# Unit - I

## Functions of a complex variables :

A function $w = f(z)$ is said to tends to limit as $z$ approaches a point $z_0$, If for every real $\epsilon$ (epsilon), we can find a positive $\delta$ (delta) such that

$$|f(z) - l| < \epsilon \quad \text{for } 0 < |z - z_0| < \delta$$

$$\lim_{z \to z_0} f(z) = l$$

**Continuity :** A function $f(z)$ is continuous at $z = z_0$ if $f(z_0)$ is defined and

$$\lim_{z \to z_0} f(z) = f(z_0)$$

**Derivative of $f(z)$ :-**

Let $w = f(z)$ be a given function defined for all $z$ in a neighbourhood of $z_0$

$$\lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z_0)}{\Delta z} \text{ exists.}$$

The function $f(z)$ is said to be derivable at the point $z_0$ then the ~~function $f(z)$ is~~ s limit is denoted by $f'(z_0)$

also can be written as

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

~~Anal~~ <u>Analytic Function</u>:- ~~$f'(z)$~~ Let a function $f(z)$ be derivable at every point $z$ in an $\epsilon$ neighbourhood of $z_0$

<u>Cauchy's Riemann Equations (C-R)</u> :-

The necessary and sufficient condition for the derivative of the function of $f(z) = w = u(x,y) + i\,v(x,y)$ for all values of $z$ in domain

R

Case i) $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are continous functions of $x$ & $y$.

ii) $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$.

**Note :-** If $f(z) = u + iv$ is analytic function then $u, v$ are satisfies Couchy's Riemann eqn's.

**# Laplace eqn's**

If $f(z) = u + iv$ is analytic in a domain $D$ then $u, v$ satisfies laplace eqn.

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$\nabla^2 v = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

**Hormonic Function :-** Solutions of laplace eqn's having continous second order partial derivatives are called Hormonic Function.

Conjugate hormonic function: If two hormonic functions $u$ & $v$ satisfies the cauchy's riemand eq$^n$ in a domain $D$ & they are the real & imaginary parts of an analytic function in $D$. Then $v$ is said to be a conjugate hormonic function of $u$ in domain $D$

ex: i) Show that $f(z) = z^2$ is analytic function

Given $f(z) = z^2$

$= (x+iy)^2$

$u + iv = x^2 + (iy)^2 + 2xiy$

$u + iv = (x^2 - y^2) + i\, 2xy$

$u = x^2 - y^2$ $\qquad$ $v = 2xy$

$\frac{\partial u}{\partial x} = 2x$ $\quad$ $\frac{\partial u}{\partial y} = -2y$ $\quad$ $\frac{\partial v}{\partial x} = 2y$ $\quad$ $\frac{\partial v}{\partial y} = 2x$

we have $\frac{\partial u}{\partial x} = 2x = \frac{\partial v}{\partial y} \Rightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$

we have $\frac{\partial u}{\partial y} = -2y = -\frac{\partial v}{\partial x} \Rightarrow \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$

Hence it satisfie CR eq$^n$

$\Rightarrow$ It is an analytic function.

ii) S.T $f(z) = z + 2\bar{z}$ is not analytic.

Given $f(z) = z + 2\bar{z}$

$= (x+iy) + 2(x-iy)$

$u + iv = 3x - iy$

$u = 3x$ $\quad$ $v = -y$

$\frac{\partial u}{\partial x} = 3$ $\quad$ $\frac{\partial u}{\partial y} = 0$ $\quad$ $\frac{\partial v}{\partial x} = 0$ $\quad$ $\frac{\partial v}{\partial y} = -1$

We ha. $\frac{\partial u}{\partial x} = 3 \neq \frac{\partial v}{\partial y} \Rightarrow \frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}$

CR not satisfied.

$\therefore$ $z + 2\bar{z}$ is not AF

iii) S.T $f(z) = \bar{z}$ is not analytic func!

G. $f(z) = \bar{z}$

$= x - iy$.

$u = x$ $\quad$ $v = -y$

$\frac{\partial u}{\partial x} = \cancel{0}\,1$ $\quad$ $\frac{\partial u}{\partial y} = 0$ $\quad$ $\frac{\partial v}{\partial x} = 0$ $\quad$ $\frac{\partial v}{\partial y} = \cancel{0}\,-1$

We have $\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}$

CR not satisfied

$\bar{z}$ is not AF.

iv) Determine wheather the Function $f(z) = 2xy + i(x^2+y^2)$ is AF

$f(z) = 2xy + i(x^2+y^2)$

$U = 2xy$ $\qquad$ $V = x^2 + y^2$

$\frac{\partial u}{\partial x} = 2y$ $\quad$ $\frac{\partial u}{\partial y} = 2x$ $\quad$ $\frac{\partial v}{\partial x} = 2x$ $\quad$ $\frac{\partial v}{\partial y} = 2y$

$$\frac{\partial u}{\partial y} = 2x \neq -\frac{\partial v}{\partial x}.$$

$$\Rightarrow \frac{\partial u}{\partial y} \neq -\frac{\partial v}{\partial x}.$$

$\therefore$ U & V Not satisfies CF

$\therefore$ Not AF

v) Find all the values of K such that

$f(z) = e^x(\cos ky + i \sin ky)$ is analytical.

$f(x) = e^x \cos ky + i e^x \sin ky$

$U + iV = e^x \cos ky + i e^x \sin ky$

$U = e^x \cos ky$ $\qquad$ $V = e^x \sin ky$

$\frac{\partial u}{\partial x} = e^x \cos ky$ $\qquad$ $\frac{\partial u}{\partial y} = -K e^x \sin ky$

$\frac{\partial v}{\partial x} = e^x \sin ky$ $\qquad$ $\frac{\partial v}{\partial y} = K e^x \cos ky$

we know

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\Rightarrow e^{x}\cos ky = ke^{x}\cos ky \quad k = 1$$

$$\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$$

$$\Rightarrow -ke^{x}\sin ky = e^{x}\sin ky \quad k = 1$$

vi) Determin P, $F(z) = \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}\left(\frac{Px}{y}\right)$

is analatic

$$F(z) = \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}\left(\frac{Px}{y}\right)$$

$$u + iv = \frac{1}{\cancel{(x+iy)}\,2}\log(x^2+y^2) + i\tan^{-1}\left(\frac{Px}{y}\right)$$

$$u = \frac{1}{\cancel{(x+iy)}\,2}\log(x^2+y^2) \qquad v = \tan^{-1}\left(\frac{Px}{y}\right)$$

$$\frac{\partial u}{\partial x} = \frac{1}{\cancel{2}}\frac{1}{x^2+y^2}\cancel{2}x \qquad \frac{\partial u}{\partial y} = \frac{1}{\cancel{2}}\frac{1}{x^2+y^2}\cancel{2}y$$

$$\frac{\partial v}{\partial x} = \frac{1}{\cancel{1}+\left(\frac{Px}{y}\right)^2}\left(\frac{P}{y}\right) = \frac{y^{\cancel{2}}}{y^2+P^2x^2}\frac{P}{\cancel{y}}$$

$$\frac{\partial v}{\partial y} = \frac{1}{1+\left(\frac{Px}{y}\right)^2}\left(\frac{-Px}{y^2}\right) = \frac{\cancel{y^2}}{y^2+P^2x^2}\frac{-Px}{\cancel{y^2}}$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \Rightarrow \frac{1}{x^2+y^2}x = \frac{-Px}{y^2+P^2x^2}$$

$p = -1$

∴ follows CF & they are AF

vii) Show that the function

$u = 2\log(x^2+y^2)$ is harmonic &

Find its harmonic conjugate.

Given $u = 2\log(x^2+y^2)$

$$\frac{\partial u}{\partial x} = \frac{2}{x^2+y^2}\,2x = \frac{4x}{x^2+y^2}$$

$$\frac{u}{v}$$

$$\boxed{\frac{vu' - uv'}{v^2}}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{(x^2+y^2)4 - (2x)(4x)}{(x^2+y^2)^2}$$

$$= \frac{4x^2+4y^2-8x^2}{(x^2+y^2)^2} = \frac{-4x^2+4y^2}{(x^2+y^2)^2}$$

$$\frac{\partial u}{\partial y} = \frac{2}{x^2+y^2}\,2y = \frac{4y}{x^2+y^2}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{(x^2+y^2)(4) - (2y)(2y)}{(x^2+y^2)^2}$$

$$= \frac{4x^2-4y^2}{(x^2+y^2)^2}$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$\Rightarrow \frac{-4x^2+4y^2+4x^2-4y^2}{(x^2+y^2)^2} = 0$$

$\therefore$ $u$ satisfies Laplace eqn

$\therefore$ $u$ is harmonic Function.

Let $v$ be the conjugate harmonic of $u$

we have $dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy$

we have C-R eqn

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \& \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

$$\Rightarrow dv = \frac{\partial u}{\partial y} dx - \frac{\partial u}{\partial x} dy.$$

$$dv = \frac{-4y}{x^2+y^2} dx + \frac{4x}{x^2+y^2} dy$$

$$dv = \frac{-4(y\,dx - x\,dy)}{x^2+y^2}$$

Integrating on B.S

$$\int dv = -4 \int \left( \frac{(y\,dx - x\,dy)}{x^2+y^2} \right)$$

$$V = -4 \tan^{-1}\left(\frac{x}{y}\right)$$

viii) S.T $U = x^2 - y^2 - 2xy - 2x + 3y$ is harmonic & find $f(z)$

Sol

$$U = x^2 - y^2 - 2xy - 2x + 3y$$

$$\frac{\partial u}{\partial x} = 2x - 2y - 2$$

$$\frac{\partial^2 u}{\partial x^2} = 2$$

$$\frac{\partial u}{\partial y} = -2y - 2x + 3$$

$$\frac{\partial^2 u}{\partial y^2} = -2$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2 - 2 = 0$$

$\Rightarrow$ Let $V$ be the conjugate harmonic of $U$

we have $dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy$

fr CF

$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$ ; $\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}$ :

$\to dv = -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy$.

$dv = -(-2y - 2x + 3) dx + (2x - 2y - 2) dy$

integrating on B.S

$\to v = \underline{2xy} + x^2 - 3x + 2xy - y^2 - 2y$

$= \cancel{x^2 - y^2 + 4xy - 3x - 2y}$

$= x^2 - y^2 + \overset{2}{4}xy - 3x - 2y$

$f(z) = \cancel{(x^2 - y^2 - 2xy - 3x \pm 3y - 2y)} +$

$= (x^2 - y^2 - 2xy - 2x + 3y) +$
$i(x^2 - y^2 + \overset{2}{4}xy - 3x - 2y)$

ix) Find k such that

$u(x,y) = x^3 + 3kxy^2$ may be harmonic

& Find its conjugate.

Sol Given $u = x^3 + 3kxy^2$

$\frac{\partial u}{\partial x} = 3x^2 + 3ky^2$ $\frac{\partial u}{\partial y} = 6kxy$

$\frac{\partial^2 u}{\partial x^2} = 6x$ $\frac{\partial^2 u}{\partial y^2} = 6kx.$

$k = -1$

$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6x - 6x = 0$

=) let v be the conjugate of harmonic

U

$dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy$

$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$ ; $\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}.$

$dv = -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy$

$dv = -(6kxy)dx + (\cancel{3x^2} + 3ky^2)\, dy$ 0

Integrals in R. S

$$V = +\frac{6x^2}{2}y + \frac{3k}{2}y^3$$

$$V = 3x^2y - y^3$$

$$f(z) = (x^3 + 3kxy^2) + i(3x^2y - y^3)$$

x) P.T, $U = x^2 - y^2$, $V = \frac{-y}{x^2+y^2}$ satisfie

Laplace eqn. but $U + iV$ is not

regular function.

$$U = x^2 - y^2$$

$$\frac{\partial U}{\partial x} = 2x \qquad \frac{\partial U}{\partial y} = -2y$$

$$\frac{\partial^2 U}{\partial x^2} = 2 \qquad \frac{\partial^2 U}{\partial y^2} = -2$$

$$\nabla V^2 = 2 - 2 = 0$$

$$\frac{\partial V}{\partial x} = \frac{-y}{x^2+y^2} = -y(x^2+y^2)^{-1}$$

$$= +y\cdot(x^2+y^2)^{-2}(2x)$$

$$\frac{\partial^2 V}{\partial x^2} = \frac{y(2x)}{(x^2+y^2)^2}$$

$$= \frac{(x^2+y^2)^2\, 2y - (2y)\, y\, (2x)(2)(x^2+y^2)}{(x^2+y^2)^4} \implies$$

$$= \frac{2x^2y + 2y^3 - 8y^2x}{(x^2+y^2)^4}$$

$$= \frac{2y^3 - 6y^2x}{(x^2+y^2)^4}$$

from

$$\frac{\partial v}{\partial y} = \frac{-y}{x^2+y^2} = \frac{(x^2+y^2)(-1) - 2y(-y)}{(x^2+y^2)^2} \implies$$

$$= \frac{-x^2-y^2+2y^2}{(x^2+y^2)^2}$$

$(a+b)(a-b)$
$= a^2-b^2$

$$= \frac{-x^2+y^2}{(x^2+y^2)^2}$$

$$\frac{\partial^2 v}{\partial y^2} = \frac{(x^2+y^2)^2(2y) - (2)(x^2+y^2)\,2y\,(-x^2+y^2)}{(x^2+y^2)^4}$$

$$\Rightarrow \frac{2y^3 - 6x^2y - 2y^3 + 6x^2y}{(x^2+y^2)^3} = 0$$

$\therefore \nabla U^2$ & $\nabla V^2$ satisfies $= 0$ / So

eqn satisfies. U & V satisfies Laplace eqn.

from CR eqn

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad ; \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

$$\Rightarrow 2x \neq \frac{-x^2+y^2}{(x^2+y^2)^2} \quad ; \quad +2y \neq + y(x^2+y^2)^{-2} 2x$$

$\therefore$ Not follows C.R. eqn;

$\therefore$ Not regular function

xi) P.T $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) |\text{Real } F(z)|^2 = 2|F'(z)|^2$

where $F(z) = w$ is analytic.

Sol Given $w = F(x) = U + iV$

we have

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad ; \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad ; \quad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

$$f(z) = u + iv$$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}.$$

$$|f'(z)| = \sqrt{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2}$$

$$= \sqrt{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2}$$

$\Rightarrow$ real part of $f(z) = u$

$$\Rightarrow \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(u)^2 = 2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right]$$

↳ RHS

$$\Rightarrow \cancel{\frac{\partial^2}{\partial x^2}(u)^2 + \frac{\partial}{\partial y^2}(u)^2}$$

$$\cancel{\Rightarrow 2\,\partial^2 u}$$

$$\frac{\partial^2}{\partial x^2}(u)^2 = 2\frac{\partial}{\partial x}\left(u\frac{\partial u}{\partial x}\right)$$

$$= 2\left(u\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x}\frac{\partial u}{\partial x}\right)$$

$$= 2\left(u\frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial u}{\partial x}\right)^2\right) \rightarrow (3$$

$$\frac{\partial^2}{\partial y^2}(u^2) = 2\left[u\frac{\partial^2 u}{\partial y^2} + \left(\frac{\partial u}{\partial y}\right)^2\right] \rightarrow (4)$$

(3) + 4

$$\rightarrow 2\left[u\frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial u}{\partial x}\right)^2 + u\frac{\partial^2 u}{\partial y^2} + \left(\frac{\partial u}{\partial y}\right)^2\right]$$

$$= 2\left[\cancel{u\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)} + \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right]$$

$$= 2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right] \rightarrow LHS$$

LHS = RHS

Hence proved.

ii) P.T $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|F(z)|^2 = 4|F'(z)|^2$

$$F(z) = w = u + iv$$

$$F'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$|F'(z)| = \sqrt{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2}$$

$$4|F'(z)|^2 = 4\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right] \rightarrow RHS$$

$$|F(z)| = \sqrt{u^2+v^2}$$

$$|F(z)|^2 = u^2+v^2$$

diff w.r.t $x$ twice

$$\left(\frac{\partial^2}{\partial x^2}(u^2+v^2)\right) = \frac{\partial}{\partial x}\left(2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x}\right)$$

$$= 2u\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial u}{\partial x}\frac{\partial u}{\partial x} + 2v\frac{\partial^2 v}{\partial x^2}$$

$$+ 2\frac{\partial v}{\partial x}\frac{\partial v}{\partial x}.$$

$$= 2\left[u\frac{\partial^2 u}{\partial x^2} + v\frac{\partial^2 v}{\partial x^2} + \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right.$$

$\hookrightarrow$ ①

diff w.r.t $y$ in

$$\left(\frac{\partial^2}{\partial y^2}(u^2+v^2)\right) = 2\left[u\frac{d^2 u}{\partial y^2} + v\frac{\partial^2 v}{\partial y^2} + \left(\frac{\partial u}{\partial y}\right)\right.$$

$$\left.+\left(\frac{\partial v}{\partial y}\right)^2\right]$$

$\hookrightarrow$ ②

① + ②

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$$\Rightarrow 2\left[u\cancel{\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)}^{\to 0}+v\cancel{\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)}^{\to 0}\right.$$

$$\left.+\left(\frac{\partial u}{\partial x}\right)^2+\underline{\left(\frac{\partial v}{\partial x}\right)^2}+\underline{\left(\frac{\partial u}{\partial y}\right)^2}+\left(\frac{\partial v}{\partial y}\right)^2\right]$$

$$\frac{\partial v}{\partial y}=\frac{\partial u}{\partial x} \qquad \left(\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}\right)$$

$$\Rightarrow 2\left[2\left(\frac{\partial u}{\partial x}\right)^2+2\left(\frac{\partial u}{\partial y}\right)^2\right]$$

$$\Rightarrow 4\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2\right] \rightarrow RHS$$

RHS = LHS

Hence proved.

Cauchy's Reimann eqn at point origin.

$$\left(\frac{\partial u}{\partial x}\right)_{(0,0)}=\lim_{x\to 0}\frac{u(x,0)-u(0,0)}{x}.$$

$$\left(\frac{\partial u}{\partial y}\right)_{(0,0)}=\lim_{y\to 0}\frac{u(0,y)-u(0,0)}{y}.$$

$$\left(\frac{\partial v}{\partial x}\right)_{(0,0)}=\lim_{x\to 0}\frac{v(x,0)-v(0,0)}{x}.$$

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$$\left(\frac{\partial v}{\partial y}\right)_{(0,0)} = \lim_{y \to 0} \frac{v(0,y) - v(0,0)}{y}$$

$$\lim_{z \to 0}$$

$$f'(0) = \lim_{x \to 0} \frac{f(z) - f(0)}{z}$$

i) P/T the function $f(x)$ defined by

$$f(z) = \begin{cases} \dfrac{x^3(1+i) - y^3(1-i)}{x^2+y^2} & (z \neq 0) \\ 0 & \text{if } z = 0 \end{cases}$$

is continuous & CR and satisfies at the origin. yet $f'(0)$ does not exist.

A: Given

$$f(z) = \begin{cases} \dfrac{x^3(1+i) - y^3(1-i)}{x^2+y^2} & (z \neq 0) \\ 0 & (z = 0) \end{cases} \quad \text{①}$$

$$\lim_{z \to 0} f(z) = \lim_{\substack{x \to 0 \\ y \to 0}} f(z) = \lim_{\substack{x \to 0 \\ y \to 0}} \frac{x^3(1+i) - y^3(1-i)}{x^2+y^2}$$

$$= \lim_{y \to 0} \frac{-y^3(1-i)}{y^2}$$

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$= 0 \rightarrow (2)$

$$\lim_{z\to 0} F(z) = \lim_{\substack{y\to 0\\ x\to 0}} F(z) = \lim_{\substack{y\to 0\\ x\to 0}} \frac{x^3(1+i) - y^3(1-i)}{x^2+y^2}$$

$$= \lim_{x\to 0} \frac{x^{\cancel{3}}(1+i)}{\cancel{x^2}}$$

$= 0 \rightarrow (3)$

let $y = mx$

$$\Rightarrow \lim_{z\to 0} F(z) = \lim_{x\to 0} \frac{x^3(1+i) - m^3x^3(1-i)}{x^2+m^2x^2}$$

$$= \lim_{x\to 0} \frac{x[(1+i) - m^3(1-i)]}{(1+m^2)} = 0 \rightarrow (4)$$

from (2) (3) & (4)

$F(z)$ is continous at $(0,0)$.

$$F(z) = u + iv = \frac{x^3(1+i) - y^3(1-i)}{x^2+y^2}$$

$$= \frac{x^3 + ix^3 - y^3 + iy^3}{x^2+y^2}$$

$$= \frac{x^3 - y^3}{x^2+y^2} + i\,\frac{(x^3+y^3)}{x^2+y^2}$$

$$u(x,y) = \frac{x^3 - y^3}{x^2 + y^2} \qquad v(x,y) = \frac{x^3 + y^3}{x^2 + y^2}$$

$$\left(\frac{\partial u}{\partial x}\right)_{(0,0)} = \lim_{x \to 0} \frac{u(x,0) - u(0,0)}{x}$$

$$= \lim_{x \to 0} \frac{x - 0}{x} = \lim_{x \to 0} 1 = 1$$

$$\left(\frac{\partial u}{\partial y}\right)_{(0,0)} = \lim_{y \to 0} \frac{u(0,y) - u(0,0)}{y}$$

$$= \lim_{y \to 0} \frac{0 - y}{y} = \lim -1 = -1$$

$$\left(\frac{\partial v}{\partial x}\right)_{(0,0)} = \lim_{x \to 0} \frac{v(x,0) - v(0,0)}{x} =$$

$$= \lim_{x \to 0} \frac{x + 0}{x} = \lim 1 = 1$$

$$\left(\frac{\partial v}{\partial y}\right)_{(0,0)} = \lim_{x \to 0} \frac{v(0,y) - v(0,0)}{y} =$$

$$= \lim_{y \to 0} \frac{0 + y}{y} = \lim 1 = 1$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

✓ ✓

$\therefore$ u, v satisfies CR eqn.

$\therefore$ f(z) also " " " at (origin).

$$f'(z) = \lim_{z \to 0} \frac{f(z) - f(0)}{z} \qquad f(0) = 0$$

$$= \lim_{z \to 0} f(z)/z$$

$$= \lim_{z \to 0} \frac{x^3(1+i) - y^3(1-i)}{(x^2+y^2)(x+iy)} \qquad \begin{array}{l} z \to 0 \\ \Rightarrow x = 0 \\ \quad y = 0 \end{array}$$

$$= 0 \rightarrow (5)$$

$y = mx$

$$f'(z) = \lim_{x \to 0} \frac{x^3(1+i) - m^3 x^3(1+i)}{(x^2+m^2x^2)(x+imx)}$$

$$= \lim_{x \to 0} \frac{(1+i) - m^3(1-i)}{(1+m^2)(1+im)} \rightarrow (6)$$

(5) & (6) are not unique.

f(z) is not analytic at the origin.

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ii) ST $f(z)=\sqrt{xy}$ is not analytic at the origin. although CR eqn are satisfied at the origin.

Given $f(z)=\sqrt{xy}$

$$\lim_{z\to 0} f(z) = \lim_{\substack{x\to 0\\ y\to 0}} \sqrt{xy} = \lim_{y\to 0} 0 = 0$$

$$\lim_{z\to 0} f(z) = \lim_{\substack{y\to 0\\ x\to 0}} \sqrt{xy} = \lim_{x\to 0} 0 = 0.$$

$y=mx$

$$\lim_{z\to 0} f(z) = \lim_{x\to 0} \sqrt{x\,mx} = \lim_{x\to} 0 = 0.$$

$f(z)$ is continuous

CR eqns

$u=\sqrt{xy}$ $\quad v=0$

$$\left(\frac{\partial u}{\partial x}\right)_{(0,0)} = \lim_{x\to 0} \frac{u(x,0)-u(0,0)}{x}$$

$$= 0$$

$$\left(\frac{\partial u}{\partial y}\right)_{(0,0)} = \lim_{y\to 0} \frac{u(0,y)-u(0,0)}{y}$$

$$= 0$$

$$\left(\frac{\partial v}{\partial x}\right)_{(0,0)} = \lim_{x\to 0} \frac{v(x,0)-v(0,0)}{x} = 0.$$

$$\left(\frac{\partial v}{\partial y}\right)_{(0,0)} = \lim_{y \to 0} \frac{v(0,y) - v(0,0)}{y} = 0$$

Change [illegible] 1st [illegible] 2nd [illegible] 3rd [illegible] rearranged.

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \ (\checkmark) \quad \& \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \ (\checkmark)$$

CR functions satisfies.

$f(z)$ is also satisfies CR eqn at origin

$$f'(z) = \lim_{z \to 0} \frac{f(z) - f(0)}{z} = \lim_{z \to 0} \frac{\sqrt{xy}}{x+iy} = 0 \quad \to (1)$$

$y = mx$

$$f'(z) = \lim_{x \to 0} \frac{\cancel{x}\sqrt{m}}{\cancel{x}(1+im)} = \frac{\sqrt{m}}{1+im} \quad \to (2)$$

(1) & (2) are not unique.

$f(z)$ is not differentiable at origin

$f(z)$ is not analytic at origin.

## Cauchy's Rieman eqn in polar form:

If $f(z) = f(re^{i\theta}) = u(r,\theta) + iv(r,\theta)$

and $f(z)$ is derivable at $z_0 = r_0 e^{i\theta}$

then

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}; \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

Proof:

Given $f(z) = f(r.e^{i\theta}) = u + iv \rightarrow (1)$

hence $z = re^{i\theta}$

$$\frac{\partial z}{\partial r} = e^{i\theta} ; \quad \frac{\partial z}{\partial \theta} = i.r e^{i\theta}$$

P.D eq (1) w.r.t $r$ on B.S

$$\frac{\partial}{\partial r}(f(z)) = f'(z)\frac{\partial z}{\partial r} = f'(z) e^{i\theta}$$

$$f'(z)e^{i\theta} = \frac{\partial f}{\partial r}$$

$$= \frac{\partial}{\partial r}(u+iv)$$

$$f'(z)e^{i\theta} = \frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}$$

P.D eq (1) w.r.t $\theta$ on B.S.

$$\frac{\partial}{\partial \theta}(f(z)) = f'(z)\frac{\partial z}{\partial \theta} = f'(z)(i.r.e^{i\theta})$$

$$\cancel{\frac{\partial}{\partial \theta}} f'(z)(i.r.e^{i\theta}) = \frac{\partial f}{\partial \theta}$$

$$= \frac{\partial}{\partial \theta}(u+iv)$$

$$= \frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta}$$

$$f'(z)\, e^{i\theta} = \frac{1}{ir}\left(\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta}\right) \rightarrow (3)$$

From (2) & (3)

$$\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r} = \frac{1}{ir}\left(\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta}\right)$$

$$\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r} = -i\frac{1}{r}\frac{\partial u}{\partial \theta} + \frac{1}{r}\frac{\partial v}{\partial \theta}.$$

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$$

&

$$\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

∴ Hence proved.

Construction of Analytic Function :-

" " " " when real & imaginary parts are known.

Suppose $f(z) = u + iv$ is analytic function.

By using Milne's Thomson's Method.

$f'(z)$ can be expressed in terms of $z$

Replacing $x = z$ and $y = 0$ in $f'(z)$

i) Find analytic function whose real part is

$u = x^2 - y^2 - x$

P.D Given eq$^n$ w.r.t $x$ & $y$ on B.S

$\Rightarrow \frac{\partial u}{\partial x} = 2x - 1 \qquad \frac{\partial u}{\partial y} = -2y$

We know $f(z) = u + iv$ be analytic

P.D w.r.t $x$ & $y$ on B.S

$\Rightarrow f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial y}$

we know from CR eq$^n$

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$.

$\Rightarrow f'(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$

$\Rightarrow f'(z) = (2x-1) - i(-2y)$

$= 2x - 1 + i2y$

By using Milne's Thomson method

$f'(z)$ can be expressed as

$x = z \qquad y = 0$

$f'(z) = 2z - 1$

Integration on B.S

$\Rightarrow$ $f(z) = z^2 - z$

we know $z = x + iy$

$= (x+iy)^2 - (x+iy)$

$= x^2 - y^2 + 2ixy - x - iy$

$f(z) = (x^2 - y^2 - x) + i(2xy - y)$

ii) Find imaginary part whose real part is

$u = e^x(x\cos y - y\sin y)$

Given $u = e^x(x\cos y - y\sin y)$

P.D w.r.t $x$ & $y$ on B.S

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left(e^x(x\cos y - y\sin y)\right)$$

$$= e^x\cos y + (x\cos y - y\sin y)e^x$$

$$\frac{\partial u}{\partial y} = e^x(-x\sin y - y\cos y - \sin y)$$

we k $f(z) = u + iv$

P.D w.r.t $x$ & $y$ on B.S

$$f'(z) \Rightarrow \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \qquad \text{w.k.t} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

$$\Rightarrow \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$$

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$f'(z) = e^{x}(x\cos y - y\sin y + \cos y)$
$- i\, e^{x}(-x\sin y - y\cos y - \sin y)$

By using millen Thomson method

$x = z$ $y = 0$

$f'(z) = e^{z}(z - 0 + 1)$
$- i e^{z}(0 - 0 - 0)$

$f'(z) = e^{z}(z+1)$

Integrate on B.S

$\cancel{f(z) = e^{z}(1) + (z+1)e^{z}}$

~~we k~~ $\cancel{z = x + iy}$

$\cancel{f(z) = e^{x+iy} + (x+iy+1)e^{z}}$

$f(z) = \int z e^{z} dz + \int e^{z} dz$

$= z e^{z} - \int \cancel{1 e^{z} dz + e^{z}}$

$= z e^{z}$

$= (x+iy)\, e^{(x+iy)}$

$= (x+iy)\, e^{x}(\cos y + i\sin y)$

U +

iii) D

$= (x+iy)\,(e^{x}\cos y + i e^{x}\sin y)$

$= x e^{x}\cos y + i x e^{x}\sin y +$

$\quad i y e^{x}\cos y + i^{2} y e^{x}\sin y$

$u+iv = e^{x}(x\cos y - y\sin y)$

$\quad + i e^{x}(x\sin y + y\cos y)$

iii) Determine Analytic function $\cos x \cosh y$

Given $u = \cos x \cosh y$.

let $f(z) = u + iv$ be analytic

$\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ & $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$.

~~dif~~ P.D $u$ w.r.t $x$ on B.S.

$\dfrac{\partial u}{\partial x} = -\sin x \cosh y$ $\quad \dfrac{\partial u}{\partial y} = \cos x \sinh y$.

$f(z) = u + iv$.

P.D w.r.t $x$ on B.S

$\rightarrow \quad f'(z) = \dfrac{\partial u}{\partial x} + i\dfrac{\partial v}{\partial \cancel{x}}$

$= \dfrac{\partial u}{\partial x} - i\dfrac{\partial u}{\partial y}$.

$= (-\sin x \cosh y) - i(\cos x \sinh y)$

MTM $\quad$ let $x = z \quad y = 0$

$$\frac{\partial v}{\partial y}$$

$f'(z) = -\sin z(1) - i(0)$

$= -\sin z$

Integral on B.S

$\frac{e^{y}+e^{-y}}{2}$

$f(z) = -\int \sin z$

$f(z) = \cos z + C$.

Let $z = x + iy$.

$= \cos(x+iy)$

$= \cos(x)\cos(iy) - \sin(x)\sin(iy)$

$= \underbrace{\cos x \cosh y}_{u} - i\underbrace{(\sin x \sinh y)}_{v}$

iii) Find analytic function whose imaginary part is

$e^{x}(x\sin y + y\cos y)$

$v = e^{x}(x\sin y + y\cos y)$.

P.D w.r.t $x$ & $y$ on B.S

$\frac{\partial v}{\partial x} = \cancel{e^{x}} +$

$e^{x}\sin y + (x\sin y + y\cos y)e^{x}$.

$= e^{x}[\sin y + x\sin y + y\cos y]$

$$\frac{\partial v}{\partial y} = e^{x}(x\cos y + \cos y - y\sin y)$$

We know $f(z) = u + iv$.

P.D w.r.t $x$ & $y$ on B.S

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$= \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x}$$

$$f'(z) = e^{x}(x\cos y + \cos y - y\sin y) + i\,e^{x}(\sin y + x\sin y + y\cos y)$$

MTM $x = z$ $y = 0$

$$f'(z) = e^{z}(z(1) + 1 - (0)(0)) + i\,e^{z}(0 + z(0) + 0)$$

$$= e^{z}(z+1)$$

$$= e^{z}z + e^{z}$$

Integral on B.S

$$f(z) = \int e^{z}z\,dz + \int e^{z}dz$$

$$= ze^{z} - \cancel{e^{z}} + \cancel{e^{z}}$$

$$= ze^{z}$$

$z = x + iy$

$$f(z) = (x+iy)\, e^{x+iy}$$

$$\frac{\partial v}{\partial y}$$

$$= (x+iy)\, e^{x}(\cos y + i \sin y)$$

$$= (x+iy)(e^{x}\cos y + e^{x} i \sin y)$$

$$= x e^{x}\cos y + x e^{x} i \sin y + iy e^{x}\cos y + i^{2} y e^{x} \sin y$$

$$u + iv = x e^{x}\cos y - y e^{x}\sin y + i e^{x}(x \sin y + y \cos y)$$

$$= \underbrace{e^{x}(x \cos y - y \sin y)}_{u} + i \underbrace{e^{x}(x \sin y + y \cos y)}_{v}$$

v) Find regular function whose imaginary part is $e^{-x}(x\cos y + y \sin y)$

$$v = e^{-x}(x \cos y + y \sin y)$$

P.D w.r.t $x$ & $y$ on B.S

$$\frac{\partial v}{\partial x} = e^{-x}\cos y + (x\cos y + y \sin y) - e^{-x}$$

$$= e^{-x}(\cos y - x\cos y - y \sin y)$$

$$\frac{\partial v}{\partial y} = e^{-x}(-x\sin y + \sin y + y\cos y)$$

m.k. $f(z) = u + iv$. P.D w.r.t. $x$

$$\Rightarrow f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$= \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x}$$

$$= e^{-x}(-x\sin y + \sin y + y\cos y)$$
$$+ i(e^{-x}(\cos y - x\cos y - y\sin y)$$

MTM $x = z$ ; $y = 0$

$$= \cancel{e^{-z}(0 + 0 + 0)}$$
$$+ i(e^{-z}(1 - z - 0)$$

$$f'(z) = ie^{-z}(1 - z)$$

Integrate on R.S

$$f(z) = i\int(e^{-z} - ze^{-z})$$

$$= i(-e^{-z} + ze^{-z} - \int 1 \int e^{-z})$$

$$= i(\cancel{e^{-z}} + ze^{-z} - \cancel{e^{-z}})$$

$$= i(ze^{-z})$$

$= i\,(z e^{-z})$

$= i\,((x+iy)\, e^{-(x+iy)})$

$= i\,((x+iy)\, e^{-x} e^{-iy})$

$= i\,((x+iy)\, e^{-x}(\cos y - i\sin y))$

$= i\,(x+iy)\,(e^{-x}\cos y - ie^{-x}\sin y)$

$= [(ix-y)(e^{-x}\cos y - ie^{-x}\sin y)$

$= ixe^{-x}\cos y + (1)xe^{-x}\sin y$
$\quad - ye^{-x}\cos y + ie^{-x}y\sin y$

$= i\,e^{-x}(x\cos y + y\sin y)$
$\quad + e^{-x}(x\sin y - y\cos y)$

$iu - v + u + i$
$u(1+i) + v(i-1)$
$u - v + i(u+v)$

CReam

$\frac{\partial u}{\partial x}$

$\Rightarrow$

vi) If $f(z) = u + iv$ is analytic function of $z$ and if

$u - v = e^{x}(\cos y - \sin y)$

find $f(z)$.

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Given

$$u - v = e^{x}(\cos y - \sin y)$$

Given $f(z) = u + iv$ is analytic function.

CR eqn

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \& \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

⇒ We k

$$f(z) = u + iv$$

$$if(z) = iu - v$$

$$\boxed{f(z) + if(z) = (u - v) + i(u + v)}$$

$$\therefore u - v = e^{x}(\cos y - \sin y)$$

$u$ ← $u + v = ?$ → imaginary part.

$v$ ←

~~u~~ U diff w.r.t $x$ & $y$ on B.S

$$\frac{\partial u}{\partial x} = e^{x}(\cos y - \sin y)$$

$$\frac{\partial u}{\partial y} = -e^{x}(+\sin y + \cos y)$$

$$f(z) = u + iv$$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$= \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$$

$= e^x(\cos y - \sin y) + ie^x(\sin y + \cos y)$ $f'(z)$

$\therefore$ from MTM. $(1+i)$

$x = z \quad y = 0$

$= e^z(1) + ie^z(0+1)$ vii) I. of

$f'(z) = e^z + ie^z$

Integrate on b.s

$f(z) = \int e^z + i\int e^z$

$= e^z + ie^z$ $(z = x+iy)$

$= e^{x+iy} + ie^{x+iy}$

$= e^x(\cos y + i\sin y) + ie^x(\cos y + i\sin y)$

$= \cancel{(e^x\cos y + e^x i\sin y)}$

$= e^x\cos y + e^x i\sin y + ie^x\cos y - e^x\sin y$

$(u+v) = e^x(\cos y - \sin y) + ie^x(\sin y + \cos y)$

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$$f(z) = (1+i)\, e^{z}$$

$$\cancel{(1+i)}\, f(z) = \cancel{(1+i)}\, e^{z}$$

$$f(z) = e^{z}$$

vii) If $f(z) = u + iv$ is analytic function of $z$ and if $u - v = (x-y)(x^2+4xy+y^2)$

$$u - v = (x-y)(x^2+4xy+y^2)$$

$f(z) = u + iv$ is analytic

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \& \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

$$f(z) = u + iv$$

$$if(z) = iu - v$$

$$f(z) + if(z) = (u-v) + i(u+v)$$

$$F(z) = U + iV$$

$$F(z) = (1+i)\, f(z)$$

$$\frac{\partial U}{\partial x} = 1(x^2+4xy+y^2) + (x-y)(2x+4y)$$

$$\frac{\partial U}{\partial y} = \cancel{-1(+4x+2y) + (x-y)(4x+}$$

$$-1(x^2+4xy+y^2) + (x-y)(4x+2y)$$

viii) I

$f(z) = u + iv$

$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

$= \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$

$= (x^2 + \cancel{4xy} + \cancel{y^2}) + (x - \cancel{y})(2x + \cancel{4y})$

$- i\left[(x - \cancel{y})(4x + \cancel{2y}) - (x^2 + \cancel{4xy} + \cancel{y^2})\right]$

$x = z \quad y = 0$

$= z^2 + 2z^2 - i(4z^2 - z^2)$

$= 3z^2 - i(3z^2)$

$= 3z^2(1 - i)$

$(1 - i \times i)$

$(i + 1)$

$f'(z) = 3z^2(1 - i)$

$f(z)\cancel{(1+i)} = z^3\cancel{(1-i)}\, i$

$(1 - i \times i)$

$i$

$f(z) = \frac{iz^3}{i}$

$i + 1$

$-(i)^2$

$-(-1)$

$+1$

$(1 - i) \times i = i - (i)^2$

$= i - (-1)$

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viii) If $f(z) = u + iv$ is analytic function of $z$ & if

$$u + v = \frac{\sin 2x}{\cosh 2y - \cos 2x}.$$

$f(z) = u + iv$

$if(z) = iu - v$

$(1+i) f(z) = u - v + i(u + v)$

$\downarrow \qquad \downarrow$

$U \qquad V$

$(1+i) f(z) = F(z)$

$$\Rightarrow \overset{V}{\cancel{u+v}} = \frac{\sin 2}{\cosh 2y - \cos 2x}.$$

diff $V$ w.r.t $x$ & $y$.

$$\frac{\partial V}{\partial x} = \frac{(\cosh 2y - \cos 2x)\cos 2x(2)}{(\cosh 2y - \cos 2x)^2}$$

# Complex Integration

Evaluate $\int_0^{1+i} (x^2 - iy)\,dz$ along the path

$y = x$

$z = x + iy$

$dz = \frac{dx}{dz} + \frac{i\,dy}{dz}$

Soln

Given

$$\int_0^{1+i} (x^2 - iy)\,dz \qquad x = y$$

we have $z = x + iy$

$dz = dx + i\,dy$

$x = y \Rightarrow dy = dx.$

$x$ var $0$ to $1$

$$\int_0^1 (x^2 - ix)\,(dx + i\,dy).$$

$$= (1+i) \int_0^1 x^2 - ix\,dx$$

$$= (1+i)\left[\frac{x^3}{3} - i\frac{x^2}{2}\right]_0^1$$

$$= (1+i)\left[\frac{1}{3} - i\frac{1}{2}\right]$$

$$\Rightarrow \frac{1}{3} + \frac{1}{2} + i\left(\frac{1}{3} - \frac{1}{2}\right)$$

$$\Rightarrow \frac{5}{6} + i\left(\frac{-1}{6}\right) //.$$

→ Evaluat $\int_0^{1+i} (x^2 - iy)\,dz$ along the path $y = x^2$

$$z = x + iy$$

$$dz = dx + i\,dy$$

$$y = x^2 \Rightarrow dy = 2x\,dx.$$

$x$ varies from $0$ to $1$

$$\Rightarrow \int_0^1 (x^2 - ix^2)(dx + i\,dy)$$

$$= (1-i)\int_0^1 x^2 (dx + i\,2x\,dx)$$

$$= (1-i)\int_0^1 x^2\,dx\,(1 + i\,2x)$$

$$\frac{1}{3} + \frac{1}{2}$$

$$i\left[-\frac{2+3}{6}\right]$$

$$= (1-i)\int_0^1 x^2\,dx + i\,2x^3\,dx.$$

$$= (1-i)\left(\left[\frac{x^3}{3}\right]_0^1 + i\left[\frac{\cancel{2}x^4}{\cancel{4}\,2}\right]_0^1\right)$$

$$= (1-i)\left[\frac{1}{3} + i\,\frac{1}{2}\right] = \frac{5}{6} + \frac{i}{6}$$

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$\rightarrow$ Evaluate $\int_{(0,0)}^{(1,-1)} (3x^2+4xy+ix^2)\, dz \qquad y = x^2$

$z = u + iv$

$dz = du + idv$

$y = x^2 \qquad dy = 2x\, dx$

$x \Rightarrow 0 \rightarrow 1$

$\Rightarrow \int_0^1 (3x^2 + 4x^3 + ix^2)\, (dx + i2x\, dx)$

$\Rightarrow \int_0^1 (3x^2 + 4x^3 + ix^2)\, dx\, (1 + i2x)$

$\int_0^1 \left[3x^2 + 4x^3 + ix^2 + i6x^3 + i8x^3 - 2x^3\right] dx$

$= \left[x^3 + x^4 + i\frac{x^3}{3} + i\frac{3x^4}{2} + i2x^4 - \frac{x^4}{2}\right]_0^1$

$= \underbrace{1 + 1 + \frac{i}{3} + 3\frac{i}{2} + 2i - \frac{1}{2}}_{}$

$2 - \frac{1}{2}$

$\frac{4-1}{2} = 3/2$

$= \frac{3}{2} + \frac{23}{6} i$ //

$\frac{3}{2} + \frac{1}{3} + 2$

$\frac{9 + 2 + 12}{6}$

$12$
$11$
$\overline{23}$

-) Evaluate $\int_{(0,1)}^{(0,1)} (3x^2+4xy+ix^2)\,dz$ along $y = x^2$

=) $\cancel{dz=}$ $z = x+iy$

$dz = dx + i\,dy.$

$y = x^2$

$dy = 2x\,dx.$ varies $0 \to 1$

=) $\int_0^1 (3x^2+4x\,x^2+ix^2)(dx+i\,dy)$

=) $\int_0^1 (3x^2+4x^3+ix^2)(dx+i2x\,dx)$

$= \int_0^1 (3x^2+4x^3+ix^2)(1+2ix)\,dx.$

$= \int_0^1 [3x^2+4x^3+ix^2+6ix^3+8ix^4-2x^3]\,dx.$

$= \left[x^3+x^4+\frac{ix^3}{3}+\frac{3ix^4}{2}+\frac{8ix^5}{5}-\frac{x^4}{2}\right]_0^1$

$2-\frac{1}{2}$

$\frac{4-1}{2}$

$\frac{3}{2}$

$= 1+1+\frac{i}{3}+\frac{3i}{2}+\frac{8i}{5}-\frac{1}{2}$

$= \frac{3}{2}+\left(\frac{10+45+48}{30}\right)i$

$$\begin{array}{r} 1\\ 45\\ 48\\ 10\\ \hline 103 \end{array}$$

$= \frac{3}{2}+\frac{103}{30}i$

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→ Evaluate $\int_{(0,0)}^{(1,1)} 3x^2 + 5y + i(x^2 - y^2)] \, dz$

along $y^2 = x$.

$\begin{array}{r} 14 \\ \times 5 \\ \hline 70 \end{array}$

$z = x + iy$

$dz = dx + idy$

$x = y^2$ $\qquad$ $y$ var $0 \to 1$

$dx = 2y\,dy$

$\Rightarrow \int_0^1 [3y^4 + 5y + i(y^4 - y^2)](dx + idy)$

$\Rightarrow \int_0^1 (3y^4 + 5y + i(y^4 - y^2))(2y\,dy + idy)$

$\Rightarrow \int_0^1 (3y^4 + 5y + i(y^4 - y^2))(dy(2y + i))$

$\Rightarrow \int_0^1 6y^5 + 10y^2 + i(2y^5 - 2y^3) + 3iy^4 + 5iy$

$- \cancel{y}y^4 + y^2)$

$\Rightarrow \left[ y^6 + \frac{10}{3}y^3 + \left(\frac{y^6}{3} - \frac{y^4}{2}\right)i + \frac{3}{5}iy^5 + \frac{5iy^2}{2} \right.$

$\left. - \frac{y^5}{5} + \frac{y^3}{3} \right]_0^1$

$\begin{array}{r} 15 \\ \times 5 \\ \hline 75 \end{array}$

$= 1 + \frac{10}{3} - \frac{1}{5} + \frac{1}{3} + i\left(\frac{1}{3} - \frac{1}{2} + \frac{3}{5} + \frac{5}{2}\right)$

$= \frac{15 + 50 - 3 + 5}{15} + i\left(\frac{10 - 15 + 18 + 75}{30}\right)$;

$\frac{67}{15} + i\left(\frac{88}{30}\right)$

$1+i$
$x+iy$
$x=1$ $y=1$

$\begin{array}{r} 13 \\ 75 \\ \hline 88 \end{array}$

→ Evaluate $\int_0^{1+i} z^2 dz$ along $y = x^2$

$z = x + iy$ $y = x^2$ $y$ var $0 \to 1$

$dz = dx + i\,dy$ $dy = 2x\,dx.$

$\Rightarrow \int_0^1 (x+iy)^2 (dx + i\,dy)$

$\Rightarrow \int_0^1 (x^2 - y^2 + 2ixy)\,dx\,(1 + i2x)$

$\Rightarrow \int_0^1 (x^2 - x^4 + 2ix^3)(1 + i2x)\,dx.$

$\Rightarrow \int_0^1 \left[x^2 - x^4 + 2ix^3 + \underline{i2x^3} \cancel{+} \overline{\underline{i2x^5}} \cancel{+} \overline{i}\,4x^4\right] dx$

$\Rightarrow \left[\frac{x^3}{3} - \frac{x^5}{5} + \frac{2ix^4}{2} + \frac{ix^4}{2} \cancel{+} \overline{i}\frac{x^6}{3} \cancel{+} \overline{i}\frac{4x^5}{5}\right]_0^1.$

$= \left(\frac{1}{3} - \frac{1}{5}\right) + i\left(\cancel{\frac{1}{2}} + \cancel{\frac{1}{2}} - \frac{1}{3}\right) \cancel{+}\overline{i}\,\frac{4}{5})$

$-5+12$
$7$

$= \cancel{\frac{2}{15} + i\left(\frac{+7}{15}\right)}$ $\cancel{\frac{1}{4} - \frac{1}{3}}$

$= \frac{-2}{15} - \frac{12}{15}$ $\cancel{\frac{3-4}{12} = \frac{-1}{12}}$

$\left(-\frac{14}{15}\right) - i\left(\frac{2}{3}\right)$ $1 - \frac{1}{3}$

$2/3$

//

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→ Evaluat $\int_{1-i}^{2+i} (2x+1+iy)\, dz$ along the straight line $(1,-i)$ & $(2,i)$

Sol: Given integral

$$\int_{1-i}^{2+i} (2x+1+iy)\, dz$$

$x_1 = 1$ ; $x_2 = 2$ ; $y_1 = -i$ ; $y_2 = i$

$$\Rightarrow \frac{y-y_1}{y_2-y_1} = \frac{x-x_1}{x_2-x_1}$$

$$\Rightarrow \frac{y+1}{2} = \frac{x-1}{1}$$

$$\Rightarrow y+1 = 2x-2$$

$$\Rightarrow 2x-y-3 = 0$$

$$y = 2x-3 \qquad x \to 1 \to 2$$

$$dy = 2dx$$

$$\Rightarrow \int_1^2 (2x+i(2x-3)+1)(1+2i)\, dx.$$

$$\Rightarrow \int_1^2 [2x+i2x-3i+1+4xi-4x+6+2i]$$

$$= \left[x^2 + ix^2 - 3xi + x + 2x^2 i - 2x^2 + 6x + 2xi\right]_1^2$$

$$= (1+2i)\left((6-2i) - (2-2i)\right]$$

$$= (1+2i)\,[4]$$

$$= 4(1+2i)$$

## Cauchy's Integral theorem :-

Let $F(z) = u + iv$ be analytic Function on and within a simple closed curve $C$ and let $F'(z)$ be continuous there, then $\int_C F(z)\,dz = 0$ //.

**Proof :-**

Given $F(z) = u + iv$ is analytic Function

$\Rightarrow u, v$ satisfie C.R eqn's

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \& \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$z = x + iy$$

$$dz = dx + i\,dy$$

$$F(z)\,dz = (u + iv)(dx + i\,dy)$$

$$= u\,dx + iu\,dy + iv\,dx - v\,dy$$

$$= u\,dx - v\,dy + i(v\,dx + u\,dy)$$

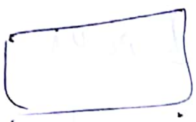
$$\int_C f(z)\,dz$$

$$= \int_C (u\,dx - v\,dy) + i\int_C (u\,dy + v\,dx)$$

Using green's theorem.

$$\boxed{\oint_C M\,dx + N\,dy = \iint_S \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy.}$$

$$\int_C f(z)\,dz = \iint_S \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy$$



$$+ i\iint_S \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\,dy.$$

$$= \iint_S \left(\frac{\partial u}{\partial y} - \frac{\partial u}{\partial y}\right) dx\,dy + i\iint_S \left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial x}\right) dx\,dy$$

$$= 0 + i\,0$$

$$\therefore \int_C f(z)\,dz = 0$$

Hence proved.

Verify Cauchy's integral theorem for

$f(z) = 3z^2 + iz - 4$ if $C$ is square

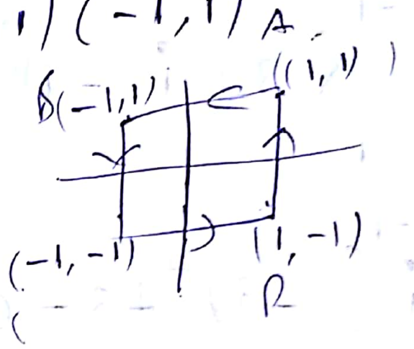
with vertices at $1 \pm i, -1 \pm i$

Given function $f(z) = 3z^2 + iz - 4$

Given

$1 \pm i, -1 \pm i$

points are $(1,1)\ (1,-1)\ (-1,1)\ (-1,1)$




W.K.T

for C I T

$$\int_C f(z)\,dz = 0$$

Along the path AB The points $(1,1)$ & $(-1,1)$

$y = 1$ $\quad dy = 0$

$x$ vary from $1$ to $-1$

$$\int_{AB} f(z)\,dz = \int_1^{-1} 3(x+iy)^2 + i(x+iy) - 4 \quad (dx + \cancel{i\,dy})$$

$$= \int_1^{-1} (3(x+i)^2 + i(x+i) - 4)\,dx$$

$$= \int_1^{-1} (3x^2 - 3 + 6xi + xi - 1 - 4)\,dx$$

$$= \left[x^3 - 8x + 3x^2 i + \frac{x^2}{2} i\right]_1^{-1}$$

$$= \left[-1 + 8 + \cancel{3i} + \cancel{\frac{i}{2}}\right] - \left[1 - 8 + \cancel{3i} + \cancel{\frac{i}{2}}\right] = 14 \rightarrow (1)$$

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Along path BC

$(-1,1)$ $(-1,-1)$

$x = -1$ $dx = 0$

$y$ vals $f$ $1$ to $-1$

$$\int_{1}^{-1} 3(x+iy)^2 + i(x+iy) - 4] [\cancel{dx}^{0} + i\,dy)$$

$$\int_{1}^{-1} 3(iy-1)^2 + i(iy-1) - 4] [i\,dy]$$

$$i\int_{1}^{-1} (-3y^2 + 3 - 6iy - y - i - 4)\,dy$$

$$i\left[\left(-\frac{\cancel{3}y^3}{\cancel{3}} + 3y - \frac{3iy^2}{} - \frac{y^2}{2} - iy - 4y\right)\right]dz$$

$$i\left[\left(-y^3 \cancel{+3y} - 3iy^2 - \frac{y^2}{2} - iy - y\right)\right]dz$$

$$i\left(\left[1 - \cancel{3i} - \cancel{\frac{1}{2}} + i + 1\right] - \left[-1 - \cancel{3i} - \cancel{\frac{1}{2}} - i - 1\right]\right.$$

$$i[4 + 2i] = 4i - 2 \rightarrow (2)$$

Along path CD $(-1,-1) \; (1,-1)$

$y = -1 \; ; \; dy = 0$

$x$ var $-1$ to $1$

$$\int_{-1}^{1} \left(3(x+iy)^2 + i(x+iy) - 4\right)(dx + i\cancel{dy})$$

$$\int_{-1}^{1} 3(x-i)^2 + i(x-i) - 4 \; dx$$

$$\int_{-1}^{1} \left(3x^2 - 3 - 6xi + xi \cancel{+1} \cancel{-4}\right) dx$$

$$\left[\frac{3x^3}{3} - \frac{6x^2 i}{2} + \frac{x^2}{2} i + 4\right]_{-1}^{1}$$

$$\Rightarrow \left[1 - \cancel{3i} + \cancel{\frac{1}{2}i} + 4\right] - \left[-1 - \cancel{\frac{6}{2}i} + \cancel{\frac{1}{2}i} + 4\right]$$

$\Rightarrow -1 \; 2 \rightarrow 3$

Along path DA $(1,-1) \rightarrow (1,1)$

$x = 1 \quad dx = 0$

$y \Rightarrow -1 \text{ to } 1$

$$\int_{-1}^{1} \left(3(\underbrace{x+iy}_{(1+iy)^2})^2 + i(\underbrace{x+iy}_{(1+iy)}) - 4\right)(\cancel{dx} + i\,dy)$$

$$i\int_{-1}^{1} \left(3 - 3y^2 + 6iy + i - y - 4\right) dy.$$

$$i\int_{-1}^{1}\left(-1 - 3y^2 + 6iy + i - y\right)$$

$$i\left[-y - y^3 + 3iy^2 + iy - \frac{y^2}{2}\right]_{-1}^{1}$$

$$\Rightarrow i\left[\left(-1 - 1 + \cancel{3i} + \cancel{i} - \cancel{\tfrac{1}{2}}\right) - \left[+1 + 1 + \cancel{3i} - iy - \cancel{\tfrac{1}{2}}\right]\right]$$

$$i\left[-4 + 2i\right] = -4i - 2 \rightarrow (4)$$

$$\int_C f(z)\,dz = \int_{AB} f(z)\,dz + \int_{BC} f(z)\,dz + \int_{CD} f(z)\,dz + \int_{DA} f(z)\,dz$$

$$\Rightarrow (1) + (2) + (3) + (4)$$

$$\cancel{14} + \cancel{4i} - \cancel{2} - \cancel{10} - \cancel{4i} - \cancel{2}$$

$$\Rightarrow 0$$

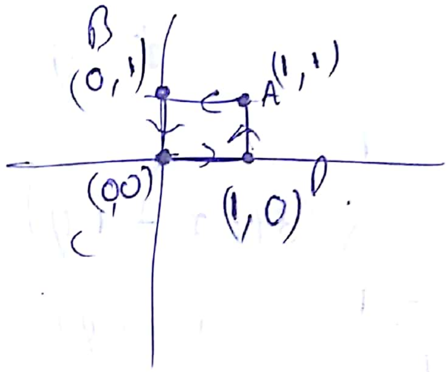
Hence CIT verified.

---

→ Verify CIT for $(f(z) = z + 1)$

if $C$ is square with $z = (0,0)$ $z = (1,0)$

$\boxed{z = 0,\ z = 1,\ z = +1 + i,\ z = i}$ $z = (+1, 1)$ $z = (0, 1)$





Sol

$$f(z) = z + 1$$
$$= x + iy + 1$$

form a squ

Along AB (1,1) & (0,1)

$y = 1 \quad dy = 0$

$x \Rightarrow 1 \to 0$

$\Rightarrow \int_1^0 (x + iy + 1)(dx + \cancel{i\,dy})$

$\Rightarrow \int_1^0 \cancel{x+1}\ (x + i + 1)\,dx$

$\Rightarrow \left[\frac{x^2}{2} + ix + x\right]_1^0 \Rightarrow \left[(0+0+0) - \left[\frac{1}{2} + i + 1\right]\right]$

$= -i - \frac{3}{2} \to (1)$

Along BC (0,1) (0,0)

$x = 0 \quad dx = 0$

$y \Rightarrow 1 \to 0$

$\int_1^0 (\cancel{x} + iy + 1)(\cancel{dx} + i\,dy)$

$i\left[\left(\frac{iy^2}{2} + y\right)\right]_1^0 \Rightarrow i(0+0) - i\left[\frac{i}{2} + 1\right]$

$\Rightarrow \cancel{-\frac{3}{2}i} + \frac{1}{2} - i \to (2)$

Along CD (0,0) (1,0)

$y = 0 \quad dy = 0$

$x \Rightarrow 0 \to 1$

$\int_0^1 (x + \cancel{iy} + 1)(dx + \cancel{i\,dy}) = \left[\frac{x^2}{2} + x\right]_0^1$

$= \frac{3}{2} \to (3)$

Along DA: $(1,0)\ (1,1)$

$x = 1 \quad dx = 0$

$y \to 0 \to 1$ ;

$$\Rightarrow \int_0^1 (x+iy+1)(dx+i\,dy)$$

$$\Rightarrow i\int_0^1 (x+iy+1) \Rightarrow \cancel{i\left[\frac{x^2}{2}+\frac{i\,y^2}{2}+x\right]_0^1}$$

$$i\int_0^1 (2+iy) = \cancel{\left[\frac{i}{2}-\frac{1}{2}+i\right]}$$

$$\Rightarrow i\left[2y+\frac{iy^2}{2}\right]$$

$$= \left[2iy-\frac{y^2}{2}\right]_0^1 = 2i-\frac{1}{2} \quad = \cancel{\left(\frac{3}{2}i-\frac{1}{2}\right)} \to (4)$$

$$\Rightarrow \int_C f(z)\,dz = \int_{AB} f(z)\,dz + \int_{BC} f(z)\,dz$$

$$\int_{CD} f(z)\,dz + \int_{DA} f(z)\,dz$$

$$= \cancel{-i}-\cancel{\frac{3}{2}}+\cancel{\frac{1}{2}}\cancel{-i}+\cancel{\frac{3}{2}} \checkmark$$

$$\cancel{+2i}-\cancel{\frac{1}{2}}$$

$= 0$

Hence CIT verified

**Cauchy's integral Formula** : Let $f(z)$ be an analytic function every where on and within a closed curve $C$

IF $z=a$ is any point within

then $f(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-a)} dz$, where the integral is taken in the positive sence around $C$

**Proof** : Let $f(z)$ be an analytic function within a close curve $C$. Let $z=a$ be any point within $C$

Chose a small suitable positive number $r_0$ and discribe a circle $C_0$ with center $a$ & radius $r_0$. So that this circle $C_0$ is entirely within $C$


consider

$z = a + r_0 e^{i\theta}$

$z - a = r_0 e^{i\theta}$

$dz = i r_0 e^{i\theta} d\theta$

consid $\int_C \frac{f(z)}{z-a)} dz = \int_{\theta=0}^{2\pi} \frac{a + r_0 e^{i\theta}}{\cancel{r_0 e^{i\theta}}} \cancel{i r_0 e^{i\theta}} d\theta$

$$= i\int_0^{2\pi} f(a + r_0 e^{i\theta})\, d\theta \quad \text{let } r_0 \to 0.$$

$$\Rightarrow \quad i \int_0^{2\pi} f(a)\, d\theta$$

$$\Rightarrow \quad = \quad i f(a) \int_0^{2\pi} d\theta$$

$$\Rightarrow \quad \int_C \frac{f(z)}{z-a}\, dz = i f(a)\, 2\pi$$

$$\therefore \quad f(a) = \frac{1}{i\, 2\pi} \int_C \frac{f(z)}{z-a}\, dz$$

Hence proved.

Generalized cauchy integral formula :-

$$f^{n}(a) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z-a)^{n+1}}$$

$n=0 \Rightarrow f(a) = \dfrac{1}{2\pi i} \displaystyle\int_C \frac{f(z)}{(z-a)} \quad \cancel{\times 2\pi i}$

$$\Rightarrow \quad \int_C \frac{f(z)}{(z-a)} = 2\pi i f(a)$$

$n=1 \Rightarrow f'(a) = \dfrac{1}{2\pi i} \displaystyle\int_C \frac{f(z)}{(z-a)^2}$

$$\Rightarrow \quad \int_C \frac{f(z)}{(z-a)^2} = 2\pi i f'(a)$$

$n = 2 \Rightarrow f''(a) = \frac{2!}{2\pi i} \int \frac{f(z)}{(z-a)^3}$

$\Rightarrow \int \frac{f(z)}{(z-a)^3} = f''(a) \frac{\cancel{2}}{\cancel{2\pi i}} \pi i$

$\rightarrow$ Evaluate $\int_C \frac{z^3 + z^2 + 2z - 1}{(z-1)^3} dz$ where c is the circle $|z| = 3$, using cauchy's integral formula.

// Given $\int_C \frac{z^3 + z^2 + 2z - 1}{(z-1)^3} dz$

$(z-1)^3 = 0 \rightarrow z - 1 = 0 \Rightarrow z = 1$
$\downarrow$
singular point

$z = 1$ is insid $|z| = 3$

By using Coushia integral formula

$\int_C \frac{f(z)}{(z-a)^3} dz = \pi i \, f(a)$

$f(z) = z^3 + z^2 + 2z - 1$ $\Rightarrow$ if $a = z$ . $f(a) = f(z)$

$a = 1$

$f'(z) = 3z^2 + 2z + 2$

$f''(z) = 6z + 2$ //

$a = z$

$= 6 + 2$

$f''(a) = 8.$

$\Rightarrow \int_c \frac{f(z)}{(z-a)^3} = \frac{2\pi i}{2} f''(a)$

$\Rightarrow \int_c \frac{z^3+z^2+2z-1}{(z-1)^3} = 8\pi i$

$\rightarrow$ Evaluate $\int_c \frac{\log z}{(z-1)^3} dz$ where c is the circle $|z-1| = \frac{1}{2}$. using cauchy's integral formula

Sol Given $\int_c \frac{\log z}{(z-1)^3} dz$

$(z-1)^3 = 0$ $\quad z-1=0$ $\quad z=1$ $\rightarrow$ singular point.

$f(z) = \log z$

$f'(z) = 1/z$

$f''(z) = -1$

$\Rightarrow f''(z) = f''(a)$

$\Rightarrow \int_c \frac{\log z}{(z-1)^3} dz = -\pi i$

→ Evaluate $\int_C \frac{z^3 - \sin 3z}{(z-\frac{\pi}{2})^3} dz$. where c is $|z| = 2$

the circle using CIF

⇒ Given $\int_C \frac{z^3 - \sin 3z}{(z-\frac{\pi}{2})^3}$ $|z| = 2$

circle

$\Rightarrow (z-\frac{\pi}{2})^3 = 0$ $\Rightarrow (z-\frac{\pi}{2}) = 0$

$z = \frac{\pi}{2} = \frac{3.14}{2}$

$= 1.57$

$z = a = \frac{\pi}{2} = 1.57$

$f(a) = f(z)$

$\Rightarrow f'(z) = 3z^2 - 3\cos 3z$

$f''(z) = 6z + 9\sin 3z$

$z = 1.57$

$= \left[6(1.57) \cancel{+} - 9 \sin \frac{3\pi}{2}\right] \pi i$

$= \left[9.42 + \cancel{0.08}\right] \pi i$ $-9$

$\cancel{= 10.15938}$ $\left[0.42\right] \pi i$

$$\begin{array}{r} .57 \\ \times 2 \\ \hline 3.14 \\ \times 3 \\ \hline 9.42 \end{array}$$

→ Evaluate $\int_C \frac{e^z}{(z-1)(z-4)} dz$ where C, $|z| = 3$ using Cauchy's integral formula.

Sol: Given $\int_C \frac{e^z}{(z-1)(z-4)}$ $|z| = 3$

$z-1=0$ $z-4=0$

$z=1$ $z=4$

$\underbrace{\qquad\qquad\qquad}$

↓

singular points

$z=1$ is inside circle. $|z| = 3$

$z=4$ " outside " " " "

∴ $f(z) = \frac{e^z}{z-4}$. $z=a=1$

$f(z) = f(a)$

⇒ $\int \frac{e^z/(z-4)}{(z-1)} dz = 2\pi i \left[ e^z/z-4 \right]_{z=1}$ $f(z)$

$= -2\pi i \frac{e}{3}$ //.

$= \frac{-2}{3}\pi i e^1$ //.

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Q, → Evaluate $\int_C \frac{dz}{z^3(z+4)}$, C, $|z|=2$

...ula.

Sol: Given $\int_C \frac{dz}{z^3(z+4)}$ $|z| = 2$

$z+4 = 0$ | $z^3 = 0$

$z = -4$ | $z = 0$

$\underbrace{\qquad\qquad\qquad}_{\text{singular points}}$

$z = -4$ is outside circle $|z| = 2$

$z = 0$ is inside circle $|z| = 2$

$f(z) = 1/(z+4)$ $z = a = 0$

$f(a) = f(z)$

$\Rightarrow \int_C \frac{1/z+4}{(z-0)^3} = 8\pi i f''(z)$

$f(z) = \left(\frac{1}{z+4}\right) \Rightarrow f'(z) = \frac{-1}{(z+4)^2}$

$-4]_{z=1}$

$= \frac{2}{(z+4)^3}$ $z=0$

$= \frac{\cancel{2}}{\cancel{64}\ 32}$

$= \frac{\pi i}{32}$

→ Evaluate $\int_c \frac{z^3-2z+1}{(z-i)^2} dz$ where c: $|z|=2$ using cauchy's integral formula.

Given $\int_c \frac{z^3-2z+1}{(z-i)^2}$

$z-i=0$

$z=i$ → singular point

⇒ $z=i$ inside $|z|=2$

⇒ $f(z)=z^3-2z+1$ $a=i=z$

⇒ $\int_c \frac{z^3-2z+1}{(z-1)^2} = 2\pi i\, f'(a)$

$f(a)=f(z)$ ; $f'(z)=3z^2-2$

$f'(z) = [2z^2-2]_{z=i}$

$= -2-2 = -5$

$\int_c \frac{z^3-2z+1}{(z-1)^2} = 2\pi i[-5]$

$= -10\pi i$

→ Evaluate : $\int_C \frac{e^z}{(z-1)(z-2)} dz$ wh c ; $|z| = 3$

using Cauchy's integral formula

Given $\int_C \frac{e^z}{(z-1)(z-2)}$

$z-1 = 0$ $\quad$ $z-3 = 0$

$z = 1$ $\quad$ $z = 2$

singular point

$z = 1$ is inside $|z| = 3$

$z = 2$ is inside $|z| = 3$

$\Rightarrow a(z-2) + b(z-1) = e^z$

$z = 2$ $\quad$ $b(1) = e^2$

$z = 1$

$-a = e^z \Rightarrow a = -e^z$

$\Rightarrow \int_C \frac{-e^z}{(z-1)} + \frac{e^z}{(z-2)} dz$

$= f(z) = -e^z$ $\quad$ $f(z) = e^z$

$a = +1$ $\quad$ $a = +2$

$2\pi i f(z)$

$= 2\pi i f(z)$ $\quad$ $= 2\pi i (e^z)_{a=+2}$

$= 2\pi i (-e^z)_{a=+1}$ $\quad$ $= 2\pi i (e^{+2})$

$2\pi i(-e^{+1})$ $\quad$ $= \frac{2\pi i}{\cancel{a}} e^2$

$= \frac{-2\pi i}{a} e$

$2\pi i f(a)$

$\int_C \frac{e^z}{(z-a)}$

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→ Evaluate $\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz$ where $|z| = 3$

C using Cauchy's integral formula

Given $\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}$

$z=1$, $z=2$ are inside circle $|z| = 3$

$$\Rightarrow \frac{1}{(z-1)(z-2)} = \frac{A}{(z-1)} + \frac{B}{(z-2)}$$

$$\frac{1}{(z-1)(z-2)} = \frac{A(z-2) + B(z-1)}{(z-1)(z-1)}$$

$z = 2$ $\quad 1 = B(1)$

$B = 1 \Rightarrow \sin \pi z^2 + \cos \pi z^2$

$z = 1$ $\quad A = -1 = -(\sin \pi z^2 + \cos \pi z^2)$

$$\int_C \left[\frac{-(\sin \pi z^2 + \cos \pi z^2)}{\cancel{z-1}}\right]_{z=1} + \left[\frac{(\sin \pi z^2 + \cos \pi z^2)}{\cancel{z-2}}\right]_{z=2}$$

$$= 2\pi i \left[-\sin \pi - \cos \pi + \sin \pi 4 + \cos \pi 4\right]$$

$$= 2\pi i \, [2] = 4\pi i$$ //