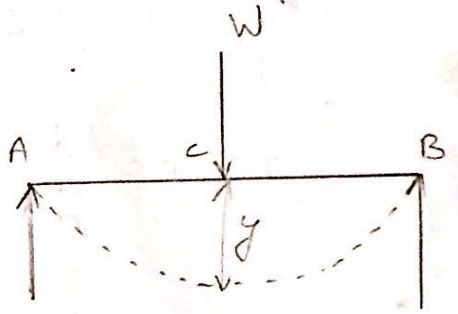
DEFLECTION OF BEAMS

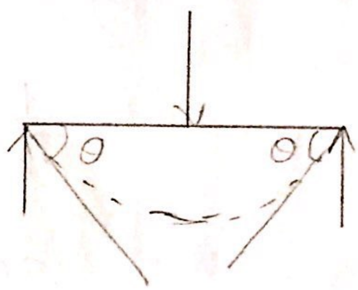


Deflection :-

It is the vertical distance of the beam measured before and after loading.

- It is denoted by 'y'
- Deflection at the supports is always zero.
- It is expressed in 'mm'.

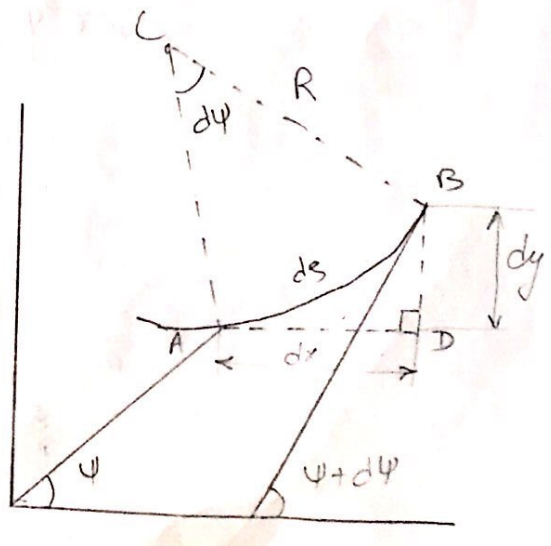
Slope :-



It is the angle measured between the tangent to the elastic curve & the original axis of the beam.

- It is denoted by 'θ' or $\frac{dy}{dx}$.
- It is expressed in radians.

Expression for Differential equation for Bending moment at any section.



we know from the diagram

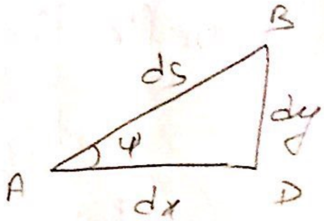
$\angle ACB = d\psi$

length of arc AB $= ds = d\psi \cdot R$

$\cancel{R = \frac{ds}{d\psi}} \Rightarrow \frac{ds}{R} = d\psi$

$\frac{1}{R} = \frac{d\psi}{ds}$ —— (1)

Consider Right Angled Δ^{le} ADB



$$\tan\psi = \frac{dy}{dx} \quad — ②$$

$$\cos\psi = \frac{dx}{ds} \quad — ③$$

$$\sec\psi = \frac{ds}{dx} \quad — ④$$

Differentiating eqn ② w.r.t. x.

$$\frac{d}{dx}(\tan\psi) = \sec^2\psi$$

$$\frac{d}{dx}(\tan\psi) = \frac{d}{dx} \times \frac{dy}{dx}$$

$$\sec^2\psi \frac{d\psi}{dx} = \frac{d^2y}{dx^2}$$

Divide and multiply by ds

$$\sec^2\psi \frac{d\psi}{ds} \cdot \frac{ds}{dx} = \frac{d^2y}{dx^2}$$

$$\sec^2\psi \cdot \sec\psi \cdot \frac{d\psi}{ds} = \frac{d^2y}{dx^2}$$

$$\therefore \frac{d\psi}{ds} = \frac{1}{R})$$

$$\sec^3\psi \cdot \frac{1}{R} = \frac{d^2y}{dx^2}$$

$$\frac{1}{R} = \frac{\frac{d^2y}{dx^2}}{\sec^3\psi}$$

$$\frac{1}{R} = \frac{\frac{d^2y}{dx^2}}{(\sec^2\psi)^{\frac{3}{2}}}$$

$$(\sec^2\psi)^{\frac{3}{2}} = \sec\psi^{\frac{2\times 3}{2}} = \sec^3\psi$$

$$\frac{1}{R} = \frac{\frac{d^2y}{dx^2}}{\left(\tan^2\psi + 1\right)^{\frac{3}{2}}} \qquad (\sec^2\theta = \tan^2\theta + 1)$$

$$\frac{1}{R} = \frac{\frac{d^2y}{dx^2}}{\left(\left(\frac{dy}{dx}\right)^2 + 1\right)^{3/2}} \qquad \tan\psi = \frac{dy}{dx}$$

$\cancel{\frac{1}{R} =}$ ∴ Denominator is very very small

neglecting the term $\left(\frac{dy}{dx}\right)^2$ we get

$$\frac{1}{R} = \frac{d^2y}{dx^2} \quad \text{——} \ (5)$$

From bending equation.

$$\frac{M}{I} = \frac{\sigma}{y} = \frac{E}{R}$$

$$\frac{M}{I} = \frac{E}{R}$$

$$\frac{M}{EI} = \frac{1}{R} \quad \text{——} \ (6)$$

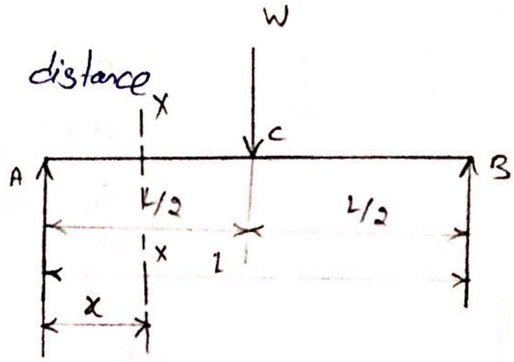
substitute eqn (6) in eqn (5)

$$\frac{d^2y}{dx^2} = \frac{M}{EI}$$

$$\boxed{M = EI\frac{d^2y}{dx^2}} \ \text{——} \ (7)$$

Deflection of a simply supported beam Carrying a point Load at the Centre.

Consider a section x-x at a distance x from A. The bending moment at this section is given by,



$$M_x = R_A \times x$$

$$\therefore R_A = \frac{W}{2} = R_B$$

$$= \frac{W}{2} \times x \quad \text{—— (1)}$$

But we know B.M at any section

$$M = EI\frac{d^2y}{dx^2} \quad \text{—— (2)}$$

Equate eqn (1) & (2)

$$EI\frac{d^2y}{dx^2} = \frac{W}{2} \times x. \quad \text{—— (3)}$$

Integrate the above equation (3) to get slope.

$$EI\int\frac{d^2y}{dx^2} = \int\frac{W}{2}\times x$$

$$EI\frac{dy}{dx} = \frac{W}{2}\left(\frac{x^2}{2}\right) + C_1 \quad \text{—— (4)}$$

For getting the C_1 value we have to apply boundary Conditions.

At $x = \frac{L}{2} \rightarrow \frac{dy}{dx} = 0$ $\qquad 0 = \frac{W}{4}\left(\frac{L^2}{2}\right) + C_1$

$$0 = \frac{WL^2}{4\times4} + C_1$$

$$\boxed{C_1 = -\frac{WL^2}{16}}$$

substitute C_1 value in eqn ④

$$EI\frac{dy}{dx} = \frac{wx^2}{4} - \frac{wL^2}{16} \quad \text{—— ⑤}$$

we know slope is maximum at A, $x = 0$.

$$EI\left(\frac{dy}{dx}\right)_{at\ A} = \frac{w(0)^2}{4} - \frac{wL^2}{16}$$

$$EI\,\theta_A = -\frac{wL^2}{16}$$

$$\theta_A = \frac{-wL^2}{16EI}$$

The slope at point will be equal to θ_A, since the load is symmetrically applied.

$$\boxed{\theta_B = \theta_A = \frac{-wL^2}{16EI}}$$

Deflection at any point.

Deflection at any point obtained by integrating the eqn ⑤

$$\int EI\frac{dy}{dx} = \int\frac{wx^2}{4} - \int\frac{wL^2}{16}$$

$$EIy = \frac{w}{4}\times\frac{x^3}{3} - \frac{wL^2}{16}x + C_2 \quad \text{—— ⑥}$$

where C_2 is another constant of integration.

At A $x = 0$, $y = 0$.

$$EI\times 0 = \frac{w(0)^3}{12} - \frac{wL^2}{16}(0) + C_2$$

$$\boxed{C_2 = 0}$$

substitute the C_2 value in eqn ⑥

$$EIy = \frac{wx^3}{12} - \frac{wxL^2}{16} + 0$$

The above eqn is known as deflection equation.

we can find the deflection at any point by substituting the x value in the above eqn.

For maximum deflection $x = \frac{L}{2}$

$$EI\,y_c = \frac{w}{12}\left(\frac{L}{2}\right)^3 - \frac{wL^2}{16}\left(\frac{L}{2}\right)$$

$$y_c = \frac{w}{12}\frac{L^3}{8} - \frac{wL^3}{32}$$

$$= \frac{wL^3}{96} - \frac{wL^3}{32}$$

48 | 48 96,
2, | 1, 2

$$= \cancel{32wL^3 - 96wL^3}$$

$$= \frac{wL^3 - 3wL^3}{96}$$

$$= \frac{-\cancel{2}wL^3}{\cancel{96}\,48}$$

$$\boxed{y_c = \frac{-wL^3}{48EI}}$$

A beam 6m long, simply supported at its ends, is carrying a point load of 50KN at its centre. The M.O.I of the beam is $78 \times 10^6 \, mm^4$. The young's modulus of the material is $2.1 \times 10^5 \, N/mm^2$. Calculate (i) deflection at the centre.
ii) slope at the supports.

Given data :-

length of the beam $(l) = 6m$

point Load $(W) = 50KN$

M.O.I $(I) = 78 \times 10^6 \, mm^4$

youngs modulus $(E) = 2.1 \times 10^5 \, N/mm^2$.

we know the deflection at the centre in case of simply supported beam carries point load it is given by

$$y = \frac{WL^3}{48EI} = \frac{50 \times 10^3 \times (6000)^3}{48 \times 2.1 \times 10^5 \times 78 \times 10^6}$$

$$= \frac{1.08 \times 10^{16}}{7.8624 \times 10^{14}}$$

$$\boxed{y = 13.73 mm}$$

$$\text{slope } (\theta) = \frac{Wl^2}{16EI} = \frac{50 \times 10^3 \times (6000)^2}{16 \times 2.1 \times 10^5 \times 78 \times 10^6}$$

$$= \frac{1.8 \times 10^{12}}{2.6208 \times 10^{14}} = 6.86 \times 10^{-3} \text{ rad} \times \frac{180}{\pi}$$

$$\boxed{\theta = 0.3935^\circ}$$

A beam 3m long, simply supported at its ends is carrying a point load W at the centre. If the slope at the ends of the beam should not exceed 1°, find the deflection at the centre of the beam.

Given data:-

$$\text{Length } (L) = 3m$$

$$\text{slope } (\theta) = 1^\circ \times \frac{\pi}{180} = 0.0174 \text{ rad.}$$

we know

$$\text{deflection } (y) = \frac{Wl^3}{48EI} = \frac{L}{3} \times \frac{Wl^2}{16EI}$$

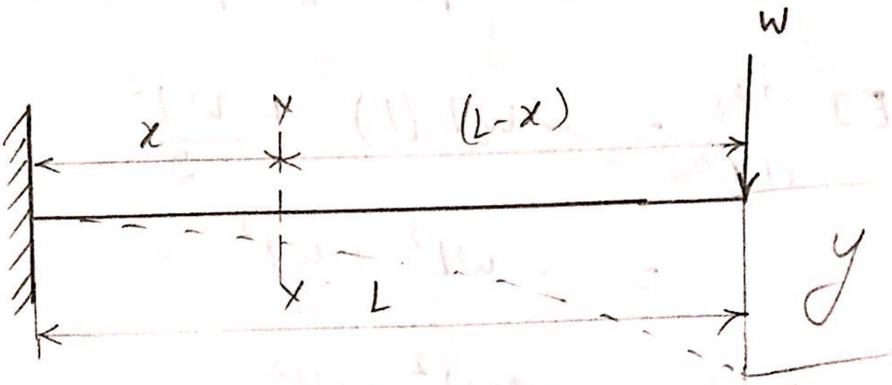
$$\theta = \frac{Wl^2}{16EI} = 0.0174 \text{ rad.}$$

$$y = \frac{\cancel{3}000}{\cancel{3}} \times 0.0174$$

$$= 1000 \times 0.0174$$

$$\boxed{y = 17.45 \text{ mm}}$$

DEFLECTION OF A CANTILEVER BEAM WITH A POINT LOAD AT THE FREE END :-



Bending moment at any section is given by

$$M = EI \frac{d^2y}{dx^2}$$

B.M at X-X section $= M_x = -W(L-x)$

$$EI\frac{d^2y}{dx^2} = -WL + Wx$$

For getting slope integrate the above eqn

$$\int EI \frac{d^2y}{dx^2} = -\int WL + \int Wx + C_1$$

$$EI\frac{dy}{dx} = -WLx + \frac{Wx^2}{2} + C_1$$

Apply boundary Conditions

$$x = 0; \quad \frac{dy}{dx} = 0$$

$$EI(0) = -WL(0) + \frac{W(0)^2}{2} + C_1$$

$$\boxed{C_1 = 0}$$

$$EI\frac{dy}{dx} = -WLx + \frac{Wx^2}{2} \qquad \text{—} \quad (1)$$

if $x = L$; then slope $\left(\frac{dy}{dx}\right)$ is maximum. For getting max. slope put $x = L$ in the above eqn.

$$EI\frac{dy}{dx} = -wL(L) + \frac{wL^2}{2}$$

$$= -wL^2 + \frac{wL^2}{2}$$

$$EI\frac{dy}{dx} = \frac{-2wL^2 + wL^2}{2}$$

$$\boxed{\frac{dy}{dx} = \frac{-wL^2}{2EI}}$$

Integrate the eqn ① to get max. deflection

$$\int EI\frac{dy}{dx} = \int -wLx + \int \frac{wx^2}{2} + C_2$$

$$EIy = -wL\frac{x^2}{2} + \frac{wx^3}{6} + C_2$$

Apply boundary conditions

if $x = 0$; $y = 0$

$$EIy(0) = -\frac{wL(0)^2}{2} + \frac{w(0)^3}{6} + C_2$$

$$\boxed{C_2 = 0}$$

$$EIy = -\frac{wLx^2}{2} + \frac{wx^3}{6}$$

~~For~~ ~~Calculating~~ The maximum obtained at the free end i.e $x = L$

$$EIy = -\frac{wL(L)^2}{2} + \frac{wL^3}{6}$$

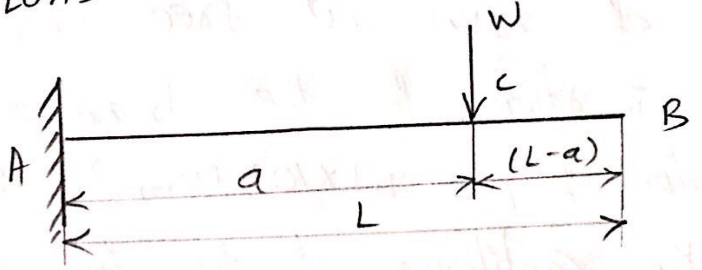
$$= -\frac{wL^3}{2} + \frac{wL^3}{6}$$

$$EIy = \frac{-6wL^3 + 2wL^3}{12}$$

$$y = -\frac{\cancel{4}^{1}\, wl^3}{\cancel{12}_{3}\, EI}$$

$$\boxed{y = -\frac{wl^3}{3EI}}$$

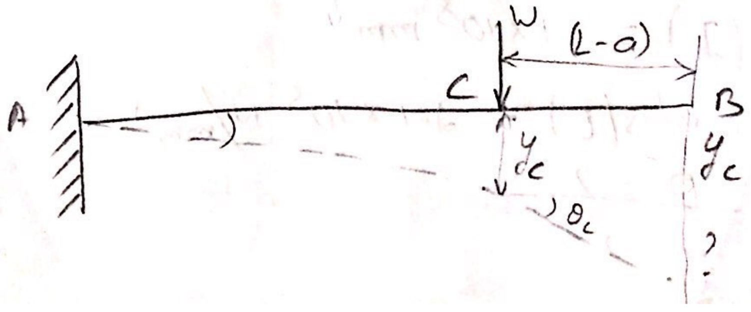
DEFLECTION OF CANTILEVER BEAM WITH A POINT LOAD AT A DISTANCE 'a' FROM FIXED END:-

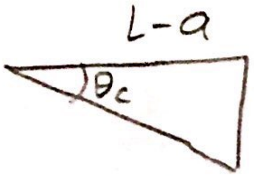


we know

$$y_c = \frac{wL^3}{3EI} = \frac{wa^3}{3EI} \qquad \boxed{\therefore L = a}$$

$$\theta_c = \frac{wL^2}{2EI} = \frac{wa^2}{2EI}$$





$$\tan\theta_c = \frac{x}{L-a}$$

$$x = \tan\theta_c \times (L-a)$$

$$x = \theta_c (L-a)$$

$$\boxed{\therefore \tan\theta = \theta}$$

$$y_B = x + y_c$$

$$= \theta(L-a) + \left(\frac{wa^3}{3EI}\right)$$

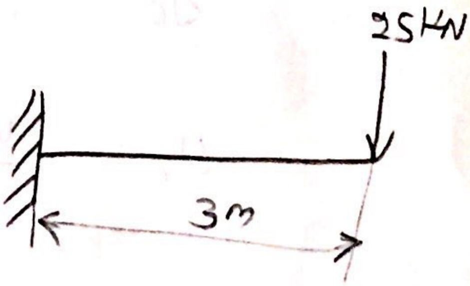
$$\boxed{y_B = \frac{wa^2}{2EI}(L-a) + \left(\frac{wa^3}{3EI}\right)}$$

Q) A Cantilever of length 3m is Carrying a point load of 25KN at free end. If the moment of inertia of the beam = 1×10^8 mm4. and the value of $E = 2.1 \times 10^5$ N/mm^2. Find.

i) slope of the Cantilever at the free end.

ii) Deflection at the free end.

Sol Given data :-



Length $(L) = 3$ m

Load $(W) = 25$KN

m.o.I $(I) = 1 \times 10^8$ mm^4

youngs modulus$(E) = 2.1 \times 10^5$ N/mm^2

$\theta = ?$

$y = ?$

we know

$$\theta = \frac{WL^2}{2EI} = \frac{25 \times 10^3 \times (3000)^2}{2 \times 2.1 \times 10^5 \times 1 \times 10^8}$$

$$\boxed{\theta = 0.005357 \text{ rad}}$$

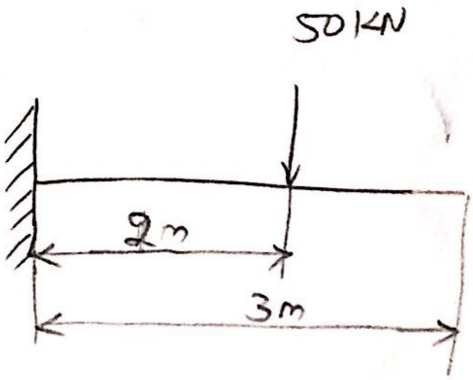
$$\text{Deflection } (y) = \frac{WL^3}{3EI} = \frac{25 \times 10^3 \times (3000)^3}{3 \times 2.1 \times 10^5 \times 10^8}$$

$$\boxed{y = 10.71 \text{ mm}}$$

A Cantilever of length 3m is Carrying a point load of 50KN at a distance of 2m from the fixed end. of $I = 1 \times 10^8 \text{ mm}^4$. and $E = 2 \times 10^5 \text{ N/mm}^2$

find 1) slope at free end.

2) Deflection at the free end.



Given data :-

moment of Inertia $(I) = 1 \times 10^8 \text{ mm}^4$

Youngs modules $(E) = 2 \times 10^5 \text{ N/mm}^2$

we know

$$\text{slope at free end} = \frac{Wa^2}{2EI}$$

$$= \frac{50000 \times (2000)^2}{2 \times 2 \times 10^5 \times 1 \times 10^8} = \frac{2 \times 10^{11}}{2 \times 10^{13} \times 2}$$

$$\boxed{\theta = 5 \times 10^{-3} \text{ rad.}}$$

Deflection at the free end

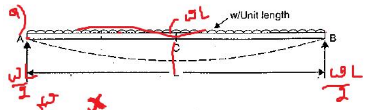
$$y = \frac{wa^3}{3EI} + \frac{wa^2}{2EI}(L-a)$$

$$= \frac{50\times10^3 \times (2000)^3}{3\times 2\times10^5 \times 10^8} + \frac{50\times10^3 \times (2000)^2}{2\times 2\times10^5 \times 10^8}(3000-2000)$$

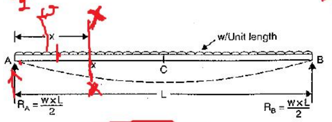
$$= 6.67 + 5$$

$$\boxed{y = 11.67 \text{ mm}}$$

a)



Slope & Deflection



$$M_x = EI\frac{d^2y}{dx^2}$$

$$R_A \times x - wx \times \frac{x}{2} = EI\frac{d^2y}{dx^2}$$

Integrating to above eqn

$$\int EI\frac{d^2y}{dx^2} = \int \frac{wL}{2} \times x - \int \frac{wx^2}{2}$$

$$EI\int \frac{d^2y}{dx^2} = \frac{wL}{2}\int x - \frac{w}{2}\int x^2$$

$$EI\frac{dy}{dx} = \frac{wL}{2} \times \frac{x^2}{2} - \frac{wx^3}{2 \times 3} + C_1 \quad \text{--- (1)}$$

By applying boundary conditions

at $x = L/2$; $\frac{dy}{dx} = 0$

$$EI(0) = \frac{wL}{4} \times \left(\frac{L}{2}\right)^2 - \frac{w}{6}\left(\frac{L}{2}\right)^3 + C_1$$

$$0 = \frac{wL^3}{16} - \frac{wL^3}{48} + C_1$$

$$0 = \frac{3wL^3 - wL^3}{48} + C_1$$

$$C_1 = -\frac{\cancel{2}wL^3}{\cancel{48}_{24}}$$

$$\boxed{C_1 = -\frac{wL^3}{24}}$$

substitute C_1 in eqn (1) to get $\frac{dy}{dx}$

$$EI\frac{dy}{dx} = \frac{wL}{2} \times \frac{x^2}{2} - \frac{wx^3}{6} - \frac{wL^3}{24} \quad \text{--- (2)}$$

max slope @ ends where $x = 0$

$$EI\frac{dy}{dx} = 0 - 0 - \frac{wL^3}{24}$$

$$\boxed{\frac{dy}{dx} = \frac{-wL^3}{24EI}}$$

Integrating to eqn (2)

$$\int EI\frac{dy}{dx} = \frac{wL}{4}\int x^2 dx - \frac{w}{6}\int x^3 dx - \frac{wL^3}{24}\int dx$$

$$EIy = \frac{wL}{4}\frac{x^3}{3} - \frac{w}{6}\frac{x^4}{4} - \frac{wL^3}{24}x + C_2$$

For calculating C_2, apply B.C

@ $x = 0$; $y = 0$

then $\boxed{C_2 = 0}$

$$EIy = \frac{wL}{4}\frac{x^3}{3} - \frac{w}{6}\frac{x^4}{4} - \frac{wL^3}{24}x$$

For getting max deflection apply B.C

@ $x = L/2$ y will be max

$$EIy_{max} = \frac{wL}{12} \times \left(\frac{L}{2}\right)^3 - \frac{w}{24}\left(\frac{L}{2}\right)^4 - \frac{wL^3}{24}\left(\frac{L}{2}\right)$$

$$EIy_{max} = \frac{wL^4}{96} - \frac{wL^4}{384} - \frac{wL^4}{48}$$

$$= \frac{4wL^4 - wL^4 - 8wL^4}{384}$$

$$EIy_{max} = \frac{4wL^4 - 9wL^4}{384}$$

$$\boxed{y_{max} = \frac{-5wL^4}{384EI}} \text{ — Deflection}$$

$$\boxed{\frac{dy}{dx} = \frac{-wL^3}{24EI}} \text{ — Slope}$$

Problem 12.5. *A beam of uniform rectangular section 200 mm wide and 300 mm deep is simply supported at its ends. It carries a uniformly distributed load of 9 kN/m run over the entire span of 5 m. If the value of E for the beam material is 1×10^4 N/mm^2, find :*

(i) the slope at the supports and *(ii) maximum deflection.*

Sol. Given :

Width, $b = 200$ mm

Depth, $d = 300$ mm

M.O.I., $I = \dfrac{bd^3}{12} = \dfrac{200 \times 300^3}{12} = 4.5 \times 10^8 \text{ mm}^4$

U.d.l., $w = 9$ kN/m $= 9000$ N/m

Span, $L = 5$ m $= 5000$ mm

$\therefore$ Total load, $W = w \,.\, L^* = 9000 \times 5 = 45000$ N

Value of $E = 1 \times 10^4$ N/mm^2

Let θ_A = Slope at the support

and y_C = Maximum deflection.

(*i*) Using equation (12.12), we get

$$\theta_A = -\frac{W \,.\, L^2}{24EI}$$

$$= -\frac{45000 \times 5000^2}{24 \times 1 \times 10^4 \times 4.5 \times 10^8} \text{ radians}$$

$$= \mathbf{0.0104 \text{ radians.}} \quad \textbf{Ans.}$$

(*ii*) Using equation (12.14), we get

$$y_C = \frac{5}{384} \cdot \frac{W \,.\, L^3}{EI}$$

$$= \frac{5}{384} \times \frac{45000 \times 5000^3}{1 \times 10^4 \times 4.5 \times 10^8}$$

$$= \mathbf{16.27 \text{ mm.}} \quad \textbf{Ans.}$$

Problem 12.6. *A beam of length 5 m and of uniform rectangular section is simply supported at its ends. It carries a uniformly distributed load of 9 kN/m run over the entire length. Calculate the width and depth of the beam if permissible bending stress is 7 N/mm² and central deflection is not to exceed 1 cm.*

Take E for beam material = 1 × 10⁴ N/mm².

Sol. Given :

Length, $L = 5 \text{ m} = 5000 \text{ mm}$

U.d.l., $w = 9 \text{ kN/m}$

∴ Total load, $W = w.L = 9 \times 5 = 45 \text{ kN} = 45000 \text{ N}$

Bending stress, $f = 7 \text{ N/mm}^2$

Central deflection, $y_C = 1 \text{ cm} = 10 \text{ mm}$

Value of $E = 1 \times 10^4 \text{ N/mm}^2$

Let b = Width of beam is mm

and d = Depth of beam in mm

∴ M.O.I., $I = \dfrac{bd^3}{12}$

Using equation (12.14), we get

$$y_C = \frac{5}{384} \cdot \frac{W \cdot L^3}{EI}$$

or
$$10 = \frac{5}{384} \times \frac{45000 \times 5000^3}{1 \times 10^4 \times \left(\dfrac{bd^3}{12}\right)}$$

or
$$bd^3 = \frac{5}{384} \times \frac{45000 \times 5000^3 \times 12}{1 \times 10^4 \times 10}$$
$$= 878.906 \times 10^7 \text{ mm}^4 \qquad ...(i)$$

The maximum bending moment for a simply supported beam carrying a uniformly distributed load is given by,

$$M = \frac{w \cdot L^2}{8} = \frac{W \cdot L}{8} \qquad (\because \quad W = w.L = \text{Total load})$$
$$= \frac{45000 \times 5}{8} \text{ Nm} = \frac{45000 \times 5}{8} \times 1000 \text{ Nmm}$$
$$= 28125000 \text{ Nmm}$$

Now using the bending equation as

$$\frac{M}{I} = \frac{f}{y}$$

or
$$\frac{28125000}{\left(\dfrac{bd^3}{12}\right)} = \frac{7}{\left(\dfrac{d}{2}\right)} \qquad \left(\because \text{ Here } y = \frac{d}{2}\right)$$

or
$$\frac{28125000 \times 12}{bd^3} = \frac{14}{d}$$

or
$$bd^2 = \frac{28125000 \times 12}{14} = 24107142.85 \text{ mm}^3 \qquad ...(ii)$$

Dividing equation (*i*) by equation (*ii*), we get

$$d = \frac{838.906 \times 10^7}{24107142.85} = \mathbf{364.58 \text{ mm.}} \quad \textbf{Ans.}$$

Substituting this value of '*d*' in equation (*ii*), we get

$$b \times (364.58)^2 = 24107142.85$$

∴
$$b = \frac{24107142.85}{364.58^2} = \mathbf{181.36 \text{ mm.}} \quad \textbf{Ans.}$$

Problem 12.7. *A beam of length 5 m and of uniform rectangular section is supported at its ends and carries uniformly distributed load over the entire length. Calculate the depth of the section if the maximum permissible bending stress is 8 N/mm² and central deflection is not to exceed 10 mm.*

Take the value of $E = 1.2 \times 10^4$ N/mm².

Sol. Given :

Length, $L = 5$ m $= 5000$ mm

Bending stress, $f = 8$ N/mm²

Central deflection, $y_C = 10$ mm

Value of $E = 1.2 \times 10^4$ N/mm²

Let $W =$ Total load

and $d =$ Depth of beam

The maximum bending moment for a simply supported beam carrying a uniformly distributed load is given by,

$$M = \frac{w.L^2}{8} = \frac{W.L}{8} \qquad (\because \quad W = w.L) \qquad ...(i)$$

Now using the bending equation,

$$\frac{M}{I} = \frac{f}{y}$$

or

$$M = \frac{f \times I}{y} = \frac{8 \times I}{(d/2)} \qquad \left(\because \; y = \frac{d}{2}\right)$$

$\therefore$

$$M = \frac{16I}{d} \qquad ...(ii)$$

Equating the two values of B.M., we get

$$\frac{W.L}{8} = \frac{16I}{d}$$

or

$$W = \frac{16 \times 8I}{L \times d} = \frac{128I}{L \times d} \qquad ...(iii)$$

Now using equation (12.14), we get

$$y_C = \frac{5}{384} \times \frac{WL^3}{EI}$$

or

$$10 = \frac{5}{384} \times \frac{128I}{L \times d} \times \frac{L^3}{EI} \qquad \left(\because \; y_C = 10 \text{ mm and } W = \frac{128I}{L \times d}\right)$$

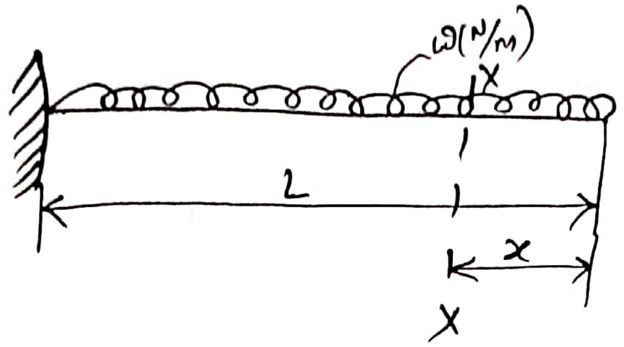
$$= \frac{5}{384} \times \frac{128 \times L^2}{d \times E}$$

or

$$d = \frac{5}{384} \times \frac{128 \times L^2}{10 \times E} = \frac{5}{384} \times \frac{128 \times 5000^2}{10 \times 1.2 \times 10^4}$$

$= 347.2$ mm $=$ **34.72 cm.** **Ans.**

Deflection of a Cantilever with a uniform Distributed Load :-



we know Bending moment at any section is given by

$$M = EI\frac{d^2y}{dx^2} \quad \text{—} (1)$$

Here Bending moment at section X-X is

$$M = -\omega(x)\left(\frac{x}{2}\right) \quad \text{—} (2)$$

Equating eqn (1) & (2)

$$EI\frac{d^2y}{dx^2} = -\frac{\omega x^2}{2} \quad \text{—} (3)$$

For getting maximum slope & Deflection we have to integrate eqn (3)

$$\int EI\frac{d^2y}{dx^2} = -\int\frac{\omega x^2}{2}$$

$$EI\frac{dy}{dx} = -\frac{\omega}{2}\left(\frac{x^3}{3}\right) + C_1 \quad \text{—} (4)$$

For calculating integration constant i.e C_1 we have to apply boundary conditions.

at $x = L$ $\quad \frac{dy}{dx} = 0$

$$EI(0) = -\frac{\omega}{2}\left(\frac{L^3}{3}\right) + C_1$$

$$\boxed{C_1 = +\frac{\omega L^3}{6}}$$

substitute C_1 in eqn (4)

$$EI\frac{dy}{dx} = -\frac{\omega x^3}{6} + \frac{\omega L^3}{6} \quad \text{—— (5)}$$

at $x=0$; slope will be maximum.

$$EI\frac{dy}{dx} = -\frac{\omega(0)^3}{6} + \frac{\omega L^3}{6}$$

$$\boxed{\frac{dy}{dx} = \frac{\omega L^3}{6EI}} \quad \text{—— slope equation.}$$

For getting maximum deflection we have to integrate eqn (5)

$$\int EI\frac{dy}{dx} = -\int\frac{\omega x^3}{6} + \int\frac{\omega L^3}{6}$$

$$EIy = -\frac{\omega}{6}\left[\frac{x^4}{4}\right] + \frac{\omega L^3}{6} \times x + C_2 \quad \text{—— (6)}$$

for calculating C_2 apply Boundary Conditions in such away that y should be zero.

at $x = L$, $y = 0$

$$EI(0) = -\frac{\omega L^4}{24} + \frac{\omega L^3 \cdot L}{6} + C_2$$

$$C_2 = \frac{\omega L^4}{24} - \frac{\omega L^4}{6}$$

$$\boxed{C_2 = -\frac{\omega L^4}{8}}$$

substitute C_2 value in eqn (6)

$$EIy = -\frac{wx^4}{24} + \frac{wL^3}{6} \times x - \frac{wL^4}{8}$$

For calculating maximum deflection apply B.C

at $x = 0$; y is max.

$$EIy = -\frac{w(0)^4}{24} + \frac{wL^3 \times 0}{6} - \frac{wL^4}{8}$$

$$\boxed{y = -\frac{wL^4}{8EI}}$$ — Deflection equation.

$$\boxed{\frac{dy}{dx} = \frac{wL^3}{6EI}}$$ — slope equation.

Problem 13.3. *A cantilever of length 2.5 m carries a uniformly distributed load of 16.4 kN per metre length over the entire length. If the moment of inertia of the beam = 7.95 × 10^7 mm^4 and value of E = 2 × 10^5 N/mm^2, determine the deflection at the free end.*

Sol. Given :

Length,	$L = 2.5 \text{ m} = 2500 \text{ mm}$
U.d.l.,	$w = 16.4 \text{ kN/m}$
∴ Total load,	$W = w \times L = 16.4 \times 2.5 = 41 \text{ kN} = 41000 \text{ N}$
Value of	$I = 7.95 \times 10^7 \text{ mm}^4$
Value of	$E = 2 \times 10^5 \text{ N/mm}^2$
Let	y_B = Deflection at the free end,

Using equation (13.6), we get

$$y_B = \frac{WL^3}{8EI} = \frac{41000 \times 2500^3}{8 \times 2 \times 10^5 \times 7.95 \times 10^7}$$

$$= \mathbf{5.036 \text{ mm.}} \quad \textbf{Ans.}$$

Problem 13.4. *A cantilever of length 3 m carries a uniformly distributed load over the entire length. If the deflection at the free end is 40 mm, find the slope at the free end.*

Sol. Given :

Length, $L = 3 \text{ m} = 3000 \text{ mm}$

Deflection at free end, $y_B = 40 \text{ mm}$

Let θ_B = slope at the free end

Using equation (13.6), we get

$$y_B = \frac{WL^3}{8EI}$$

or

$$40 = \frac{WL^2 \times L}{8EI} = \frac{WL^2 \times 3000}{8EI}$$

or

$$\frac{WL^2}{EI} = \frac{40 \times 8}{3000} \qquad \text{...}(i)$$

Slope at the free end is given by equation (13.5),

$$\therefore \qquad \theta_B = -\frac{WL^2}{6EI} = -\frac{WL^2}{EI} \times \frac{1}{6} = -\frac{40 \times 8}{3000} \times \frac{1}{6}$$

$$\left[\because \text{ From equation } (i), \frac{WL^2}{EI} = \frac{40 \times 8}{3000}\right]$$

$= \mathbf{0.01777 \text{ rad.}}$ **Ans.**

Problem 13.4 (A). *A cantilever 120 mm wide and 200 mm deep is 2.5 m long. What is the uniformly distributed load which the beam can carry in order to produce a deflection of 5 mm at the free end ? Take E = 200 GN/m².*

Sol. Given :

Width, $b = 120$ mm

Depth, $d = 200$ mm

Length, $L = 2.5 \text{ m} = 2.5 \times 1000 = 2500$ mm

Deflection at free end, $y_B = 5$ mm

Value of $E = 200 \text{ GN/m}^2 = 200 \times 10^9 \text{ N/m}^2$ $\quad (\because \; \text{G} = \text{Giga} = 10^9)$

$$= \frac{200 \times 10^9 \text{ N}}{(1000)^2 \text{ mm}^2} \qquad [\because \; 1 \text{ m}^2 = (1000 \text{ mm})^2]$$

$$= 2 \times 10^5 \text{ N/mm}^2$$

Moment of inertia, $$I = \frac{bd^3}{12} = \frac{120 \times 200^3}{12} = 8 \times 10^7 \text{ mm}^4$$

Let w = uniformly distributed load per m length in N

W = Total load

$= w \times L$ (Here L is in metre)

$= w \times 2.5 = 2.5 \times w$ N

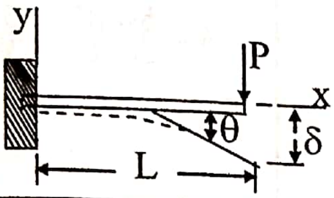
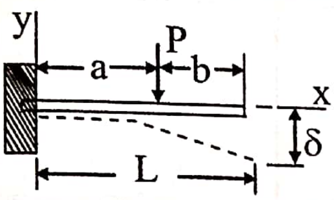
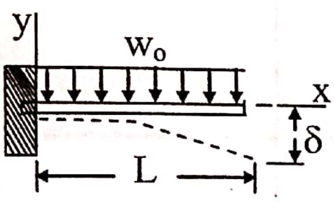
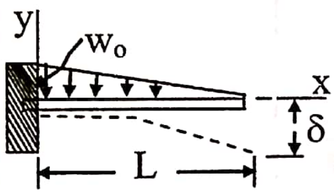
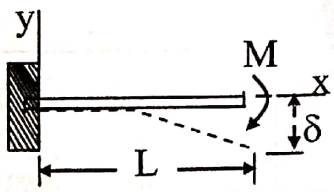
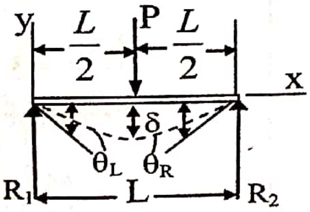
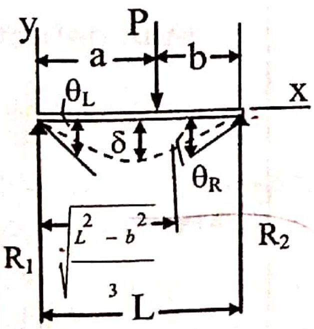
Using equation (13.6), we get

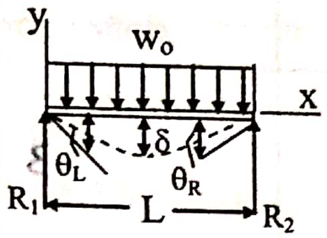
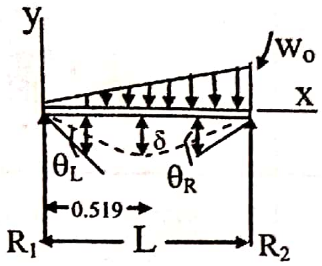
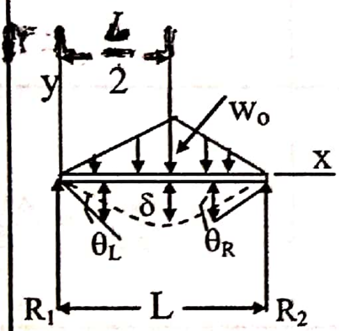
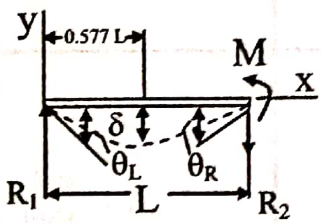
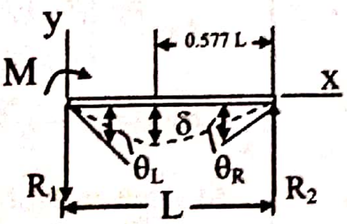
$$y_B = \frac{WL^3}{8EI}$$

or $$5 = \frac{2.5\,w \times 2500^3}{8 \times 2 \times 10^5 \times 8 \times 10^7}$$

$$w = \frac{5 \times 8 \times 2 \times 10^5 \times 8 \times 10^7}{2.5 \times 2500^3} = 16384 \text{ N/m}$$

$= \mathbf{16.384 \text{ kN/m.}}$ **Ans.**

Type of load	Max. moment (+ sagging)	Slope at end	Maximum deflection
	$M = -PL$	$\theta = \frac{PL^2}{2EI}$	$\delta = \frac{PL^3}{3EI}$
	$M = -Pa$	$\theta = \frac{Pa^2}{2EI}$	$\delta = \frac{Pa^3}{3EI} + \frac{Pa^2}{2EI}(b)$
	$M = \frac{w_o L^2}{2} = -\frac{WL}{2}$ Where $W = w_o a$	$\theta = \frac{w_o L^3}{6EI}$ $= \frac{WL^2}{6EI}$	$\delta = \frac{w_o L^4}{8EI} = \frac{WL^3}{8EI}$
	$M = \frac{w_o L^2}{6} = -\frac{WL}{3}$ (Where $W = \frac{1}{2} w_o \ell$)	$\theta = \frac{w_o L^3}{24EI}$ $= \frac{WL^2}{12EI}$	$\delta = \frac{w_o L^4}{30EI} = \frac{WL^3}{15EI}$
	$M = -M$	$\theta = \frac{ML}{EI}$	$\delta = \frac{ML^2}{2EI}$
	$M = \frac{PL}{4}$	$\theta_L = \theta_R = \frac{PL^2}{16EI}$	$\delta = \frac{PL^3}{48EI}$
	$M = \frac{Pab}{L}$ at $x = a$	$\theta_L = \frac{pb(L^2 - b^2)}{6EIL}$ $\theta_R = \frac{Pa(L^2 - a^2)}{6EIL}$	Maximum deflection $\delta = \frac{Pb(L^2 - b^2)^{3/2}}{9\sqrt{3}\,EIL}$ at $x = \sqrt{\frac{L^2 - b^2}{3}}$ At center (not max) $\delta = \frac{Pb}{48EI}(3L^2 - 4b^2)$ where $a \geq b$

	$M = \frac{w_o L^2}{8}$ $= \frac{WL}{8}$ Here, $W = w_o L$	$\theta_L = \theta_R = \boxed{\frac{w_o L^3}{24EI}}$	$\delta = \frac{5 w_o L^4}{384EI} = \frac{5WL^3}{384EI}$
	$M = \frac{w_o L^2}{9\sqrt{3}}$ $= \frac{2WL}{9\sqrt{3}}$ Here, $W = w_o L/2$	$EIy = \frac{w_o x}{24}(L^2 - 2Lx^2 + x^3)$	$\delta = \frac{1}{2}\left[\frac{5 w_o \ell^4}{384EI}\right]$
	$M = \frac{w_o L^2}{12}$ $= \frac{WL}{6}$	$\theta_L = \theta_R = \frac{5 w_o L^3}{192EI}$	$\delta = \frac{w_o L^4}{120EI} = \frac{WL^3}{60EI}$
	$M = M$	$\theta_L = \frac{ML}{6EI}$ $\theta_R = \frac{ML}{3EI}$	Max. deflection $\delta = \frac{ML^2}{9\sqrt{3}EI}$ at $x = \frac{1}{\sqrt{3}}$ At centre (not max) $\delta = \frac{ML^2}{16EI}$
	$M = M$	$\theta_L = \frac{ML}{3EI}$ $\theta_R = \frac{ML}{6EI}$	Max. deflection $\delta = \frac{ML^2}{9\sqrt{3}EI}$ At centre (not max) $\delta = \frac{ML^2}{16EI}$