Unit-2

Ordinary Diff. Eqns:

Exact diff. eqns:

$M(x,y)\,dx + N(x,y)\,dy = 0 \longrightarrow (1)$ is said to be

Exact if $\boxed{\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}}$.

Then the General sol'n of (1) is given by

$$\phi(x,y) = \int_{y\ \text{constant}} M\,dx + \int (\text{terms independent of } x \text{ in } N)\,dy = c.$$

(i.e without x)

(1). $e^y\,dx + (xe^y + 2y)\,dy = 0 \longrightarrow (1)$

$\therefore M = e^y, \qquad N = xe^y + 2y$

$\dfrac{\partial M}{\partial y} = e^y, \qquad \dfrac{\partial N}{\partial x} = e^y \cdot 1 + 0 = e^y$

$\because \dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x} \Rightarrow \therefore$ (1) is exact.

$\therefore$ The General soln of (1) is given by

$$\phi(x,y) = \int_{y\ \text{constant}} e^y\,dx + \int 2y\,dy = c$$

$$= e^y \cdot x + \cancel{2} \cdot \frac{y^2}{\cancel{2}} = c$$

$$\boxed{\phi(x,y) = xe^y + y^2 = c}$$

(2) $\left(3x^2y + \frac{y}{x}\right)dx + (x^3 + \log x)\,dy = 0$

$M = 3x^2y + \frac{y}{x}$, $N = x^3 + \log x$

$\frac{\partial M}{\partial y} = 3x^2 + \frac{1}{x}$, $\frac{\partial N}{\partial x} = 3x^2 + \frac{1}{x}$

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$\therefore$ (1) is Exact

$\therefore$ The G.S. is

$$\phi(x,y) = \int_{y\ \text{const}} \left(3x^2y + \frac{y}{x}\right)dx + \int (0)\,dy = C$$

$$= 3y\int x^2\,dx + y\int \frac{1}{x}\,dx = C$$

$$= \cancel{3}y\,\frac{x^3}{\cancel{3}} + y\cdot\log x = C$$

$$\boxed{\phi = y(x^3 + \log x) = C}$$

~~Also~~ Integrating factor: Let $M\,dx + N\,dy = 0$ is not exact diff. eqn. — (1) The eqn (1) can be made exact by multiplying both sides with suitable function $u(x,y)$. This suitable fn $u(x,y)$ is called as Integrating factor.

Method I:

Some Imp. Integrating factors.

$d(\sin x) = \cos x$

$\int \cos x \, dx = \sin x$

1. $x\,dy + y\,dx = d(xy)$

2. $\dfrac{y\,dx - x\,dy}{y^2} = d\left(\dfrac{x}{y}\right)$

3. $\dfrac{x\,dy - y\,dx}{x^2} = d\left(\dfrac{y}{x}\right)$

4. $x\,dx + y\,dy = d\left(\dfrac{x^2+y^2}{2}\right)$.

5. $\dfrac{dy}{y} - \dfrac{dx}{x} = d\left[\log\left(\dfrac{y}{x}\right)\right]$

$$\int f(x)^n f'(x)\,dx = \frac{f(x)^{n+1}}{(n+1)}$$

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① $y(y^3 - x)\,dx + x(y^3 + x)\,dy = 0 \quad —①$

$M = y^4 - xy, \quad N = x^2 + xy^3$

$\dfrac{\partial M}{\partial y} = 4y^3 - x, \quad \dfrac{\partial N}{\partial x} = 2x + y^3$

$\because \dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x}$

$\therefore$ ① is not exact

$y(y^3 - x)\,dx + x(y^3 + x)\,dy = 0$

$y^4\,dx + xy^3\,dy - xy\,dx + x^2\,dy = 0$

$\Rightarrow y^3(y\,dx + x\,dy) - x[y\,dx - x\,dy] = 0$

$\Rightarrow y\,dx + x\,dy - \dfrac{x}{y^3}(y\,dx - x\,dy) = 0$

$\Rightarrow \cancel{d(xy) - x\left(\dfrac{dx}{y^2} - \dfrac{x\,dy}{y^3}\right) = 0}$

$d(xy) + \dfrac{x}{y^3}(x\,dy - y\,dx) = 0$

$d(xy) + \dfrac{x \cdot x^2}{y^3}\left(\dfrac{x\,dy - y\,dx}{x^2}\right) = 0$

$d(xy) + \dfrac{x^3}{y^3}\, d\left(\dfrac{y}{x}\right) = 0$

$d(xy) + \left(\dfrac{y}{x}\right)^{-3} d\left(\dfrac{y}{x}\right) = 0$

Integrating w.r.g.

$\int d(xy) = xy + \dfrac{\left(\frac{y}{x}\right)^{-3+1}}{-3+1} = c$

$= xy - \dfrac{1}{2}\left(\dfrac{y}{x}\right)^{-2} = c$

$= xy - \dfrac{1}{2}\left(\dfrac{x}{y}\right)^2 = c$ #

(2) $(x^3y^3+1)\,dx + x^4y^2\,dy = 0 \longrightarrow (1)$

$M = x^3y^3+1 \quad , \quad N = x^4y^2$

$\dfrac{\partial M}{\partial y} = 3y^2x^3 \quad , \quad \dfrac{\partial N}{\partial x} = 4x^3y^2$

$\because \dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x} \Rightarrow \therefore$ (1) is not exact

(1) can be written as

$x^3y^3\,dx + dx + x^4y^2\,dy = 0$

$\Rightarrow (x^2y^2)(xy\,dx + x^2\,dy) + dx = 0$

$\Rightarrow x(x^2y^2)(y\,dx + x\,dy) + dx = 0$

Divide both by x. we get

$(xy)^2\,d(xy) + \dfrac{1}{x}\,dx = 0$

On integrating w.r.t.

$$\int f(x)^n f'(x)\,dx = \frac{f(x)^{n+1}}{n+1} + C$$

$\therefore f(x,y) = \dfrac{(xy)^{2+1}}{2+1} + \log x = c \Rightarrow \dfrac{(xy)^3}{3} + \log x = c$

Method II: ~~when~~ If $M\,dx + N\,dy = 0$, both M and N are homogeneous fns then $I.F = \dfrac{1}{Mx+Ny}$, $Mx+Ny \neq 0$.

① $y(y^2-2x^2)\,dx + x(2y^2-x^2)\,dy = 0$ — ①

$M = y^3 - 2x^2y$, $N = 2xy^2 - x^3$

Since both M and N are homo. fns.

then $Mx + Ny = xy^3 - 2x^3y + 2xy^3 - x^3y$

$= 3xy^3 - 3x^3y$

$= 3yx(y^2-x^2) \neq 0$.

$\frac{\partial M}{\partial y} = 3y^2 - 2x^2$

$\frac{\partial N}{\partial x} = 2y^2 - 3x^2$

$\because \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$.

$\therefore$ ① is not exact

$\therefore$ I.F $= \frac{1}{Mx+Ny} = \frac{1}{3xy(y^2-x^2)}$.

Multiply ① by $\frac{1}{3xy(y^2-x^2)}$

$$\frac{y(y^2-2x^2)}{3xy(y^2-x^2)}\,dx + \frac{x(2y^2-x^2)}{3xy(y^2-x^2)}\,dy = 0$$

$$\Rightarrow \left(\frac{y^2}{3x(y^2-x^2)} - \frac{2x^2}{3x(y^2-x^2)}\right)dx + \left(\frac{2y^2}{3y(y^2-x^2)} - \frac{x^2}{3y(y^2-x^2)}\right)dy$$

$M_1\,dx + N_1\,dy = 0$

$M_1 = \frac{y^2}{3x(y^2-x^2)} - \frac{2x}{3(y^2-x^2)}$, $N_1 = \frac{2y}{3(y^2-x^2)} - \frac{x^2}{3y(y^2-x^2)}$

$$\frac{\partial M_1}{\partial y} = \frac{1}{3x}\left[\frac{(y^2-x^2)2y - y^2\cdot 2y}{(y^2-x^2)^2}\right] - \frac{2x}{3}\left(\frac{-2y}{(y^2-x^2)^2}\right)$$

$$= \frac{1}{3x}\,\frac{(\cancel{y^2}-x^2-\cancel{y^2})2y}{(y^2-x^2)^2} + \frac{4x}{3}\,\frac{y}{(y^2-x^2)^2} = \frac{-2x^2y}{3x(y^2-x^2)^2} + \frac{4xy}{3(y^2-x^2)^2} = \frac{2xy}{3(y^2-x^2)^2}$$

$$\boxed{\frac{\partial M_1}{\partial y} = \frac{2xy}{3(y^2-x^2)^2}}$$

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$\therefore$

$\therefore \phi(x,y) = \int_{y\,\text{const}} M_1\,dx + \int \left(\text{[illegible]} \; x \text{ in } N_1 \right) dy = C$

$= \int_{y\,\text{const}} \frac{y^2 - x}{}$