Unit - I

Numerical Methods :-

Bisection Method :- It is a simple iteration method to solve an equation.

Suppose, we know an equation of the form $f(x) = 0$ has exactly one real root b/n two real numbers (x_0, x_1).

- The number is choosen such that $f(x_0)$ & $f(x_1)$ will have opposite sign.

Let us bisect the interval $[x_0, x_1]$ into two half intervals & find the midpoint $x_2 = \frac{x_0 + x_1}{2}$.

If $f(x_2) = 0$ then x_2 is a root.

If $f(x_1)$ & $f(x_2)$ has same sign then the root lies b/n x_0 & x_2 $[x_0, x_2]$ interval.

Otherwise the root lies b/n interval $[x_2, x_1]$

Repeating the process of bisection we obtain successive sub intervals which are smaller.

At each iteration we get the midpoint as a better approximation of the root.

ex:

i) Find a root of the eqn $x^3 - 5x + 1 = 0$ using Bisection method upto 5 stages

Sol Let ~~a root~~ $f(x) = x^3 - 5x + 1$

Let $x = 0 \Rightarrow 0 - 0 + 1 = 1 > 0$

Let $x = 1 \Rightarrow 1 - 5 + 1 = -3 < 0$

$f(x_0) > 0$ & $f(x_1) < 0$.

$$x_2 = \frac{x_0 + x_1}{2} = \frac{0+1}{2} = 0.5$$

$$f(x_2) = (0.5)^3 - 5(0.5) + 1 = -1.37 < 0$$

$f(x_2) < 0$ & $f(x_0) > 0$

$$x_3 = \frac{x_0 + x_2}{2} = \frac{0 + 0.5}{2} = 0.25$$

$F(x_3) = (0.25)^3 - 5(0.25) + 1 = -0.23 < 0$

$F(x_3) < 0$ & $F(x_0) > 0$

$$x_4 = \frac{x_0 + x_3}{2} = \frac{0 + 0.25}{2} = 0.125$$

$F(x_4) = (0.125)^3 - 5(0.125) + 1 = 0.1815 > 0$

$F(x_4) > 0$ & $F(x_3) < 0$

$$x_5 = \frac{x_3 + x_4}{2} = \frac{0.25 + 0.125}{2} = 0.1875$$

$F(x_5) = 0.069 > 0$ > 0

$F(x_5) > 0$ & $F(x_3) < 0$

$$x_6 = \frac{x_3 + x_5}{2} = \frac{0.25 + 0.1875}{2} = 0.2187$$

the approximate root of given eqn $= 0.2187$

ii) Find a root of the eqn $x^3 - x - 1 = 0$ using Bisection method upto 5 stages.

Given $F(x) = x^3 - x - 1$

let $x = 0 \Rightarrow 0 - 0 - 1 \Rightarrow -1 < 0$

let $x = 1 \Rightarrow 1 - 1 - 1 \Rightarrow -1 < 0$

~~$x_2 = \frac{x_0 + x_1}{2} = -1 < 0$~~

let $x = 2$ in ① $\Rightarrow 8 - 2 - 1 = 5 > 0$

$x_0 = 1$ & $x_1 = 2$

root b/n x_0 & x_1

$\Rightarrow x_2 = \dfrac{1+2}{2} = \dfrac{3}{2} = 1.5$

$f(x)_2 = (1.5)^3 - 1.5 - 1 \Rightarrow 0.875 > 0$

root b/n x_2 & x_0

$\Rightarrow x_3 = \dfrac{1.5+1}{2} = 1.25$

-0.296875

$f(x_3) = (1.25)^3 - 1.25 - 1$

$\Rightarrow -0.296875 < 0$

root b/n x_3 & x_2

$\Rightarrow x_4 = \dfrac{1.25+1.5}{2} = 1.375$

$f(x_4) = (1.375)^3 - 1.375 - 1$

$\Rightarrow 0.224 > 0$

root b/n x_4 & x_3

$x_5 = \dfrac{1.375+1.25}{2} = 1.3125$

$f(x_5) = (1.3125)^3 - 1.3125 - 1 =$

-0.05151

< 0

root b/n x_5 & x_4

$$x_6 = \frac{1.375 + 1.3125}{2} = 1.34375$$

iii) Find a root of 25 given $x_0 = 2.0$ & $x_1 = 7.0$ using Bisection method.

Let $x = \sqrt{25}$

$x^2 = 25$

$x^2 - 25 = 0$

$f(x) = x^2 - 25$

given $x_0 = 2 \Rightarrow 4 - 25 = -21 < 0$

$x_1 = 7 \Rightarrow 49 - 25 = 24 > 0$

root b/n x_0 & x_1

$$x_2 = \frac{2+7}{2} = 4.5$$

$f(x_2) = (4.5)^2 - 25 = -4.75 < 0$

root b/n x_2 & x_1

$$x_3 = \frac{4.5+7}{2} = 5.75$$

$f(x_3) = (5.75)^2 - 25 = 8.0625 > 0$

root b/n x_2 & x_3

$$x_4 = \frac{4.5 + 5.75}{2} = 5.125$$

$$F(x_4) = (5.125)^2 - 25 = 1.265625 > 0$$

root b/n x_4 & x_2

$$x_5 = \frac{5.125 + 4.5}{2} = 4.8125$$

$$F(x_5) = (4.8125)^2 - 25 = -1.839 < 0$$

root b/n x_5 & x_4

$$x_6 = \frac{5.125 + 4.8125}{2} = 4.96875$$

$$F(x_6) = -0.311 < 0$$

root b/n x_6 & x_4

$$x_7 = \frac{\cancel{4}\ 4.96875 + 5.125}{2} = 5.04695$$

H.W $x^3 - 4x - 9 = 0$

$x^3 - 5x + 3 = 0$

$x^3 - x - 11 = 0$

$x \log x = 1.2$

b/n 2 & 3

$\sin x = \frac{1}{x}$

b/n 1 & 1.5

Newton Raphson method

$$(x_{i+1}) = x_i - \frac{F(x_i)}{F'(x_i)} \rightarrow ①$$

Taking $i = 0$, $x_1 = x_0 - \frac{F(x_0)}{F'(x_0)}$

$i = 1$, $x_2 = x_1 - \frac{F(x_1)}{F'(x_1)}$

⋮

$i = 3$ $x_4 = x_3 - \frac{F(x_3)}{F'(x_3)}$

3.125

0.28632

⋮

ex: i) Find root of the eqn

$x^3 - 3x - 5 = 0$ by using

NRM

Let $F(x) = x^3 - 3x - 5 \rightarrow ①$

Taking $x = 0$ in ① $F(0) = -5 < 0$

$x = 1$ in ① $F(1) = 1 - 3 - 5 = -7 < 0$

$x = 2$ in ① $F(2) = 8 - 6 - 5 = -3 < 0$

$x = 3$ in ① $F(3) = 27 - 9 - 5 = 13 > 0$

$F(2) < 0$ & $F(3) > 0$

root b/n 2 & 3

Here $x_0 = \frac{2+3}{2} = 2.5$

$$F(x) = x^3 - 3x - 5$$

$$F'(x) = 3x^2 - 3$$

By using Newton RM

$$x_1 = x_0 - \frac{F(x_0)}{F'(x_0)}$$

$$F(x_0) = F(2.5) = 3.125$$

$$F'(x_0) = F'(2.5) = 15.75$$

$$x_1 = 2.5 - \frac{3.125}{15.75} = 2.3015$$

$$F(x_1) = F(2.3015) = \cancel{(2.3015)^2}\ 0.2863$$

$$F'(x_1) = F'(2.3015) = 12.8907$$

$$x_2 = 2.3015 - \frac{0.3863}{12.8907} = 2.2792$$

$$F(x_2) = F(2.2792) = 0.0022$$

$$F'(x_2) = F'(2.2792) = 12.5846$$

$$x_3 = \cancel{x_2}\ 2.2792 - \frac{0.0022}{12.5840} = 2.2792$$

$x_2 = x_3 = 2.2792$

$x = 2.2792$ is an approximate root of eqn.

ii) Find root of the eqn

$x \sin x + \cos x = 0$ by using NRM

$\Rightarrow$ Let $f(x) = x \sin x + \cos x$.

$x = 0$ in ① $0 \sin(0) + \cos(0) = 1 > 0$

$x = 1$ in ① $1 \sin(1) + \cos(1) = 1.38 > 0$

$x = 2$ in ① $2 \sin(2) + \cos(2) = 1.40 > 0$

$x = 3$ in ① $3 \sin(3) + \cos 3 = -0.57 < 0$

$f(2) > 0$ & $f(3) < 0$

the root lie b/n 2 & 3

take $x_0 = \frac{2+3}{2} = 2.5$

$f'(x) = x \cos x + \sin x - \sin x = x \cos x$

By using NRM

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 2.5 - \frac{(2.5) \sin(2.5) + \cos(2.5)}{2.5 \cos(2.5)}$$

$$= 2.8470$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 2.7991$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_3)}$$

$$= 2.7983$$

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)}$$

$$= 2.7983$$

$x_3 = x_4$

$\therefore$ 2.7983 is an approximate root of eq^n.

iii) Find root of a number (N) using NRM?

Sol: $x = \sqrt{N} \Rightarrow x^2 = N$

$\Rightarrow x^2 - N = 0$

$x^2 - 23 = 0$

(let $N = 23$)

$f'(x) = 2x$

$x = 4 \Rightarrow -7 < 0$

$x = 5 \Rightarrow 2 > 0$

root b/n 4 & 5

$$x_0 = \frac{4+5}{2} = 4.5$$

$$x_1 = 4.5 - \frac{4.5^2 - 23}{2 \times 4.5} = 4.8055$$

$$x_2 = 4.8055 - \frac{(4.8055)^2 - 23}{2 \times 4.8055}$$

$$= 4.\cancel{7896}\ 7958$$

$$x_3 = 4.7958 - \frac{(4.7958)^2 - 23}{2 \times 4.7958}$$

$$= \underline{4.7958}$$

$\Rightarrow x_2 = x_3$

4.7958 is root//

HW NRM

$x = e^{-x}$ (real root)

$x + \log_{10} x = 3.375$

$x e^{x} - \cos x = 0$ (real root)

Gauss - Seidel iteration Method

i) Using GSEIM to solve the system of eqn's

$$10x + y + z = 12$$
$$2x + 10y + z = 13$$
$$2x + 2y + 10z = 14$$

Sol:

$10x + y + z = 12 \Rightarrow x = \frac{1}{10}(12 - y - z) \rightarrow ①$

$2x + 10y + z = 13 \Rightarrow y = \frac{1}{10}(13 - 2x - z) \rightarrow ②$

$2x + 2y + 10z = 14 \Rightarrow z = \frac{1}{10}(14 - 2x - 2y) \rightarrow ③$

Taking $y^{(0)} = 0$ & $z^{(0)} = 0$ in ①

$\Rightarrow x^{(1)} = \frac{12}{10} = 1.2$

Taking $x^{(1)} = 1.2$ & $z = 0$ in ②

$\Rightarrow y^{(1)} = \frac{1}{10}(13 - 2(1.2)) = 1.06$

Take $x^{(1)} = 1.2$ & $y^{(1)} = 1.06$ in ③

$\Rightarrow z^{(1)} = \frac{1}{10}(14 - 2.4 - 2(1.06))$

$= 0.948$

sub $y^{(1)} = 1.06$ & $z^{(1)} = 0.948$ in ①

$x^{(2)} = \frac{1}{10}(12 - 1.06 - 0.948) = 0.999$

sub $x^{(2)}, z^{(1)}$ in ②

$y^{(2)} = \frac{1}{10}(13 - 2(0.999) - 0.948)$

$= 1.005$

Sub $x^{(2)}, y^{(2)}$ in 3

$$z^{(2)} = \frac{1}{10}[14 - 2(0.999) - 2(1.005)]$$

$$= 0.9984$$

Sub $y^{(2)}, z^{(2)}$ in 1

$$x^{(3)} = \frac{1}{10}(12 - 1.005 - 0.9984)$$

$$= 0.9996$$

Sub $x^{(3)}, z^{(2)}$ in 2

$$y^{(3)} = \frac{1}{10}[13 - 2(0.9996) - 0.99984]$$

$$= 1.0003$$

Sub $x^{(3)}, y^{(3)}$ in 3

$$z^{(3)} = \frac{1}{10}[14 - 2(0.9996) - 2(1.0003)]$$

$$= 1$$

Sub $y^{(3)}, z^{(3)}$ in 1

$$x^{(4)} = 0.999.$$

Sub $x^{(4)}, z^{(3)}$ in 2

$$y^{4} = 1.00$$

$x = y = z = 1$

ii) $8x - 3y + 2z = 20$

$4x + 11y - z = 33$ GSIM

$6x + 3y + 12z = 36$

Sol

$\Rightarrow 8x - 3y + 2z = 20 \Rightarrow x = (20 + 3y - 2z)\frac{1}{8}$

$\Rightarrow 4x + 11y - z = 33 \Rightarrow y = (33 - 4x + z)\frac{1}{11}$

$\Rightarrow 6x + 3y + 12z = 36 \Rightarrow z = (36 - 6x - 3y)\frac{1}{12}$

$y^{(0)} = 0$ & $z^{(0)} = 0$ in ①

$x^{(1)} = \frac{20}{8} = 2.5$

$x^{(1)} = 2.5$ & $z^{(0)} = 0$

$y^{(1)} = \frac{1}{11}(33 - 4(2.5)) = 2.0909$

$y^{(1)} = 2.0909$ & $x^{1} = 2.5$

$z^{(1)} = \frac{1}{12}(36 - 6(2.5) - 3(2.0909))$

$z^{(1)} = 1.2272$

$y^{(1)} = 2.909$ & $z^{(1)} = 1.2272$

$x^{(2)} = \frac{1}{8}[20 + 3(2.909) - 2(1.2272)]$

$x^{(2)} = 2.9772.$

Sub $x^{(2)}, z^{(1)}$ in ②

$$y^{(2)} = \frac{1}{11}\left[33 - 4(2.972) + 1.2272\right]$$

$= 2.0289$

Sub $x^{(2)} = 2.9772$ $y^{(2)} = 2.0289$

$$z^{(2)} = \frac{1}{12}(36 - 6(2.9772) - 3(2.0289))$$

$= 1.0041$

Sub $y^{(2)} = 2.0289$ $z^{(2)} = 1.0041$

$$x^{(3)} = \frac{1}{8}(20 + 3(2.0289) - 2(1.0041))$$

$= 3.0098.$

Sub $x^{(3)} = 3.0098$ & $z^{(2)} = 1.0041$

$$y^{(3)} = \frac{1}{11}33 - 4(3.0098) + 1.0041$$

$= \cancel{0.9959}$ 1.9968

Sub $x^{(3)} = 3.0098$ & $y^{(3)} = 1.9968$

$$z^{(3)} = \frac{1}{12}(36 - 6(3.0098) - 3(1.9968))$$

$= 0.9959$

$x \simeq 3 \quad y \simeq 2 \quad z \simeq 1$

H.W GS IM

i) $x + 10y + z = 6$

$10x + y + z = 6$

$x + y + 10z = 6$

$x = 0.5$

$y = 0.5$

$z = 0.5$

Newtons Forward Interpolation Formula :-

$$y = f(x) = y_0 + P\Delta y_0 + \frac{P(P-1)}{2!}\Delta^2 y_0 + \frac{P(P-1)(P-2)}{3!}\Delta^3 y_0 + \cdots$$

$$P = \frac{x - x_0}{h}$$, h is the length of interval.

Newtons backward Interpolation formula :-

$$y = f(x) = y_n + P\nabla y_n + \frac{P(P+1)}{2!}\nabla^2 y_n + \frac{P(P+1)(P+2)}{3!}\nabla^3 y_n + \cdots$$

$$P = \frac{x - x_n}{h}$$, h is the length of interval.

Note: NFIF symbolically denoted by Δ & NBIF symbolically denoted by ∇

ex: i) Using NFIF find $y(1.4)$ from the following table

x	1.1	1.3	1.5	1.7	1.9
y	0.21	0.69	1.25	1.89	2.61

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1.1	0.21				
		0.48			
1.3	0.69		0.08		
		0.56		0	
1.5	1.25		0.08		0
		0.64		0	
1.7	1.89		0.08		
		0.72			
1.9	2.61				

Newtons Forward interpolation Formula

$$y = f(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \dots \rightarrow (1)$$

$p = \frac{x - x_0}{h}$ h is the length of interval

Here $x = 1.4$ from question

$x_0 = 1.1$ " "

$h = 0.2$ (ie ⇒ $1.3 - 1.1 = 0.2$)

$$p = \frac{1.4 - 1.1}{0.2} = \frac{0.3}{0.2} = 1.5$$

$y_0 = 0.21$ $\Delta y_0 = 0.48$ $\Delta^2 y_0 = 0.08$

$\Delta^3 y_0 = 0$ $\Delta^4 y_0 = 0$

Sub the values

$$\Rightarrow y = 0.21 + (1.5)(0.48) + \frac{(1.5)(0.5)}{2} \times 0.08 + 0 + 0$$

$\Rightarrow y(1.4) = \underline{\underline{0.96}}$

<u>Note</u>: By using Newton Backword interpolation Formula

$$y = F(x) = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \frac{p(p+1)(p+2)(p+3)}{4!}\nabla^4 y_n \ldots$$

$$p = \frac{x - x_n}{h} \hookrightarrow ②$$

h = length of interval.

$x = 1.4$, $x_n = 1.9$, $h = 0.2$ $y_n = 2.61$

$\nabla y_n = 0.72$ $\nabla^2 y_n = 0.08$ $\nabla^3 y_n = 0$ $\nabla^4 y_n = 0$

$$y(1.4) = 2.61 + (-2.5)(0.72) + \frac{(2.5)(-1.5)}{2}(0.08) + 0 + 0$$

$$= 0.96$$

ii) Using WFIF Find the polynomial eq^n for following table

x	0	1	2	3
$f(x)$	1	3	7	13

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
0	1			
		2		
1	3		2	
		4		0
2	7		2	
		6		
3	13			

$\Rightarrow$ we know

$$y = f(x) = y_0 + p\,\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y \ldots$$

$x_0 = 0 \quad h = 1 \quad p = \frac{x-0}{1} = x$

$y_0 = 1 \quad \Delta y_0 = 2 \quad \Delta^2 y_0 = 2 \quad \Delta^3 y_0 = 4$

$$y = 1 + x(2) + \frac{x(x-1)}{2}(2) + 0$$

$$y = x^2 + x + 1$$

iii) Using NFIF

x	0	1	2	3	4
$F(x)$	1	14	15	5	6

$F(2.5)$

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	1				
		13			
1	14		-12		
		1		1	
2	15		11		+21
		-10		20 2	
3	5		-11		
		1			
4	6				

=) We know

$$y = F(x) = y_0 + P\Delta y_0 + \frac{P(P-1)}{2!}\Delta^2 y_0 + \frac{P(P-1)(P-2)}{3!}\Delta^3 y + \frac{P(P-1)(P-2)(P-3)}{4!}\Delta^4 y \ldots$$

$x_0 = 0$; $h = 1$; $x = 2.5$

$$p = \frac{2.5 - 0}{1} = 2.5$$

$$y = 1 + 2.5(13) + \frac{(2.5)(1.5)}{2}(-12) + \frac{(2.5)(1.5)(0.5)}{6}(1) + \frac{(2.5)(1.5)(0.5)(-0.5)}{24}(21) \ldots$$

$y = 10.49$

Lagrange's Interpolation formula :-

x	x_0	x_1	x_2	x_3
$y = f(x)$	$f(x_0)$	$f(x_1)$	$f(x_2)$	$f(x_3)$

$$y = f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} f(x_0)$$

$$+ \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} f(x_1)$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} f(x_2)$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} f(x_3)$$

ex i) Find $f(2)$ by using LIF

x	x_0 0	x_1 1	x_2 3	x_3 4
$y = f(x)$	5 $f(x_0)$	6 $f(x_1)$	50 $f(x_2)$	105 $f(x_3)$

$x = 2$

Sub value in above eqn

$$\Rightarrow y = f(2) = \frac{(2-1)(2-3)(2-4)}{(0-1)(0-3)(0-4)} 5 + \frac{(2-0)(2-3)(2-4)}{(1-0)(1-3)(1-4)}(6) + \frac{(2-0)(2-1)(2-4)}{(3-0)(3-1)(3-4)}(50) + \frac{(2-0)(2-1)(2-3)}{(4-0)(4-1)(4-3)} 105$$

$$= -\frac{5}{6} + 4 + \frac{100}{3} - \frac{35}{2}$$

$$= \cancel{21.5} \quad 19$$

ii) Find $f(10)$ by using LIF

$x = 10 \quad 5 \quad 6 \quad 9 \quad 11$

$y = f(x) \quad 12 \quad 13 \quad 14 \quad 16$

$$\Rightarrow \cancel{\frac{(-4)(-7)(-9)}{(-1)(-4)(-6)}(12)} + \frac{(4)(1)(-1)}{(-1)(-4)(-6)}(12) + \frac{(5)(1)(-1)}{(1)(-3)(-5)}(13) + \frac{(5)(4)(-1)}{(4)(3)(-2)}(14) + \frac{(5)(4)(1)}{(6)(5)(2)}(16)$$

$$= \cancel{23.33} \quad 14.67$$

Find polynomial by using Lagrange I F

x	0	1	2	5
$y - f(x)$	2	3	12	147

$$\Rightarrow \frac{(x-1)(x-2)(x-5)}{(0-1)(0-2)(0-5)}(2) + \frac{(x-0)(x-2)(x-5)}{(1-0)(1-2)(1-3)}(3)$$

$$+ \frac{(x-0)(x-1)(x-5)}{(2-0)(2-1)(2-5)}(12) + \frac{(x-0)(x-1)(x-2)}{(5-0)(5-1)(5-2)}(147)$$

$$\Rightarrow -\frac{(x-1)(x^2-7x+10)}{5} \Rightarrow -\frac{[x^3-7x^2+10x-x^2+7x-10]}{5}$$

$$= -\frac{[x^3-8x^2+17x-10]}{5}$$

$$\Rightarrow \frac{3x(x^2-7x+10)}{2} = \frac{3x^3-21x^2+30x}{2}$$

$$\Rightarrow -2x(x^2-6x+5) = -[2x^3-12x^2+10x]$$

$$\Rightarrow \frac{49}{20}x(x^2-3x+2) = \frac{49x^3-147x^2+98x}{20}$$

$$\Rightarrow \left(-\frac{1}{5}+\frac{3}{2}-2+\frac{49}{20}\right)x^3$$

$$+\left(+\frac{8}{5}-\frac{21}{2}+12-\frac{147}{20}\right)x^2$$

$$+\left(-\frac{17}{5}+\frac{30}{2}-10+\frac{98}{20}\right)x$$

$$+\left(\frac{2\cancel{10}}{\cancel{8}}\right]$$

$$\Rightarrow \frac{7}{4}x^3-\frac{17}{4}x^2+\frac{13}{2}x+2$$

But $\left(\frac{x^3}{2}+x^2-x+2\right)$

iii)

x =	1,	2,	3	4
$f(x)$ =	1,	2,	9,	28

(3.5) $\Rightarrow$ using LIF

f(3.5)

$$= \frac{(1.5)(0.5)(-0.5)}{(-1)(-2)(-3)}(1)+\frac{(2.5)(0.5)(-0.5)}{(+1)(-1)(-2)}(2)$$

$$\frac{(2.5)(1.5)(-0.5)}{(2)(1)(-1)}(9)+\frac{(2.5)(1.5)(0.5)}{(3)(2)(1)}(28)$$

$$=\frac{3}{32}+\left(-\frac{5}{8}\right)+\frac{135}{16}+\frac{35}{4} = 16.65625$$

H.W i) $f(5) = ?$ use LIF

x	0	1	3	8
$f(x) = y$	1	3	13	128

ii) polynomial eq^n - - - or LIF

x	0	1	3	4
$f(x) = y$	-12	0	6	12

& also $f(2)$

Taylor's method :

$$y_1 = y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \ldots$$

$\boxed{y' = f(x, y)}$

$$y_2 = y_1 + \frac{h}{1!} y_1' + \frac{h^2}{2!} y_1'' + \frac{h^3}{3!} y_1''' + \ldots$$

$$y' = f(x, y)$$

$$x_1 = x_0 + h$$

$$x_2 = x_1 + h$$

$$x_3 = x_2 + h$$

i) Using TSM solve

$$\frac{dy}{dx} = x^2 + y^2, \text{ given that } y = 1 \text{ at } x = 0$$

Find $y(0.1)$ & $y(0.2)$

Sol: Given $y' = x^2 + y^2$

Given $y = 1$ at $x = 0 \Rightarrow x_0 = 0, y_0 = 1$

hence $x_1 = 0.1$, $x_2 = 0.2$, $h = 0.1$

$y' = x^2 + y^2 \Rightarrow y_0' = x_0^2 + y_0^2 = 0 + 1 = 1$

$y'' = 2x + 2yy'$.

$y_0'' = 2(0) + 2(1)(1) = 2$

$y''' = 2 + 2(y')^2 + 2yy''$

$y_0''' = 8$

$y'''' = 0 + 2.2(y')y'' + 2yy''' + 2y'y''$

$y_0'''' = 28$

By using Taylor's method

$$y = y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \frac{h^4}{4!} y_0^{iv} + \dots$$

$y_1 = y(x_1) = y(0.1) = 1.1114$

Now $x_1 = 0.1$ $y_1 = 1.1114$, $h = 0.1$

$y' = x^2 + y^2 \Rightarrow y_1' = x_1^2 + y_1^2$

$\Rightarrow (0.1)^2 + (1.1114)^2$

$y_1' = 1.2452$

$y'' = 2x + 2yy'$

$y_1'' = 2x_1 + 2y_1 y_1'$

$= 2(0.1) + 2(1.1114)(1.2452)$

$y_1'' = 2.9678.$

$y''' = 2 + 2(y')^2 + 2yy''$

$y_1''' = 2 + 2(y_1')^2 + 2y_1 y_1''$

$= 2 + 2(1.2452)^2 + 2(1.1114)$
(2.9678)

$y_1''' = 11.6978$

$y'''' = 0 + 2.2(y')y'' + 2yy''' + 2y'y''$

$y_1'''' = 4(1.2452)(2.9678) + 2.(1.1114)$
$(11.6978) + 2(1.2452)(2.9678)$

$y_1'''' = 48.1748$

By un TM $y_2 = y_1 + \frac{h}{1!} y_1' + \frac{h^2}{2!} y_1'' + \frac{h^3}{3!} y_1'''$

$y_2 = 1.2529$

$y_2 = y(x_2) = y(0.2) = 1.2529$

ii) Using TM solve $y' = x^2 - y$

$y(0) = 1$ Find $y(0.1)$ & $y(0.2)$

Sol Given $y' = x^2 - y$ $y(0) = 1$

$x_0 = 0, \ y_0 = 1$

$x_1 = 0.1, \ x_2 = 0.2, \ h = 0.1$

$y' = x^2 - y$

$y'_0 = 0 - 1 = -1$

$y'' = 2x - y'$

$y''_0 = 2(0) + 1 = 1$

$y''' = 2 - y''$

$y'''_0 = 2 - 1 = 1$

$y'''' = 0 - y'''$

$y''''_0 = 0 - 1 = -1$

Sub in TM

$$y_1 = y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \frac{h^4}{4!} y_0''''$$

$$= 1 + \frac{0.1}{1}(-1) + \frac{(0.1)^2}{2}(1) + \frac{(0.1)^3}{6}(1) + \frac{(0.1)^4}{24}(-1)$$

$$y_1 = 0.9051625 = \underline{\underline{y(0.1)}}$$

To be continued ... $\underline{\underline{[\,y(0.2)\,]}}$

$x_1 = 0.1,\ y_1 = 0.9051,\ h = 0.1.$

$y' = x^2 - y$

$y_1' = (0.1)^2 - 0.9051 = \cancel{-0.8092}\ -0.8951$

$y'' = 2x - y'$

$y_1'' = 1.0951$

$y''' = 2 - y''$

$= 2 - 1.0951 = 0.9049$

$y'''' = 0 - y'''$

$= -0.9049$

$y_2 = 0.9051 + \frac{0.1}{1}(-0.8951)$

$+ \frac{(0.1)^2}{2}(1.0951) + \frac{(0.1)^3}{6}(0.9049)$

$+ \frac{(0.1)^4}{24}(-0.9049)$

$= 0.8212$

$y_2 = y(0.2) = 0.8212$

iii) Using TM $y' = x + y$ $y(1) = 0$

$y(1.1)$ & $y(1.2)$

Sol: Gn $y' = x + y$

$y(1) = 0$; $x_0 = 1$; $y_0 = 0$

$x_1 = 1.1$ $x_2 = 1.2$, $h = 0.1$

$$y' = x + y$$

$$y_0' = 1$$

$$y'' = 1 + y'$$

$$y_0'' = 1 + 1 = 2$$

$$y''' = 0 + y''$$

$$y_0''' = 0 + 2 = 2$$

$$y'''' = 0 + y'''$$

$$y_0'''' = 0 + 2 = 2$$

$$y_1 = 0 + \frac{(0.1)^1}{1}(1) + \frac{(0.1)^2}{2}(2) + \frac{(0.1)^3}{6}(2) + \frac{(0.1)^4}{24}(2)$$

$$y_1 = 0.1103$$

$$y_1 = y(1.1) = 0.1103$$

$$y' = x + y$$

$$y_1' = (1.1) + 0.1103$$

$$= 1.2103$$

$$y'' = 1 + y'$$

$$= 1 + 1.2103$$

$$= 2.2103$$

$$y''' = 0 + y''$$

$$= 2.2103$$

$$y'''' = 0 + y'''$$

$$= 2.2103$$

$$y_2 = 0.1103 + \frac{(0.1)^1}{1}(1.2103) + \frac{(0.1)^2}{2}(2.2103) + \frac{(0.1)^3}{6}(2.2103) + \frac{(0.1)^4}{24}(2.2103)$$

$$y_2 = 0.2427$$

$$y_2 = y(2.2) = 0.2427$$

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HW i) $y' = x^2 y - 1$, $y(0) = 1$
$y(0.1)$, $y(0.2)$

ii) $y' = 3x + y^2$, $y(0) = 1$
$y(0.1)$, $y(0.2)$

Eulers method:

$$y_{n+1} = y_n + h\,F(x_n, y_n), \quad n = 0, 1, 2, 3$$

$y' = F(x, y)$

$$y_1 = y_0 + h\,F(x_0, y_0)$$
$$y_2 = y_1 + h\,F(x_1, y_1)$$
$$y_3 = y_2 + h\,F(x_2, y_2)$$

i) Solve EM $y' = x + y$, $y(0) = 1$, Find $y(0.2)$ talking step size $h = 0.1$

Given $y' = F(x, y) = x + y$

Given $y(0) = 1$

$\Rightarrow$ $x_0 = 0$; $y_0 = 1$; $h = 0.1$

$x_1 = x_0 + h \Rightarrow x_1 = 0 + 0.1 \Rightarrow x_1 = 0.1$

$x_2 = x_1 + h \Rightarrow x_2 = 0.1 + 0.1 \Rightarrow x_2 = 0.2$

By usin. EM

$$y_1 = y_0 + hf(x_0, y_0)$$
$$= 1 + (0.1)(x_0 + y_0)$$
$$= 1 + (0.1)(1) = 1.1$$

$$y_2 = y_1 + hf(x_1, y_1)$$
$$= 1.1 + (0.1)(1.2)$$
$$= 1.1 + 0.12 = 1.22$$

$y(x_2) = y(0.2) = 1.22$

$y(x_1) = y(0.1) = 1.1$

ii) Solve eulers method $y' = x^2 - y$

$y(0) = 1$ find $y(0.1), y(0.2)$

Gin $y' = f(x, y) = (x^2 - y)$

$y(0) = 1$

$x_0 = 0$ $y_0 = 1$ $h = 0.1$

$x_1 = 0 + 0.1 = 0.1$

$x_2 = 0.1 + 0.1 = 0.2$

By using EM

$y_1 = y_0 + hF(x_0, y_0)$

$= 1 + (0.1)(0 - 1)$

$= 1 - 0.1$

$= 0.9$

$y_2 = 0.9 + (0.1)(x_1, y_1)$

$= 0.9 + (0.1)(0.1, 0.9)$

$= 0.9 + (0.1)(-0.89)$

$= 0.811$

$y_1 = y(0.1) = 0.9$

$y_2 = y(0.2) = 0.811$

iii) Solve euler's method $y' = x^2 + xy^2$

$y(0) = 1$ & $y(0.1), y(0.2)$

$\Rightarrow$ Given $y' = x^2 + xy^2$

$y(0) = 1$

$x_0 = 0, \; y_0 = 1, \; h = 0.1$

$x_1 = 0 + 0.1 = 0.1$

$x_2 = 0.1 + 0.1 = 0.2$

$y_1 = y_0 + h F(x_0, y_0)$

$= 1 + (0.1)(0)$

$= 1$

$y_2 = 1 + (0.1)((0.1)^2 + (0.1)1)$

$= 1.011$

$y_1 = y(0.1) = 1$

$y_2 = y(0.2) = 1.011$

HW EM

i) $y' = y + e^x$, $y(0) = 1$, $y(0.1)$ & $y(0.2$

ii) compute y at 0.25 by EM

$y' = 2xy$ $y(0) = 1$ $y(0.25)$ $h = \ldots$

iii) $y' = y^2 + x$, $y(0) = 1$ & $y(0.1)$

& $y(0.2$

Ruge - Kutta Method of Fourth order

(R - K Method)

$$y_1 = y_0 + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$$

when $K_1 = h\,f(x_0, y_0)$

$$K_2 = h\,f\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}\right)$$

$$K_3 = h\,f\left(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}\right)$$

$$K_4 = h\,f(x_0 + h, y_0 + K_3)$$

$$y_2 = y_1 + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$$

when $K_1 = h\,f(x_1, y_1)$

$$K_2 = h\,f\left(x_1 + \frac{h}{2}, y_1 + \frac{K_1}{2}\right)$$

$$K_3 = h\,f\left(x_1 + \frac{h}{2}, y_1 + \frac{K_2}{2}\right)$$

$$K_4 = h\,f(x_1 + h, y_1 + K_3)$$

ex: i) using R-K method solve $y' = x+y$

$y(0) = 1$. Find $y(0.1)$ & $y(0.2)$

Given $y' = x+y$

$y(0) = 1$

$x_0 = 0$, $y_0 = 1$, $h = 0.1$

$x_1 = x_0 + h$

$= 0 + 0.1 = 0.1$

$K_1 = h\, f(x_0, y_0)$

$= (0.1)(1) = 0.1$

$K_2 = (0.1)\, f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$

$= (0.1)\left[0.05 + 1.05\right]$ $\left(1 + \frac{0.11}{2}\right.$

$= (0.1)(1.1) = 0.11$

$K_3 = (0.1)\, f(0.05 + 1.055)$

$= 0.1(1.105)$

$= 0.1105$

$K_4 = (0.1)\, f(0.1, 1.1105)$

$= (0.1)(1.2105)$

$= 0.12105$

$$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= 1 + \frac{1}{6}\Big(0.1 + 2(0.11) + 2(0.11051) + 0.12105\Big)$$

$$y(0.1) = y_1 = 1.11034$$

$x_1 = 0.1$
$y_1 = 1.11034$

$$k_1 = h f(x_1, y_1)$$

$$= (0.1) f(0.1, 1.11034)$$

$$= (0.1)(1.21034)$$

$$= 0.121034$$

$$k_2 = h f\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right)$$

$$= (0.1) f\left(\underset{0.7}{0.1 + 0.05}, 1.11034 + \frac{0.121034}{2}\right)$$

$$= (0.1)(0.105 + 1.170857)$$

$$= (0.1)(1.320857)$$

$$= 0.1320857$$

$$K_3 = hF\left(x_1 + \frac{h}{2}, y_1 + \frac{K_2}{2}\right)$$

$$= (0.1)\, F(0.15,\ 1.11034 + 0.06604285)$$

$$= 0.132638285$$

$$K_4 = hF(x_1 + h, y_1 + K_3)$$

$$= (0.1)\, F(0.2 + 1.242978285)$$

$$= 0.1442978285.$$

$$y_2 = 1.11034 + \frac{1}{6}(0.121034 + 2.(0.1320857) + 2(0.132638285 + 0.1442978285)$$

$$y_2 = y(0.2) = 1.2428033$$

ii) Apply the Fourth order R-K method to find y at $x = 1.2$ in step of 0.1 given that $y' = x^2 + y^2$, $y(1) = 1.5$

1 $\xrightarrow{0.1}$ 1.2

$1 \to 1.1 \to 1.2$

Sol Given $y' = x^2 + y^2$

Gn $y(1) = 1.5$

$x_0 = 1 \quad y_0 = 1.5 \quad h = 0.1$

$x_1 = x_0 + h = 1 + 0.1 = 1.1$

$x_2 = x_1 + h = 1.1 + 0.1 = 1.2$

$$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$k_1 = hF(x_0, y_0)$

$= 0.1(1^2 + 1.5^2)$

$= 0.325$

$k_2 = hF\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$

$= 0.1\, F(1.05, 1.6625)$

$= 0.386640625$

$k_3 = hF\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$

$= 0.1\, F(1.05, 1.693320313$

$= 0.39698336(8)$

$K_4 = hf(x_0 + h, y_0 + K_3)$

$= 0.1\, f(1.1,\ 1.8969833681)$

$= 0.4808545899$

$y_1 = 1.5 + \frac{1}{6}\left(0.325 + 2(0.38664062 5)\right.$

$\left. + 2(0.3969833681) + 0.4808545899\right)$

$y_1 = y(1.1) = 1.89551709 6$

$y_2 = y_1 + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$

$K_1 = hf(x_1, y_1) = 0.1(1.1,\ 1.895517096)$

$= 0.4802985061$

$K_2 = hf(x_1 + \frac{h}{2} + y_1 + \frac{K_1}{2}) = 0.1(1.15,\ \cancel{1.9455170 69})\ 2.135666322$

$= 0.5883570639\ \cancel{= 0.51075367 7}$

$K_3 = hf(x_1 + \frac{h}{2},\ y_1 + \frac{K_2}{2})\ \cancel{= 0.1(1.15,\ 0.56075}$

$= 0.1(1.15,\ 2.189695628)$

$= 0.6117266943$

$$k_4 = hf(x_1+h, y_1+k_3)$$

$$= 0.1 f(1.2, 2.50724379)$$

$$= 0.7726271424$$

$$y_2 = y_1 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$\cancel{= 1.895517096 + \frac{1}{6}(}$$

$$= 2.504365957$$

H.W

RK meth

i) $y' = y - x$, $y(0) = 2$, $h = 0.2$

Find $y(0.2)$

ii) $y(0.1)$ & $y(0.2)$, $y' = x^2 - y$

$y(0) = 1$, $h = 0.1$

Newton's divided & differnce formulas

$$F(x) = F(x_0) + (x - x_0) F(x_0 - x_1) + (x - x_0)$$

$$(x - x_1)\ F(x_0, x_1, x_2) + (x - x_0)(x - x_1)$$

$$(x - x_2)\ F(x_0, x_1, x_2, x_3) \ldots\ldots$$

ex! i) Compute $f'(4)$

x	1	2	4	8	10
$f(x)$	0	1	5	21	27

Using WDDF

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
1	0				
		$\frac{1-0}{2-1}=1$			
2	1		$\frac{2-1}{4-1}=\frac{1}{3}$		
		$\frac{5-1}{4-2}=2$		$\frac{\frac{1}{3}-\frac{1}{3}}{8-1}=0$	
4	5		$\frac{4-2}{8-2}=\frac{1}{3}$		$\frac{\frac{1}{16}-0}{10-1}=\frac{-1}{144}$
		$\frac{21-5}{8-4}=4$		$\frac{-\frac{1}{6}-\frac{1}{3}}{10-2}=\frac{-1}{16}$	
8	21		$\frac{3-4}{10-4}=-\frac{1}{6}$		
		$\frac{27-21}{10-8}=3$			
10	27				

By using NDDF

$$f(x) = f(x_0) + (x-x_0)\,f(x_0,x_1) + (x-x_0)(x-x_1)$$
$$f(x_0,x_1,x_2) + (x-x_0)(x-x_1)(x-x_2)$$
$$f(x_0\,x_1\,x_2\cdot x_3 + \ldots$$

$x_0 = 1, x_1 = 2, x_2 = 4, x_3 = 8, x_4 = 10$

sub the value in (1)

$$f(x) = (x-1) + \frac{1}{3}(x^2 - 3x + 2)$$

$$- \frac{1}{144}(x^2 - 3x + 2)(x^2 - 2x + 32)$$

$$f'(x) = 1 + \frac{1}{3}(2x - 3) - \frac{1}{144}[4\cancel{x^3} - 30x^2$$

$$+ 64x - 9x^2 + 72x - 96 + 4x$$

$$- 24]$$

$$= 2.833333$$

ii) Compute $f'(6)$

x	0	2	3	4	7	9
$f(x)$	4	26	58	112	466	922

x	$F(x)$	$\Delta F(x)$	$\Delta^2 F(x)$	$\Delta^3 F(x)$	$\Delta^4 F(x)$	$\Delta^5 F(x)$
0	4					
		11				
			7			
2	26			1		
		32			0	
			11			0
3	58			1		
		54				
4	112		16		0	
		118		1		
7	466		22			
		228				
9	922					

$x_0 = 0,\ x_1 = 2,\ x_2 = 3,\ x_3 = 4,\ x_4 = 7,\ x_5 = 9$

By NDDF

$$F(x) = F(x_0) + (x - x_0)\, F(x_0, x_1) + (x - x_0)(x - x_1)$$
$$F(x_0, x_1, x_2) + (x - x_0)(x - x_1)(x - x_2)$$
$$F(x_0, x_1, x_2, x_3) + (x - x_0)(x - x_1)(x - x_2)$$
$$(x - x_3)\, F(x_0, x_1, x_2, x_3) + \ldots$$

$\Rightarrow 4 + x(11) + x(x-2)7 + x(x-2)(x-3)$
$+ 0x.$

$\Rightarrow 4 + 11x + 7x^2 - 14x + x(x^2 - 5x + 6)$

$\Rightarrow 4 + 11x + 7x^2 - 14x + x^3 - 5x^2 + 6x$

$f(x) \Rightarrow x^3 + 2x^2 + 3x + 4$

$f'(x) = 3x^2 + 4x + 3$

$= 3(6)^2 + 4(6) + 3$

$= 135$

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