sigma_0 + \sigma_b$$

or $\sigma_{max} = 31.25 \times 10^6 + 134.3 \times 10^6$ N/m^2

$= 166.55 \times 10^6$ N/m^2 = **166 .55 MN/m^2. Ans.**

HIGHLIGHTS

1. A vertical member of a structure, which is subjected to axial compressive load and is fixed at both of its ends, is known as a column.
2. Strut is a member of a structure which is not vertical or whose one or both of its ends are hinged or pin joined.
3. All short columns fail due to crushing whereas long columns fail due to buckling and crushing.
4. The load at which the column just buckles is known as buckling load or critical load or crippling load.
5. The crippling load for a column by Euler's formula for different end conditions is given by

$$P = \frac{\pi^2 EI}{l^2} \quad \text{when both ends are hinged}$$

$$= \frac{\pi^2 EI}{4l^2} \quad \text{when one end is fixed and other is free}$$

$$= \frac{4\pi^2 EI}{l^2} \quad \text{when both ends are fixed}$$

$$= \frac{2\pi^2 EI}{l^2} \quad \text{when one end is fixed and other is hinged}$$

where l = Actual length of column
E = Young's modulus of the material of the column
I = Least moment of inertia of the column.

6. The effective length of a given column with given end conditions is the length of an equivalent column of the same material and cross-section with hinged ends, and having the value of crippling load equal to that of the given column.

7. The crippling load for any type of end condition is given by

$$P = \frac{\pi^2 EI}{L_e^2}$$

where L_e = Effective length
$= l$ when both ends hinged
$= 2l$ when one end is fixed and other is free
$= \dfrac{l}{2}$ when both ends are fixed
$= \dfrac{l}{\sqrt{2}}$ when one end fixed and other is hinged

where l = Actual length of the column.

8. Crippling load and crippling stress in terms of effective length and radius of gyration are given by

Crippling loads, $P = \dfrac{\pi^2 E.A}{\left(\dfrac{L_e}{k}\right)^2}$

and Crippling stress $= \dfrac{\pi^2 E}{\left(\dfrac{L_e}{k}\right)^2}$

where L_e = Effective length and k = Least radius of gyration = $\sqrt{\dfrac{I}{A}}$

where I = Least moment of Inertia.

9. Slenderness ratio is the ratio of the effective length of the column to the least radius of gyration. Mathematically,

Slenderness ratio = $\dfrac{L_e}{k}$.

10. The crippling load by Rankine's formula is given by

$$P = \frac{\sigma_c \times A}{1 + a\left(\dfrac{L_e}{k}\right)^2}$$

where σ_c = Ultimate crushing stress
A = Area of cross-section of column
a = Rankine's constant
L_e = Effective length
k = Least radius of gyration.

11. The crippling load by straight line formula is given by

$$P = \sigma_c \,.\, A - n\left(\frac{L_e}{k}\right).\, A$$

where σ_c = Compressive yield stress
A = Area of cross-section of column
$\left(\dfrac{L_e}{k}\right)$ = Slenderness ratio and n = A constant.

12. Johnson's parabolic formula for crippling load is given by

$$P = \sigma_c \cdot A - r\left(\frac{L_e}{k}\right)^2 \cdot A$$

where σ_c, A and $\frac{L_e}{k}$ are compressive yield stress, area and slenderness ratio respectively and

$$r = A \text{ constant} = \frac{\sigma_c^2}{4\pi^2 E}.$$

EXERCISE 19

(A) Theoretical Questions

1. Explain the assumptions made in Euler's column theory. How far are the assumptions valid in practice ? *(AMIE, Summer 1982)*
2. Define the terms : column, strut and crippling load.
3. Explain how the failure of a short and of a long column takes place ?
4. What do you mean by end conditions of a column ? What are the important end conditions for a column ? Explain them.
5. What is 'equivalent length of a column' ? How is the concept used in the column theory ? *(AMIE, Winter 1982)*
6. What is 'equivalent length of a column' ? Give the ratios of equivalent length and actual length of columns with various end conditions. *(AMIE, Summer 1985)*
7. Derive an expression for the Euler's crippling load for a long column with following end conditions.
   (*a*) Both ends are hinged (*b*) Both ends are fixed.
8. Explain how Rankine-Gordon formula is used to calculate the intensity of stress in short, intermediate and long columns. *(AMIE, Winter 1981)*
9. Prove that the crippling load by Euler's formula for a column having one end fixed and other end free is given by

$$P = \frac{\pi^2 EI}{4l^2}$$

   where l = Actual length of the column,
   E = Young's modulus, and
   I = Least moment of inertia.
10. Find an expression for crippling load for a long column when one end of the column is fixed and other end is hinged.
11. Prove that crippling stress by Euler's formula is given by $f_c = \dfrac{\pi^2 E}{\left(\frac{L_e}{k}\right)^2}$.
12. Define slenderness ratio. State the limitations of Euler's formula.
13. How will you justify that Rankine's formula is applicable for all lengths of columns, ranging from short to long columns.
14. What is a Rankine's constant ? What is the approximate value of Rankine's constant for cast iron column ?
15. Deduce an expression for the Euler's crippling load of an 'ideal column' pin-joined at each end. Explain the limitations, if any, in using the formula.

16. Derive the equation for the Euler's crippling load for a column with one end fixed and the other is free. *(Annamalai University, 1991)*
17. Derive expression for Euler's buckling load for a long column of length L with both ends fixed, from first principle. Mention the assumptions made in the derivation. *(Bangalore University, 1991)*

(B) Numerical Problems

1. A solid round bar 4 m long and 6 cm in diameter is used as a strut with both ends hinged. Determine the crippling load. Take $E = 2 \times 10^5$ N/mm^2. **[Ans.** 78.486 kN]
2. For the problem 1, determine the crippling loads when the given strut is used with the following conditions :
   (*i*) One end is fixed and other end is free (*ii*) Both the ends are fixed and
   (*iii*) One end is fixed and other is hinged. **[Ans.** (*i*) 19.621 kN (*ii*) 313.94 kN (*iii*) 156.96 kN]
3. A column of timber section 10 cm × 15 cm is 5 m long both ends being fixed. If the Young's modulus for timber = 17.5 kN/mm^2, determine :
   (*i*) Crippling load, and (*ii*) Safe load for the column if factor of safety = 3.
   [Ans. (*i*) 45.4 kN and (*ii*) 115.1 kN]
4. A hollow mild steel tube 5 m long, 4 cm internal diameter and 5 mm thick is used as a strut with both ends hinged. Find the crippling load and safe load taking factor of safety as 3. Taking $E = 2 \times 10^5$ N/mm^2. **[Ans.** 14.99 N and 4766 N]
5. A solid circular bar 5 m long and 4 cm in diameter was found to extend 4.5 mm under a tensile load of 48 kN. The bar is used as a strut with both ends hinged. Determine the buckling load for the bar and also the safe load taking factor of safety as 3.0. **[Ans.** 2105.5 N and 701.8 N]
6. Calculate the safe compressive load on a hollow cast iron column (one end rigidly fixed and other hinged) of 10 cm external diameter, 7 cm internal diameter and 8 m in length. Use Euler's formula with a factor of safety of 4 and $E = 95$ kN/mm^2. **[Ans.** 27.3 kN]
7. Determine Euler's crippling load for an I-section joist 30 cm × 15 cm × 2 cm and 5 m long which is used as a strut with both ends fixed. Take Young's modulus for the joist as 2×10^5 N/mm^2.
   [Ans. 3.6 MN]
8. Determine the crippling load for a T-section of dimensions 12 cm × 12 cm × 2 cm and of length 6 cm when it is used as a strut with both of its ends hinged. Take $E = 2 \times 10^5$ N/mm^2.
   [Ans. 161.57 kN]
9. Determine the ratio of buckling strengths of two columns one hollow and the other solid. Both are made of the same material and have the same length, cross-sectional area and end conditions. The internal diameter of hollow column is 2/3rd of its external diameter. **[Ans.** 2.6 : 1]
10. The external and internal diameter of a hollow cast iron column are 5 cm and 3 cm respectively. If the length of this column is 4 m and both of its ends are fixed, determine the crippling load using Rankine's formula. Take the value of f_c = 550 N/mm^2 and $a = \frac{1}{1600}$ in Rankine's formula.
    [Ans. 540.09 kN]
11. A hollow cylindrical cast iron column is 6 m long with both ends fixed. Determine the minimum diameter of the column if it has to carry a safe load of 300 kN with a factor of safety of 4. Take the internal diameter as 0.7 times the external diameter. Take f_c = 550 N/mm^2 and $a = \frac{1}{1600}$ in Rankine's formula. **[Ans.** D = 9.53 cm, d = 6.67 cm]
12. A 2.0 m long column has a circular cross-section of 6 cm diameter. One of the ends of the column is fixed in direction and position and other end is free. Taking factor of safety as 3, calculate the safe load using