UNIT – II

Fundamentals of Information:

* The information theory is related to the concepts of statistic properties of messages / sources / channels, noise interference etc.
* The information theory is used for mathematical modeling and analysis of the communication systems.

Definition of Information (measure of Information):

Let the communication system transmits message $m_1, m_2 \ldots$ with probabilities of occurrence $P_1, P_2, P_3 \ldots$ The amount of information transmitted through the message m_K with probability P_K is given as

Amount of Information: $I_K = \log_2\left(\frac{1}{P_K}\right)$

Properties of Information:

Units: bit

1. If there is more uncertainty about the message, info carried is also more.
2. If the Rxer knows the message being transmitted, the amount of info carried is zero.

Ex: Calculate the amount of information if $P_K = \frac{1}{4}$

Sol: $I_K = \log \frac{1}{P_K} = \frac{\log_{10}\left(\frac{1}{P_K}\right)}{\log_{10} 2} = \frac{\log_{10} 4}{\log_{10} 2}$

$= 2$ bits.

Entropy :- Average Information content of symbols in Long Independent Sequences :

Consider that we have m-different messages. Let these messages be $m_1, m_2, m_3 \dots m_m$ and they have probabilities of occurrence as $P_1, P_2 \dots P_m$. Suppose that a sequence of L messages is transmitted.

Information due to message m_1 will be

$$I_1 = \log_2\left(\frac{1}{P_1}\right)$$

Since there are $P_1 L$ number of messages of m_1, the total information due to all message of m_1 will be.

$$I_1(\text{total}) = P_1 L \cdot \log_2\left(\frac{1}{P_1}\right)$$

$$I_2(\text{total}) = P_2 \cdot L \cdot \log_2\left(\frac{1}{P_2}\right)$$

Thus $I(\text{total}) = I_1(\text{total}) + I_2(\text{total}) + \dots + I_m(\text{total})$

$$\therefore I_{total} = P_1 L \log_2\left(\frac{1}{P_1}\right) + P_2 \cdot L \cdot \log_2\left(\frac{1}{P_2}\right) + \dots + P_m L \log_2\left(\frac{1}{P_m}\right)$$

The average information per message = $\frac{\text{total information}}{\text{Number of messages}}$

$$= \frac{I(\text{total})}{L}$$

Average information is represented by Entropy.

$$\text{Entropy}(H) = \frac{I(\text{total})}{L}$$

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Entropy $(H) = P_i \log_2\left(\frac{1}{P_1}\right) + P_2 \log_2\left(\frac{1}{P_2}\right) + \cdots + P_m \log_2\left(\frac{1}{P_m}\right)$

$$\therefore \boxed{\text{Entropy}: H = \sum_{K=1}^{M} P_K \log_2\left(\frac{1}{P_K}\right)}$$

Properties of Entropy:

① Entropy is zero if the event is sure or it is impossi

$H = 0$ if $P_K = 0$ or 1

② when $P_K = \frac{1}{M}$ for all M symbols Then The symbols a equally likely. For such $H = \log_2 M$

Information Rate : $R = rH$

'R' is info Rate, H entropy, 'r' rate at which mess are generated.

$R = (r$ in messages/second $) \times (H$ in information bits / message.

$=$ Information bits/second.

The source coding :- A conversion of the output of a discrete memoryless source (DMS) into a sequence of binary symbols is called source coding.

* Objective of source coding is to minimize the average bit rate required for representation of the source by reducing the redundancy of the information source.

* Terms related to source coding:

(i) Code word length: let X be a DMS with finite entropy H(x) and an alphabet ($x_1, x_2 --- x_m$) with correspon probabilities of occurrence $P(x_i)$ ($i=1 --- m$). let the binary codeword assigned to symbol x_i by the encoder have length n_i measured in bits. The length of a codeword is the number of binary digits in the codeword.

(ii) Average codeword lengths: the average codeword length L, per source symbol is given by

$$L = \sum_{i=1}^{m} P(x_i)\, n_i$$

(iii) Code Efficiency : The code efficiency η

$$\eta = \frac{L_{min}}{L}$$

(iv) Code Redundancy :

$$\gamma = 1 - \eta$$

Digits
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The source coding theorem states that for a DMS X, with entropy $H(x)$, the average code word length L per symbol is bounded as

$$L \geq H(x) \quad : \quad \eta = \frac{H(x)}{L}$$

SHANNON – FANO CODING :

S1: The messages are first written in the order of non-incre probabilities.

S2: The message set then is partitioned into two equi-probable subsets $[x_1]$ and $[x_2]$.

S3: A '0' is assigned to each message contained in one subset and a '1' to each message contained in the other set

S4: The same procedure is repeated for the subsets of $[x_1]$ and $[x_2]$

i.e. $[x_1]$ — $[x_{11}]$ 00, $[x_{12}]$ 01

$[x_2]$ — $[x_{21}]$ 10, $[x_{22}]$ 11

S5: the procedure is continued until each subset contains only one message.

Ex: $[x] = [x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5 \quad x_6 \quad x_7 \quad x_8]$

$[P] = [1/4 \quad 1/8 \quad 1/16 \quad 1/16 \quad 1/16 \quad 1/4 \quad 1/16 \quad 1/8]$

Sol:

Message	Probability	Encoded message	length (n_i)
x_1	0.25	0 0	2
x_6	0.25	0 1	2
x_2	0.125	1 0 0	3
x_8	0.125	1 0 1	3
x_3	0.0625	1 1 0 0	4
x_4	0.0625	1 1 0 1	4
x_5	0.0625	1 1 1 0	4
x_7	0.0625	1 1 1 1	4

Average length $L = \sum_{K=1}^{8} P_K n_K = \left(\frac{1}{4}\times 2\right) + \left(\frac{1}{8}\times 3\right) + \left(\frac{1}{16}\times 4\right)$

$+ \left(\frac{1}{16}\times 4\right) + \left(\frac{1}{16}\times 4\right) + \left(\frac{1}{4}\times 2\right) + \left(\frac{1}{16}\times 4\right) + \left(\frac{1}{8}\times 3\right)$

$= 2.75$ letters / message.

Entropy $H = -\sum_{K=1}^{8} P_K \log P_K$

$= -\left[\frac{1}{4}\log\frac{1}{4} + \frac{1}{8}\log\frac{1}{8} + \frac{1}{16}\log\frac{1}{16} + \frac{1}{16}\log\frac{1}{16}\right.$

$\left. + \frac{1}{16}\log\frac{1}{16} + \frac{1}{4}\log\frac{1}{4} + \frac{1}{16}\log\frac{1}{16} + \frac{1}{8}\log\frac{1}{8}\right]$

$= 2.75$ ~~letters~~ bits / message.

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$\log M = \log 2 = 1$ bit/letter. (M=2 since bin...

efficiency $\eta = \frac{H}{L \log m} = \frac{2.75}{2.75 \times 1} = 100\%$

Ex 2:

$[x] =$	x_1	x_2	x_3	x_4	x_5	x_6	x_7
$[P] =$	0.4	0.2	0.12	0.08	0.08	0.08	0.04

Sol: (i)

message	Probability	Encoded message	length
x_1	0.4	0 0	2
x_2	0.2	0 1	2
x_3	0.12	1 0 0	3
x_4	0.08	1 0 1	3
x_5	0.08	1 1 0	3
x_6	0.08	1 1 1 0	4
x_7	0.04	1 1 1 1	4

$$L = \sum_{K=1}^{7} P_K n_K = (0.4 \times 2) + (0.2 \times 2) + (0.12 \times 3) + (0.08 \times 3) + (0.08 \times 3) + (0.08 \times 4) + (0.04 \times 4)$$

$$= 2.52 \text{ letters/message}$$

$$H = -\sum_{K=1}^{7} P_K \log_2 P_K = -\left[(0.4 \log 0.4) + (0.2 \log 0.2) + (0.12 \log 0.12) + (0.08 \log 0.08) + (0.08 \log 0.08) + (0.08 \log 0.08) + (0.04 \log 0.04) \right]$$

$$= 2.42 \text{ bits/message.}$$

(ii)

message	Probability	Encoded message	length
x_1	0.4	0	1
x_2	0.2	1 0 0	3
x_3	0.12	1 0 1	3
x_4	0.08	1 1 0 0	4
x_5	~~0.0~~8	1 1 0 1	4
x_6	0.08	1 1 1 0	4
x_7	0.04	1 1 1 1	4

$$L = \sum_{k=1}^{7} P_k n_k = (0.4\times1) + (0.2\times3) + (0.12\times3) + (0.08\times4) + (0.08\times4) + (0.08\times4) + (0.04\times4)$$

$$= 2.4\cancel{2}\ \text{bits}\,|\,\text{message}$$

In second method the average word length is less.

$$\therefore \text{Efficiency } \eta = \frac{H}{L \log M} = \frac{2.42}{2.48 \log 2} = 97.6\%.$$

message	probability	Encoded message	length
x_1	0.4	0	1
x_2	0.2	1 0	2
x_3	0.12	1 1	2
x_4	0.08	2 0	2
x_5	0.08	2 1	2
x_6	0.08	2 2 0	3
x_7	0.04	2 2 1	3

$$L = \sum_{k=1}^{7} = 1.72\ \text{letters}\,|\,\text{message} \qquad \eta = \frac{H}{L \log M}$$

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HUFFMAN CODING.

S1: N messages are arranged in an order of non-increasing probability.

S2: The probabilities of least likely messages are combined and re arranged in a non-increasing manner.

Pb:

Message	Probability	1st reduction	2nd	3rd	4th	5th
x_1	0.4	0.4	0.4	0.4	0.4	0.6 1
x_2	0.2	0.2	0.2	0.24	0.36 1	0.4 0
x_3	0.12	0.12	0.16	0.2	0.24 0	
x_4	0.08	0.12	0.12 1	0.16 0		
x_5	0.08	0.08 1	0.12 0			
x_6	0.08	0.08 0				
x_7	0.04					

Code	Length
$C_1 = 0$	1
$C_2 = 111$	3
$C_3 = 101$	3
$C_4 = 1101$	4
$C_5 = 1100$	4
$C_6 = 1001$	4
$C_7 = 1000$	4.

$L = \sum_{k=1}^{7} = (0.4\times1) + (0.2\times3) + \dots$

$= 2.48$ letters/message

$H = -\sum_{k=1}^{7} P_k \log P_k = \qquad = 2.42$ bits/message.

$\therefore \eta = \frac{H}{L \log M} = \frac{2.42}{2.48 \times \log_2 2} = 97.6\%$

Discrete Communication Channel :

* the discrete communication channel has i/p x and o/p 'Y'. Both x and Y are the random variables. the channel discrete when both x and Y are discrete.

* the channel is called memoryless (zero memory) when current o/p depends only on current input.

* the transition probability $P(y_j/x_i)$ is the conditional probability of y_j received given that x_i was t[ransmitted]

If $i = j$ then $P(y/x)$ represent conditional probability ~~y_i ~~ ~~received~~ correct reception.

If $i \neq j$ then $P(y/x)$ represents error conditional probability.

PTM

$$P = \begin{bmatrix} P(y_1/x_1) & P(y_2/x_1) & \cdots & P(y_m/x_1) \\ P(y_1/x_2) & P(y_2/x_2) & \cdots & P(y_m/x_2) \\ \vdots & & & \\ P(y_1/x_n) & P(y_2/x_n) & \cdots & P(y_m/x_n) \end{bmatrix}$$

The above matrix is called as channel matrix, noise matrix or probability transition matrix.

where along a row

$$P(y_1/x_1) + P(y_2/x_1) + P(y_3/x_1) + \cdots P(y_m/x_1) = 1$$

$P(x_i, y_j)$ is the joint probability of x_i, y_j like.

$$P(x_i y_j) = P(y_j/x_i) \cdot P(x_i)$$

If we add all joint probabilities for fixed y_j then we get $P(y_j)$ i.e $\sum_{i=1}^{n} P(x_i, y_j) = P(y_j)$.

i.e $P(y_j) = \sum_{i=1}^{n} P(y_j/x_i) \cdot P(x_i)$

thus if we are given the probabilities of input symbols and transition probabilities, then it is possible to calculate the probabilities of output symbols.

$$\boxed{\text{Error probability} \quad P_e = \sum_{\substack{j=1 \\ j \neq i}}^{m} \sum_{i=1}^{n} P(y_j/x_i) \cdot P(x_i)}$$

probability of correct reception $P_c = 1 - P_e$.

PTM

$$P = P(y_j|x_i) = \begin{bmatrix} P(y_1|x_1) & P(y_2|x_1) & \cdots & P(y_m|x_1) \\ P(y_1|x_2) & P(y_2|x_2) & \cdots & P(y_m|x_2) \\ \vdots & & & \vdots \\ P(y_1|x_n) & \cdots & \cdots & P(y_m|x_n) \end{bmatrix}$$

multiply first row by $P(x_1)$ and second row by $P($

---- last with $P(x_n)$

$$P(y_j|x_i)\,P(x_i) = \begin{bmatrix} P(y_1|x_1)\cdot P(x_1) & P(y_2|x_1)\cdot P(x_1) & \cdots & P(y_m \\ \vdots & \cdots & & \vdots \\ P(y_1|x_n)\cdot P(x_n) & \cdots & & P(y_m|x_n) \end{bmatrix}$$

as per above discussion.

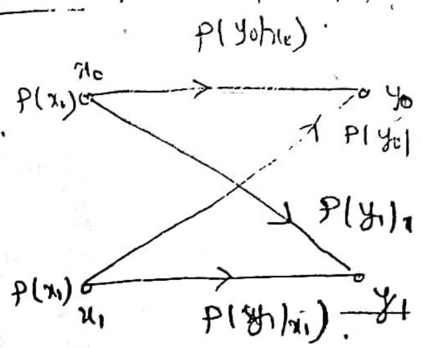
$$P(x_i, y_j) = \begin{array}{c|cccc} i \backslash j & 1 & 2 & \cdots & m \\ \hline 1 & P(x_1, y_1) & P(x_1, y_2) & \cdots & P(x_1, y_m) \\ 2 & P(x_2, y_1) & P(x_2, y_2) & \cdots & P(x_2, y_m) \\ \vdots & \vdots & & & \\ n & P(x_n, y_1) & P(x_n, y_2) & \cdots & P(x_n, y_m) \end{array}$$

Properties

① sum of all element of JPM is equal to unity.

② sum of " " j^{th} column of JPM gives probability of j^{th} o/p.

Binary Communication Channel BCC :



$$P(y_0) = P(y_0/x_0)\,P(x_0) + P(y_0/x_1)\,P(x_1).$$

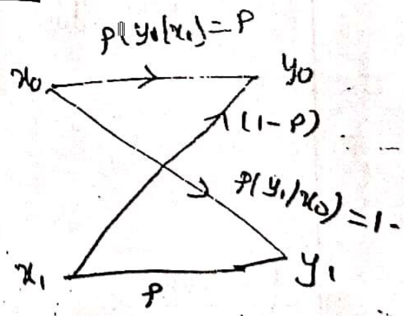
$$P(y_1) = P(y_1/x_1)\,P(x_1) + P(y_1/x_0)\cdot P(x_0).$$

Above equation in matrix form

$$\begin{bmatrix} P(y_0) \\ P(y_1) \end{bmatrix} = \begin{bmatrix} P(x_0) & P(x_1) \end{bmatrix} \begin{bmatrix} P(y_0/x_0) & P(y_0/x_0) \\ P(y_1/x_1) & P(y_1/x_1) \end{bmatrix}$$

2×2 matrix is Probability transition matrix (PTM).

Binary Symmetric Channel BSC :



BCC is said to be symmetric

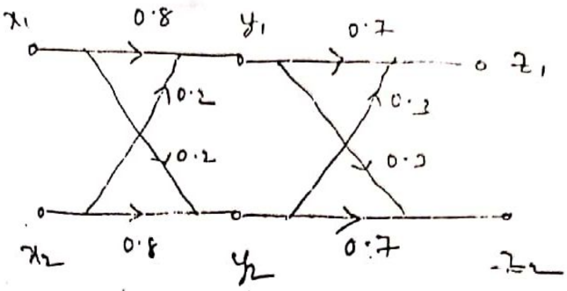
if $P(y_0/x_0) = P(y_1/x_1) = P.$

$$\begin{bmatrix} P(y_0) \\ P(y_1) \end{bmatrix} = \begin{bmatrix} P(x_0) & P(x_1) \end{bmatrix} \begin{bmatrix} P & 1-P \\ 1-P & P \end{bmatrix}.$$

... BSC's are connected in cascade

(i) Find the channel matrix of resultant channel

(ii) Find $P(z_1)$ and $P(z_2)$ if $P(x_1) = 0.6$, $P(x_2) = 0.4$.



Sol: (i)

$$P(Y|X) = \begin{bmatrix} P(y_1|x_1) & P(y_2|x_1) \\ P(y_1|x_2) & P(y_2|x_2) \end{bmatrix} = \begin{bmatrix} 0.8 & 0.2 \\ 0.2 & 0.8 \end{bmatrix}$$

$$P(Z|Y) = \begin{bmatrix} P(z_1|y_1) & P(z_2|y_1) \\ P(z_1|y_2) & P(z_2|y_2) \end{bmatrix} = \begin{bmatrix} 0.7 & 0.3 \\ 0.3 & 0.7 \end{bmatrix}$$

Hence resultant channel matrix

$$P(Z|X) = P(Y|X) \cdot P(Z|Y)$$

$$= \begin{bmatrix} 0.8 & 0.2 \\ 0.2 & 0.8 \end{bmatrix} \begin{bmatrix} 0.7 & 0.3 \\ 0.3 & 0.7 \end{bmatrix} = \begin{bmatrix} 0.62 & 0 \\ 0.38 & \end{bmatrix}$$

(ii) To obtain $P(z_1)$ and $P(z_2)$

$$P(Z) = P(X) \cdot P(Z|X) = \begin{bmatrix} P(x_1) & P(x_2) \end{bmatrix} \begin{bmatrix} 0.62 & 0.38 \\ 0.38 & 0.62 \end{bmatrix}$$

$$= \begin{bmatrix} 0.6 & 0.4 \end{bmatrix} \begin{bmatrix} 0.62 & 0.38 \\ 0.38 & 0.62 \end{bmatrix} = \begin{bmatrix} 0.524 & 0.476 \end{bmatrix}$$

thus $P(z_1) = 0.524$

$P(z_2) = 0.476$.

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Rate of Information Transmission over a Discrete Channel

Let entropy of a symbol is 'H', generated at a rate of 'r' symbols/second.

Average rate of info in channel $D_{in} = r\,H(x)$ bits

Errors are introduced in the data during transmission. Some data is lost. Conditional entropy $H(x|y)$ is the measure of info lost.

$$\text{Transmitted info} = H(x) - H(x|y)$$

$$\therefore \quad D_t = \left[H(x) - H(x|y)\right] \cdot r \text{ bits/sec.}$$

The channel capacity :- $C_s = \max\limits_{\{P(x_i)\}} I(x:y)$. bits/sym

where the maximization is over all possible input probability distributions $\{P(x_i)\}$ on X. C_s is function of only channel transition probabilities which define the chan

BSC :

$$I(x:y) = H(y) + p\log_2 p + (1-p)\log_2 p$$

$$\therefore \quad C_s = 1 + p\log_2 p + (1-p)\log_2 p$$

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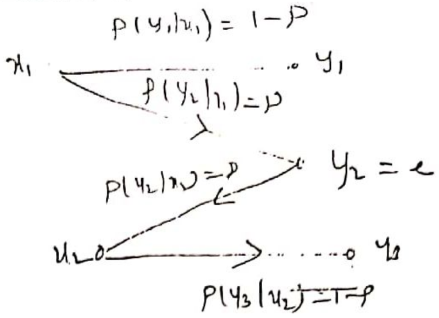
The channel with two i/p symbols and Three o/p symbols the output symbol y_2 represents error (e). Such channel is called Binary Erasure channel.



Pb:
$$P(Y/X) = \begin{bmatrix} 1-P & P & 0 \\ 0 & P & 1-P \end{bmatrix}$$

If sources has equally likely o/p, compute the probabilities associated with channel o/p for $P = 0.2$

$$P(Y|X) = \begin{bmatrix} P(y_1|x_1) & P(y_2|x_1) & P(y_3|x_1) \\ P(y_1|x_2) & P(y_2|x_2) & P(y_3|x_2) \end{bmatrix}$$

Given $P(x_1) = 1/2 \quad P(x_2) = 1/2$

The output probabilities are given as

$$\begin{bmatrix} P(y_1) \\ P(y_2) \\ P(y_3) \end{bmatrix} = \begin{bmatrix} P(x_1) & P(x_2) \end{bmatrix} \begin{bmatrix} 1-P & P & 0 \\ 0 & P & 1-P \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0.8 & 0.2 & 0 \\ 0 & 0.2 & 0.8 \end{bmatrix} = \begin{bmatrix} 0.4 \\ 0.2 \\ 0.4 \end{bmatrix}$$

Thus $P(y_1) = 0.4 \quad P(y_2) = 0.2 \quad P(y_3) = 0.4$

Noise-free channel:

lossless channel: A channel described by a channel matrix with only one non-zero element in each column

$$P(Y|X) = \begin{bmatrix} 3/4 & 1/4 & 0 & 0 & 0 \\ 0 & 0 & 1/3 & 2/3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

~~Deterministic~~

Deterministic channel: A channel described by a channel matrix with only one non-zero element in each row.

$$P(Y|X) = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

A channel is called noise-less if it is both lossless and deterministic.

The matrix of noise-free channel is

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Def: The mutual information is defined as the amount information transferred when x_i is transmitted and y received.

$$\boxed{I(x_i, y_j) = \log\left(\frac{P(x_i/y_j)}{P(x_i)}\right)} \quad \text{bits}.$$

$P(x/y)$ is the conditional probability that x_i tran and y_j received.

Average mutual information is the amount of source information gained per received symbol. (Different from

$$I(x;y) = \sum_{i=1}^{n}\sum_{j=1}^{m} P(x_i, y_j)\, I(x_i, y_j)$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{m} P(x_i, y_j) \log_2 \frac{P(x_i/y_j)}{P(x_i)}$$

Properties

1. Mutual Info is symmetric $I(x;y) = I[y;x]$
2. " " is always +ve $I[x;y] \geq 0$
3. " " is related to the joint entropy $H(x$

$$I[x;y] = H(x) + H(y) - H(x,y).$$

P: For a channel whose matrix is $P(Y/X) = \begin{bmatrix} 0.6 & 0.2 & 0.2 \\ 0.2 & 0.6 & 0 \\ 0.2 & 0.2 & \end{bmatrix}$

Find $I(X;Y)$ and channel capacity,

Given the i/p symbols occur with equal probability.

Sol: S1: To obtain probabilities of o/p symbols.

$P(x_1) = P(x_2) = P(x_3) =$

$$P(Y) = P(Y/X) \cdot P(X)$$

$$= \begin{bmatrix} 0.6 & 0.2 & 0.2 \\ 0.2 & 0.6 & 0.2 \\ 0.2 & 0.2 & 0.6 \end{bmatrix} \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix}$$

Thus $P(y_1) = P(y_2) = P(y_3) = \frac{1}{3}$ because channel is symm

S2: To obtain entropy of o/p $H(Y)$

$$H(y) = \sum_{i=1}^{3} P(y_i) \log_2 \frac{1}{P(y_i)} = \frac{1}{3} \log_2 3 = 1.585$$

bits/sy

S3: To obtain $P(x_i, y_j)$

We know that $P(x_i, y_j) = P(y_j/x_i) \cdot P(x_i)$

so

$$P(y_1/x_1) = P(y_2/x_2) = P(y_3/x_3) = 0.6$$

$$P(y_2/x_1) = P(y_3/x_1) = P(y_3/x_2) = 0.2$$

$$P(y_3/x_1) = P(y_3/x_2) = P(y_3/x_3) = 0.2$$

$$\therefore P(x_1, y_1) = P(y_1/x_1) \cdot P(x_1) = 0.6 \times \frac{1}{3} = 0.2$$

Similarly P(x ...

Conditional entropy $H(Y|X) = \sum_{i=1}^{n} \sum_{j=1}^{m} P(x_i, y_j) \cdot \log_2 \frac{1}{P(y_j|x_i)}$

$$= \sum_{i=1}^{3} \sum_{j=1}^{3} P(x_i, y_j) \cdot \log_2 \frac{1}{P(y_j/x_i)}$$

$$= 3 \times \left(0.2 \log_2 \frac{1}{0.6}\right) + 6 \times \left(0.0667 \log_2 \frac{1}{0.2}\right)$$

$$= 1.37 \text{ bits/symbol}.$$

S5: To obtain mutual information

$$I(Y;X) = H(Y) - H(Y/X)$$

$$= 1.585 - 1.37 = 0.215 \text{ bits/symbol}.$$

S6 To obtain channel capacity.

Given is a symmetric channel.

Capacity $C = [\log_2 M - h] \cdot r$ bits/sec.

Here $h = \sum_{j=1}^{M} P(y_j/x_i) \log_2 \frac{1}{P(y_j/x_i)}$ for any i

$$= \sum_{j=1}^{3} P(y_j/x_i) \log_2 \frac{1}{P(y_j/x_i)} \text{ for } i=1$$

$$= -\left[P(y_1/x_1) \log_2 P(y_1/x_1) + P(y_2/x_1) \log P(y_2/x_1) + P(y_3/x_1) \log_2 P(y_3/x_1)\right]$$

$$= 0.6 \log_2 \frac{1}{0.6} + \left(0.2 \log_2 \frac{1}{0.2}\right) \times 2$$

$$= 1.37 \text{ bits/symbol.}$$

Since $M = 3$ symbol.

$$C = \log_2 [3 - 1.37] \cdot r \text{ bits/s.}$$

$$= 0.215 \times r \text{ bits/sec.}$$ here 'r' is symbol rate

① An event has six possible outcomes with probabilit

$P_1 = 1/2 \quad P_2 = 1/4 \quad P_3 = 1/8 \quad P_4 = 1/16 \quad P_5 = 1/32 \quad P_6 = 1/32$

Find ⓐ entropy ⓑ rate of information if there are 16

Sol: The entropy H is $H = \sum_{K=1}^{6} P_K \log \frac{1}{P_K}$

$= \frac{1}{2}\log 2 + \frac{1}{4}\log 4 + \frac{1}{8}\log 8 + \frac{1}{16}\log 16 + \frac{1}{32}\log 32 + \frac{1}{32} l$

$= \frac{31}{16}$ bits/message

Now $r = 16$ outcome/sec

∴ Rate of information $R = r \cdot H = 16 \times \frac{31}{16} = 31$ bits/

② A continous signal is bandlimited to 5KHz. the signal is quantized in 8 levels of a PCM system wi the probabilities 0.25, 0.2, 0.2, 0.1, 0.1, 0.05, 0.05 and 0.

ⓐ Entropy $H = -(0.25 \log 0.25 + 0.2 \log 0.2 + 0.2 \log 0.2 + \dots$

$\dots 0.05 \log 0.05)$

$= 2.74$ bits/message.

ⓑ As the signal $f_m = 5KHz$, sampled frequency $f_s = 2f_m$

$= 10KH$

$= 10,000$

$R = r \cdot H$

$= 10,000 \times 2.74$

$= 27,400$ bits/sec.

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3. Complete the following "Probability matrix in all possible way.

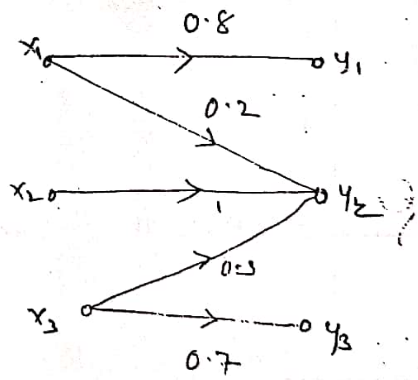
$$\begin{bmatrix} 0.1 & a & 0.2 & 0.4 \\ 0.3 & 0.1 & b & 0.5 \\ c & 0.4 & 0.1 & 0.3 \\ 0.2 & 0.2 & 0.1 & d \end{bmatrix}$$

Sol: * Sum of all entries of the matrix is 2.9 (which is greater than 1). so it can not be a joint probability matrix.

* As sum of all the given entries in each row of the matrix is less than 1, it can be a conditional probability matrix $P(Y/x)$. Thus when $a = 0.3$, $b = 0.1$, $c = 0.2$, $d = 0.5$

4. A discrete source transmits messages $x_1\ x_2\ x_3$ with the probabilities 0.3, 0.4, 0.3. The source is connected to the channel given in fig. calculate all the entropies.

Sol: From the fig the Conditional - probability matrix



$$P(Y/x) = \begin{array}{c|ccc} & y_1 & y_2 & y_3 \\ \hline x_1 & 0.8 & 0.2 & 0 \\ x_2 & 0 & 1 & 0 \\ x_3 & 0 & 0.3 & 0.7 \end{array}$$

also given $P(x) = [0.3\ \ 0.4\ \ 0.3]$

joint probability matrix $P(x, y)$ can be obtained by

$$\begin{bmatrix} 0 & & 0 \\ 0 & 1\times 0.4 & 0 \\ 0 & 0.3\times 0.3 & 0.7\times 0.3 \end{bmatrix}$$

$$= \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{ccc} y_1 & y_2 & y_3 \end{array} \\ \begin{bmatrix} 0.24 & 0.06 & 0 \\ 0 & 0.4 & 0 \\ 0 & 0.09 & 0.21 \end{bmatrix} \end{array}$$

The probabilities $P(y_1)$, $P(y_2)$ and $P(y_3)$ can be obtained by adding the columns of $P(x,y)$ ~~by~~ $P(y_1)$, ~~$P(y_2)$~~, $P(y_3)$ of

$P(y_1) = 0.24$ $\quad P(y_2) = 0.06 + 0.4 + 0.09 = 0.55$ $\quad P(y_3) = 0.21$

The conditional-probability matrix $P(x|y)$ can be obtained by dividing the columns of $P(x,y)$ by $P(y_1)$, $P(y_2)$ and $P(y_3)$.

$$P(x|y) = \begin{bmatrix} 1 & 0.109 & 0 \\ 0 & 0.727 & 0 \\ 0 & 0.164 & 1 \end{bmatrix}$$

Entropy

$$H(x) = -\sum_{j=1}^{3} P(x_j) \log P(x_j)$$

$$= -(0.3 \log 0.3 + 0.4 \log 0.4 + 0.3 \log 0.3)$$

$$= 1.571 \text{ bits/message.}$$

$$H(y) = -\sum_{k=1}^{3} P(y_k) \log P(y_k)$$

$$= -[0.024 \log 0.24 + 0.55 \log 0.55 + 0.21 \log 0.21$$

$$H(x,y) = -\sum_{j=1}^{3}\sum_{k=1}^{3} P(x_j, y_k) \cdot \log P(x_j, y_k)$$

$$= -[0.24 \log 0.24 + 0.06 \log 0.06 + 0.4 \log 0.4$$
$$+ 0.09 \log 0.09 + 0.21 \log 0.21] = 2.053 \text{ bits/mess}$$

$$H(x/y) = -\sum_{j=1}^{3}\sum_{k=1}^{3} P(x_j, y_k) \log P(x_j/y_k)$$

$$= -\{0.24 \log 1 + 0.06 \log 0.109 + 0.4 \log 0.727 + 0.09 \log 0$$
$$+ 0.21 \log 1\} = 0.612 \text{ bits/message.}$$

$$H(y/x) = -\sum_{j=1}^{3}\sum_{k=1}^{3} P(x_j, y_k) \log P(y_k/x_j)$$

$$= -[0.24 \log 0.8 + 0.06 \log 0.2 + 0.4 \log 1 +$$
$$0.09 \log 0.3 + 0.21 \log 0.7] = 0.482 \text{ bit/messa}$$

Calculate all the entropies

$$P(x,y) = \begin{array}{c|ccc} & y_1 & y_2 & y_3 \\ \hline x_1 & 0.3 & 0.05 & 0 \\ x_2 & 0 & 0.25 & 0 \\ x_3 & 0 & 0.15 & 0.1 \\ x_4 & 0 & 0.05 & 0. \end{array}$$

Sol:

The probabilities of transmitter symbols are founded by Summation of rows $P(x) = [0.35 \quad 0.25 \quad 0.2 \quad 0$

The probabilities of the receiver symbols are found Summation of column $P(y) = [0.3 \quad 0.5 \quad 0.2]$

Entropies

$$H(x) = -\sum_{j=1}^{4} P(x_j) \cdot \log P(x_j)$$

$$= [0.35 \log 0.35 + 0.25 \log 0.25 + 0.2 \log 0.2 + 0.2 \log$$

$$= 1.96 \text{ bits/message}$$

$$H(y) = -\sum_{k=1}^{3} P(y_k) \log P(y_k)$$

$$= -[0.3 \log 0.3 + 0.5 \log 0.5 + 0.2 \log 0.2] = 1.49 \text{ bits/}$$

$$H(x,y) = -\sum_{j=1}^{4}\sum_{k=1}^{3} P(x_j, y_k) \log P(x_j, y_k)$$

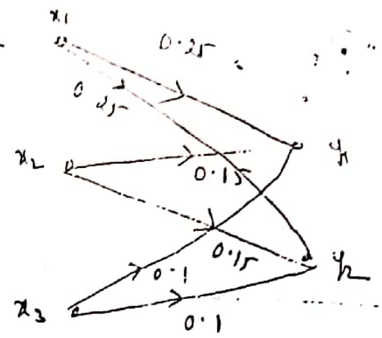
$$= -[0.3 \log 0.3 + 0.05 \log 0.05 + 0.25 \log 0.25 + 0.15 \log$$
$$+ 0.05 \log 0.05 + 0.05 \log 0.05 + 0.15 \log 0.15]$$

$$= 2.49 \text{ bits/message}$$

$$H(x|y) = H(x,y) - H(y) = 2.49 - 1.49 = 1.00 \text{ bit/m}$$

$$H(y|x) = H(x,y) - H(x) = 2.49 - 1.96 = 0.53 \text{ bit/}$$

(6) Find the mutual information for the channel as shown in fig



Sol: The joint Probability matrix

$$P(x,y) = \begin{array}{c|cc} & y_1 & y_2 \\ \hline x_1 & 0.25 & 0.25 \\ x_2 & 0.15 & 0.15 \\ x_3 & 0.1 & 0.1 \end{array}$$

$P(x_1) = 0.25 + 0.25 = 0.5$; $P(x_2) = 0.15 + 0.15 = 0.3$; $P(x_3) = 0.1 + 0.1 =$.

and $P(y_1) = P(y_2) = 0.25 + 0.15 + 0.1 = 0.5$

Hence

$$H(x) = -\sum_{j=1}^{3} P(x_j) \cdot \log P(x_j)$$

$$= -[0.5 \log 0.5 + 0.3 \log 0.3 + 0.2 \log 0.2]$$

$$= 1.485 \text{ bits/mess.}$$

$$H(y) = -\sum_{K=1}^{2} P(y_K) \log P(y_K) = -[0.5 \log 0.5 + 0.5 \log 0.5] = 1 \text{ bit},$$

$$H(x,y) = -\sum_{j=1}^{3} \sum_{K=1}^{2} P(x_j, y_K) \log (P(x_j, y_K))$$

$$= -[0.25 \log 0.25 + 0.25 \log 0.25 + 0.15 \log 0.15 + 0.15 \log 0.15 + 0.1 \log 0.1 + 0.1 \log 0.1] = 2.485 \text{ bits/mess}$$

Hence

$$I(x; y) = H(x) + H(y) - H(x,y)$$

$$= 1.485 + 1 - 2.485 = 0$$

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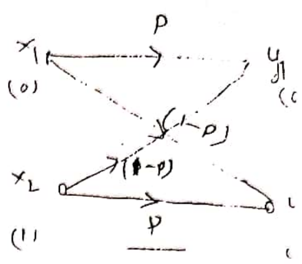
... shown in Fig, find the

channel capacity for (i) $p = 0.9$

(ii) $p = 0.6$



Sol: For a symmetric channel.

$$I(x;y) = H(y) - H(y|x)$$

$$= H(y) - \sum_{j=1}^{m} H(y|x_j) \cdot P(x_j) = H(y) - A \sum_{j=1}^{m}$$

where $A = H(y|x_j)$ is independent of j, hence is take out of summation. Also $\sum_{j=1}^{m} P(x_j) = 1$

Hence $I(x;y) = H(y) - A$.

The channel capacity of a symmetric channel is

$$C = \max I(x;y) = \max [H(y) - A]$$

$$= \max [H(y)] - A$$

$$= \log n - A \quad \because \max H(y) = \log n.$$

where n is total no of receiver.

So, in the given problem

$$P(y|x) = \begin{bmatrix} p & 1-p \\ 1-p & p \end{bmatrix}$$

$n = 2$

$$C = \log n - A = \log 2 - H(Y|x_j)$$

$$= \log 2 - \left[- \sum_{j=1}^{2} P(y_k|x_j) \log P(y_k|x_j)\right]$$

$$= \log 2 + p \log p + (1-p) \log (1-p)$$

$$= 1 - (p \log p + q \log q) = 1 - H(p) = 1 - H(q)$$

(i) for $p = 0.9$

$$C = 1 + 0.9 \log 0.9 + 0.1 \log 0.1 = 0.531 \text{ bit/message}$$

(ii) for $p = 0.6$

$$C = 1 + 0.6 \log 0.6 + 0.4 \log 0.4 = 0.029 \text{ bit/message}$$

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