1–37.

UNIT-I

- Review of vector analysis
- Different coordinate systems
  - Cartesian
  - Cylindrical
  - Spherical
- Coulomb's law
- Electric Field Intensity ($\bar{E}$) due to different charge distributions.
- $\bar{E}$ due to infinite line charge
- $\bar{E}$ due to infinite sheet of charge.
- Electric Flux Density ($\bar{D}$)
- Gauss's law & its applications
- Divergence Theorem
- Problems.

Electromagnetics(EM) is a branch of physics (or) electrical Engineering in which electric and magnetic phenomena are studied.

EM principles find applications in various allied disciplines such as microwave, antennas, electric machines, Satellite commn, plasmas, fibre optics, and remote sensing.

EM devices includes Transformers, relays, radio/TV, Transmission lines, antennas and lasers. The design of these devices requires Through knowledge of laws and principles of EM.

Electric and magnetic fields are closely related to each other. Ex. Magnet 

magnetic field.

There are two types of charges +ve and -ve. Such an electric charge produces a field around it is called electric field.

Moving charges produces current and current $\left(\frac{dq}{dt} = i\right)$ Carrying conductor produces a magnetic field. In such case electric and magnetic fields are related to each other. Such a field is called electromagnetic field (EM).

The Comprehensive study of Characteristics of electric, magnetic and combined fields is nothing but Engineering electromagnetics. Such fields may be time dependent or time independent.

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- A field is a function that specifies a particular quantity everywhere in a region.

- Vector analysis is a mathematical tool with which EM concepts are most conviniently expressed and best comprehended.

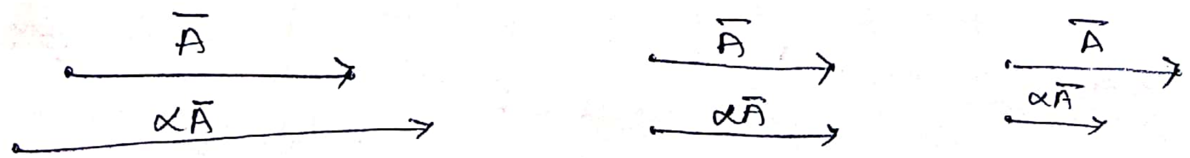
- The various quantities involved in Engineering electromagnetics can be classified as

1) Scalars → it has only magnitude.
   Ex: time, mass, distance, temp, potential, population.
2) Vectors
   ↳ it has both magnitude and direction.
   Ex: Velocity, Force, $\bar{E}$, $\bar{D}$, displacement.

Vector Algebra:

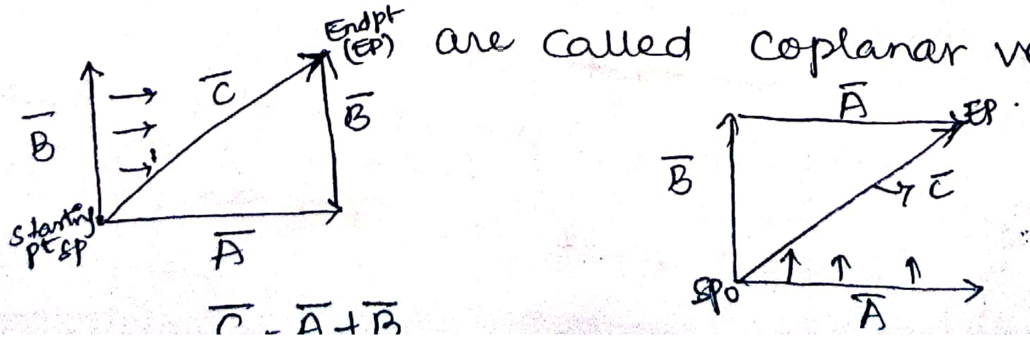
1) Scaling
2) Addition
3) Subtraction
4) Multiplication.

1) Scaling



a) $\alpha > 1$ b) $\alpha = 1$ c) $\alpha < 1$

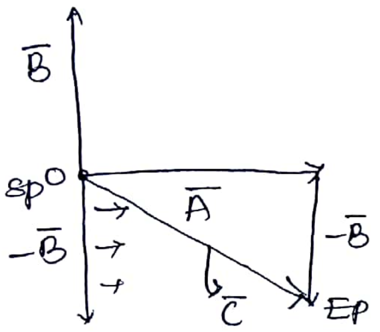
2) Addition vectors which lie in the same plane are called coplanar vectors.



$\bar{C} = \bar{A} + \bar{B}$

iii) Subtraction

$$\bar{C} = \bar{A} - \bar{B} = \bar{A} + (-\bar{B})$$



- Identical vectors : Two vectors are said to be identical if there difference is zero.
(equal)

$$\boxed{\bar{A} - \bar{B} = 0} \quad \text{ie. } \bar{A} = \bar{B}.$$

iv) Vector multiplication

1) Dot (or) Scalar product

$$\bar{A} \cdot \bar{B} = |\bar{A}||\bar{B}| \cos\theta_{AB}$$

$$\bar{a}_x \cdot \bar{a}_x = \bar{a}_y \cdot \bar{a}_y = \bar{a}_z \cdot \bar{a}_z = 1$$

$$\bar{a}_x \cdot \bar{a}_y = \bar{a}_y \cdot \bar{a}_z = \bar{a}_z \cdot \bar{a}_x = 0$$

2) Cross (or) Vector product

$$\bar{A} \times \bar{B} = |\bar{A}||\bar{B}| \sin\theta_{AB} \, \bar{a}_N$$

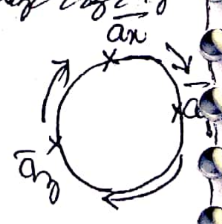
$$\bar{A} \times \bar{B} = \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

$$\bar{a}_x \times \bar{a}_x = \bar{a}_y \times \bar{a}_y = \bar{a}_z \times \bar{a}_z = 0$$

$$\bar{a}_x \times \bar{a}_y = \bar{a}_z$$

$$\bar{a}_y \times \bar{a}_z = \bar{a}_x$$

$$\bar{a}_z \times \bar{a}_x = \bar{a}_y$$



3) Scalar triple product

$$\boxed{\bar{A} \cdot (\bar{B} \times \bar{C}) = \bar{B} \cdot (\bar{C} \times \bar{A}) = \bar{C} \cdot (\bar{A} \times \bar{B})}$$

4) Vector triple product ['bac – cab" rule]

$$\boxed{\bar{A} \times \bar{B} \times \bar{C} = \bar{B}(\bar{A} \cdot \bar{C}) - \bar{C}(\bar{A} \cdot \bar{B})}$$

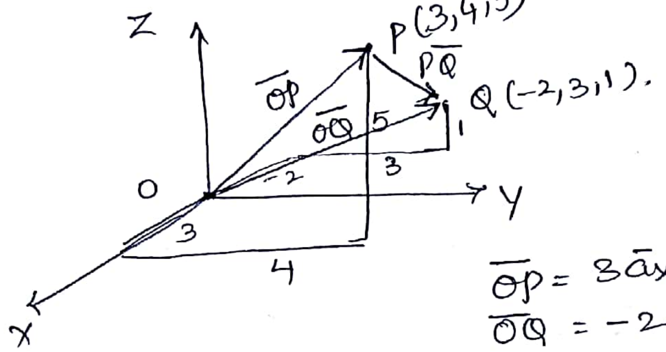
$(\bar{A} \cdot \bar{B})\bar{C} \neq \bar{A}(\bar{B} \cdot \bar{C})$

but $(\bar{A} \cdot \bar{B})\bar{C} = \bar{C}(\bar{A} \cdot \bar{B})$

Position vector : (or radius vector)

This is the distance travelled by the point 'p' from the origin i.e ($\overline{OP}$)

Distance vector : It is the displacement from one point to another.



$\overline{OP} = 3\bar{a}_x + 4\bar{a}_y + 5\bar{a}_z$ } position vector

$\overline{OQ} = -2\bar{a}_x + 3\bar{a}_y + \bar{a}_z$

$\overline{PQ} = \overline{OQ} - \overline{OP}$ } distance vector

$= -5\bar{a}_x - \bar{a}_y - 4\bar{a}_z$

Problems:

1) If $\bar{A} = 10\bar{a}_x - 4\bar{a}_y + 6\bar{a}_z$

$\bar{B} = 2\bar{a}_x + \bar{a}_y$

Find i) Component of $\bar{A}$ along $\bar{a}_y$

ii) magnitude of $3\bar{A} - \bar{B}$

iii) unit vector along $\bar{A} + 2\bar{B}$

Sol: i) Component of A along $\bar{a}_y = -4$

ii) $3\bar{A} - \bar{B} = 3(10\bar{a}_x - 4\bar{a}_y + 6\bar{a}_z) - (2\bar{a}_x + \bar{a}_y)$

$= 28\bar{a}_x - 13\bar{a}_y + 18\bar{a}_z$

$|3\bar{A} - \bar{B}| = \sqrt{(28)^2 + (-13)^2 + (18)^2} = 35.7$ units

iii) $\bar{A} + 2\bar{B} = 10\bar{a}_x - 4\bar{a}_y + 6\bar{a}_z + 2(2\bar{a}_x + \bar{a}_y)$

$= 14\bar{a}_x - 2\bar{a}_y + 6\bar{a}_z$

$|\bar{A} + 2\bar{B}| = \sqrt{(14)^2 + (-2)^2 + (6)^2} = 15.36$ units

Unit vector along $\bar{A} + 2\bar{B} = \dfrac{\bar{A} + 2\bar{B}}{|\bar{A} + 2\bar{B}|} = \dfrac{14\bar{a}_x - 2\bar{a}_y + 6\bar{a}_z}{15.36}$

$= 0.911\bar{a}_x - 0.13\bar{a}_y + 0.39\bar{a}_z$

2) Points P and Q are located at (0,2,4) and (-3,1,5). Calculate the following

i) position vector P

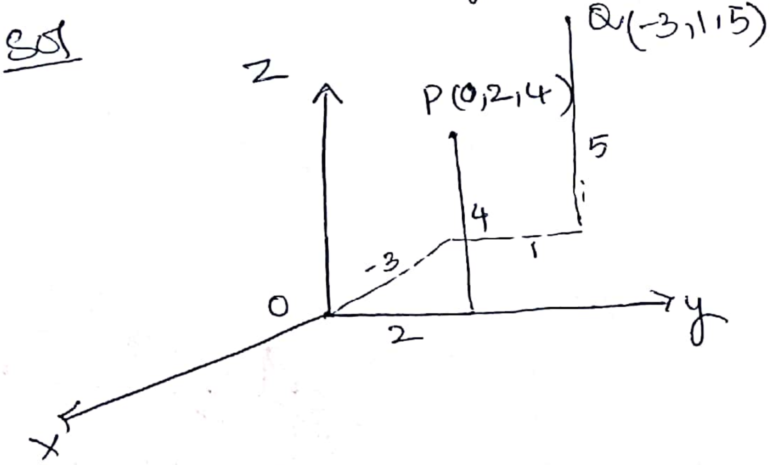
ii) distance vector from P to Q

iii) distance vector from Q to P

IV) distance ~~vector~~ between P and Q

V) a vector parallel to PQ with magnitude of 10

<u>Sol</u>



i) $\bar{OP} = 2\bar{a}_y + 4\bar{a}_z$

ii) $\bar{PQ} = \bar{OQ} - \bar{OP}$

$= (-3\bar{a}_x + \bar{a}_y + 5\bar{a}_z) - (2\bar{a}_y + 4\bar{a}_z)$

$= -3\bar{a}_x - \bar{a}_y + \bar{a}_z$

iii) $\bar{QP} = \bar{OP} - \bar{OQ}$

$= -3\bar{a}_x - \bar{a}_y + \bar{a}_z$ //.

iv) $|\bar{PQ}| = \sqrt{(-3)^2 + (-1)^2 + (1)^2} = \sqrt{11}$

V) A vector parallel to $\bar{PQ}$ with magnitude of 10

$\bar{PQ} = -3\bar{a}_x - \bar{a}_y + \bar{a}_z$

Unit vector $= \frac{\bar{PQ}}{|\bar{PQ}|} = \frac{-3\bar{a}_x - \bar{a}_y + \bar{a}_z}{\sqrt{11}}$

A vector parallel to PQ with magnitude of 10 is

$= \frac{10[-3\bar{a}_x - \bar{a}_y + \bar{a}_z]}{\sqrt{11}}$

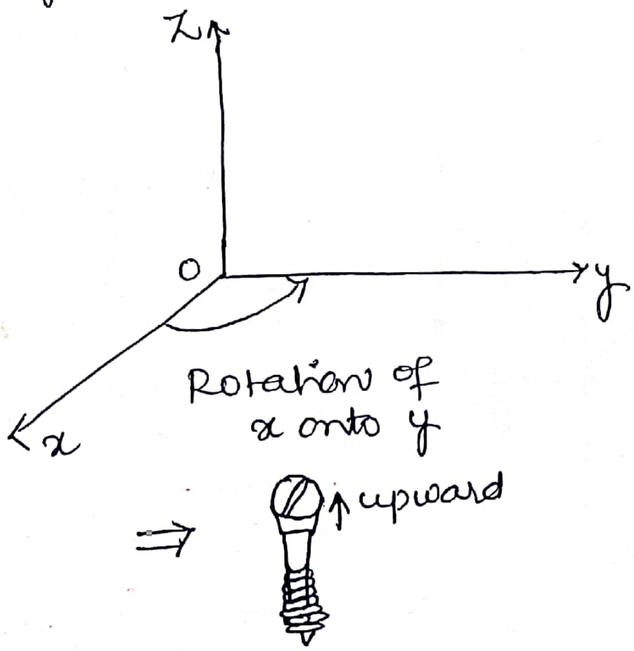
$= -99.3\bar{a}_x - 33.1\bar{a}_y + 33.1\bar{a}_z$

Coordinate System

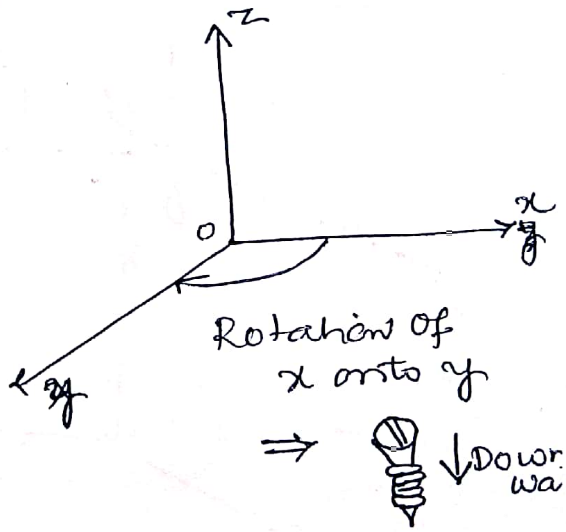
To express a vector interms of its component the following coordinate systems are generally prefered.

1) Cartesian (or) Rectangular coordinate system
2) cylindrical coordinate system
3) Spherical coordinate system.

i) Right handed system



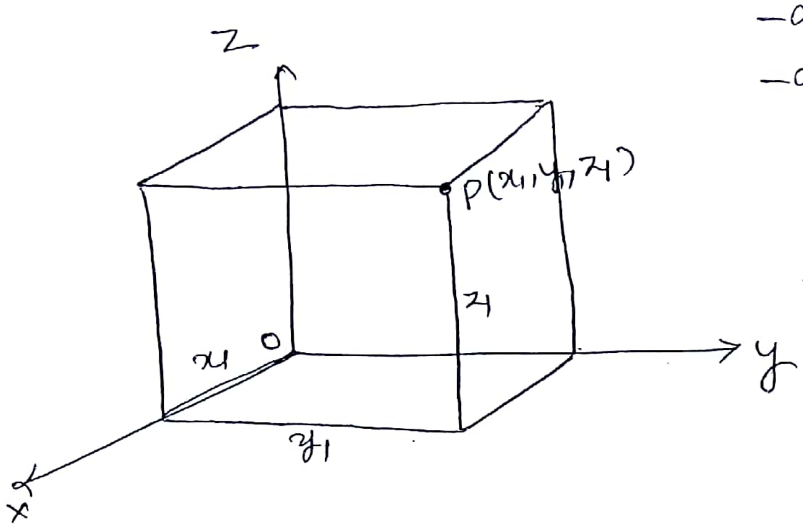
ii) Left handed system



Thumb → x-axis
fore finger → y-axis
Middle finger → z-axis

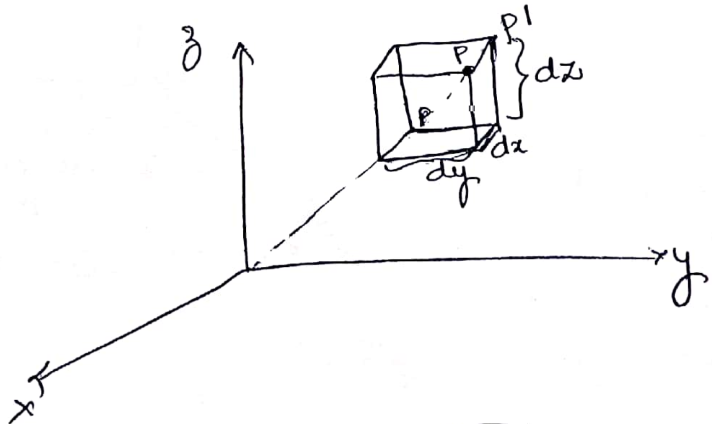
1) Cartesian Coordinate System

$-\infty < x < \infty$
$-\infty < y < \infty$
$-\infty < z < \infty$.



Differential elements

If 'P' is shifted to 'P'' $P'(x+dx, y+dy, z+dz)$.



$\left.\begin{array}{l} dx \\ dy \\ dz \end{array}\right\}$ Differential elements

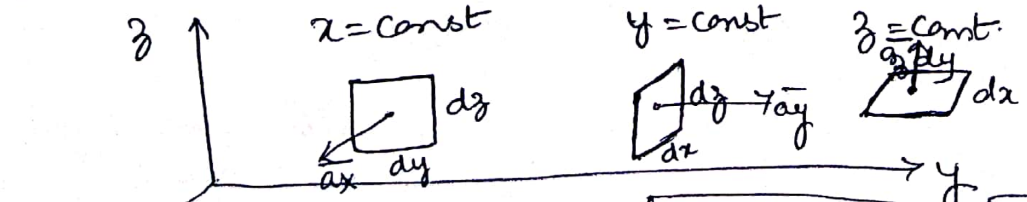
differential length ($\bar{dL}$)

$$\boxed{\bar{dL} = dx\,\bar{a}_x + dy\,\bar{a}_y + dz\,\bar{a}_z}$$

differential volume (dV)

$$\boxed{dV = dx\,dy\,dz}$$

differential surface ($\bar{ds}$)



$\boxed{\bar{dS_x} = dy\,dz\,\bar{a}_x}$ $\boxed{\bar{dS_y} = dx\,dz\,\bar{a}_y}$ $\boxed{\bar{dS_z} = dx\,dy\,\bar{a}_z}$

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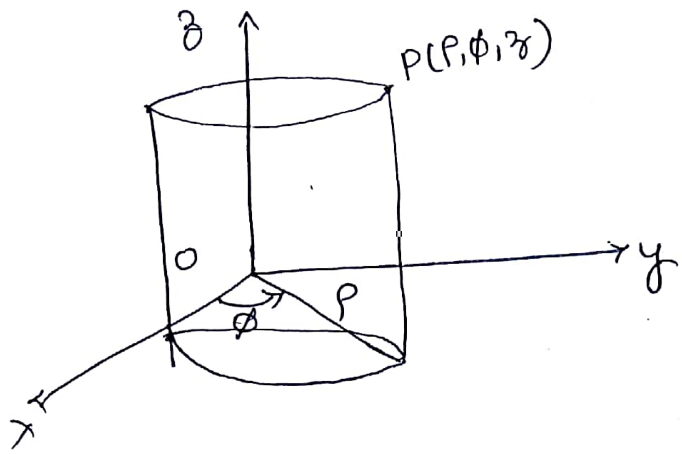
2) cylindrical Coordinate System

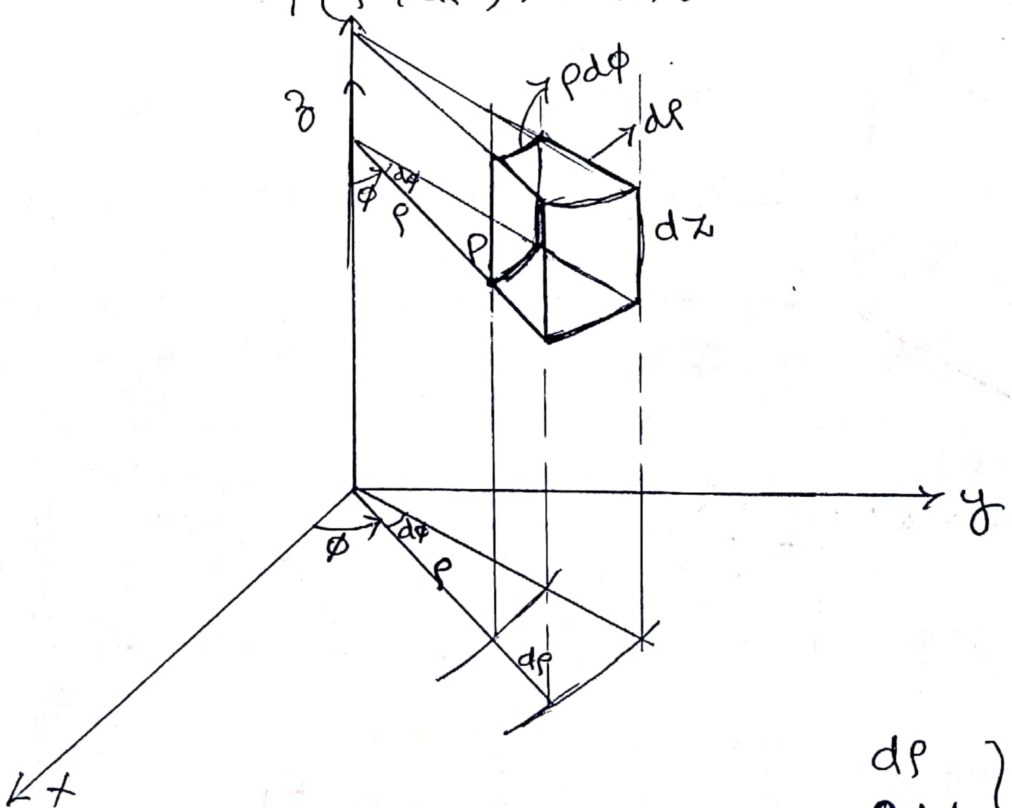


$0 \le \rho \le \infty$
$0 \le \phi \le 2\pi$
$-\infty \le z \le \infty.$

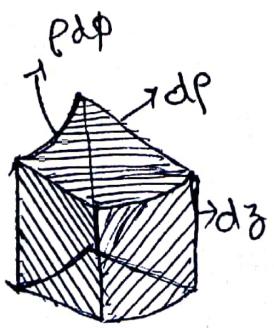
Differential Elements

If 'p' is shifted to 'p''

$p'(\rho + d\rho, \phi + d\phi, z + dz).$



$\left.\begin{matrix} d\rho \\ \rho d\phi \\ dz \end{matrix}\right\}$ differential elements.



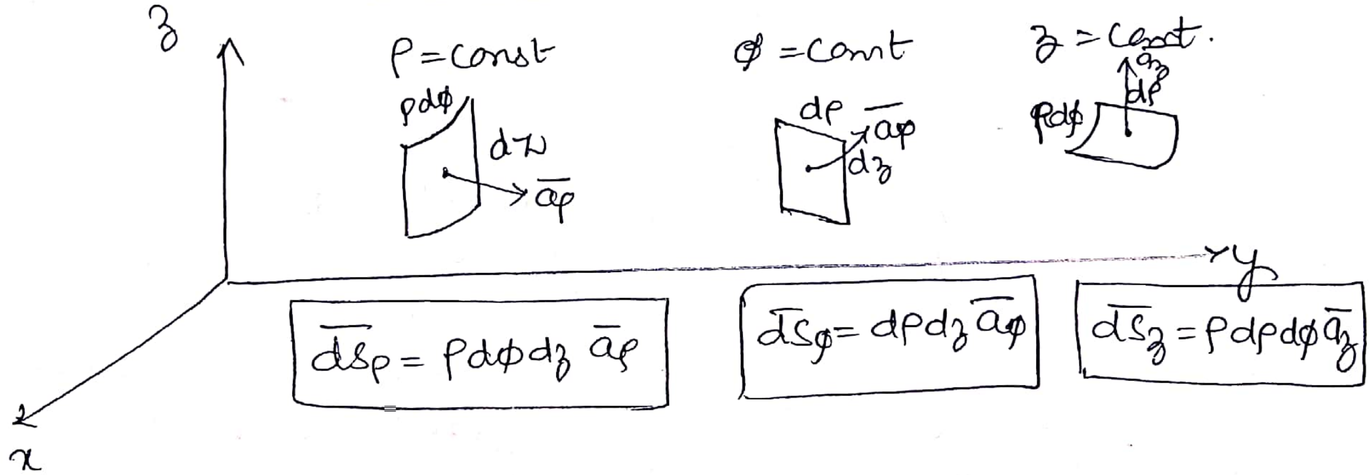
Differential Length ($\overline{dL}$)

$$\boxed{\overline{dL} = d\rho\,\overline{a_\rho} + \rho\,d\phi\,\overline{a_\phi} + dz\,\overline{a_z}}$$

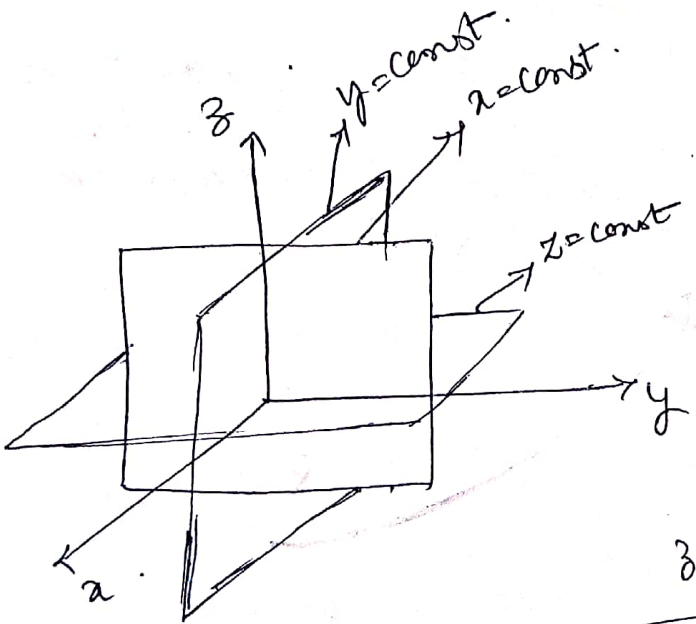
Differential Volume (dv)

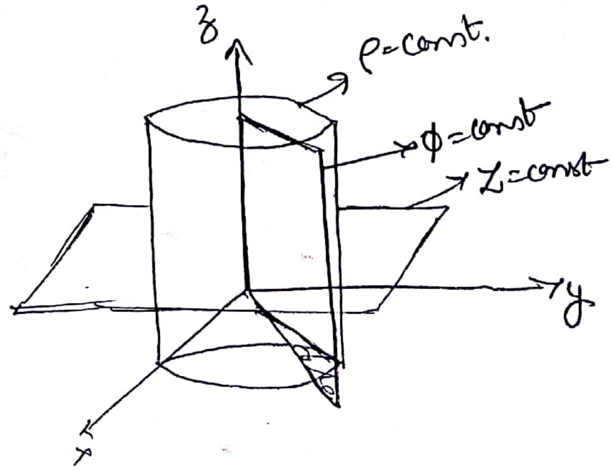
$$\boxed{dv = \rho\,d\rho\,d\phi\,dz}$$

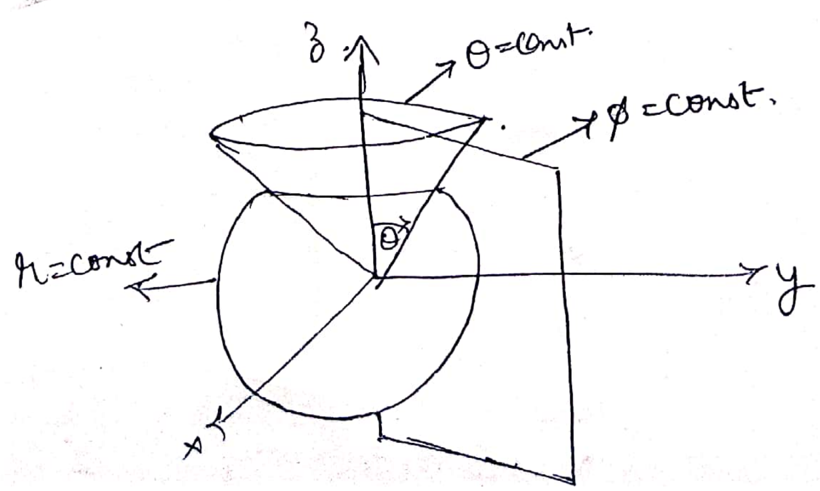
Differential Surface ($\overline{ds}$)



$$\boxed{\overline{ds_\rho} = \rho\,d\phi\,dz\,\overline{a_\rho}} \quad \boxed{\overline{ds_\phi} = d\rho\,dz\,\overline{a_\phi}} \quad \boxed{\overline{ds_z} = \rho\,d\rho\,d\phi\,\overline{a_z}}$$







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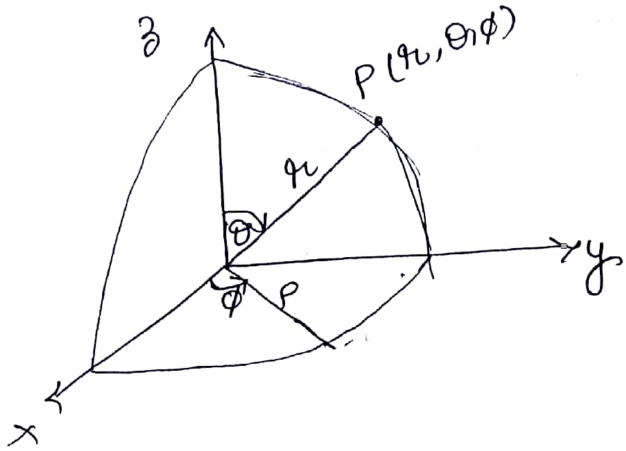
3) Spherical Coordinate System

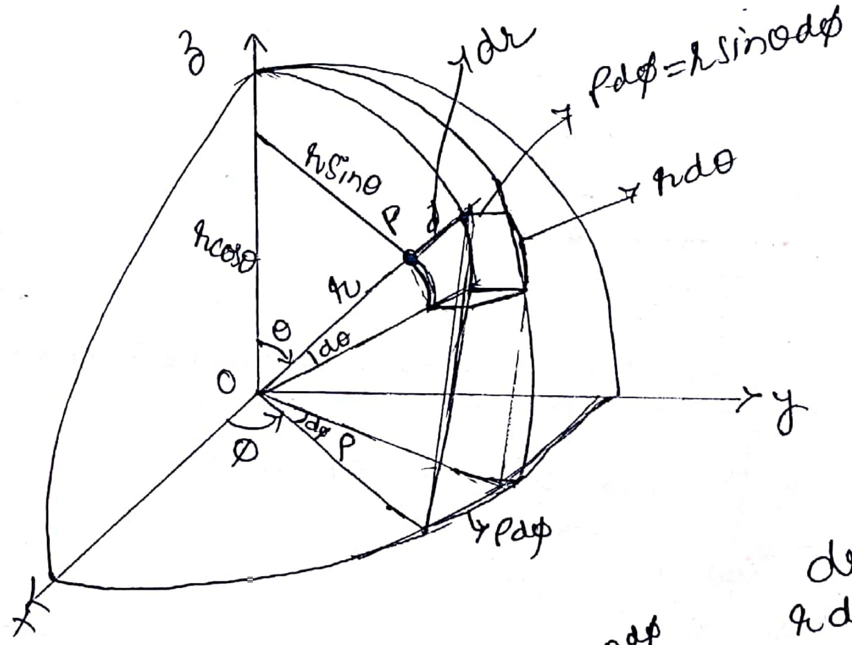


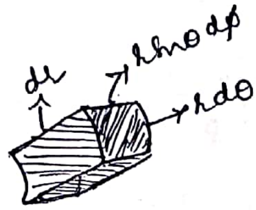
$0 \leq r \leq \infty$

$0 \leq \theta \leq \pi$

$0 \leq \phi \leq 2\pi$

Differential elements





$\left.\begin{matrix} dr \\ r\,d\theta \\ r\sin\theta\, d\phi \end{matrix}\right\}$ differential element

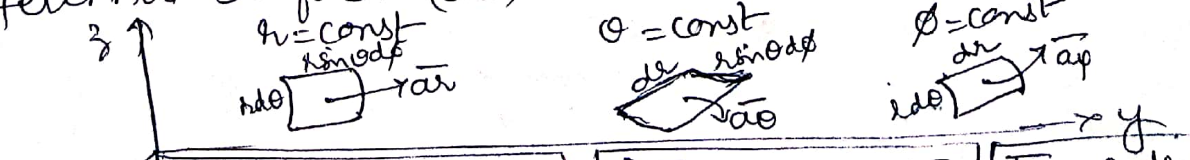
Differential Length ($\overline{dL}$)

$$\boxed{\overline{dL} = dr\,\bar{a}_r + r\,d\theta\,\bar{a}_\theta + r\sin\theta\, d\phi\,\overline{a_\phi}}$$

Differential Volume (dV)

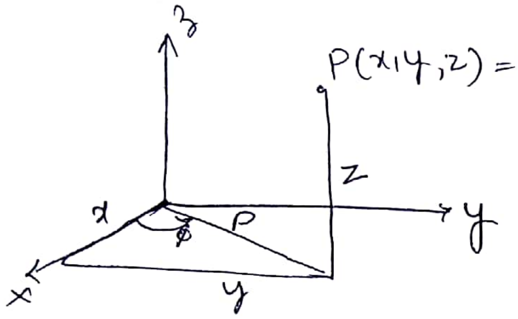
$$\boxed{dV = r^2 \sin\theta\, dr\, d\theta\, d\phi}$$

Differential Surface ($\overline{ds}$)



$\boxed{\overline{ds_r} = r^2 \sin\theta\, d\theta\, d\phi\, \bar{a}_r}$ $\boxed{\overline{ds_\theta} = r\sin\theta\, dr\, d\phi\, \bar{a}_\theta}$ $\boxed{\overline{ds_\phi} = r\, dr\, d\theta\, \bar{a}_\phi}$

Transformation of i) cylindrical to cartesian coordinates



$\cos\phi = \frac{Adj}{hyp.}$

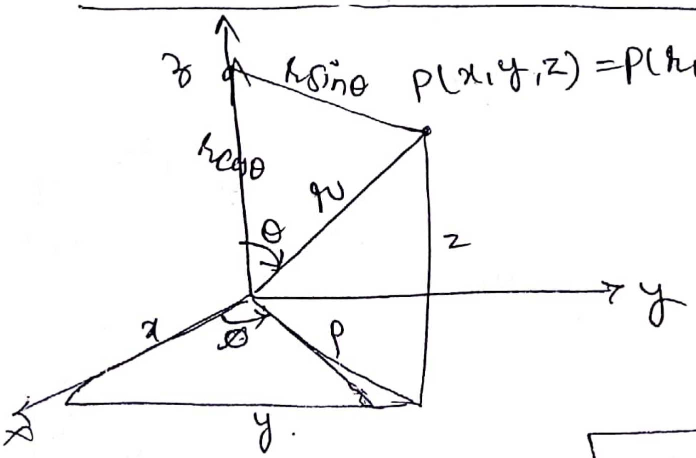
$\sin\phi = \frac{OPP}{hyp.}$

$$\boxed{\begin{aligned} x &= \rho\cos\phi \\ y &= \rho\sin\phi \\ z &= z \end{aligned}}$$

Cartesian to cylindrical

$$\boxed{\begin{aligned} \rho &= \sqrt{x^2+y^2} \\ \phi &= \tan^{-1}\frac{y}{x} \\ z &= z \end{aligned}}$$

ii) Spherical to Cartesian coordinates



$$\boxed{\begin{aligned} x &= \rho\cos\phi = r\sin\theta\cos\phi \\ y &= \rho\sin\phi = r\sin\theta\sin\phi \\ z &= r\cos\theta \end{aligned}}$$

Cartesian to spherical

$$\boxed{\begin{aligned} r &= \sqrt{x^2+y^2+z^2} \\ \theta &= \cos^{-1}\frac{z}{r} = \cos^{-1}\frac{z}{\sqrt{x^2+y^2+z^2}} \\ \phi &= \tan^{-1}\frac{y}{x} \end{aligned}}$$

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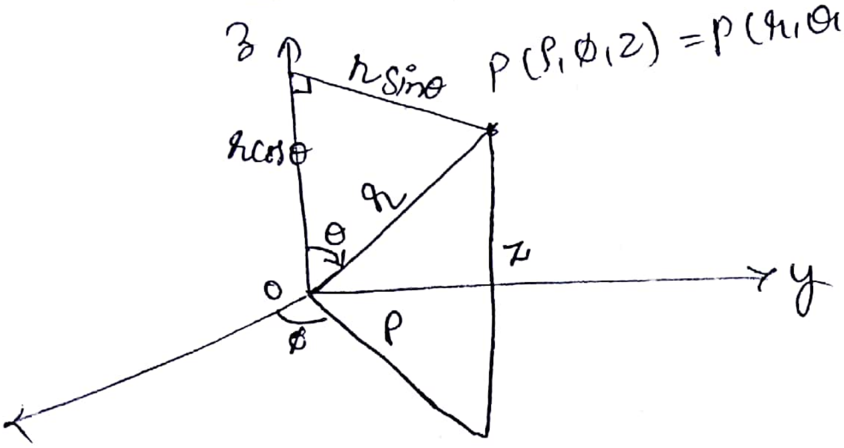
iii) cylindrical to spherical coordinates



$$r = \sqrt{\rho^2 + z^2}$$

$$\theta = \tan^{-1}\frac{r\sin\theta}{r\cos\theta} = \tan^{-1}\frac{\rho}{z}$$

$$\phi = \phi$$

Spherical to cylindrical Coordinates

$$\rho = r\sin\theta$$

$$\phi = \phi$$

$$z = r\cos\theta$$

1) Find the volume of cylinder having radius 'R' and height 'h'.

Sol.

$$dv = \rho \, d\rho \, d\phi \, dz$$

$$V = \int_V dv = \int_{\rho=0}^{R}\int_{\phi=0}^{2\pi}\int_{z=0}^{h} \rho \, d\rho \, d\phi \, dz$$

$$= \left[\frac{\rho^2}{2}\right]_0^R [\phi]_0^{2\pi} [z]_0^h$$

$$= \frac{R^2}{\cancel{2}} \cdot \cancel{2}\pi \cdot h$$

$$= \pi R^2 h$$

2) Find the surface area for $\rho = R_1$ and height 'h'

Sol:

$$\bar{ds}_\rho = \rho \, d\phi \, dz$$

$$S = \int_{\phi=0}^{2\pi}\int_{z=0}^{h} \rho \, d\phi \, dz$$

$$= R_1 [\phi]_0^{2\pi} [z]_0^h$$

$$= 2\pi R_1 h$$

3) (OR) $\bar{ds} = \rho \, d\phi \, dz \, \bar{a}_\rho + d\rho \, dz \, \bar{a}_\phi + \rho \, d\phi \, d\rho \, \bar{a}_z$ top

$$S = \int_{\phi=0}^{2\pi}\int_{z=0}^{h} \rho \, d\phi \, dz \text{ (lateral)} + \iint d\rho \, dz + \iint_{top} \rho \, d\phi \, d\rho$$

$$S_{top} = 2\pi R_1 h + [\rho]_0^{R_1} [z]_0^h + \left[\frac{\rho^2}{2}\right]_0^{R_1} [\phi]_0^{2\pi}$$

$$= 2\pi R_1 h + R_1 h + \frac{R_1^2}{\cancel{2}} \cdot \cancel{2}\pi$$

$$= 2\pi R_1 \left[h + \frac{h}{2\pi} + \right]$$

$S_{top} = \pi R_1^2$, $S_{bottom} = \pi R_1^2$

$\therefore S = 2\pi R_1 h + 2\pi R_1^2$

$= 2\pi R_1 [h + R_1]$

3) Find the volume of Sphere having Radius 'R'. (8)
using integration (or) spherical system.

Sol:

$$dV = dr \cdot r d\theta \cdot r\sin\theta \, d\phi$$

$$dV = r^2 \sin\theta \, dr \, d\theta \, d\phi$$

$$\therefore V = \int_V dV$$

$$= \int_{r=0}^{R}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} r^2 \sin\theta \, dr \, d\theta \, d\phi$$

$$= \left[\frac{r^3}{3}\right]_0^R \left[-\cos\theta\right]_0^\pi \left[\phi\right]_0^{2\pi}$$

$$= \frac{R^3}{3} \cdot \left[-(-1-1)\right]\left[2\pi\right]$$

$$= \frac{4}{3}\pi R^3$$

4) Find the surface area of a sphere of radius R, by integration.

$$d\bar{s}_r = r^2 \sin\theta \, d\theta \, d\phi$$

$$S = \int_S ds_r = \int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} r^2 \sin\theta \, d\theta \, d\phi$$

$$= R^2 \left[-\cos\theta\right]_0^\pi \left[\phi\right]_0^{2\pi}$$

$$= R^2 \left[-(-1-1)\right] 2\pi$$

$$= 4\pi R^2$$

1) Dot product (or Scalar)

$$\bar{A}\cdot\bar{B} = |\bar{A}||\bar{B}|\cos\theta_{AB}$$

Cartesian : $\bar{a}_x, \bar{a}_y, \bar{a}_z$

$$\bar{a}_x\cdot\bar{a}_x = \bar{a}_y\cdot\bar{a}_y = \bar{a}_z\cdot\bar{a}_z = 1$$

$$\bar{a}_x\cdot\bar{a}_y = \bar{a}_y\cdot\bar{a}_z = \bar{a}_z\cdot\bar{a}_x = 0.$$

cylindrical : $\bar{a}_\rho, \bar{a}_\phi, \bar{a}_z$

$$\bar{a}_\rho\cdot\bar{a}_\rho = \bar{a}_\phi\cdot\bar{a}_\phi = \bar{a}_z\cdot\bar{a}_z = 1$$

$$\bar{a}_\rho\cdot\bar{a}_\phi = \bar{a}_\phi\cdot\bar{a}_z = \bar{a}_z\cdot\bar{a}_\rho = 0$$

Spherical : $\bar{a}_r, \bar{a}_\theta, \bar{a}_\phi$

$$\bar{a}_r\cdot\bar{a}_r = \bar{a}_\theta\cdot\bar{a}_\theta = \bar{a}_\phi\cdot\bar{a}_\phi = 1$$

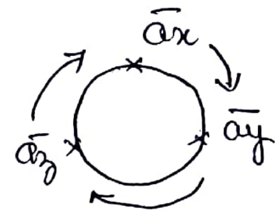
$$\bar{a}_r\cdot\bar{a}_\theta = \bar{a}_\theta\cdot\bar{a}_\phi = \bar{a}_\phi\cdot\bar{a}_r = 0$$

2) Vector (or cross) product

$$\bar{A}\times\bar{B} = |\bar{A}||\bar{B}|\sin\theta_{AB}\,\bar{a}_N$$

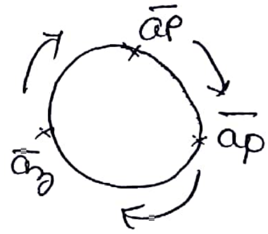
cartesian :

$$\bar{a}_x\times\bar{a}_y = \bar{a}_z$$
$$\bar{a}_y\times\bar{a}_z = \bar{a}_x$$
$$\bar{a}_z\times\bar{a}_x = \bar{a}_y$$



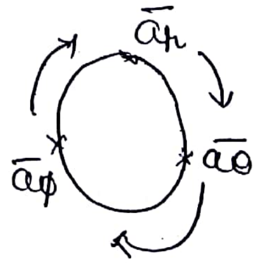
cylindrical

$$\bar{a}_\rho\times\bar{a}_\phi = \bar{a}_z$$
$$\bar{a}_\phi\times\bar{a}_z = \bar{a}_\rho$$
$$\bar{a}_z\times\bar{a}_\rho = \bar{a}_\phi$$



Spherical

$$\bar{a}_r\times\bar{a}_\theta = \bar{a}_\phi$$
$$\bar{a}_\theta\times\bar{a}_\phi = \bar{a}_r$$
$$\bar{a}_\phi\times\bar{a}_r = \bar{a}_\theta$$



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Transformation of Vector from i) Cartesian to Cylindrical coordinate System:

Let $\bar{A} = A_x \bar{a}_x + A_y \bar{a}_y + A_z \bar{a}_z$ (known)

$\bar{A} = A_\rho \bar{a}_\rho + A_\phi \bar{a}_\phi + A_z \bar{a}_z$ (unknown) —— (1)

$A_\rho = \bar{A} \cdot \bar{a}_\rho = (A_x \bar{a}_x + A_y \bar{a}_y + A_z \bar{a}_z) \cdot \bar{a}_\rho$

$= A_x (\bar{a}_x \cdot \bar{a}_\rho) + A_y (\bar{a}_y \cdot \bar{a}_\rho) + A_z (\bar{a}_z \cdot \bar{a}_\rho)$ —— (2)

$=$

$A_\phi = \bar{A} \cdot \bar{a}_\phi = (A_x \bar{a}_x + A_y \bar{a}_y + A_z \bar{a}_z) \cdot \bar{a}_\phi$

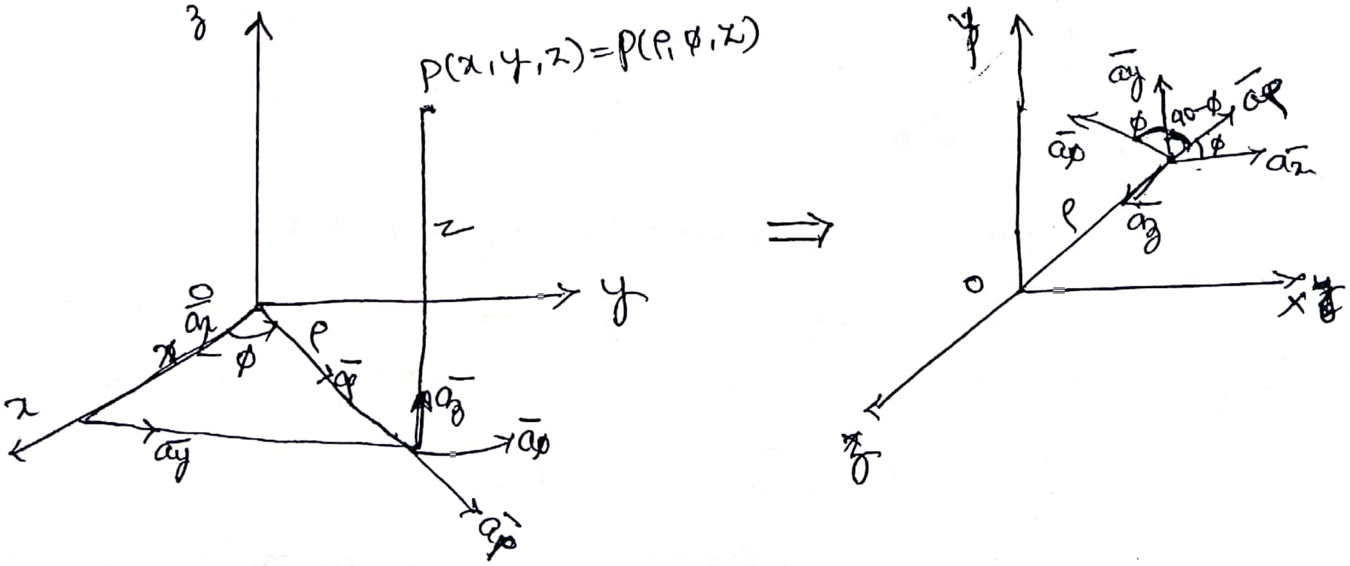
$= A_x (\bar{a}_x \cdot \bar{a}_\phi) + A_y (\bar{a}_y \cdot \bar{a}_\phi) + A_z (\bar{a}_z \cdot \bar{a}_\phi)$ —— (3)

$=$

$A_z = \bar{A} \cdot \bar{a}_z = (A_x \bar{a}_x + A_y \bar{a}_y + A_z \bar{a}_z) \cdot \bar{a}_z$

$= A_x (\bar{a}_x \cdot \bar{a}_z) + A_y (\bar{a}_y \cdot \bar{a}_z) + A_z (\bar{a}_z \cdot \bar{a}_z)$

$= A_x (0) + A_y (0) + A_z (1) = A_z$ —— (4)



$\bar{a}_x \cdot \bar{a}_\rho = |\bar{a}_x||\bar{a}_\rho| \cos\theta_{x\rho}$

$= 1 \cdot 1 \cdot \cos\phi = \cos\phi$

$\bar{a}_y \cdot \bar{a}_\rho = 1 \cdot 1 \cdot \cos(90-\phi) = \sin\phi$

$\bar{a}_z \cdot \bar{a}_\rho = 1 \cdot 1 \cdot \cos 90^\circ = 0$

$\bar{a}_x \cdot \bar{a}_\phi = 1 \cdot 1 \cdot \cos(90+\phi) = -\sin\phi$

$\bar{a}_y \cdot \bar{a}_\phi = 1 \cdot 1 \cdot \cos\phi = \cos\phi$

$\bar{a}_z \cdot \bar{a}_\phi = 1 \cdot 1 \cdot \cos 90^\circ = 0$.

$\cdot$	$\bar{a}_\rho$	$\bar{a}_\phi$	$\bar{a}_z$
$\bar{a}_x$	$\cos\phi$	$-\sin\phi$	0
$\bar{a}_y$	$\sin\phi$	$\cos\phi$	0
$\bar{a}_z$	0	0	1

Sub in (2) (3)(4)

$$A_\rho = A_x \cos\phi + A_y \sin\phi + A_z (0)$$

$$A_\phi = -A_x \sin\phi + A_y \cos\phi + A_z (0).$$

$$A_z = A_z.$$

Sub in (1)

$$\therefore \bar{A} = [A_x \cos\phi + A_y \sin\phi]\,\bar{a}_\rho + [-A_x \sin\phi + A_y \cos\phi]\,\bar{a}_\phi + A_z.$$

$$\begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

ii) Cylindrical to Cartesian system

$$\bar{A} = A_x \bar{a}_x + A_y \bar{a}_y + A_z \bar{a}_z \text{ (unknown)} \quad \text{—— (1)}$$

$$A_x = \bar{A} \cdot \bar{a}_x = (A_\rho \bar{a}_\rho + A_\phi \bar{a}_\phi + A_z \bar{a}_z) \cdot \bar{a}_x$$

$$= A_\rho (\bar{a}_\rho \cdot \bar{a}_x) + A_\phi (\bar{a}_\phi \cdot \bar{a}_x) + A_z (\bar{a}_z \cdot \bar{a}_x).$$

$$= A_\rho \cos\phi + A_\phi (-\sin\phi) + 0 \quad \text{—— (2)}$$

$$A_y = \bar{A} \cdot \bar{a}_y$$

$$= (A_\rho \bar{a}_\rho + A_\phi \bar{a}_\phi + A_z \bar{a}_z) \cdot \bar{a}_y$$

$$= A_\rho \sin\phi + A_\phi \cos\phi + 0 \quad \text{—— (3)}$$

$$A_z = \bar{A} \cdot \bar{a}_z = (A_\rho \bar{a}_\rho + A_\phi \bar{a}_\phi + A_z \bar{a}_z) \cdot \bar{a}_z$$

$$= A_\rho (0) + A_\phi (0) + A_z \quad \text{—— (4)}$$

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix} //$$

Transformation of vector from Cartesian to Spherical

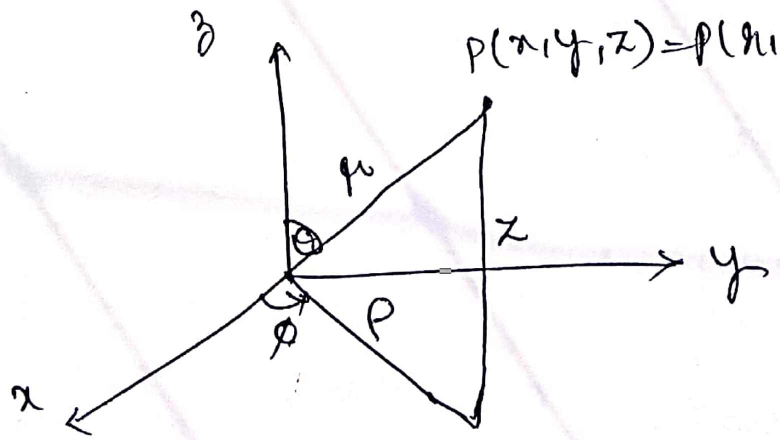
Let $\bar{A} = A_x \bar{a_x} + A_y \bar{a_y} + A_z \bar{a_z}$ (Known)

$\bar{A} = A_r \bar{a_r} + A_\theta \bar{a_\theta} + A_\phi \bar{a_\phi}$ (unknown) —(1)

$$A_r = \bar{A} \cdot \bar{a_r} = (A_x \bar{a_x} + A_y \bar{a_y} + A_z \bar{a_z}) \cdot \bar{a_r}$$
$$= A_x (\bar{a_x} \cdot \bar{a_r}) + A_y (\bar{a_y} \cdot \bar{a_r}) + A_z (\bar{a_z} \cdot \bar{a_r}) \quad —(2)$$

$$A_\theta = \bar{A} \cdot \bar{a_\theta} = (A_x \bar{a_x} + A_y \bar{a_y} + A_z \bar{a_z}) \cdot \bar{a_\theta}$$
$$= A_x (\bar{a_x} \cdot \bar{a_\theta}) + A_y (\bar{a_y} \cdot \bar{a_\theta}) + A_z (\bar{a_z} \cdot \bar{a_\theta}) \quad —(3)$$

$$A_\phi = \bar{A} \cdot \bar{a_\phi} = (A_x \bar{a_x} + A_y \bar{a_y} + A_z \bar{a_z}) \cdot \bar{a_\phi}$$
$$= A_x (\bar{a_x} \cdot \bar{a_\phi}) + A_y (\bar{a_y} \cdot \bar{a_\phi}) + A_z (\bar{a_z} \cdot \bar{a_\phi}) \quad —(4)$$



$\cdot$	$\bar{a}_r$	$\bar{a}_\theta$	$\bar{a}_\phi$
$\bar{a}_x$	$\sin\theta\cos\phi$	$\cos\theta\cos\phi$	$-\sin\phi$
$\bar{a}_y$	$\sin\theta\sin\phi$	$\cos\theta\sin\phi$	$\cos\phi$
$\bar{a}_z$	$\cos\theta$	$-\sin\theta$	0

$$\begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

ii) Spherical to cartesian

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix}$$

Transformation of a vector from spherical to Cylindrical

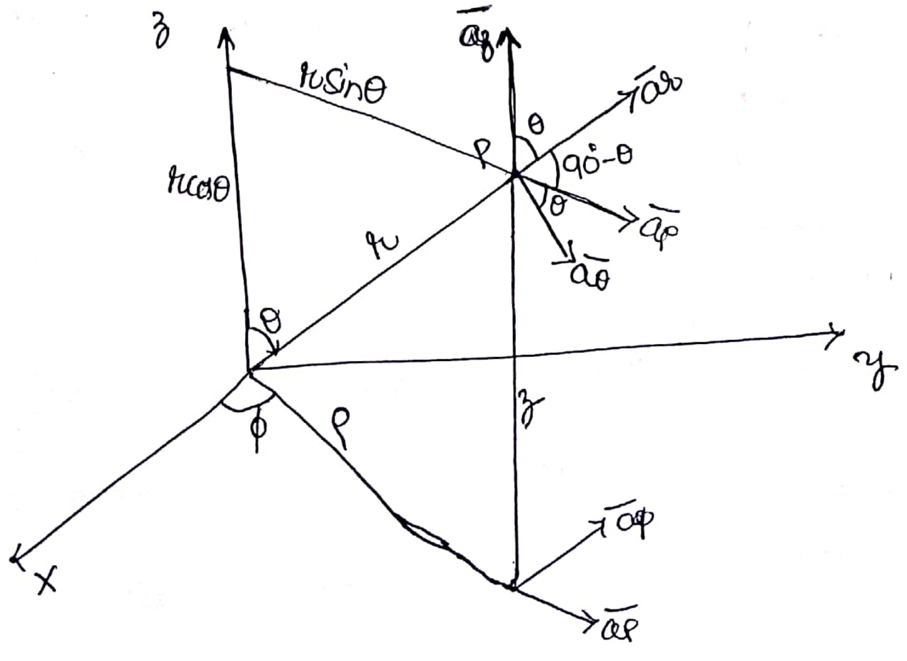
Let $\bar{A} = A_r \bar{a}_r + A_\theta \bar{a}_\theta + A_\phi \bar{a}_\phi$ (known)

$\bar{A} = A_\rho \bar{a}_\rho + A_\phi \bar{a}_\phi + A_z \bar{a}_z$ (unknown).

$$A_\rho = \bar{A} \cdot \bar{a}_\rho$$
$$= (A_r \bar{a}_r + A_\theta \bar{a}_\theta + A_\phi \bar{a}_\phi) \cdot \bar{a}_\rho$$
$$= A_r(\bar{a}_r \cdot \bar{a}_\rho) + A_\theta (\bar{a}_\theta \cdot \bar{a}_\rho) + A_\phi (\bar{a}_\phi \cdot \bar{a}_\rho) \quad \text{— (1)}$$

$$A_\phi = \bar{A} \cdot (\bar{a}_\phi)$$
$$= (A_r \bar{a}_r + A_\theta \bar{a}_\theta + A_\phi \bar{a}_\phi) \cdot \bar{a}_\phi$$
$$= A_r (\bar{a}_r \cdot \bar{a}_\phi) + A_\theta (\bar{a}_\theta \cdot \bar{a}_\phi) + A_\phi (\bar{a}_\phi \cdot \bar{a}_\phi) \quad \text{— (2)}$$

$$A_z = \bar{A} \cdot \bar{a}_z$$
$$= (A_r \bar{a}_r + A_\theta \bar{a}_\theta + A_\phi \bar{a}_\phi) \cdot \bar{a}_z$$
$$= A_r (\bar{a}_r \cdot \bar{a}_z) + A_\theta (\bar{a}_\theta \cdot \bar{a}_z) + A_\phi (\bar{a}_\phi \cdot \bar{a}_z) \quad \text{— (3)}$$



$\bar{a}_r \cdot \bar{a}_\rho = 1 \cdot 1 \cdot \cos(90-\theta) = \sin\theta$

$\bar{a}_r \cdot \bar{a}_\phi = 1 \cdot 1 \cdot \cos 90° = 0$

$\bar{a}_r \cdot \bar{a}_z = 1 \cdot 1 \cdot \cos\theta = \cos\theta$

$\bar{a}_\theta \cdot \bar{a}_\rho = 1 \cdot 1 \cdot \cos\theta = \cos\theta$

$\bar{a}_\theta \cdot \bar{a}_\phi = 1 \cdot 1 \cdot \cos 90° = 0$

$\bar{a}_\theta \cdot \bar{a}_z = 1 \cdot 1 \cdot \cos(90+\theta)$
$= -\sin\theta$

$\bar{a}_\phi \cdot \bar{a}_\rho = 1 \cdot 1 \cdot$

$\bar{a}_\phi \cdot \bar{a}_\phi =$

$\bar{a}_\phi \cdot \bar{a}_z =$

$$\begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix} = \begin{bmatrix} \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix}$$

$\cdot$	$\bar{a}_\rho$	$\bar{a}_\phi$	$\bar{a}_z$
$\bar{a}_r$	$\sin\theta$	0	$\cos\theta$
$\bar{a}_\theta$	$\cos\theta$	0	$-\sin\theta$
$\bar{a}_\phi$	0	1	0

ii) cylindrical to spherical

$$\begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta & 0 & \cos\theta \\ \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix}$$

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Problems :

1) If a point $A(2,3,-1)$ and $B(4,50^\circ,2)$

i) Convert points Cartesian to cylindrical and cylindrical to Cartesian system.

ii) find the distance between origin to point A and origin to point 'B', distance between A to B.

Sol: $A(2,3,-1) \rightarrow$ cartesian

$B(4,50^\circ,2) \rightarrow$ cylindrical

i) Cartesian to cylindrical

$\rho = \sqrt{x^2+y^2} = \sqrt{2^2+3^2} = 3.6$

$\phi = \tan^{-1}\frac{y}{x} = \tan^{-1}\frac{3}{2} = 56.3^\circ$

$z = z = -1$

$\therefore A(2,3,-1) = A(3.6, 56.3^\circ, -1)$

cylindrical to Cartesian

$x = \rho\cos\phi = 4\cos 50^\circ = 2.57$

$y = \rho\sin\phi = 4\sin 50^\circ = 3.06$

$z = z = 2$

$\therefore B(4,50^\circ,2) = B(2.57, 3.06, 2)$

ii) distance between $(0,0,0)$ to $A(2,3,-1)$

$$|\overline{OA}| = \sqrt{(2-0)^2+(3-0)^2+(-1-0)^2} = 3.74.$$

$$|\overline{OB}| = \sqrt{(2.5)^2+(3.06)^2+(2)^2} = 4.42$$

distance between A & B.

$$|\overline{AB}| = \sqrt{(2.57-2)^2+(3.06-3)^2+(2+1)^2}$$

$$= 3.054$$

2) Transform the following vector to cylindrical at a point specified $\bar{P} = 4\bar{a_x} - 2\bar{a_y} - 4\bar{a_z}$ the point $A(2,3,5)$

Sol

$$\bar{P} = 4\bar{a_x} - 2\bar{a_y} - 4\bar{a_z} \quad \text{(Cartesian)}$$

$$\bar{P} = P_r\bar{a_r} + P_\phi\bar{a_\phi} + P_z\bar{a_z} \quad \text{(cylindrical)}.$$

$$P_r = \bar{P}\cdot\bar{a_r}$$
$$= (4\bar{a_x} - 2\bar{a_y} - 4\bar{a_z})\cdot\bar{a_r}$$
$$= 4(\bar{a_x}\cdot\bar{a_r}) - 2(\bar{a_y}\cdot\bar{a_r}) - 4(\bar{a_z}\cdot\bar{a_r})$$
$$= 4\cos\phi - 2\sin\phi - 4(0).$$

$A(2,3,5) = A(r,\phi,z)$

$\rho = r = \sqrt{x^2+y^2} = 3.6$

$\phi = \tan^{-1}\frac{y}{x} = \tan^{-1}3/2 = 56.3°.$

$z = 5.$

$A(3.6, 56.3°, 5).$

$$= 4\cos 56.3° - 2\sin 56.3°$$

$$\boxed{P_r = 0.555}$$

$$P_\phi = \bar{P}\cdot\bar{a_\phi}$$
$$= (4\bar{a_x} - 2\bar{a_y} - 4\bar{a_z})\cdot\bar{a_\phi}$$
$$= 4(\bar{a_x}\cdot\bar{a_\phi}) - 2(\bar{a_y}\cdot\bar{a_\phi}) - 4(\bar{a_z}\cdot\bar{a_\phi})$$
$$= -4\sin\phi - 2\cos\phi - 4(0)$$
$$= -4\sin 56.3° - 2\cos 56.3°$$
$$= -4.43$$

$$P_z = \bar{P}\cdot\bar{a_z}$$
$$= (4\bar{a_x} - 2\bar{a_y} - 4\bar{a_z})\cdot\bar{a_z}$$
$$= 0 - 0 - 4(1)$$
$$= -4$$

$$\therefore \bar{P} = 0.555\,\bar{a_r} - 4.43\,\bar{a_\phi} - 4\bar{a_z}$$

3) Given the point $A(2,3,-1)$ and $B(4,25^\circ,120^\circ)$ (13)

Find i) the spherical coordinates of A

ii) Cartesian coordinates of 'B'

iii) distance between A and B.

Sol

i) Spherical coordinates of $A(r,\theta,\phi)$.

$$r = \sqrt{x^2+y^2+z^2} = \sqrt{2^2+3^2+(-1)^2} = 3.74$$

$$\theta = \cos^{-1}\frac{z}{\sqrt{x^2+y^2+z^2}} = 105.5^\circ$$

$$\phi = \tan^{-1}\frac{y}{x} = 56.3^\circ.$$

$\therefore A(3.74, 105.5^\circ, 56.3^\circ)$

ii) Cartesian coordinates of 'B' (x,y,z).

$$x = \rho\cos\phi = r\sin\theta\cos\phi = -0.845$$

$$y = \rho\sin\phi = r\sin\theta\sin\phi = 1.463$$

$$z = r\cos\theta = 3.625$$

$B(-0.845, 1.463, 3.625)$

iii) distance between A & B

$$|\overline{AB}| = \sqrt{(-0.845-2)^2+(1.463-3)^2+(3.625+1)^2}$$

$=$.

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4) Obtain the spherical vector $10\,a_x$ at $P(-3,2,4)$

Sol $\bar{A} = A_r\bar{a_r} + A_\theta\bar{a_\theta} + A_\phi\bar{a_\phi}$ (spherical)

$P(-3,2,4) = P(r,\theta,\phi)$

$r = \sqrt{3^2+2^2+4^2} = 5.38$

$\theta = \cos^{-1}\frac{z}{r} = 42^\circ$

$\phi = \tan^{-1}\frac{y}{x} = -33.69^\circ$

$= 180^\circ - 33.69^\circ$

$= 146.31^\circ$.

* x is $-ve$
y is $+ve$.

$A_r = \bar{A}\cdot\bar{a_r}$

$= 10\bar{a_x}\cdot\bar{a_r}$

$= 10\sin\theta\cos\phi$

$= 10\sin 42^\circ\cos(-33.69^\circ)$

$= +5.57$

$A_\theta = \bar{A}\cdot\bar{a_\theta}$

$= 10\bar{a_x}\cdot\bar{a_\theta}$

$= 10\cos\theta\cos\phi$

$= +6.18$

$A_\phi = \bar{A}\cdot\bar{a_\phi}$

$= 10\bar{a_x}\cdot\bar{a_\phi}$

$= -10\sin\phi$

$= +5.536$

$\therefore \bar{A} = +5.57\,\bar{a_r} + 6.18\,\bar{a_\theta} + 5.53\,\bar{a_\phi}$

5) Express the vector B in Cartesian and cylindrical systems then find $\bar{B}$ at $(-3,4,0)$ and $(5,\frac{\pi}{2},-2)$. The given vector $\bar{B} = \frac{10}{r}\bar{a_r} + r\cos\phi\,\bar{a_\theta} + \bar{a_\phi}$

Sol $\bar{B} = B_x\bar{a_x} + B_y\bar{a_y} + B_z\bar{a_z}$

$B_x = \bar{B}\cdot\bar{a_x}$

$= \left[\frac{10}{r}\bar{a_r} + r\cos\phi\,\bar{a_\theta} + \bar{a_\phi}\right]\cdot\bar{a_x}$

$= \frac{10}{r}\sin\theta\cos\phi + r\cos\phi\cos\theta\cos\phi - \sin\phi$ —— (i).

$B_y = \bar{B}\cdot\bar{a_y}$

$= \left(\frac{10}{r}\bar{a_r} + r\cos\phi\,\bar{a_\theta} + \bar{a_\phi}\right)\cdot\bar{a_y}$

$B_y = \frac{10}{r} \sin\theta \sin\phi + r\cos\phi \cos\theta \sin\phi + \cos\phi$ — (2)

$B_z = \bar{B} \cdot \bar{a_z}$

$= \left(\frac{10}{r} \bar{a_r} + r\cos\phi \, \bar{a_\theta} + \bar{a_\phi} \right) \cdot \bar{a_z}$

$= \frac{10}{r} \cos\theta + r\cos\phi (-\sin\theta) + 0$ — (3)

(B) At $(-3, 4, 0)$

$r = \sqrt{x^2+y^2+z^2} = 5$

$\theta = \cos^{-1} \frac{z}{r} = 90^\circ$

$\phi = \tan^{-1} \frac{y}{x} = -53.13^\circ = 180^\circ - 53.13^\circ = 126.87^\circ$

$B_x = 1.99$

$B_y = -0.99$

$B_z = -1.2$

$\therefore \bar{B} = 1.99 \, \bar{a_x} - 0.99 \, \bar{a_y} - 1.2 \, \bar{a_z}$

At $(5, \frac{\pi}{2}, -2)$.

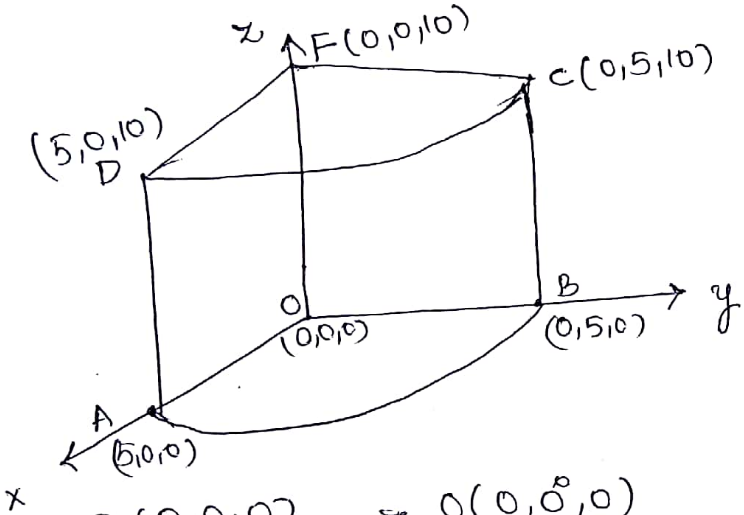
(ρ, ϕ, z)

6) Consider object shown in fig, Calculate

i) distance BC ii) distance CD iii) Surface area of ABCD

iv) Surface area of ABO v) Surface area of AOFD

vi) Volume of ABCDFO



$\rho = \sqrt{x^2+y^2}$

$\phi = \tan^{-1}\frac{y}{x}$

$z = z$.

$O(0,0,0) \rightarrow O(0,0^\circ,0)$

$A(5,0,0) \rightarrow A(5,0^\circ,0)$

$B(0,5,0) \rightarrow B(5,90^\circ,0)$

$C(0,5,10) \rightarrow C(5,90^\circ,10)$

$D(5,0,10) \rightarrow D(5,0^\circ,10)$

$F(0,0,10) \rightarrow F(0,0^\circ,10)$

i) distance BC

$$BC = \int_{z=0}^{10} dl = [z]_0^{10} = 10$$

ii) distance CD

$$CD = \int dl = \int_{\phi=0}^{90^\circ} r\,d\phi = 5[\phi]_0^{\pi/2} = 2.5\pi$$

iii) surface area of ABCD (r const)

$$= \iint dS_r = \iint_{\phi=0, z=0}^{\pi/2, 10} r\,d\phi\,dz$$

$$= 5[\phi]_0^{\pi/2}[z]_0^{10}$$

$$= 5 \cdot \frac{\pi}{2} \cdot 10 = 25\pi$$

iv) Surface area of ABO (z-const).

$$= \iint ds_z = \int_{r=0}^{5}\int_{\phi=0}^{\pi/2} r\, dr\, d\phi$$

$$= \left[\frac{r^2}{2}\right]_0^5 \left[\phi\right]_0^{\pi/2} = \frac{25}{2} \cdot \pi/2 = \frac{25\pi}{4}.$$

v) Surface area of AOFD (ϕ-const)

$$= \iint ds_\phi = \int_{r=0}^{5}\int_{z=0}^{10} dr\, dz$$

$$= \left[r\right]_0^5 \left[z\right]_0^{10}$$

$$= 5.(10) = 50$$

vi) Volume of ABCDFO

$$V = \iiint dV$$

$$= \int_{r=0}^{5}\int_{\phi=0}^{\pi/2}\int_{z=0}^{10} dr\, r d\phi\, dz$$

$$= \left[\frac{r^2}{2}\right]_0^5 \left[\phi\right]_0^{\pi/2} \left[z\right]_0^{10}$$

$$= \frac{25}{2} \cdot \frac{\pi}{\cancel{2}} \cdot \cancel{10}\,5 = 62.5\pi$$

Electrostatic force

Electrostatic is related to the electric charges which are static i.e. charges are at rest.

Coulomb's law :

The coulomb's law states that force between the two point charges Q_1 and Q_2

$$F \propto Q_1 Q_2$$
$$\propto \frac{1}{R^2}$$

$$\therefore F \propto \frac{Q_1 Q_2}{R^2}$$

$$\boxed{F = K \frac{Q_1 Q_2}{R^2}}$$

$$K = \frac{1}{4\pi\varepsilon}$$

$\varepsilon \rightarrow$ permittivity of medium

$$\boxed{\varepsilon = \varepsilon_0 \varepsilon_r}$$

ε_r = relative permittivity

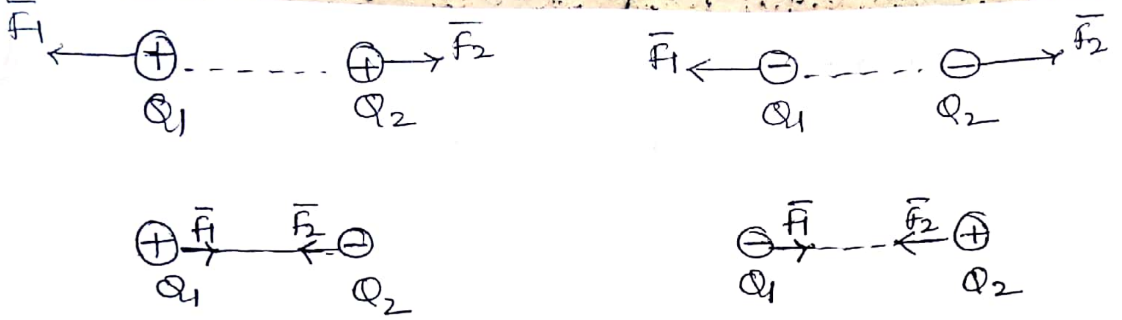
ε_0 = permittivity in free space

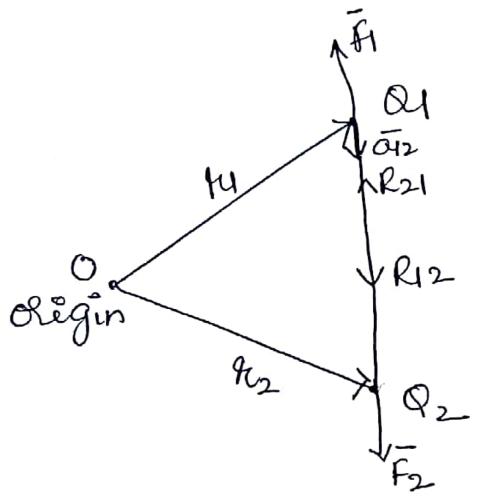
$$\boxed{\begin{aligned} \varepsilon_0 &= \frac{1}{36\pi} \times 10^{-9} \\ &= 8.854 \times 10^{-12} \, F/m \end{aligned}}$$

$$\therefore F = \frac{Q_1 Q_2}{4\pi\varepsilon R^2}$$

Vector form of Coulomb's law

$$\boxed{\bar{F} = \frac{Q_1 Q_2}{4\pi\varepsilon R^2} \bar{a}_R}$$





Force acting on Q_2 due to Q_1

i.e. $\boxed{\bar{F}_2 = \dfrac{Q_1 Q_2}{4\pi\varepsilon R_{12}^2}\, \bar{a}_{R12}}$

$$\bar{a}_{R12} = \frac{\bar{R}_{12}}{|\bar{R}_{12}|} = \frac{\bar{r}_2 - \bar{r}_1}{|\bar{r}_2 - \bar{r}_1|}$$

$$\therefore \boxed{\bar{F}_2 = \frac{Q_1 Q_2}{4\pi\varepsilon R_{12}^2} \frac{\bar{r}_2 - \bar{r}_1}{|\bar{r}_2 - \bar{r}_1|}}$$

Force acting on Q_1 due to Q_2

i.e. $\bar{F}_1 = \dfrac{Q_1 Q_2}{4\pi\varepsilon R_{21}^2}\, \bar{a}_{R21}$

$$\bar{a}_{R21} = \frac{\bar{R}_{21}}{|\bar{R}_{21}|} = \frac{-\bar{r}_2 + \bar{r}_1}{|\bar{r}_1 - \bar{r}_2|}$$

$$\therefore \bar{F}_1 = \frac{Q_1 Q_2}{4\pi\varepsilon R_{21}^2} \frac{\bar{r}_1 - \bar{r}_2}{|\bar{r}_1 - \bar{r}_2|}$$

$$= -\frac{Q_1 Q_2}{4\pi\varepsilon R_{21}^2} \frac{\bar{r}_2 - \bar{r}_1}{|\bar{r}_1 - \bar{r}_2|} = -\bar{F}_2$$

Hence force exerted by the two charges on each other is equal and opp...

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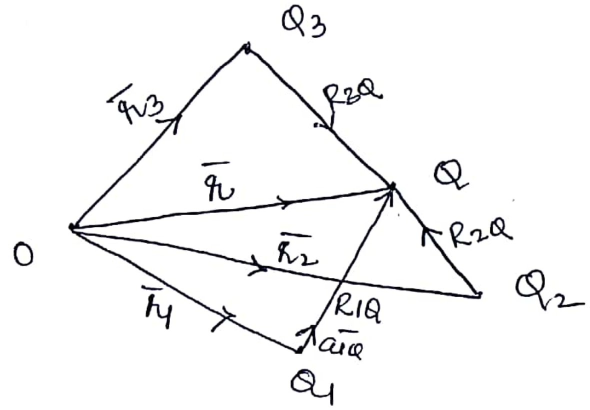
Super Position Principle :

If there are more than two point charges, then each will exerts force on the other.

The net force acting on charge Q ~~due~~ is equal to the force exerted by the individual charges.



$$\bar{F}_{1Q} = \frac{Q_1 Q}{4\pi\varepsilon R_{1Q}^2}\,\bar{a}_{1Q} = \frac{Q_1 Q}{4\pi\varepsilon R_{1Q}^2}\,\frac{\bar{r}-\bar{r}_1}{|\bar{r}-\bar{r}_1|} \quad \text{—— (1)}$$

$$\bar{F}_{2Q} = \frac{Q_2 Q}{4\pi\varepsilon R_{2Q}^2}\,\bar{a}_{2Q} = \frac{Q_2 Q}{4\pi\varepsilon R_{2Q}^2}\,\frac{\bar{r}-\bar{r}_2}{|\bar{r}-\bar{r}_2|} \quad \text{—— (2)}$$

$$\bar{F}_{3Q} = \frac{Q_3 Q}{4\pi\varepsilon R_{3Q}^2}\,\bar{a}_{3Q} = \frac{Q_3 Q}{4\pi\varepsilon R_{3Q}^2}\,\frac{\bar{r}-\bar{r}_3}{|\bar{r}-\bar{r}_3|} \quad \text{—— (3)}$$

$\therefore$ Net force acting on Q

$$\bar{F} = \bar{F}_{1Q} + \bar{F}_{2Q} + \bar{F}_{3Q} \cdots$$

$$= \frac{Q}{4\pi\varepsilon}\left[\frac{Q_1}{R_{1Q}^2}\,\frac{\bar{r}-\bar{r}_1}{|\bar{r}-\bar{r}_1|} + \frac{Q_2}{R_{2Q}^2}\,\frac{\bar{r}-\bar{r}_2}{|\bar{r}-\bar{r}_2|} + \frac{Q_3}{R_{3Q}^2}\,\frac{\bar{r}}{|\bar{r}}\right.$$

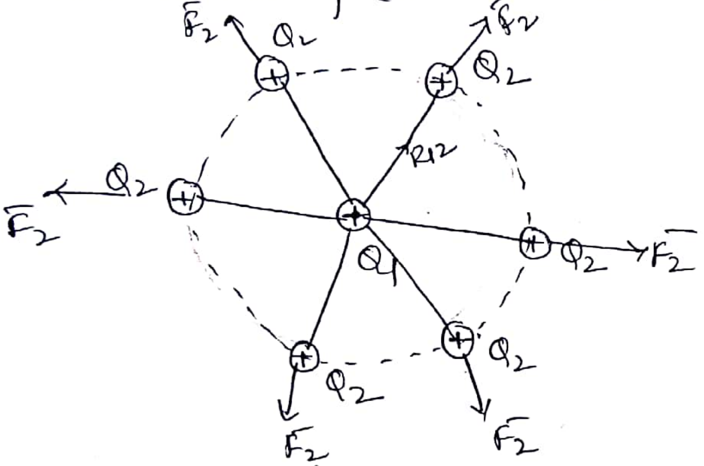
$$\boxed{\bar{F} = \frac{Q}{4\pi\varepsilon}\sum_{i=1}^{n}\frac{Q_i}{R_{iQ}^2}\,\frac{\bar{r}-\bar{r}_i}{|\bar{r}-\bar{r}_i|}}$$

or

$$\boxed{\bar{F} = \frac{Q}{4\pi\varepsilon}\sum_{i=1}^{n}\frac{Q_i}{R_{iQ}^2}\,\bar{a}_{iQ}}$$

Electric field Intensity ($\bar{E}$)

Force exerted per unit charge is called electric field Intensity (or) Electric field strength.



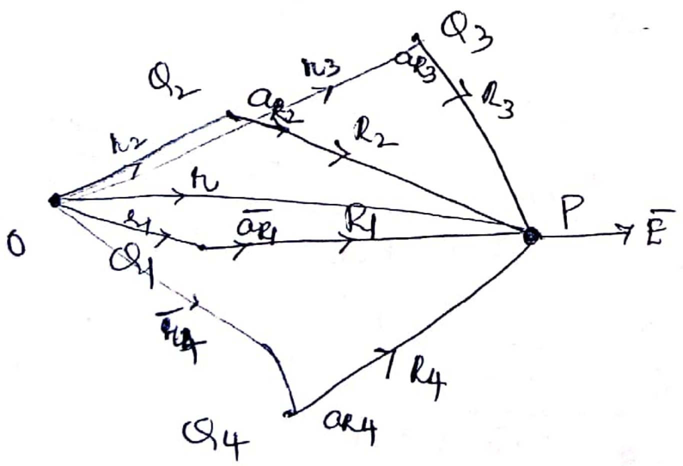
$$\bar{F}_2 = \frac{Q_1 Q_2}{4\pi\varepsilon R_{12}^2}\,\bar{a}_{12}$$

$$\frac{\bar{F}_2}{Q_2} = \frac{Q_1}{4\pi\varepsilon R_{12}^2}\,\bar{a}_{12}$$

$$\boxed{\bar{E} = \frac{Q_1}{4\pi\varepsilon R_{12}^2}\,\bar{a}_{12}}$$

Units: N/C (or) Volt/meter.

Super position principle ($\bar{E}$ due to discrete charges)



$$\bar{E} = \bar{E}_1 + \bar{E}_2 + \bar{E}_3 + \bar{E}_4 + \cdots + \bar{E}_n$$

$$= \frac{Q_1}{4\pi\varepsilon R_1^2}\bar{a}_{R_1} + \frac{Q_2}{4\pi\varepsilon R_2^2}\bar{a}_{R_2} + \frac{Q_3}{4\pi\varepsilon R_3^2}\bar{a}_{R_3} + \frac{Q_4}{4\pi\varepsilon R_4^2}$$

$$\boxed{\bar{E} = \frac{1}{4\pi\varepsilon}\sum_{i=1}^{n}\frac{Q_i}{R_i^2}\,\bar{a}_{R_i}}$$

or $$\boxed{\bar{E} = \frac{1}{4\pi\varepsilon}\sum_{i=1}^{n}\frac{Q_i}{R_i^2}\,\frac{\bar{r}-\bar{r}_i}{|\bar{r}-\bar{r}_i|}}$$

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Types of charge distribution:

(i) Point charge:

The point charge has a position but not the dimension. It gives the location of charge and the geometrical dimensions of charge is very small.

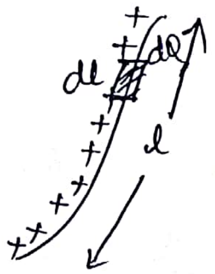
ii) Line charge distribution:

Total charge distributed along a line is line charge distribution.

$$\boxed{\rho_l = \frac{\text{total charge }(Q)}{\text{length of line}}} = C/m \text{ or } C/cm$$

Coulomb

ρ_l = line charge density



$$\rho_l = \frac{dQ}{dl}$$

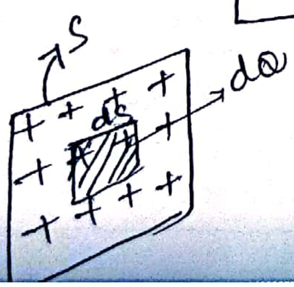
$$dQ = \rho_l\, dl$$

$$\boxed{Q = \int_L \rho_l\, dl}$$

iii) Surface charge distribution:

Distribution of charge along the surface is called surface charge distribution.

$$\boxed{\rho_s = \frac{\text{total charge }(Q)}{\text{Surface of Sheet}}} = C/m^2 \text{ or } C/cm^2$$



$$\rho_s = \frac{dQ}{ds}$$

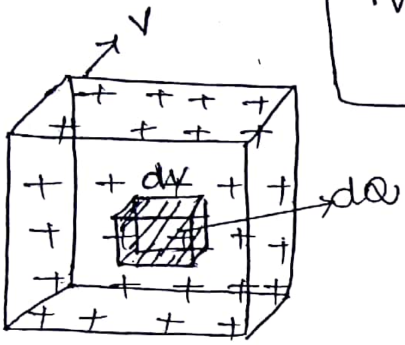
$$dQ = \rho_s\, ds$$

$$\boxed{Q = \int \rho_s\, ds}$$

IV) Volume charge distribution :

Total charge distribution in a volume is volume charge distribution.

$$\boxed{\rho_v = \frac{\text{total Charge }(Q)}{\text{total volume}}} = C/m^3 \text{ (or) } C/cm^3$$



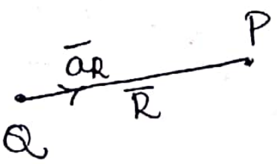
$$\rho_v = \frac{dQ}{dV}$$

$$dQ = \rho_v \, dV$$

$$\boxed{Q = \int_V \rho_v \, dV}$$

Electric Field Intensity (E) due to different charge distribution:

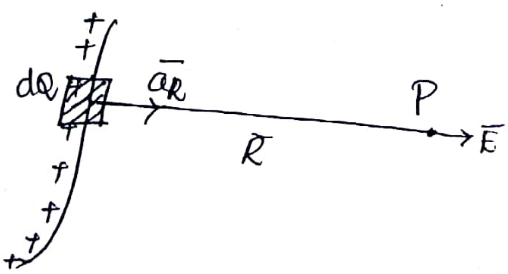
(i) $\bar{E}$ due to point Charge



$\bar{E}$ at point 'p' i.e.

$$\boxed{\bar{E} = \frac{Q}{4\pi\varepsilon R^2}\bar{a}_R}$$

ii) $\bar{E}$ due to line charge distribution:



$$dQ = \rho_l \, dL \quad \text{—(1)}$$

$\bar{E}$ at point 'p' due to dQ

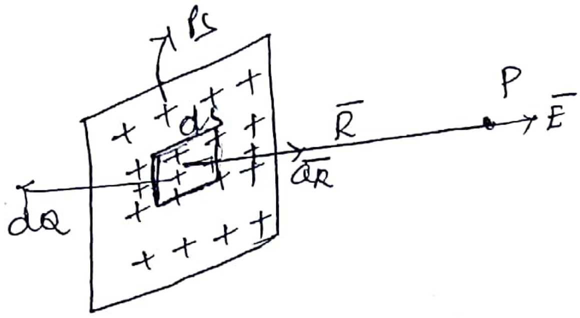
$$\therefore d\bar{E} = \frac{dQ}{4\pi\varepsilon R^2}\bar{a}_R$$

$$d\bar{E} = \frac{\rho_l \, dL}{4\pi\varepsilon R^2}\bar{a}_R$$

$$\therefore \boxed{\bar{E} = \int_L \frac{\rho_l \, dL}{4\pi\varepsilon R^2}\bar{a}_R}$$

iii) $\bar{E}$ due to Surface charge distribution :

$$dQ = \rho_s \, ds \quad \text{—(1)}$$



$\bar{E}$ at point 'p' due to dQ

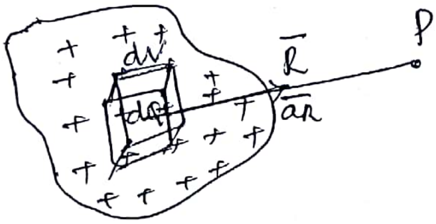
$$\bar{dE} = \frac{dQ}{4\pi\epsilon R^2}\,\bar{a}_R$$

$$\bar{dE} = \frac{\rho_s \, ds}{4\pi\epsilon R^2}\,\bar{a}_R$$

$$\boxed{\bar{E} = \int_S \frac{\rho_s \, ds}{4\pi\epsilon R^2}\,\bar{a}_R}$$

iv) $\bar{E}$ due to volume charge distribution :

$$dQ = \rho_v \, dv \quad \text{—(1)}$$



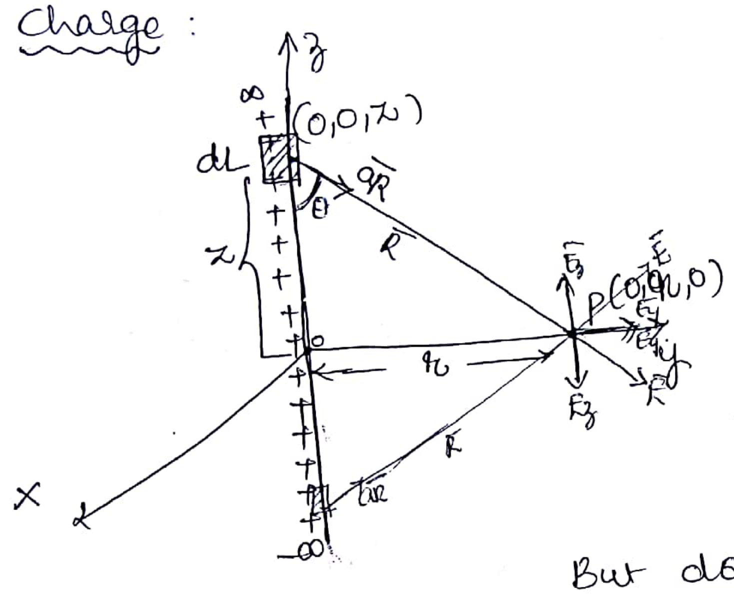
$\bar{E}$ at point 'p' due to dQ

$$\bar{dE} = \frac{dQ}{4\pi\epsilon R^2}\,\bar{a}_R$$

$$\bar{dE} = \frac{\rho_v \, dv}{4\pi\epsilon R^2}\,\bar{a}_R$$

$$\boxed{\bar{E} = \int_V \frac{\rho_v \, dv}{4\pi\epsilon R^2}\,\bar{a}_R}$$

Electric field Intensity ($\bar{E}$) due to infinite line Charge :



$\bar{E}$ at point 'p'

$$d\bar{E} = \frac{dQ}{4\pi\epsilon R^2}\bar{a}_R \quad -(1)$$

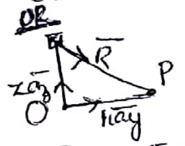
$$|\bar{R}| = \sqrt{(0)^2+(r)^2+(0-z)^2}$$

$$= \sqrt{r^2+z^2} \quad -(2)$$

$$\bar{a}_R = \frac{\bar{R}}{|\bar{R}|} = \frac{r\bar{a}_y - z\bar{a}_z}{\sqrt{r^2+z^2}} \quad -(3)$$

But $dQ = \rho_l\, dL$ — (4)

OR



$\bar{R} = -z\bar{a}_z + r\bar{a}_y$

Sub (2)(3)(4) in (1)

$$\therefore d\bar{E} = \frac{\rho_l\, dL}{4\pi\epsilon(r^2+z^2)}\frac{r\bar{a}_y - z\bar{a}_z}{(\sqrt{r^2+z^2})}$$

$$d\bar{E} = \frac{\rho_l\, dL}{4\pi\epsilon(r^2+z^2)^{3/2}}[r\bar{a}_y - \cancel{z\bar{a}_z}] \quad —— (5)$$

<u>NOTE</u>: For every charge on +ve z-axis, there is equal charge present on −ve z-axis. Hence the z-component of electric field Intensity at 'p' cancels each other. Hence there will not be any z-component of $\bar{E}$ at 'p'.

$$\therefore d\bar{E} = \frac{\rho_l\, dL}{4\pi\epsilon(r^2+z^2)^{3/2}} r\bar{a}_y$$

$$\bar{E} = \int \frac{\rho_l\, dl}{4\pi\epsilon(r^2+z^2)^{3/2}} r\bar{a}_y$$

$$= \int_{z=-\infty}^{\infty} \frac{\rho_l\, dz}{4\pi\epsilon(r^2+z^2)^{3/2}} r\bar{a}_y$$

let $\tan\theta = \frac{r}{z} \Rightarrow z = r\cot\theta$

$dz = -r\,\mathrm{cosec}^2\theta\, d\theta$

$z = -\infty \Rightarrow -\infty = \cot\theta \Rightarrow \theta = \pi$

$z = \infty \Rightarrow \infty = \cot\theta \Rightarrow \theta = 0°$

$$\bar{E} = \int_{\theta=\pi}^{\theta=0^\circ} \frac{-\rho_L\, r\, \bar{a_y}\, r\, \mathrm{cosec}^2\theta\, d\theta}{4\pi\varepsilon\,(r^2+r^2\cot^2\theta)^{3/2}}$$

$$= -\int_{\theta=\pi}^{0} \frac{\rho_L\, r^2\, \bar{a_y}\, \mathrm{cosec}^2\theta\, d\theta}{4\pi\varepsilon\,(r^2)^{3/2}(1+\cot^2\theta)^{3/2}}$$

$$= -\frac{\rho_L\,\bar{a_y}}{4\pi\varepsilon} \int_{\theta=\pi}^{0} \frac{r^2\,\mathrm{cosec}^2\theta\, d\theta}{r^3\,(\mathrm{cosec}^2\theta)^{3/2}}$$

$$= -\frac{\rho_L}{4\pi\varepsilon}\,\bar{a_y}\; \frac{1}{r}\int_{\pi}^{0} \sin\theta\, d\theta$$

$$= -\frac{\rho_L}{4\pi\varepsilon r}\,\bar{a_y}\,\cdot\left[-\cos\theta\right]_{\pi}^{0}$$

$$= -\frac{\rho_L}{4\pi\varepsilon r}\,\bar{a_y}\left[-(1+1)\right]$$

$$= \frac{\rho_L}{2\pi\varepsilon r}\,\bar{a_y}$$

$$\therefore \boxed{\bar{E} = \frac{\rho_L}{2\pi\varepsilon r}\,\bar{a_y}} \quad \text{volt/meter}$$

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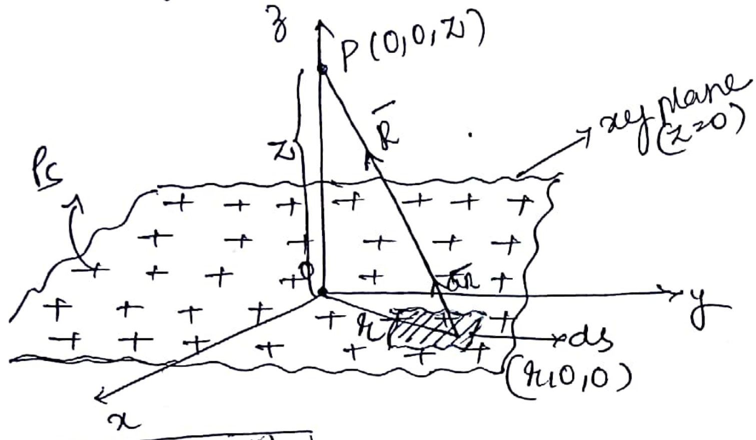
Electric field Intensity ($\bar{E}$) due to infinite sheet of charge :



$\bar{E}$ at point 'p'

$$d\bar{E} = \frac{dQ}{4\pi\epsilon R^2}\,\bar{a}_R \quad -(1)$$

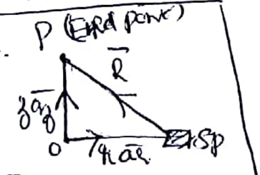
$$dQ = \rho_s\, ds = \rho_s\, dr\, r\, d\phi \quad -(2)$$

$$\bar{R} = -r\bar{a}_r + z\bar{a}_z$$

$$|\bar{R}| = \sqrt{r^2+z^2}$$

$$\bar{a}_R = \frac{-r\bar{a}_r + z\bar{a}_z}{\sqrt{r^2+z^2}}$$

OR

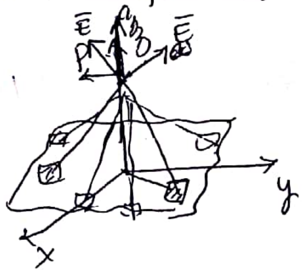


$\bar{R} = -r\bar{a}_r + z\bar{a}_z$

Sub in (1)

$$\therefore d\bar{E} = \frac{\rho_s r\, dr\, d\phi}{4\pi\epsilon(r^2+z^2)} \cdot \frac{\left[\cancel{-r\bar{a}_r} + z\bar{a}_z\right]}{\sqrt{r^2+z^2}} \quad -(3)$$

NOTE: As there is Symmetry about Z-axis from all radial Components, all $\bar{a}_r$ Component of $\bar{E}$ going to be cancel each other and the net $\bar{E}$ will not have any radial component. Therefore $\bar{a}_r$ component neglected.



$$\therefore d\bar{E} = \frac{\rho_s r\, dr\, d\phi\;\; z\bar{a}_z}{4\pi\epsilon(r^2+z^2)^{3/2}}$$

$$\bar{E} = \int \frac{\rho_s r\, dr\, d\phi\;\; z\bar{a}_z}{4\pi\epsilon(r^2+z^2)^{3/2}}$$

$$= \int_{r=0}^{\infty}\int_{\phi=0}^{2\pi} \frac{\rho_s\, r\, dr\, d\phi\;\; z\bar{a}_z}{4\pi\epsilon(r^2+z^2)^{3/2}}$$

$$= \frac{\rho_s\bar{a}_z}{4\pi\epsilon}\int_{r=0}^{\infty}\int_{\phi=0}^{2\pi} \frac{r\, dr\, d\phi\; z}{(r^2+z^2)^{3/2}}$$

Let $r^2+z^2=u^2$

$2r\,dr = 2u\,du$

$r=0 \Rightarrow z^2=u^2 \Rightarrow u=z$

$r=\infty \Rightarrow \infty=u^2 \Rightarrow u=\infty$

$$\bar{E} = \frac{\rho_s \bar{a}_z}{4\pi\varepsilon} \int_{u=z}^{\infty}\int_{\phi=0}^{2\pi} \frac{u\, du\, d\phi\, z}{(u^{\cancel{2}})^{3/\cancel{2}}}$$

$$= \frac{\rho_s \bar{a}_z z}{4\pi\varepsilon} \int_{u=z}^{\infty} \frac{1}{u^2}\, du \int_{\phi=0}^{2\pi} \cancel{z}\, d\phi$$

$$= \frac{\rho_s z \bar{a}_z}{4\pi\varepsilon} \left[-\frac{1}{u}\right]_z^{\infty} \left[\phi\right]_0^{2\pi}$$

$$= \frac{\rho_s z \bar{a}_z}{4\pi\varepsilon} \left[-\left(\frac{1}{\infty} - \frac{1}{z}\right)\right] \left[2\pi\right]$$

$$= \frac{\rho_s \cancel{z} \bar{a}_z}{\frac{4\pi}{2}\varepsilon} \left[\frac{1}{\cancel{z}}\right] \left[\cancel{2\pi}\right]$$

$$\boxed{\bar{E} = \frac{\rho_s}{2\varepsilon} \bar{a}_z}$$

For free space $\varepsilon = \varepsilon_0 \varepsilon_r$

$\varepsilon = \varepsilon_0 (1)$

$$\therefore \boxed{\bar{E} = \frac{\rho_s}{2\varepsilon_0} \bar{a}_z}$$

Electric Flux:

Michael Faraday performed the experiment on Electric field
Electric field should be assumed to be composed of very small bunches containing a fixed no. of electric lines of force.

The total no. of force ~~of~~ lines in any particular electric field is called as the electric flux.

Properties of flux lines:

Flux is nothing but the lines of force, around a charge.

i) The flux lines start from positive charge and terminate on -ve charge.

ii) If -ve charge is absent, flux lines terminates at ∞.

iii) If more no. of lines are there, then electric field is stronger.

iv) Electric flux lines are parallel and never cross each other.

v) These lines are independent of medium.

$$\boxed{\text{Electric Flux} = \psi = Q}$$

vi) flux is a scalar quantity

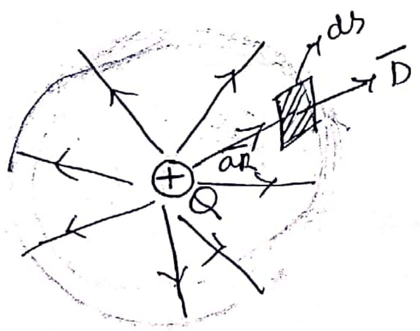
Electric Flux Density ($\bar{D}$)

The net flux passing through the unit surface area is called the electric flux density ($\bar{D}$).

$$\boxed{D = \frac{\psi}{S} = \frac{Q}{S}} = C/m^2$$

Vector notation $\boxed{\bar{D} = \frac{d\psi}{ds}\,\bar{a}_n}$

$\bar{D}$ due to point charge :



$$\bar{D} = \frac{\psi}{S}\,\bar{a}_R$$

$$= \frac{Q}{S}\,\bar{a}_R$$

Surface area $= S = 4\pi R^2$

$$\therefore \boxed{\bar{D} = \frac{Q}{4\pi R^2}\,\bar{a}_R}$$

Relationship between $\bar{D}$ & $\bar{E}$

$$\bar{E} = \frac{Q}{4\pi \varepsilon R^2}\,\bar{a}_R \quad \text{---(1)}$$

$$\bar{D} = \frac{Q}{4\pi R^2}\,\bar{a}_R \quad \text{---(2)}$$

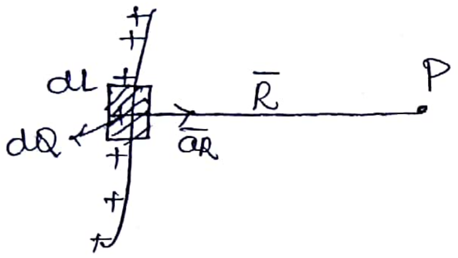
from (1) & (2)

$$\bar{E} = \frac{Q}{4\pi \varepsilon R^2}\,\bar{a}_R$$

$$= \frac{\bar{D}}{\varepsilon}$$

$$\therefore \boxed{\bar{D} = \varepsilon \bar{E}}$$

$\bar{D}$ due to Line charge distribution



$$dQ = \rho_l \, dl \quad \text{— (1)}$$

$$d\bar{E} = \frac{dQ}{4\pi\varepsilon R^2} \bar{a}_R$$

$$d\bar{E} = \frac{\rho_L \, dl}{4\pi\varepsilon R^2} \bar{a}_R$$

$$\bar{E} = \int_L \frac{\rho_l \, dl}{4\pi\varepsilon R^2} \bar{a}_R \quad \text{—(2)}$$

But $\bar{D} = \varepsilon \bar{E}$

$$= \cancel{\varepsilon} . \int_L \frac{\rho_L \, dl}{4\pi \cancel{\varepsilon} R^2} \bar{a}_R \quad \text{(from(2))}$$

$$\boxed{\bar{D} = \int_L \frac{\rho_l \, dl}{4\pi R^2} \bar{a}_R}$$

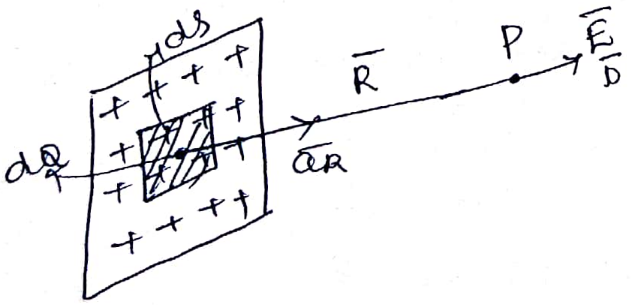
If the charge is distributed along an infinite line

$$\bar{D} = \varepsilon \bar{E}$$

$$= \cancel{\varepsilon} . \frac{\rho_L}{2\pi \cancel{\varepsilon} r} \bar{a}_r$$

$$\boxed{\bar{D} = \frac{\rho_L}{2\pi r} \bar{a}_r}$$

$\bar{D}$ due to Surface charge distribution



$$dQ = \rho_s \, ds \quad \text{—(1)}$$

$$d\bar{E} = \frac{dQ}{4\pi\varepsilon R^2} \bar{a}_R$$

$$\bar{E} = \int_S \frac{dQ}{4\pi\varepsilon R^2} \bar{a}_R$$

But $\bar{D} = \varepsilon \bar{E}$

$$= \cancel{\varepsilon} . \int_S \frac{dQ}{4\pi \cancel{\varepsilon} R^2} \bar{a}_R$$

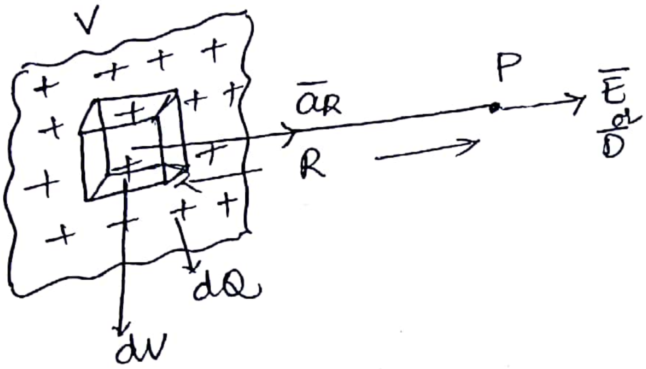
$$\boxed{\bar{D} = \int_S \frac{\rho_s \, ds}{4\pi R^2} \bar{a}_R}$$

If sh charge is distribution on infinite sheet

$$\bar{E} = \frac{\rho_s}{2\varepsilon} \bar{a}_R \qquad \boxed{\bar{D} = \varepsilon \bar{E} = \frac{\rho_s}{2} \bar{a}_R} \quad C/m^2$$

$\bar{D}$ due to volume charge distribution :

$$dQ = \rho_v dV \quad \text{---(1)}$$



$$\bar{dE} = \frac{dQ}{4\pi\varepsilon R^2}\bar{a}_R$$

$$\bar{D} = \frac{Q}{4\pi R^2}\bar{a}_R$$

$$\boxed{\bar{D} = \frac{\int_V \rho_v dV}{4\pi R^2}\bar{a}_R}$$

$$\bar{D} = \varepsilon\bar{E}$$

$$\therefore \boxed{\bar{E} = \frac{\int_V \rho_v dV}{4\pi\varepsilon R^2}\bar{a}_R}$$

Problems :

(1) Charge $Q_1 = 1mC$ is located at $P(0,0,0)$ and charge $Q_2 = 2\mu C$ is located at $M(0,4,3)$. Find the force on Q_2 due to Q_1.

Sol

$Q_1 = 1mC \quad P(0,0,0)$

$Q_2 = 2\mu C \quad M(0,4,3)$

$$R = \sqrt{(0-0)^2 + (4-0)^2 + (3-0)^2} = 5$$

$$\bar{F} = \frac{Q_1 Q_2}{4\pi\varepsilon R^2}\bar{a}_R = \frac{Q_1 Q_2}{4\pi\varepsilon R^2}\frac{\bar{r}_2 - \bar{r}_1}{|\bar{r}_2 - \bar{r}_1|}$$

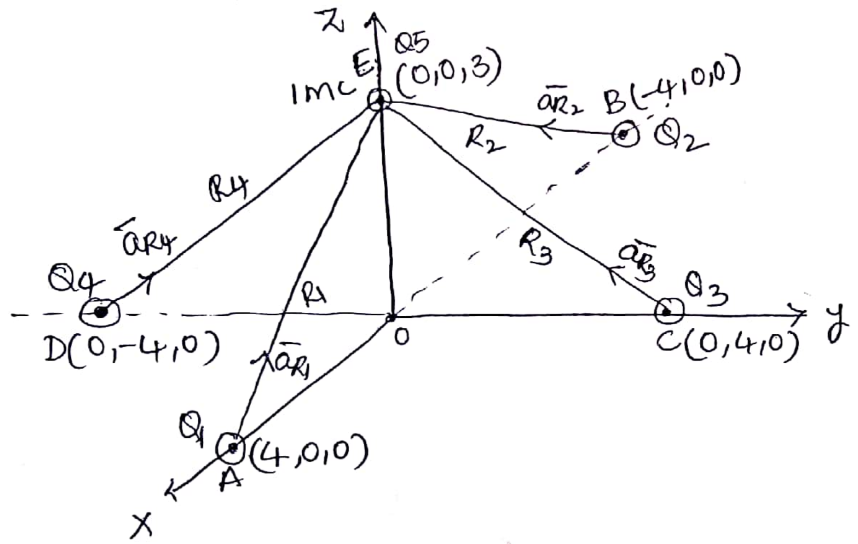
$$= \frac{1\times10^{-3}\times2\times10^{-6}}{4\pi\times8.854\times10^{-12}\times(5)^2}\cdot\frac{[(4-0)\bar{a}_y + (3-0)\bar{a}_z]}{\sqrt{4^2+3^2}}$$

$$= \frac{2\times10^{-9}}{4\pi\times8.854\times10^{-12}\times25}\frac{[4\bar{a}_y + 3\bar{a}_z]}{5}$$

$$= 0.56\,\bar{a}_y + 0.43\,\bar{a}_z$$

2) Four point charges of 1μc is located on z=0 plane, x and y on ±4mts. Find the force acting 1mc charge located on z-axis at 3 meters.

Sol.



Net Force on point E

$$\bar{F} = \bar{F}_1 + \bar{F}_2 + \bar{F}_3 + \bar{F}_4$$

$$= \frac{Q_1 Q_5}{4\pi\varepsilon R_1^2}\bar{a}_{R_1} + \frac{Q_2 Q_5}{4\pi\varepsilon R_2^2}\bar{a}_{R_2} + \frac{Q_3 Q_5 \bar{a}_{R_3}}{4\pi\varepsilon R_3^2} + \frac{Q_4 Q_5 \bar{a}_{R_4}}{4\pi\varepsilon R_4^2}$$

$$= \frac{Q_5}{4\pi\varepsilon}\left[\frac{Q_1}{R_1^2}\bar{a}_{R_1} + \frac{Q_2}{R_2^2}\bar{a}_{R_2} + \frac{Q_3}{R_3^2}\bar{a}_{R_3} + \frac{Q_4}{R_4^2}\bar{a}_{R_4}\right]$$

$$= \frac{1\times10^{-3}}{4\pi\times8.854\times10^{-12}}\left[\frac{1\times10^{-6}}{(5)^2}\frac{(-4\bar{a}_x+3\bar{a}_z)}{5} + \frac{1\times10^{-6}}{(5)^2}\frac{(4\bar{a}_y+3\bar{a}_z)}{5} + \frac{1\times10^{-6}}{(5)^2}\frac{(-4\bar{a}_y+3\bar{a}_z)}{5} + \frac{1\times10^{-6}}{(5)^2}\frac{(4\bar{a}_x+3\bar{a}_z)}{5}\right]$$

$$= \frac{10^{-9}}{4\pi(8.854\times10^{-12})(5)^3}\left[\cancel{-4\bar{a}_x} + 3\bar{a}_z + \cancel{4\bar{a}_y} + 3\bar{a}_z \cancel{- 4\bar{a}_y} + 3\bar{a}_z + \cancel{4\bar{a}_x} + 3\bar{a}_z\right]$$

$$\bar{F} = \cancel{0.086}\,\bar{a}_z \quad 0.86\,\bar{a}_z$$

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3) Charge is distributed on x-axis in Cartesian system having a line charge density of $3x^2 \mu C/m$. Find the total charge over the length of 10mts.

sol

$\rho_l = 3x^2 \mu C/m$

$Q = ?$

$l = 10mts.$

$$\rho_l = \frac{dQ}{dl}$$

$dQ = \rho_l \, dl$

$Q = \int_L \rho_l \, dl$

$= \int_0^{10} 3x^2 \times 10^{-6} \, dx = 3\left[\frac{x^3}{3}\right]_0^{10} \times 10^{-6}$

$= \frac{\cancel{3} \times 10^3}{\cancel{3}} \times 10^{-6}$

$Q = 10^{-3} = \underline{\underline{1 mC}}.$

4). Find the total charge inside a volume having volume charge density as $10z^2 e^{-0.1x} \sin \pi y \; C/m^3$. The volume is defined between $-2 \le x \le 2$, $0 \le y \le 1$, $3 \le z \le 4$.

sol

$\rho_v = \frac{dQ}{dv} \Rightarrow dQ = \rho_v \, dv$

$$\boxed{Q = \int_V \rho_v \, dv}$$

$$Q = \int_{-2}^{2}\int_{0}^{1}\int_{3}^{4} 10z^2 e^{-0.1x} \sin \pi y \; dx\, dy\, dz$$

$$= 10 \int_{-2}^{2} e^{-0.1x} dx \int_0^1 \sin \pi y \, dy \int_3^4 z^2 dz$$

$$= 10\left[\frac{e^{-0.1x}}{-0.1}\right]_{-2}^{2}\left[\frac{-\cos \pi y}{\pi}\right]_0^1 \left[\frac{z^3}{3}\right]_3^4$$

$$= 10\left[\frac{e^{-0.2} - e^{0.2}}{\ldots}\right]\left[-\left(\frac{\cos\pi - 1}{\pi}\right)\right]\left[\frac{4^3 - 3^3}{3}\right]$$

5) Three point charges are located at each corner of equilateral triangle. If the charges are 3Q, −2Q & Q. Find electric field Intensity (E) at mid point of 3Q and 1Q side.

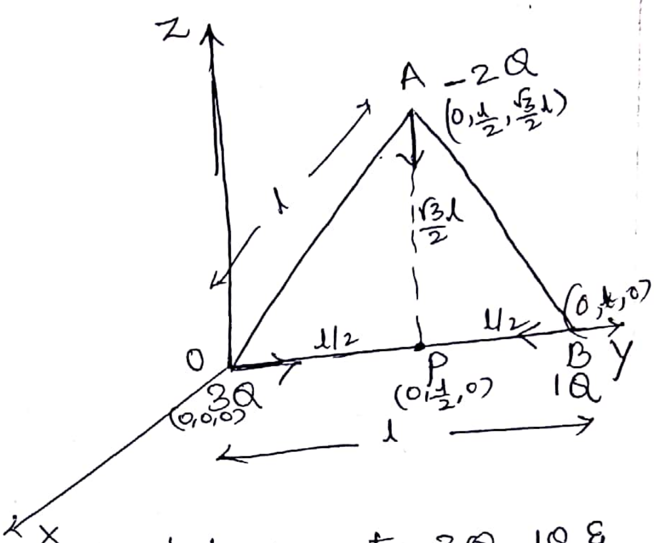
Sol:

O(0,0,0)

$A\left(0, \frac{l}{2}, \frac{\sqrt{3}}{2}l\right)$

$B(0, l, 0)$

$P\left(0, \frac{l}{2}, 0\right)$



Electric field Intensity at point 'p' due to 3Q, 1Q & −2Q.

$$\text{Net } \bar{E} \text{ at 'p'} = \bar{E}_{OP} + \bar{E}_{BP} + \bar{E}_{AP} \quad \text{— I}$$

$$\bar{E}_{OP} = \frac{Q}{4\pi\varepsilon R^2}\bar{a}_R$$

$$= \frac{3Q}{4\pi\varepsilon\left(\frac{l}{2}\right)^2}\frac{\left(\frac{l}{2}\bar{a}_y\right)}{\sqrt{\left(\frac{l}{2}\right)^2}} = \frac{3Q}{4\pi\varepsilon\frac{l^2}{4}}\bar{a}_y \quad \text{— (1)}$$

$$\bar{E}_{BP} = \frac{1Q}{4\pi\varepsilon\left(\frac{l}{2}\right)^2}\frac{-\frac{l}{2}\bar{a}_y}{\sqrt{\left(\frac{l}{2}\right)^2}} = \frac{-1Q}{4\pi\varepsilon\frac{l^2}{4}}\bar{a}_y \quad \text{— (2)}$$

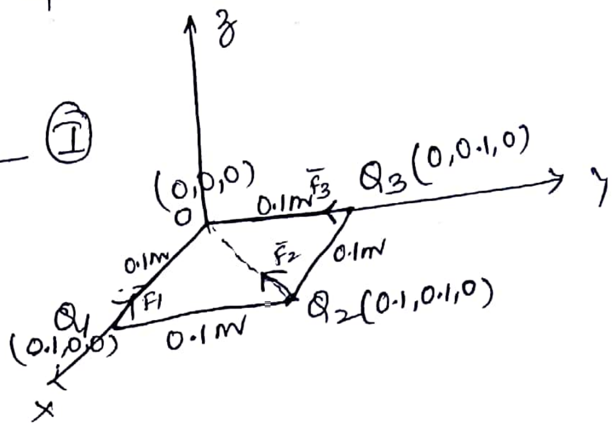
$$\bar{E}_{AP} = \frac{-2Q}{4\pi\varepsilon\left(\frac{\sqrt{3}}{2}l\right)^2}\frac{\left(-\frac{\sqrt{3}}{2}l\,\bar{a}_z\right)}{\left(\sqrt{\left(\frac{\sqrt{3}l}{2}\right)^2}\right)} = +\frac{2Q}{4\pi\varepsilon\frac{3}{4}l^2}\bar{a}_z \quad \text{— (3)}$$

Sub (1)(2)(3) in (I)

$$\therefore \bar{E} = \frac{3Q}{4\pi\varepsilon\frac{l^2}{4}}\bar{a}_y + \frac{-1Q}{4\pi\varepsilon\frac{l^2}{4}}\bar{a}_y + \frac{2Q}{4\pi\varepsilon\frac{3}{4}l^2}\bar{a}_z$$

$$\bar{E} = \frac{2Q}{\pi\varepsilon l^2}\bar{a}_y + \frac{2Q}{\pi\varepsilon 3l^2}\bar{a}_z \quad V/m.$$

6) Three equal charges of 1μC are placed at the corners of square of length 10cms. Find the magnitude and direction of $\bar{E}$ at vacant corners.



sol:

$\bar{E}$ at origin $= \bar{E}_1+\bar{E}_2+\bar{E}_3$ — (I)

$$\bar{E}_1 = \frac{Q}{4\pi\varepsilon R_1^2}\bar{a}_{R_1}$$

$$= \frac{1\times10^{-6}}{4\pi\varepsilon(0.1)^2}\frac{(-\cancel{0.1}\bar{a}_x)}{\sqrt{(\cancel{0.1})^2}}$$

$$= \frac{1\times10^{-6}(-\bar{a}_x)}{4\pi\varepsilon(0.1)^2} \quad —(1)$$

$$\bar{E}_2 = \frac{1\times10^{-6}}{4\pi\varepsilon((0.1)^2+(0.1)^2)}\frac{(-0.1\bar{a}_x-0.1\bar{a}_y)}{\sqrt{(0.1)^2+(0.1)^2}} \quad \left(\because R_2=\sqrt{(0.1)^2+(0.1)^2}\right)$$

$$= \frac{10^{-6}(-0.1\bar{a}_x-0.1\bar{a}_y)}{4\pi\varepsilon((0.1)^2+(0.1)^2)^{3/2}} \quad —(2)$$

$$\bar{E}_3 = \frac{1\times10^{-6}}{4\pi\varepsilon(0.1)^2}\frac{(-\cancel{0.1}\bar{a}_y)}{\sqrt{(\cancel{0.1})^2}}$$

$$= \frac{1\times10^{-6}}{4\pi\varepsilon(0.1)^2}(-\bar{a}_y) \quad —(3)$$

Sub (1)(2)(3) in (I)

$$\bar{E} = \frac{1\times10^{-6}}{4\pi\varepsilon}\left[-\frac{\bar{a}_x}{(0.1)^2}+\frac{(-0.1\bar{a}_x-0.1\bar{a}_y)}{((0.1)^2+(0.1)^2)^{3/2}}-\frac{\bar{a}_y}{(0.1)^2}\right]$$

$$= \frac{1\times10^{-6}}{4\pi\times8.854\times10^{-12}}\left[-100\bar{a}_x-100\bar{a}_y-35.35\bar{a}_x-35.35\bar{a}_y\right]$$

$$= -1216.49\bar{a}_x-1216.49\bar{a}_y \text{ KV/m}$$

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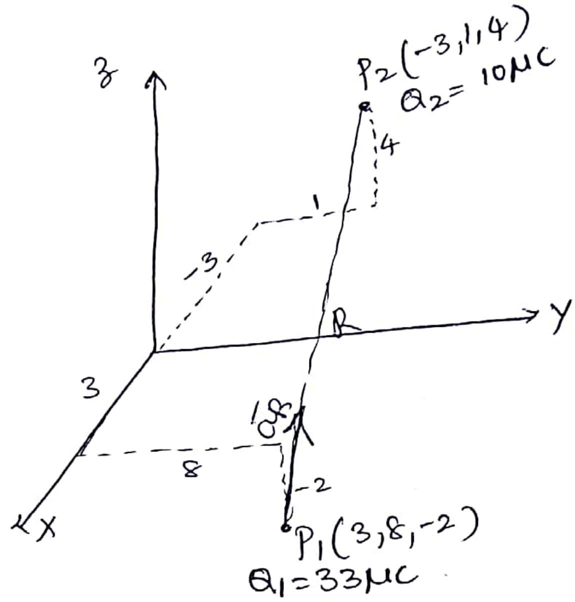
7) A Charge $Q_2 = 10\mu C$ is located in an air at point of $P_2(-3,1,4)$ m. Find the force on Q_2 due to $Q_1 = 33\mu C$ located at $P_1(3,8,-2)$ m.

Sol



$$R = \sqrt{(-3-3)^2 + (1-8)^2 + (4+2)^2}$$

$$= \sqrt{6^2 + 7^2 + 6^2}$$

$$= 11$$

$$\bar{F} = \frac{Q_1 Q_2}{4\pi \epsilon R^2} \bar{a}_R$$

$$= \frac{33\times 10^{-6} \times 10 \times 10^{-6}}{4\pi \times 8.854 \times 10^{-12} \times (11)^2} \left[\frac{-6\bar{a}_x - 7\bar{a}_y + 6\bar{a}_z}{\sqrt{6^2+7^2+6^2}}\right]$$

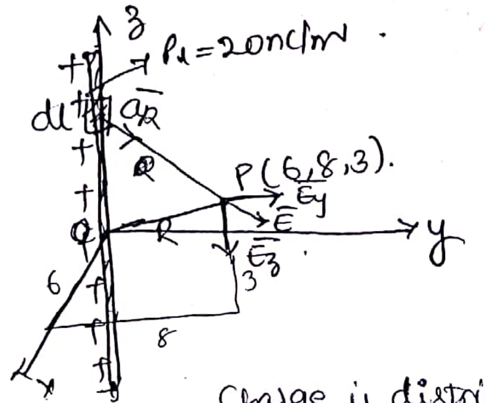
$$= -0.0133\,\bar{a}_x + 0.0154\,\bar{a}_y + 0.0133\,\bar{a}_z$$

8) A uniform line charge of infinite length with $\rho_l = 20$ nc/m lies along z-axis. Find $\bar{E}$ at (6, 8, 3) m.

$$R = \sqrt{(6-0)^2 + (8-0)^2 + (3-0)^2}$$

$$= 10$$

Sol



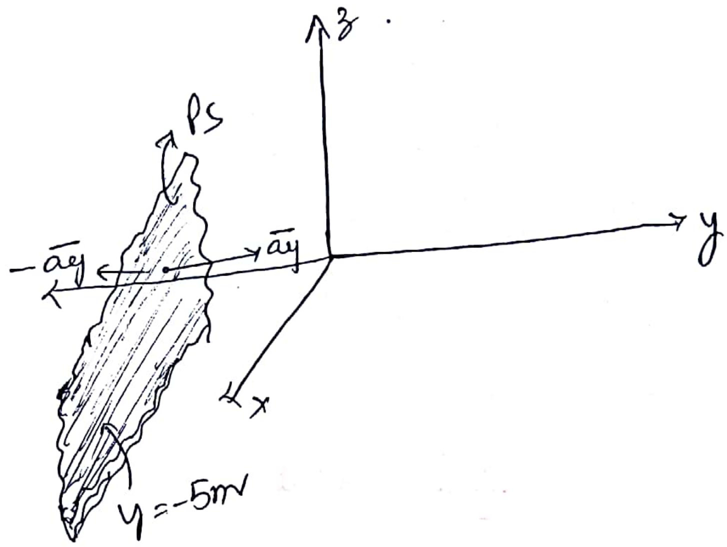
$$\bar{E} = \frac{\rho_l}{2\pi \epsilon R} \bar{a}_R$$

$$= \frac{20\times 10^{-9}}{2\pi \times 8.854 \times 10^{-12} \times (10)} \left[6\bar{a}_x + 8\bar{a}_y\right]$$

$$= 10.79\,\bar{a}_x + 14.5\,\bar{a}_y$$

(NOTE: As Charge is distributed along z-axis ∴ no field exist along z axis
$\bar{E}$ along only on x-axis & y axis.)

9) A charge lies in $y=-5m$ plane in the form of an infinite square sheet with a uniform charge density of $\rho_s = 20 nC/m^2$. Determine $\bar{E}$ at all points. (26)



sol

$$\bar{E} = \frac{\rho_s}{2\varepsilon}\bar{a}_r$$

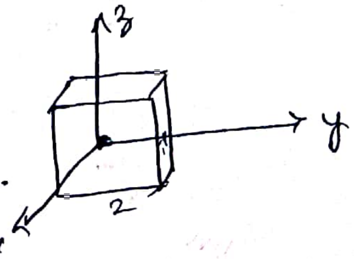
$$\bar{E}_1 = \frac{20\times10^{-9}}{2\times8.854\times10^{-12}}\bar{a}_y$$

$$\bar{E}_2 = \frac{20\times10^{-9}}{2\times8.854\times10^{-12}}(-\bar{a}_y)$$

$$\bar{E} = \bar{E}_1 + \bar{E}_2$$

∴ At any point to the left or right of the plane $|\bar{E}|$ is constant and acts normal to the plane.

10) Find the charge enclosed in a cube of having side of 2m with the edges of the cube parallel to axis x, y, z. While origin is in the centre of the cube the charge density within the cube is $50x^2\cos(\frac{\pi}{2}y)\ \mu C/m^3$



sol

$$Q = \int \rho_v \, dV$$

$$= \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} 50x^2\cos(\tfrac{\pi}{2}y)\,10^{-6}\,dx\,dy\,dz.$$

$$= 10^{-6}\times50\left[\frac{x^3}{3}\right]_{-1}^{1}\left[\frac{\sin \pi/2\, y}{\pi/2}\right]_{-1}^{1}\left[z\right]_{-1}^{1}$$

$$= \frac{50\times10^{-6}}{3}[1-(-1)]\left[\frac{2}{\pi}\left(\sin\frac{\pi}{2} - \sin(-\pi/2)\right)\right][1-(-1)]$$

$$= \frac{50\times10^{-6}}{3}(2)\left(\frac{2}{\pi}(2)\right)(2) = \frac{50\times10^{-6}\times16}{3\pi} = 85\,\mu C$$

Gauss's Law

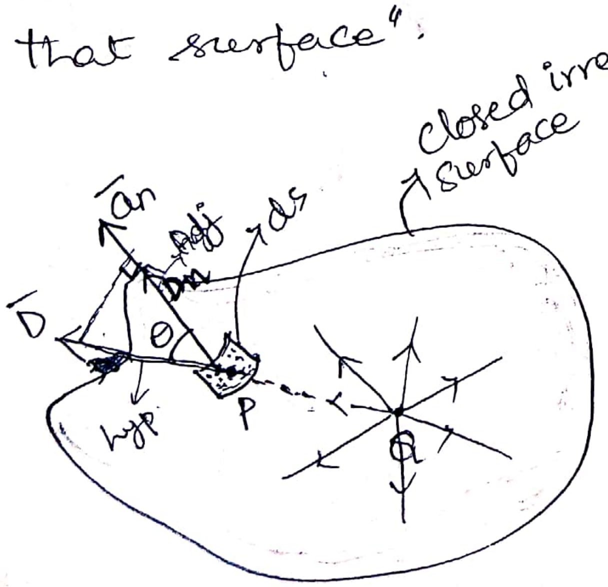
The charge Q radiates the flux φ which is equal to the charge Q. This is proved by Faraday's experiment.

Consider a sphere of radius 'r' and a point charge Q is located at its centre, then the total flux radiated outwards and passing through the total surface area of the sphere is same as the charge +Q, which is enclosed by the sphere.

Instead of point charge consider a line charge & surface charge. Similarly instead of sphere, any irregular closed surface is considered.

Statement of Gauss's law

Gauss's law states that "the net electric flux passing through any closed surface which is equal to the total charge enclosed by that surface".



The flux $d\varphi$ passing through the surface ds

$$D = \frac{d\varphi}{ds}$$

$$d\varphi = D_n \, ds \quad —(1)$$

Where D_n = Component of $\bar{D}$

$$\cos\theta = \frac{D_n}{D} \Rightarrow D_n = D\cos\theta \quad —(2)$$

Sub (2) in (1)

$$d\varphi = D\cos\theta \, ds \quad —(3)$$

from dot product

$$\bar{A}\cdot\bar{B} = |\bar{A}||\bar{B}|\cos\theta_{AB}$$

$$\therefore \bar{D}\cdot\bar{ds} = |\bar{D}||\bar{ds}|\cos\theta$$

$\therefore$ eqn (3) becomes

$$d\psi = \bar{D}\cdot\bar{ds}$$

$$\psi = \int_S \bar{D}\cdot\bar{ds}$$

This is the flux passing through the closed surface

$$\therefore \boxed{\psi = \oint_S \bar{D}\cdot\bar{ds} = Q}$$

Closed surface is also called as Gaussian surface

The charge enclosed may take any of the following forms

1. If there are number of point charges $Q_1, Q_2 \cdots Q_n$ enclosed by the surface then

$$Q = Q_1 + Q_2 + \cdots Q_n = \Sigma Q_n$$

$$\boxed{\psi = Q = \Sigma Q_n}$$

2. If the charge is distributed along the line [line charge distribution]

$$\boxed{\psi = Q = \int_L \rho_L dL}$$

from Gauss's law

$$\boxed{\psi = \oint_S \bar{D}\cdot\bar{ds} = Q = \oint_L \rho_L dL}$$

3. If the charge is distributed along the surface [surface charge distribution]

$$\boxed{\psi = Q = \int_S \rho_s ds}$$

From Gauss's law

$$\boxed{\psi = \oint_S \bar{D}\cdot\bar{ds} = Q = \int_S \rho_s ds}$$

4. If the charge is distributed ~~along the~~ inside volume [volume charge distribution]

$$\boxed{\psi = Q = \int_V \rho_v dV}$$

from Gauss's law

$$\boxed{\psi = \oint_S \bar{D}\cdot\bar{ds} = Q = \int_V \rho_v dV}$$

Applications of Gauss's law.

(i) It is used to find $\bar{E}$ and $\bar{D}$ for symmetrical charge distribution like point charge, infinite line charge, infinite sheet charge and spherical charge distribution.

ii) Gauss's law cannot be used to find $\bar{E}$ or $\bar{D}$ if charge distribution is not symmetric.

Gaussian Surface:

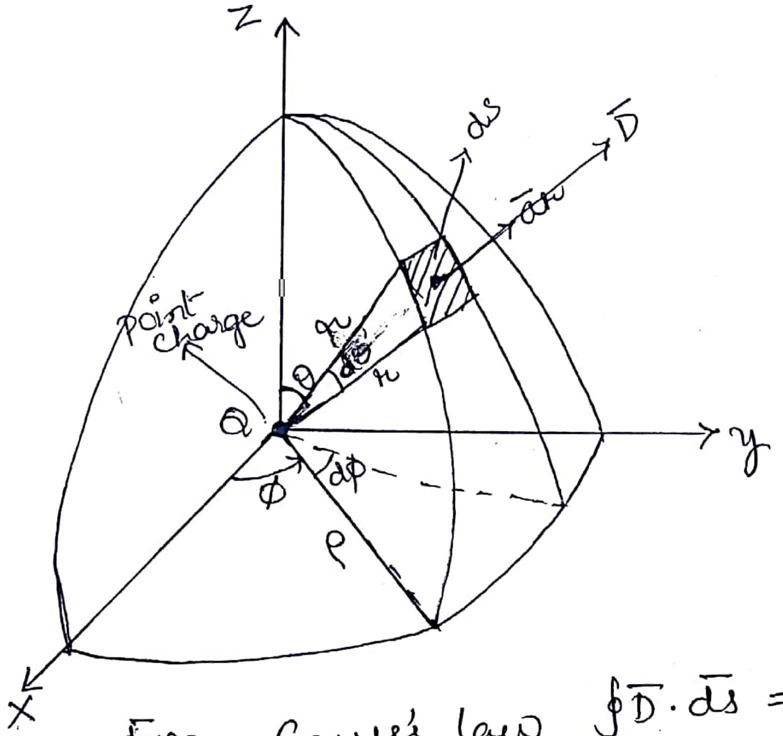
The surface over which the Gauss's law is applied is called Gaussian surface.

Conditions:

1. The surface may be irregular but should be sufficiently large so as to enclose the entire charge.
2. The surface must be closed.
3. At each point of the surface $\bar{D}$ is either normal or tangential to the surface.
4. The $\bar{D}$ is constant over the surface at which $\bar{D}$ is normal.

Proof of Gauss's law / Gauss's law in Integral form / Gauss's law applied to point charge

(28)



Spherical coord.

$$\left.\begin{array}{l} dr \\ r d\theta \\ r\sin\theta\, d\phi \end{array}\right\}$$

From Gauss's law $\oint_S \bar{D}\cdot\bar{ds} = Q$ —— (1)

$\bar{D}$ due to point charge $= \bar{D} = \dfrac{Q}{4\pi r^2}\,\bar{a_r}$ —— (2)

Surface element $= \bar{ds} = r^2 \sin\theta\, d\theta\, d\phi\, \bar{a_r}$ —(3)
(r=const)

From eqn (1)

$$L.H.S = \oint_S \bar{D}\cdot\bar{ds} = \oint_S \frac{Q}{4\pi \cancel{r^2}}\,\bar{a_r}\cdot \cancel{r^2}\sin\theta\, d\theta\, dr\, \bar{a_r}$$

$$= \oint \frac{Q}{4\pi}\sin\theta\, d\theta\, d\phi$$

$$= \int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} \frac{Q}{4\pi}\sin\theta\, d\theta\, d\phi = \frac{Q}{4\pi}\left[-\cos\theta\right]_0^{\pi}\left[\phi\right]_0^{2\pi}$$

$$= \frac{Q}{\cancel{4\pi}}\left[\cancel{2}\right]\left[\cancel{2\pi}\right]$$

$$= Q = R.H.S.$$

Proved.

Q: Gauss's law used to obtain $\bar{D}$ & $\bar{E}$.

From Gauss's law.

$\oint_S \bar{D}\cdot\bar{ds} = Q$ —— (1)

$\bar{D} = D_r\, \bar{a_r}$ —— (2)

$\bar{ds} = r^2 \sin\theta\, d\theta\, d\phi\, \bar{a_r}$ —(3)

Sub (2)(3) in (1)

$$\oint D_r \bar{a_r}\cdot r^2 \sin\theta\, d\theta\, d\phi\, \bar{a_r} = \int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} D_r\, r^2 \sin\theta\, d\theta\, d\phi = Q$$

$$D_r r^2 [-\cos\theta]_0^{\pi} [\phi]_0^{2\pi} = Q$$

$$D_r r^2 [2][2\pi] = Q$$

$$D_r = \frac{Q}{4\pi r^2} \quad \text{—— (4)}$$

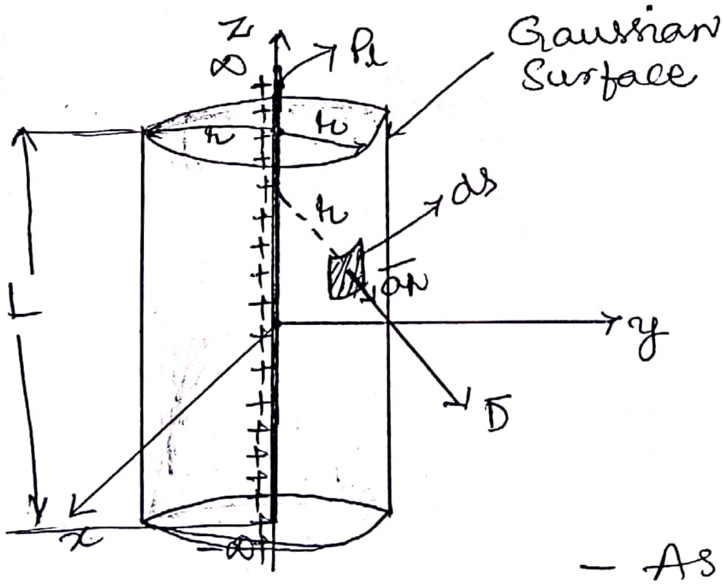
Sub (4) in (2)

$$\therefore \bar{D} = D_r \bar{a}_r$$

$$\boxed{\bar{D} = \frac{Q}{4\pi r^2} \bar{a}_r}$$

lly $$\boxed{\bar{E} = \frac{Q}{4\pi \varepsilon r^2} \bar{a}_r}$$

Electric Field Intensity ($\bar{E}$) due to infinite line charge using Gauss's law



- Consider ~~a line~~ an infinite line charge ρ_L C/m along the z-axis from $-\infty$ to $+\infty$.
- Consider Gaussian Surface is cylinder (closed).
- As the line charge along the z-axis, there cannot be any component of $\bar{D}$ in z-direction so, $\bar{D}$ has only radial component.

From Gauss's law

$$\oint \bar{D} \cdot \bar{ds} = Q$$

$$\underset{\text{top}}{\oint \bar{D} \cdot \bar{ds}} + \underset{\text{bottom}}{\oint \bar{D} \cdot \bar{ds}} + \underset{\text{lateral (side)}}{\oint \bar{D} \cdot \bar{ds}} = Q \quad \text{—— (1)}$$

$$\oint_{top} \bar{D}\cdot\bar{ds} = \oint_{bottom} \bar{D}\cdot\bar{ds} = 0 \quad \left[\because \text{no field component along z-axis}\right]$$

For lateral surface

$$\oint_{lateral} \bar{D}\cdot\bar{ds} = \oint_{lateral} D_r \bar{a_r} \cdot r\,d\phi\,dz = Q \quad \left\{\begin{array}{l}\because \bar{D} = D_r \bar{a_r} \\ \bar{ds} = r\,d\phi\,dz \\ (r = \text{const})\end{array}\right\}$$

$$\Rightarrow D_r \int_{\phi=0}^{2\pi}\int_{z=0}^{L} r\,d\phi\,dz = Q$$

$$\Rightarrow D_r\, r\, [\phi]_0^{2\pi} [z]_0^{L} = Q$$

$$\Rightarrow D_r\, r(2\pi)(L) = Q$$

$$D_r = \frac{Q}{2\pi r L}$$

$$D_r = \frac{\rho_L}{2\pi r} \qquad \left(\because \frac{Q}{L} = \rho_L \ C/m\right)$$

But $\bar{D} = D_r \bar{a_r}$

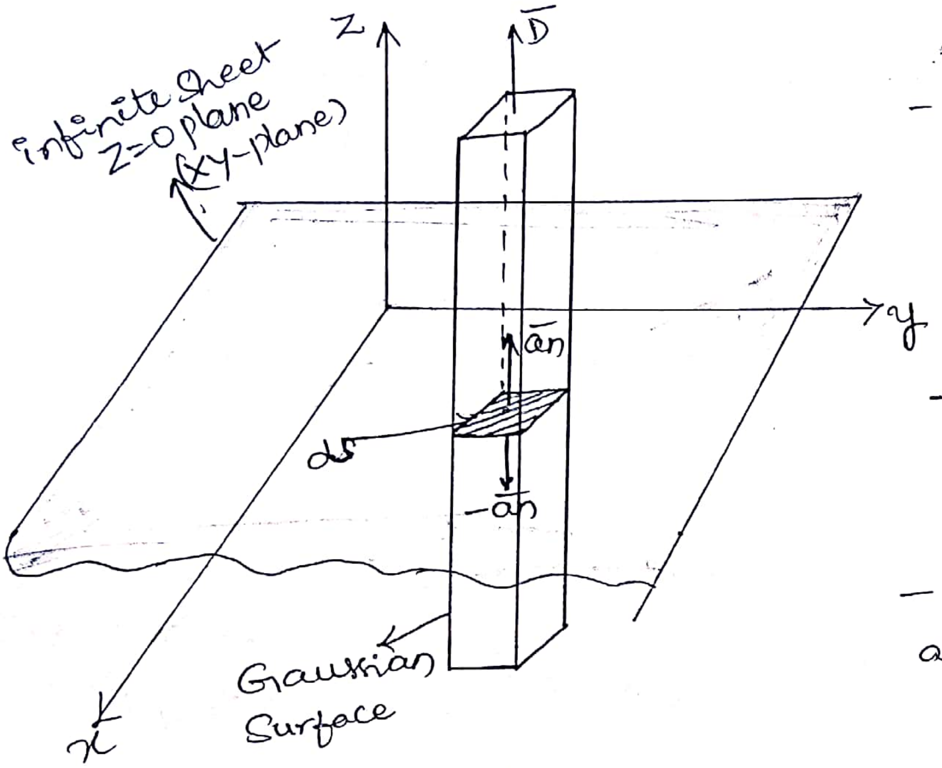
$$\bar{D} = \frac{\rho_L}{2\pi r}\bar{a_r} \quad C/m^2$$

$$\bar{D} = \varepsilon \bar{E}$$

$$\bar{E} = \frac{\bar{D}}{\varepsilon}$$

$$\boxed{\bar{E} = \frac{\rho_L}{2\pi \varepsilon r}\bar{a_r}} \qquad V/m \text{ (or) } N/C$$

Electric Field Intensity ($\bar{E}$) due to infinite sheet of Charge using Gauss's law:



- Consider infinite sheet of charge of uniform charge density ρ_s C/m²
- differential surface element i.e. $ds = dx\,dy$
- $\bar{D}$ has no component along x and y-axis.

From Gauss's law

$$\oint \bar{D}\cdot\bar{ds} = Q$$

$$\oint_{top} \bar{D}\cdot\bar{ds} + \oint_{bottom} \bar{D}\cdot\bar{ds} + \oint_{lateral} \bar{D}\cdot\bar{ds} = Q \quad \text{——— (1)}$$

$$\oint_{top} \bar{D}\cdot\bar{ds} = \oint_{top} D_z \bar{a}_z \cdot dx\,dy\,\bar{a}_z$$

$$= \oint_{top} D_z\, dx\,dy = D_z \oint dx\,dy \quad \text{— (2)}$$

$$\oint_{bottom} \bar{D}\cdot\bar{ds} = \oint_{bottom} D_z(-\bar{a}_z)\cdot dx\,dy(-\bar{a}_z)$$

$$= \oint_{bottom} D_z\, dx\,dy = D_z \oint_{bottom} dx\,dy \quad \text{——— (3)}$$

$$\oint_{lateral} \bar{D}\cdot\bar{ds} = 0 \quad \left(\text{no component in } x \text{ \& } y \text{ direction}\right) \quad \text{——— (4)}$$

Sub (2)(3)(4) in (1)

$$D_z \oint dx\,dy + D_z \oint dx\,dy + 0 = Q \qquad \left[\because \oint dx\,dy = A\right] \text{ Area of Surface}$$

$$D_z A + D_z A = Q$$

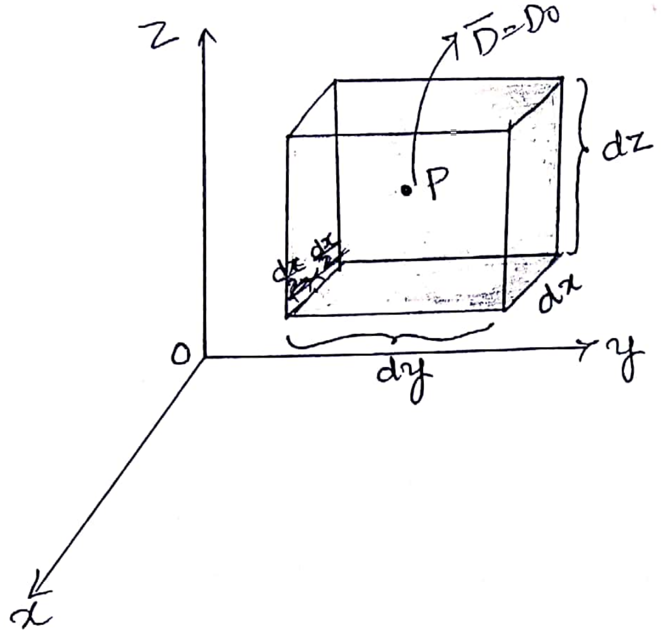
$$2 D_z A = Q \Rightarrow D_z = \frac{Q}{2A} \Rightarrow D_z = \frac{\rho_s}{2}$$

But $\bar{D} = D_z \bar{a}_z \Rightarrow \boxed{\bar{D} = \frac{\rho_s}{2}\bar{a}_z}$ C/m²

$E = \frac{\bar{D}}{\varepsilon}$

$\boxed{\bar{E} = \frac{\rho_s}{2\varepsilon}\bar{a}_z}$ N/m

Gauss's law applied to Differential Volume Element



- If there does not exist a symmetry and Gaussian surface can not be chosen such that the normal component of $\bar{D}$ is constant or zero everywhere on the surface, Gauss's law cannot be applied directly.

- In such a case a differential ~~surface~~ closed Gaussian surface is considered. The closed surface is so small that $\bar{D}$ is almost constant everywhere on the surface.
- Finally results can be obtained by decreasing the volume enclosed by Gaussian surface to approach to zero.

At point 'P'

$$\bar{D} = D_x \bar{a}_x + D_y \bar{a}_y + D_z \bar{a}_z \quad \text{——— (1)}$$

The point 'P' is at the centre of the element and treated to be origin.

Hence $\bar{D}$ at 'P' can be denoted as $\bar{D}_0$

$$\therefore \bar{D} = \bar{D}_0 = D_{x0} \bar{a}_x + D_{y0} \bar{a}_y + D_{z0} \bar{a}_z \quad \text{——— (2)}$$

From Gauss's law

$$\oint \bar{D} \cdot \bar{ds} = Q$$

$$\oint_{top} \bar{D} \cdot \bar{ds} + \oint_{bottom} \bar{D} \cdot \bar{ds} + \oint_{front} \bar{D} \cdot \bar{ds} + \oint_{back} \bar{D} \cdot \bar{ds} + \oint_{left} \bar{D} \cdot \bar{ds} + \oint_{right} \bar{D} \cdot \bar{ds} = Q \quad \text{——— (3)}$$

For front surface

$$\oint_{front} \bar{D}_f \cdot \bar{ds}$$

$$\bar{D}_{front} = D_{x,front}\, \bar{a}_x$$

$$\bar{ds} = dy\, dz\, \bar{a}_x$$

$$\therefore \oint_{front} \bar{D}_f \cdot \bar{ds} = \oint_{front} D_{x,front}\, \bar{a}_x \cdot dy dz\, \bar{a}_x$$

$$= \oint_{front} D_{x,front}\, dy dz \quad \text{——} (4)$$

$D_{x,front}$ is changing in x-direction. At P, it is D_{x0} while on the front surface it will change and given by

$$D_{x,front} = D_{x0} + \left[\begin{matrix}\text{Rate of change} \\ \text{of } D_x \text{ with } x\end{matrix}\right] \times \left[\begin{matrix}\text{distance of} \\ \text{Surface from} \\ \text{'P'}\end{matrix}\right]$$

$$= D_{x0} + \frac{\partial D_x}{\partial x} \cdot \frac{dx}{2} \quad \text{——} (5)$$

NOTE: The point 'P' is at centre, so distance of surface in x-direction from 'P' is $\frac{dx}{2}$

Sub (5) in (4)

$$\oint_{front} \bar{D}_f \cdot \bar{ds} = \oint_{front} \left[D_{x0} + \left(\frac{\partial D_x}{\partial x}\right)\left(\frac{dx}{2}\right)\right] dy\, dz \quad \text{——} (6)$$

For back surface

$$\oint_{back} \bar{D}_{back} \cdot \bar{ds}$$

$$\bar{D}_{back} = D_{x,back}\, (+\bar{a}_x)$$

$$\bar{ds} = dy\, dz\, (+\bar{a}_x)$$

[Assume flux entering from back side & leaving from front] i.e. -ve axis to +ve axis

$$\therefore \oint_{back} \bar{D}_{back} \cdot \bar{ds} = \oint_{back} -D_{x,back}\, \bar{a}_x \cdot dy dz\, \bar{a}_x$$

$$= -\oint D_{x,back}\, dy dz \quad \text{——} (7)$$

$$D_{x,back} = D_{x0} + \begin{bmatrix}\text{Rate of change} \\ \text{of } D_x \text{ with } x\end{bmatrix}\begin{bmatrix}\text{distance of} \\ \text{surface from} \\ P\end{bmatrix}$$

$$= D_{x0} - \frac{\partial D_x}{\partial x} \cdot \frac{dx}{2} \quad \text{——(8)}$$

Sub (8) in (7)

$$\oint_{back} \bar{D}_{back} \cdot \bar{ds} = -\oint_{back}\left[D_{x0} - \left(\frac{\partial D_x}{\partial x}\right)\left(\frac{dx}{2}\right)\right] dy\,dz$$

$$= \oint_{back}\left[-D_{x0} + \frac{\partial D_x}{\partial x} \cdot \frac{dx}{2}\right] dy\,dz \quad \text{——(9)}$$

Ily For top surface

$$\oint_{top} \bar{D} \cdot \bar{ds} = \oint_{top}\left[D_{z0} + \frac{\partial D_z}{\partial z} \frac{dz}{2}\right] dx\,dy \quad \text{——(10)}$$

$$\oint_{bottom} \bar{D} \cdot \bar{ds} = \oint_{bottom}\left[-D_{z0} + \frac{\partial D_z}{\partial z} \frac{dz}{2}\right] dx\,dy \quad \text{——(11)}$$

$$\oint_{right} \bar{D} \cdot \bar{ds} = \oint_{right}\left[D_{y0} + \frac{\partial D_y}{\partial y} \frac{dy}{2}\right] dx\,dz \quad \text{——(12)}$$

$$\oint_{left} \bar{D} \cdot \bar{ds} = \oint_{left}\left[-D_{y0} + \frac{\partial D_y}{\partial y} \frac{dy}{2}\right] dx\,dz \quad \text{——(13)}$$

Sub (6) (9) (10) (11) (12) (13) in (3)

$$\oint_{front}\left[\cancel{D_{x0}} + \frac{dx}{2}\frac{\partial D_x}{\partial x}\right] dy\,dz + \oint_{back}\left[\cancel{-D_{x0}} + \frac{dx}{2}\frac{\partial D_x}{\partial x}\right] dy\,dz +$$

$$\oint_{top}\left[\cancel{D_{z0}} + \frac{\partial z}{2}\frac{\partial D_z}{\partial z}\right] dx\,dy + \oint_{bottom}\left[\cancel{-D_{z0}} + \frac{dz}{2}\frac{\partial D_z}{\partial z}\right] dx\,dy +$$

$$\oint_{right}\left[\cancel{D_{y0}} + \frac{dy}{2}\frac{\partial D_y}{\partial y}\right] dx\,dz + \oint_{left}\left[\cancel{-D_{y0}} + \frac{\partial D_y}{\partial y}\frac{dy}{2}\right] dx\,dz =$$

$$\Rightarrow \cancel{2}\frac{\partial D_x}{\partial x}\frac{dx}{\cancel{2}} dy\,dz + \cancel{2}\frac{\partial D_z}{\partial z}\frac{dz}{\cancel{2}} dx\,dy + \cancel{2}\cdot\frac{\partial D_y}{\partial y}\frac{dy}{\cancel{2}} dx\,dz = Q$$

$$\Rightarrow \left[\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}\right] dx\,dy\,dz = Q$$

$$\left[\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}\right] dv = Q \quad — (14)$$

From Gauss's law

$$\boxed{\oint \bar{D}\cdot \bar{ds} = Q = \left[\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}\right] dv}$$

∴ The above eqn is charge enclosed ~~by~~ in a volume Δv (or) dv.

Divergence:

From the above eqn

$$\oint \bar{D}\cdot \bar{ds} = Q = \left[\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}\right] dv$$

To apply Gauss's law, we have assumed a differential volume element as the Gaussian surface over which $\bar{D}$ is constant.

As $dv \to 0$

$$\oint_S \bar{D}\cdot \bar{ds} = \lim_{dv \to 0} \left[\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}\right] dv$$

$$\lim_{dv\to 0} \frac{Q}{dv} = \lim_{dv \to 0} \frac{\oint_S \bar{D}\cdot \bar{ds}}{dv} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}$$

In general if $\bar{A}$ is any vector

$$\boxed{\lim_{dv \to 0} \frac{\oint \bar{A}\cdot \bar{ds}}{dv} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}}$$

The mathematical operation on A is called a divergence.

$\nabla \to$ del $\to$ vector quantity

$$\nabla = \frac{\partial}{\partial x}\bar{a_x} + \frac{\partial}{\partial y}\bar{a_y} + \frac{\partial}{\partial z}\bar{a_z}$$

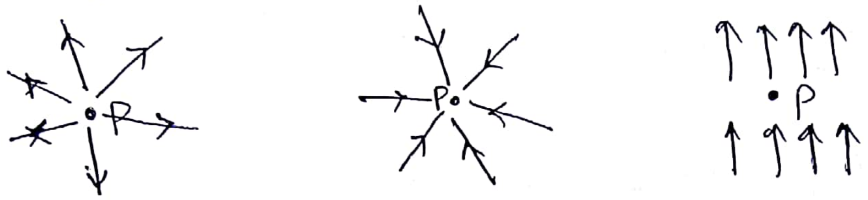
If $\bar{A} = A_x \bar{a_x} + A_y \bar{a_y} + A_z \bar{a_z}$

$$\nabla\cdot\bar{A} = \frac{\partial A_x}{\partial x} + \frac{\partial}{\partial y} A_y + \frac{\partial}{\partial z} A_z.$$

$$\lim_{dv \to 0} \frac{\oint \bar{A}\cdot \bar{ds}}{dv} = \text{div}\,\bar{A}$$

$$\therefore \boxed{\text{div}\,\bar{A} \text{ or } \nabla\cdot\bar{A} = \lim_{dv\to 0} \frac{\oint \bar{A}\cdot\bar{ds}}{dv}}$$

Divergence at 'p' if field is diverging | converging



(i) positive (ii) Negative (iii) Zero.

Properties of Divergence :

(i) Divergence is a scalar quantity.

(ii) Divergence of a scalar has no meaning.

$\nabla \cdot n \rightarrow$ no meaning.

iii) $\nabla \cdot (\bar{A}+\bar{B}) = \nabla \cdot \bar{A} + \nabla \cdot \bar{B}$

Divergence in different coordinate systems :

$$\nabla \cdot \bar{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \quad \text{(Cartesian)}$$

$$\nabla \cdot \bar{A} = \frac{1}{r}\frac{\partial}{\partial r}(r A_r) + \frac{1}{r}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z} \quad \text{(Cylindrical)}$$

$$\nabla \cdot \bar{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta A_\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial \phi}$$

(Spherical)

Maxwell's eqn (first eqn)

From divergence

$$\nabla \cdot \bar{D} = \operatorname{div} \bar{D} = \lim_{dv \to 0} \frac{\oint \bar{D}\cdot \bar{ds}}{dv} \quad \text{—— (1)}$$

From Gauss's law

$$\oint_s \bar{D}\cdot \bar{ds} = Q \quad \text{—— (2)}$$

divide with dv on both sides

$$\lim_{dv \to 0} \frac{\oint_s \bar{D}\cdot \bar{ds}}{dv} = \lim_{dv \to 0} \frac{Q}{dv}$$

$$\boxed{\nabla \cdot \bar{D} = \rho_v}$$

This is differential form of Gauss's law. (or) point form of Gauss's law (or) Maxwell's ~~first~~ eqn in point

Divergence Theorem :

From Gauss's law

$$\oint \bar{D} \cdot \bar{ds} = Q \quad \text{--- (1)}$$

Charge enclosed in a volume is

$$Q = \int_V \rho_v \, dV \quad \text{--- (2)}$$

From Gauss's law in point form

$$\nabla \cdot \bar{D} = \rho_v \quad \text{--- (3)}$$

Sub (2) in (1)

$$\oint_S \bar{D} \cdot \bar{ds} = \int_V \rho_v \, dV$$

$$\boxed{\oint_S \bar{D} \cdot \bar{ds} = \int_V (\nabla \cdot \bar{D}) \, dV} \quad \text{(from (3))}$$

Divergence theorem states that "Surface integral can be converted into volume integral provided that the closed surface encloses the certain volume."

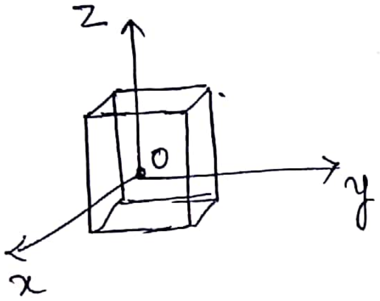
In general

$$\boxed{\oint_S \bar{A} \cdot \bar{ds} = \int_V (\nabla \cdot \bar{A}) \, dV}$$

Problems:

1) Find the charge enclosed in a cube of having side of 2m with the edges of the cube parallel to axis x, y, z. While origin is in the Centre of the cube, charge density within the cube is $50x^2\cos(\frac{\pi}{2}y)$ $\mu C/m^3$

Sol:



$x = 2$
$y = 2$
$z = 2$

$$Q = \int_V \rho_v \, dv$$

$$= \int_V 50x^2 \cos(\tfrac{\pi}{2}y) \times 10^{-6} \, dx\,dy\,dz .$$

$$= 10^{-6} \times 50 \int_{x=-1}^{1} x^2 dx \int_{y=-1}^{1} \cos(\tfrac{\pi}{2}y)\, dy \int_{z=-1}^{1} dz .$$

$$= 50 \times 10^{-6} \left[\frac{x^3}{3}\right]_{-1}^{1} \left[\frac{\sin(\frac{\pi}{2}y)}{\pi/2}\right]_{-1}^{1} \left[z\right]_{-1}^{1}$$

$$= 50 \times 10^{-6} \left[\frac{1}{3}+\frac{1}{3}\right] \left[\frac{1+1}{\pi/2}\right] \left[1+1\right]$$

$$= 50 \times 10^{-6} \left[\frac{2}{3}\right]\left[\frac{4}{\pi}\right]\left[2\right]$$

$$= \frac{800 \times 10^{-6}}{3\pi} \text{ Coulomb.}$$

2) A flat conducting surface has $\rho_s = 1\, C/m^2$. What would be the value of electric field strength at surface ?

Sol:

$$E = \frac{\rho_s}{2\varepsilon} \bar{a}_N$$

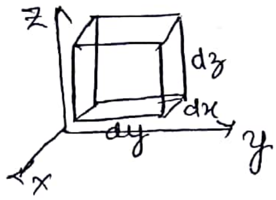
$$= \frac{1}{2 \times 8.854 \times 10^{-12}} \bar{a}_n$$

$$= 5.64 \times 10^{10} \bar{a}_n \, V/m .$$

3) Find the total charge in a volume defined by 6 planes for which $1 \le x \le 2$, $2 \le y \le 3$, $3 \le z \le 4$ if $\bar{D} = 4x\bar{a}_x + 3y^2\bar{a}_y + 2z^3\bar{a}_z$ C/m².

Sol: Method 1. From Gauss's law

$$\oint \bar{D} \cdot \bar{ds} = Q$$



$$\left[\oint_{front} + \oint_{back} + \oint_{top} + \oint_{bottom} + \oint_{left} + \oint_{right}\right] \bar{D} \cdot \bar{ds} = Q \quad \text{---(1)}$$

$\bar{D} = 4x\bar{a}_x + 3y^2\bar{a}_y + 2z^3\bar{a}_z$

$\bar{D} = D_x\bar{a}_x + D_y\bar{a}_y + D_z\bar{a}_z$

$$\oint_{front} \bar{D} \cdot \bar{ds} = \oint D_x \bar{a}_x \cdot dy\,dz\,\bar{a}_x$$

$$= \int_{y=2}^{3}\int_{z=3}^{4} 4x\,dy\,dz$$

$$= 4x[y]_2^3[z]_3^4 = 4x[3-2][4-3] = 4x$$

Front surface at $x=2 \Rightarrow \oint_{front} \bar{D} \cdot \bar{ds} = 4(2) = 8$ //.

$$\int_{back} \bar{D} \cdot \bar{ds} = \oint D_x\bar{a}_x \cdot dy\,dz\,(-\bar{a}_x)$$

$$= -\int_{y=2}^{3}\int_{z=3}^{4} 4x\,dy\,dz = -4x[y]_2^3[z]_3^4$$

$$= -4x$$

Back surface at $x=1 \Rightarrow \oint_{back} \bar{D} \cdot \bar{ds} = -4(1) = -4$ //

$$\oint_{top} \bar{D} \cdot \bar{ds} = \oint D_z\bar{a}_z \cdot dx\,dy\,\bar{a}_z$$

$$= \int_{x=1}^{2}\int_{y=2}^{3} 2z^3\,dx\,dy = 2z^3[x]_1^2[y]_2^3$$

$$= 2z^3$$

top surface at $z=4 \Rightarrow \oint_{top} \bar{D} \cdot \bar{ds} = 2(4)^3 = 128$ //

$$\oint_{bottom} \bar{D} \cdot \bar{ds} = \oint D_z\bar{a}_z \cdot dx\,dy\,(-\bar{a}_z)$$

$$= -\int_{x=1}^{2}\int_{y=2}^{3} 2z^3\,dx\,dy = -2z^3[x]_1^2[y]_2^3 = -2z^3$$

Bottom surface at $z=3 \Rightarrow \oint_{bottom} \bar{D} \cdot \bar{ds} = -2(3)^3 = -54$ //

$$\oint_{right} \bar{D} \cdot \bar{ds} = \oint D_y\bar{a}_y \cdot dx\,dz\,\bar{a}_y$$

$$= \int_{x=1}^{2}\int_{z=3}^{4} 3y^2\,dx\,dz = 3y^2[x]_1^2[z]_3^4 = 3y^2$$

Right surface at $y=3 \Rightarrow \oint_{right} \bar{D} \cdot \bar{ds} = 3(3)^2 = 27$ //

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$$\oint_{left} \bar{D}\cdot\bar{ds} = \oint D_y \bar{a_y}\cdot dx dz(-\bar{a_y})$$

$$= -\int_{x=1}^{2}\int_{z=3}^{4} 3y^2\, dx dz = -3y^2[x]_1^2[z]_3^4 = -3y^2$$

Left surface at $y=2 \Rightarrow \oint_{left} \bar{D}\cdot\bar{ds} = -3(2)^2 = -12$

Sub. all these values in (I)

$$8+(-4)+128+(-54)+27+(-12) = Q$$

$$\therefore \boxed{Q = 93C}$$

Method 2: using divergence theorem

$$\oint_S \bar{D}\cdot\bar{ds} = \int_V (\nabla\cdot\bar{D})\, dv = Q$$

$$\int_V (\nabla\cdot\bar{D})\, dv = Q$$

$$L.H.S = \nabla\cdot\bar{D} = \left[\frac{\partial}{\partial x}\bar{a_x} + \frac{\partial}{\partial y}\bar{a_y} + \frac{\partial}{\partial z}\bar{a_z}\right]\cdot\left[4x\bar{a_x} + 3y^2\bar{a_y} + 2z^3\bar{a_z}\right]$$

$$= 4 + 6y + 6z^2$$

$$\therefore \int_V (\nabla\cdot\bar{D})\, dv = \int_{x=1}^{2}\int_{y=2}^{3}\int_{z=3}^{4}[4+6y+6z^2]\, dx dy dz$$

$$= 4\int_1^2\int_2^3\int_3^4 dx dy dz + 6\int_1^2\int_2^3\int_3^4 y\, dx dy dz + 6\int_1^2\int_2^3\int_3^4 z^2\, dx dy dz$$

$$= 4[x]_1^2[y]_2^3[z]_3^4 + 6[x]_1^2\left[\frac{y^2}{2}\right]_2^3[z]_3^4 + 6[x]_1^2[y]_2^3\left[\frac{z^3}{3}\right]_3^4$$

$$= 4 + 6(1)\left(\frac{9}{2}-\frac{4}{2}\right)(1) + 6(1)(1)\left(\frac{64}{3}-\frac{27}{3}\right)$$

$$= 4 + \cancel{6}^{3}\left(\frac{5}{\cancel{2}}\right) + \cancel{6}^{2}\left(\frac{37}{\cancel{3}}\right)$$

$$= 4 + 15 + 74$$

$$= 93\, C$$

4) Find the divergence of a vector $\bar{A}$ at point $P(5, \frac{\pi}{2}, 1)$ where $\bar{A} = rz\sin\phi\, \bar{a}_r + 3rz^2\cos\phi\, \bar{a}_\phi$

Sol
$$\nabla \cdot \bar{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

$$= \frac{1}{r}\frac{\partial}{\partial r}(r^2 z\sin\phi) + \frac{1}{r}\frac{\partial}{\partial \phi}(3rz^2\cos\phi) + 0$$

$$= \frac{1}{\cancel{r}} z\sin\phi\, 2\cancel{r} + \frac{1}{\cancel{r}} 3\cancel{r}z^2(-\sin\phi)$$

$$\nabla \cdot \bar{A} = 2z\sin\phi - 3z^2\sin\phi$$

at 'p' $\nabla \cdot \bar{A} = 2(1)\sin\pi/2 - 3(1)\sin\pi/2$

$$= 2 - 3 = \underline{\underline{-1}}$$

5) Write an expression for divergence of $\bar{D}$

(i) $\nabla \cdot \bar{D}$ in cylindrical coordinates

(ii) find $\nabla \cdot \bar{D}$ where $\bar{D} = 10\rho\, \bar{a}_\rho$ C/m².

Sol (i) $\nabla \cdot \bar{D} = \frac{1}{r}\frac{\partial}{\partial r}(rD_r) + \frac{1}{r}\frac{\partial D_\phi}{\partial \phi} + \frac{\partial D_z}{\partial z}$

(ii) $\nabla \cdot \bar{D} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho\, 10\rho) + \frac{1}{\rho}(0) + 0$

$$= \frac{1}{\cancel{\rho}} 10(2\cancel{\rho})$$

$$= \underline{\underline{20}}$$

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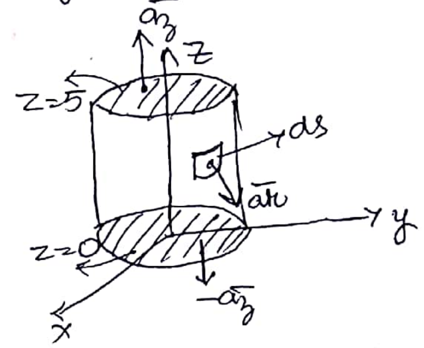
6) Given that $\bar{A} = 30e^{-r}\bar{a}_r - 2z\,\bar{a}_z$ in the cylindrical coordinates. Evaluate both sides of the divergence theorem for the volume enclosed by $r=2, z=0$ and $z=5$. (35)



Sol:

From divergence theorem

$$\oint_S \bar{A}\cdot\bar{ds} = \int_V (\nabla\cdot\bar{A})\,dv$$

$$L.H.S. = \oint_S \bar{A}\cdot\bar{ds}$$

$$= \oint_{top} \bar{A}\cdot\bar{ds} + \oint_{bottom} \bar{A}\cdot\bar{ds} + \oint_{side} \bar{A}\cdot\bar{ds} \quad \text{---(1)}$$

differential elements $\left.\begin{matrix} dr \\ r\,d\phi \\ dz \end{matrix}\right\}$.

Given $\bar{A} = 30e^{-r}\bar{a}_r - 2z\,\bar{a}_z$

$$\bar{A} = A_r\bar{a}_r + A_\phi a_\phi + A_z\bar{a}_z$$

$$\oint_{top} \bar{A}\cdot\bar{ds} = \oint_{top} A_z\bar{a}_z\cdot r\,dr\,d\phi\,\bar{a}_z$$

$$= \int_{r=0}^{2}\int_{\phi=0}^{2\pi} -2z\,r\,dr\,d\phi = -2z\left[\frac{r^2}{2}\right]_0^2 [\phi]_0^{2\pi}$$

$$= -2z\left[\frac{4}{2}\right][2\pi]$$

$$= -8\pi z$$

top surface at $z=5 \Rightarrow \oint_{top} \bar{A}\cdot\bar{ds} = -8\pi(5) = -40\pi$ //

$$\oint_{bottom} \bar{A}\cdot\bar{ds} = \oint_{bottom} A_z\bar{a}_z\cdot r\,dr\,d\phi(-\bar{a}_z)$$

$$= -\int_{r=0}^{2}\int_{\phi=0}^{2\pi} -2z\,r\,dr\,d\phi = +2z\left[\frac{r^2}{2}\right]_0^2[\phi]_0^{2\pi}$$

$$= 2z\left[\frac{4}{2}\right][2\pi] = 8\pi z$$

Bottom surface at $z=0 \Rightarrow \oint_{bottom} \bar{A}\cdot\bar{ds} = 0$

$$\oint_{side} \bar{A}\cdot\bar{ds} = \oint_{side} A_r\bar{a}_r\cdot r\,d\phi\,dz\,\bar{a}_r$$

$$= \int_{\phi=0}^{2\pi}\int_{z=0}^{5} 30e^{-r}\,r\,d\phi\,dz = 30e^{-r}r\,[\phi]_0^{2\pi}[z]_0^5$$

$$= 30e^{-r}r\,[2\pi][5]$$

Side surface at $r=2 \Rightarrow \oint_{side} \bar{A}\cdot\bar{ds} = 30e^{-2}\cdot(2)(10\pi)$

$$= 255.1$$

Sub. these values in eqn (1)

$$L.H.S = -40\pi + 0 + 255.1 = 129.44 \;//$$

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$$R.H.S. = \int_V (\nabla \cdot \bar{A})\, dv$$

$$\nabla \cdot \bar{A} = \frac{1}{r}\frac{\partial}{\partial r}(r A_r) + \frac{1}{r}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

$$= \frac{1}{r}\frac{\partial}{\partial r}(r\, 30 e^{-r}) + \frac{1}{r}\frac{\partial}{\partial \phi}(0) + \frac{\partial}{\partial z}(-2z)$$

$$= \frac{30}{r}\left[r e^{-r}(-1) + e^{-r}.1\right] + (-2)$$

$$\nabla \cdot \bar{A} = -30 e^{-r} + \frac{30 e^{-r}}{r} - 2$$

$$RHS = \int_V \left[-30e^{-r} + \frac{30e^{-r}}{r} - 2\right] dv$$

$$= \int_{r=0}^{2}\int_{\phi=0}^{2\pi}\int_{z=0}^{5} \left[-30e^{-r} + \frac{30e^{-r}}{r} - 2\right] dr\, r d\phi\, dz$$

$$= \int_0^2\int_0^{2\pi}\int_0^5 \frac{(-30re^{-r} + 30e^{-r} - 2r)}{\cancel{r}}\, dr\, \cancel{r} d\phi\, dz.$$

$$= \int_0^2 \left[-30\underset{u}{r}\underset{dv}{e^{-r}} + 30e^{-r} - 2r\right] dr \int_0^{2\pi} d\phi \int_0^5 dz$$

$$\boxed{\int u dv = uv - \int v du}$$

$$= \left\{-30\left\{r\frac{e^{-r}}{-1} - \int \frac{e^{-r}}{-1}.1\right\}_0^2 + 30\left[\frac{e^{-r}}{-1}\right]_0^2 - 2\left[\frac{r^2}{2}\right]_0^2\right\}[\phi]_0^{2\pi}[z]_0^5$$

$$= \left\{\left[30re^{-r} + 30e^{-r}\right]_0^2 + 30\left[\frac{e^{-r}}{-1}\right]_0^2 - \frac{\cancel{2}}{\cancel{2}}[r^2]_0^2\right\}(2\pi)(5)$$

$$= \left\{\left[30(2)e^{-2} + \cancel{30e^{-2}} - 0 - \cancel{30}\right] - 30\left[\cancel{e^{-2}} - \cancel{1}\right] - [4]\right\} 10\pi$$

$$= \left[60e^{-2} - 4\right]10\pi$$

$$= 129.44$$

$\therefore$ L.H.S = R.H.S.

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7) The flux density $\bar{D} = \frac{r}{3} \bar{a}_r \; nC/m^2$ is in the free space. (36)

a) Find $\bar{E}$ at $r = 0.2m$

b) find the total electric flux leaving the sphere of $r = 0.2m$

c) Find the total charge within the sphere of $r = 0.3m$

Sol:

a) $O \xrightarrow{r=0.2m} \bar{E}$

$$\bar{E} = \frac{\bar{D}\bar{a}_r}{\varepsilon_0} = \frac{r}{3\varepsilon_0} \bar{a}_r \times 10^{-9}$$

$$= \frac{0.2 \times 10^{-9}}{3 \times 8.854 \times 10^{-12}} \bar{a}_r = 7.53 \; \bar{a}_r \; V/m.$$

b) $\oint_S \bar{D} \cdot \bar{ds} = Q = \psi$

at $r = 0.2m \Rightarrow ds = r^2 \sin\theta \, d\theta \, d\phi \, \bar{a}_r$ $\left.\begin{matrix} dr \\ r d\theta \\ r\sin\theta \, d\phi \end{matrix}\right\}$.

$$\psi = \oint_S \frac{r}{3} \bar{a}_r 10^{-9} \cdot r^2 \sin\theta \, d\theta \, d\phi \, \bar{a}_r$$

$$= \int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} \frac{r^3}{3} 10^{-9} \sin\theta \, d\theta \, d\phi = \frac{r^3}{3} 10^{-9} \left[-\cos\theta\right]_0^{\pi} \left[\phi\right]_0^{2\pi}$$

$$= \frac{(0.2)^3}{3} \cdot 10^{-9} \left[-\cos\theta\right]_0^{\pi} \left[\phi\right]_0^{2\pi}$$

$$= \frac{0.008}{3} \cdot 10^{-9} \cdot (2)(2\pi) = 33.51 \, pC$$

c) $Q = \oint \bar{D} \cdot \bar{ds}$

$$= \oint \frac{r}{3} \bar{a}_r 10^{-9} \cdot r^2 \sin\theta \, d\theta \, d\phi \, \bar{a}_r$$

$$= \int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} \frac{r^3}{3} \cdot 10^{-9} \sin\theta \, d\theta \, d\phi$$

At $r = 0.3m$

$$Q = \frac{(0.3)^3}{3} 10^{-9} \left[-\cos\theta\right]_0^{\pi} \left[\phi\right]_0^{2\pi}$$

$$= 113.1 \, pC$$

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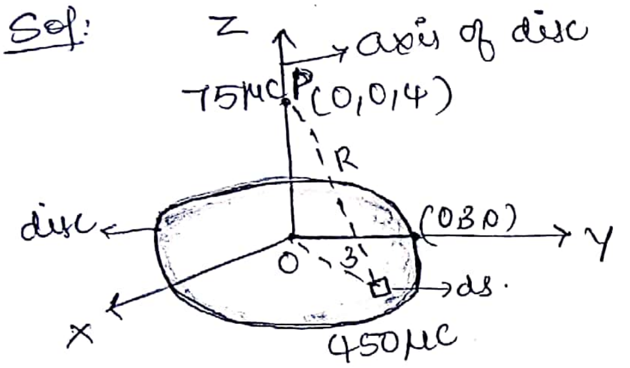
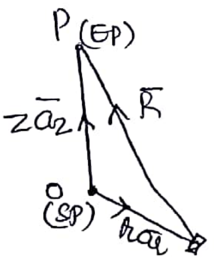
8) A circular disc of radius 3m carries the uniformly distributed charge of 450 μC. Calculate the force on 75 μC located on the axis of the disc and it is at 4m from its centre.

Sol:



$$\bar{F} = \frac{Q_1 Q_2}{4\pi\varepsilon R^2}\,\bar{a}_R$$

Force on 75μC is nothing but $\bar{E}$ at 'p' due to 450μC.



$$ds = r\,dr\,d\phi$$

$$\bar{R} = -r\bar{a}_r + z\bar{a}_z$$

$$\bar{a}_R = \frac{\bar{R}}{|R|} = \frac{-r\bar{a}_r + z\bar{a}_z}{\sqrt{r^2+z^2}}$$

$$\rho_s = \frac{Q}{\text{Area}} = \frac{450\times10^{-6}}{\pi r^2} = \frac{450\times10^{-6}}{\pi(3)^2} = 15.92\times10^{-6}\ C/m^2$$

$$\therefore \bar{E} = \int_S \frac{dQ}{4\pi\varepsilon R^2}\,\bar{a}_R$$

$$= \int_S \frac{\rho_s\,ds}{4\pi\varepsilon R^2}\,\bar{a}_R \cancel{\neq \iint \frac{15.92\times10^{-6}}{4\pi\times8.854\times10^{-12}} \times r\,dr\,d\phi}$$

$$= \int_{\phi=0}^{2\pi}\int_{r=0}^{3} \frac{\rho_s\, r\,dr\,d\phi}{4\pi\varepsilon_0 (r^2+z^2)} \cdot \frac{(\cancel{-r\bar{a}_r}+z\bar{a}_z)}{\sqrt{r^2+z^2}}$$

$$= \frac{\rho_s z}{4\pi\varepsilon_0}\int_{\phi=0}^{2\pi}\int_{r=0}^{3} \frac{r\,dr\,d\phi}{(r^2+z^2)^{3/2}}\,\bar{a}_z$$

(radial components of $\bar{E}$ at 'p' cancel each others)

Let $r^2+z^2=u^2$ | $r=0 \Rightarrow u_1 = z$

$\cancel{2}r\,dr = \cancel{2}u\,du$ | $r=3 \Rightarrow u_2 = \sqrt{3^2+z^2}$

$$= \frac{\rho_s z}{4\pi\varepsilon_0}\int_0^{2\pi}\int_{u_1}^{u_2} \frac{\cancel{u}\,du\,d\phi}{u^{3/2}}\,\bar{a}_z = \frac{\rho_s z}{4\pi\varepsilon_0}\left[-\frac{1}{u}\right]_{u_1}^{u_2}\left[\phi\right]_0^{2\pi}\bar{a}_z$$

but $z = 4m$, $u_1 = 4$, $u_2 = 5$

$$\therefore \bar{E} = \frac{15.92\times10^{-6}\times4}{4\pi\times8.854\times10^{-12}}\left[-\frac{1}{5}+\frac{1}{4}\right][2\pi]\,\bar{a}_z = 0.179\,\bar{a}_z\ \text{M...}$$

9) $\bar{D} = 12x^2\bar{a}_x - 3z^3\bar{a}_y - 9yz^2\bar{a}_z$ C/m² in free space. specified the point in cube is $1 \le x, y, z \le 2$. At which the following quantities are maximum and give the maximum value.

i) magnitude of $\bar{D}$

ii) $|\rho_v|$

iii) ρ_v

Sol: i) $|\bar{D}| = \sqrt{(12x^2)^2 + (3z^3)^2 + (9yz^2)^2}$.

$$= \sqrt{(12(2)^2)^2 + (3(2)^3)^2 + (9(2)(2)^2)^2}$$

$$= 89.8 \text{ C/m}^2.$$

ii) $\nabla \cdot \bar{D} = \rho_v$

$$\therefore \rho_v = \frac{\partial}{\partial x}(12x^2) + \frac{\partial}{\partial y}(-3z^3) + \frac{\partial}{\partial z}(-9yz^2)$$

$$\rho_v = 24x + 0 - 18yz.$$

iii) ρ_v max

at $x=y=z=1 \Rightarrow \rho_v = 24(1) - 18(1)(1) = \underline{6}$

at $x=y=z=2 \Rightarrow \rho_v = 24(2) - 18(2)(2) = \underline{-24}$

10) A Sphere of 200mm radius contains the electrical charge density $\frac{2}{r\sin\theta}$ C/m³. What is the total charge contained within the sphere.

Sol: $r = 200$mm , $\rho_v = \frac{2}{r\sin\theta}$

$$= \frac{200}{1000} = 0.2m.$$

$$Q = \int_V \rho_v \, dV$$

$$= \int_{r=0}^{0.2}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} \frac{2}{r\sin\theta} r^2\sin\theta \, dr\, d\theta\, d\phi$$

$$= 2\left[\frac{r^2}{2}\right]_0^{0.2}\left[\theta\right]_0^{\pi}\left[\phi\right]_0^{2\pi}$$

$$= 2\left[\frac{(0.2)^2}{2}\right]\left[\pi\right]\left[2\pi\right]$$

$$Q = \underline{0.79 \text{ C}}$$

~~$Q = \frac{\rho_v}{\text{Volume}}$~~

~~$\frac{Q}{\frac{4}{3}\pi r^2} = \frac{2}{r\sin}$~~

~~$Q = \frac{2 \cdot 4\pi r^2}{3\sin}$~~

~~$= \frac{8\pi(0.2)}{3}$~~

UNIT–II

- Work done and potential.
- Potential due to different charge distribution
- potential due to infinite line charge.
- potential gradient
- Energy density
- Boundary conditions between
  - i) conductor and free space
  - ii) conductor and dielectric
  - iii) two dielectrices.
- Capacitance & its calculations for different cases.
- current and current density
- Continuity equation
- Poisson's and Laplace's equation
- Uniqueness Theorem.

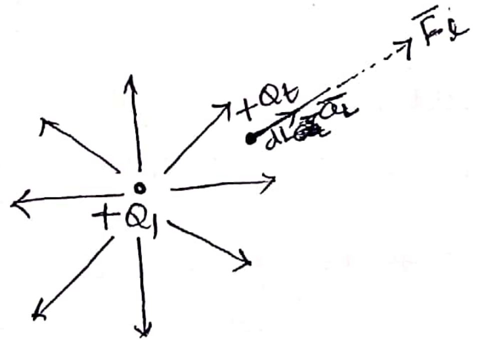
Electric Work & Potential

Coulomb's law & Gauss's law is used to determine the Electric Field Intensity ($\bar{E}$). Similarly electric scalar potential ~~potential~~ can be used to obtain $\bar{E}$. This is another method of obtaining $\bar{E}$ from scalar potential.

Work done:

"The work is said to be done when the test charge is moved against the electric field."

Ex: Earths gravitational field towards down. If an object is to be moved up then external source required i.e. work is done.



- Let +ve charge Q_1 and its Field is $\bar{E}$.
- If a +ve test charge Q_t is placed in the field, it will move due to force of repulsion.
- Let the movement of Q_t is dL.

According to Coulomb's law

$$\bar{F} = Q_t \bar{E} \quad \text{——— (1)}$$

Component of $\bar{F}$ in the direction of unit vector $\bar{a}_L$

$$F_l = \bar{F} \cdot \bar{a}_L = Q_t \bar{E} \cdot \bar{a}_L \quad \text{——— (2)}$$

This is the force responsible to move the charge Q_t through the distance dL, in the direction of the field. To keep the charge in equilibrium, it is necessary to apply equal and opposite force to the force exerted by the field in the direction dL.

$\bar{F}_{applied} = -\bar{F}_e = -Q_t \bar{E}\cdot\bar{a}_L$ —(3)

In this case the work is said to be done. Thus the work done is nothing but the product of force and the distance.

" The differential work done by an external source in moving the charge Q_t through a distance dl, against the direction of field $\bar{E}$ ".

$$dW = \bar{F}_{applied} \times \bar{dL}$$

$$= -Q_t \bar{E}\cdot\bar{a}_L \, dL$$

$dL\,\bar{a}_L = \bar{dL}$ = distance vector

$$\boxed{dW = -Q_t \bar{E}\cdot\bar{dL}} \text{ Joules.}$$

$dW \rightarrow$ Scalar quantity

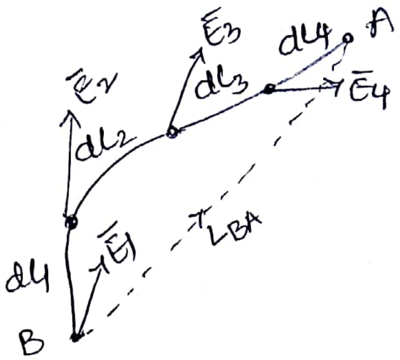
- If charge Q is moved from initial to final position against the direction of Electric field $\bar{E}$, Then the total work done

$$W = \int_{initial}^{final} dW = \int_{initial}^{final} -Q\, \bar{E}\cdot\bar{dL}$$

$$\therefore \boxed{W = -Q \int_{initial}^{final} \bar{E}\cdot\bar{dL}} \text{ Joules.}$$

NOTE: At initial & final the charge is at rest, and not moving.

The Line integral :



$$W = -Q \int_B^A \bar{E} \cdot \bar{dL}$$

1) choose path B to A.
2) Break up the path into no. of very small segments.
3) Find $\bar{E}$ along each differential length (segment)
4) Add all components and multiply by charge.

$$W = -Q \left[\bar{E_1} \cdot \bar{dL_1} + \bar{E_2} \cdot \bar{dL_2} + \bar{E_3} \cdot \bar{dL_3} + \bar{E_4} \cdot \bar{dL_4} \right]$$

But $\bar{E_1} = \bar{E_2} = \bar{E_3} = \bar{E_4} = \bar{E}$ (Electric field is uniform)

$$W = -Q\bar{E} \left[\bar{dL_1} + \bar{dL_2} + \bar{dL_3} + \bar{dL_4} \right]$$

$$\boxed{W = -Q \, \bar{E} \cdot \bar{L}_{BA}}$$

$\bar{dL} = dx \, \bar{a}_x + dy \, \bar{a}_y + dz \, \bar{a}_z$ — Cartesian
$\bar{dL} = dr \, \bar{a}_r + r d\phi \, \bar{a}_\phi + dz \, \bar{a}_z$ — Cylindrical
$\bar{dL} = dr \, \bar{a}_r + r d\theta \, \bar{a}_\theta + r \sin\theta \, d\phi \, \bar{a}_\phi$ — Spherical.

- movement of charge Q is against the direction of $\bar{E}$, then workdone is +ve, which indicates external source has done the work.
- movement of charge in the direction of $\bar{E}$ then work done is −ve, no external source is required.
- If the path is a closed contour i.e. starting and ending point is same then the workdone is zero.

Potential Difference :

Work done $W = -Q\int_B^A \bar{E}\cdot \bar{dl}$

"work done in moving unit charge from B to A in the field $\bar{E}$ is called potential difference" b/w B & A points.

$$\frac{W}{Q} = -\int_B^A \bar{E}\cdot \bar{dl}$$

$$\boxed{V = -\int_B^A \bar{E}\cdot \bar{dl}}$$

$$\boxed{V_{AB} = -\int_B^A \bar{E}\cdot \bar{dl}}$$

B → initial point
A → Final point .

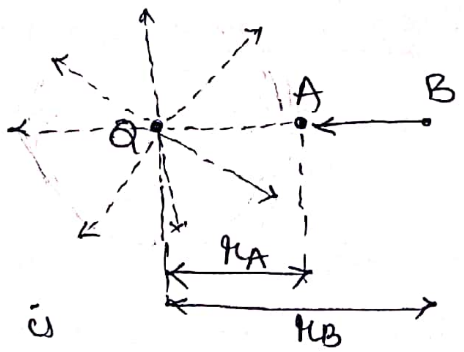
Units : $V = \frac{W}{Q} = \frac{\text{Joules}}{\text{Coulomb}} = \text{volt}.$

J/C (or) Volt (V).

1. Potential due to point charge

Assuming free space,
$\bar{E}$ due to point charge

$$\bar{E} = \frac{Q}{4\pi\epsilon_0 r^2}\bar{a}_r \quad \text{—— (1)}$$



Let a unit charge which is placed at a point B. It is moved against the direction of $\bar{E}$ from B to A point.

Differential length in spherical system

$$\bar{dl} = dr\,\bar{a}_r + r\,d\theta\,\bar{a}_\theta + r\sin\theta\,d\phi\,\bar{a}_\phi$$

potential difference between A & B is V_{AB}

$$V_{AB} = -\int_B^A \bar{E}\cdot \bar{dl}$$

$$= -\int_{r_B}^{r_A} \frac{Q}{4\pi\epsilon_0 r^2}\bar{a}_r \cdot (dr\,\bar{a}_r + r\,d\theta\,\bar{a}_\theta + r\sin\theta\,d\phi\,\bar{a}_\phi)$$

$$V_{AB} = -\int_{r_B}^{r_A} \frac{Q}{4\pi\epsilon_0 r^2}\, dr$$

$$= -\frac{Q}{4\pi\epsilon_0}\left[-\frac{1}{r}\right]_{r_B}^{r_A}$$

$$= -\frac{Q}{4\pi\epsilon_0}\left[-\frac{1}{r_A} + \frac{1}{r_B}\right]$$

$$\boxed{V_{AB} = \frac{Q}{4\pi\epsilon_0}\left[\frac{1}{r_A} - \frac{1}{r_B}\right]} \quad \text{Volts}$$

When $r_B > r_A$ i.e $\frac{1}{r_B} < \frac{1}{r_A}$ then $V_{AB} = +ve$. This indicates the work is done by external source in moving charge from B to A point.

Absolute potential :

if $r_B = \infty$

$$V_{AB} = \frac{Q}{4\pi\epsilon_0 r_A}$$

This is nothing but potential of point A

$$\boxed{V_A = \frac{Q}{4\pi\epsilon_0 r_A}} \quad V.$$

This is also called absolute potential of point A.

Similarly absolute potential of point B

$$\boxed{V_B = \frac{Q}{4\pi\epsilon_0 r_B}} \quad \text{Volts}$$

This is work done in moving unit charge from infinity at point B.

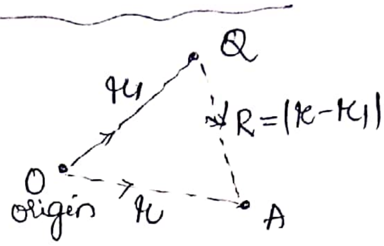
potential difference can be expressed as the difference between the absolute potentials of the two points.

$$\boxed{V_{AB} = V_A - V_B}$$

"Absolute potential at any point in an electric field is defined as the work done in moving a charge from infinity to the point, against the direction of field".

Potential due to point charge Not at origin

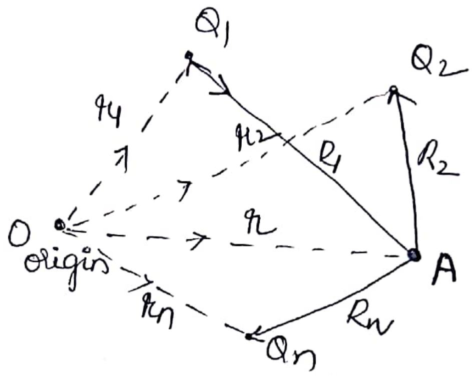
$$V_A = \frac{Q}{4\pi\epsilon_0 R}$$



$$\boxed{V_A = \frac{Q}{4\pi\epsilon_0 (|r - r_1|)}}$$

Potential due to several point charges :

Superposition principle



$$V_{A1} = \frac{Q_1}{4\pi\epsilon_0 R_1} = \frac{Q_1}{4\pi\epsilon_0 |r - r_1|}$$

$$V_{A2} = \frac{Q_2}{4\pi\epsilon_0 R_2} = \frac{Q_2}{4\pi\epsilon_0 |r - r_2|}$$

$$V_{An} = \frac{Q_n}{4\pi\epsilon_0 R_n} = \frac{Q_n}{4\pi\epsilon_0 |r - r_n|}$$

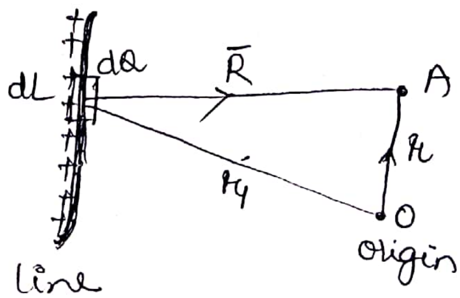
$$\therefore V_A = V_{A1} + V_{A2} + \dots + V_{An}$$

$$= \frac{Q_1}{4\pi\epsilon_0 |r - r_1|} + \frac{Q_2}{4\pi\epsilon_0 |r - r_2|} + \dots + \frac{Q_n}{4\pi\epsilon_0 |r - r_n|}$$

$$\boxed{V_A = \sum_{i=1}^{n} \frac{Q_i}{4\pi\epsilon_0 |r - r_i|} = \sum_{i=1}^{n} \frac{Q_i}{4\pi\epsilon_0 R_i}}$$

2. Potential due to line charge distribution

$$V_A = \frac{Q}{4\pi\epsilon_0 R} \quad \text{—— (1)}$$



$$dV_A = \frac{dQ}{4\pi\epsilon_0 R} \quad \text{—— (2)}$$

But charge is distributed on line

$$\rho_L = \frac{dQ}{dL} \Rightarrow dQ = \rho_L dL$$

$\therefore$ eqn (2) becomes

$$dV_A = \frac{\rho_L dL}{4\pi\epsilon_0 R}$$

$$\boxed{V_A = \int_L \frac{\rho_L dL}{4\pi\epsilon_0 R}}$$

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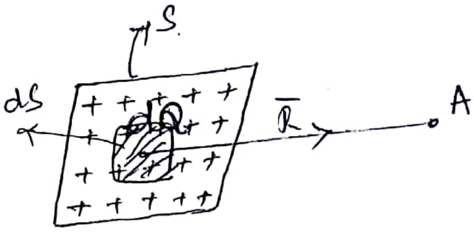
3. Potential due to Surface charge distribution:

$$V_A = \frac{Q}{4\pi\varepsilon_0 R} \quad \text{—— (1)}$$



$$dV_A = \frac{dQ}{4\pi\varepsilon_0 R} \quad \text{—— (2)}$$

But $\rho_s = \frac{dQ}{dS} \Rightarrow dQ = \rho_s \, dS$

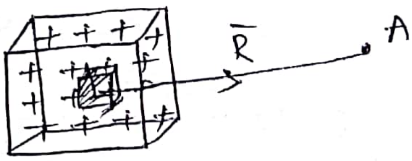
∴ eqn (2) becomes

$$dV_A = \frac{\rho_s \, dS}{4\pi\varepsilon_0 R}$$

$$\boxed{V_A = \int_S \frac{\rho_s \, dS}{4\pi\varepsilon_0 R}}$$

4. Potential due to Volume charge distribution

$$V_A = \frac{Q}{4\pi\varepsilon_0 R} \quad \text{—— (1)}$$



$$dV_A = \frac{dQ}{4\pi\varepsilon_0 R} \quad \text{—— (2)}$$

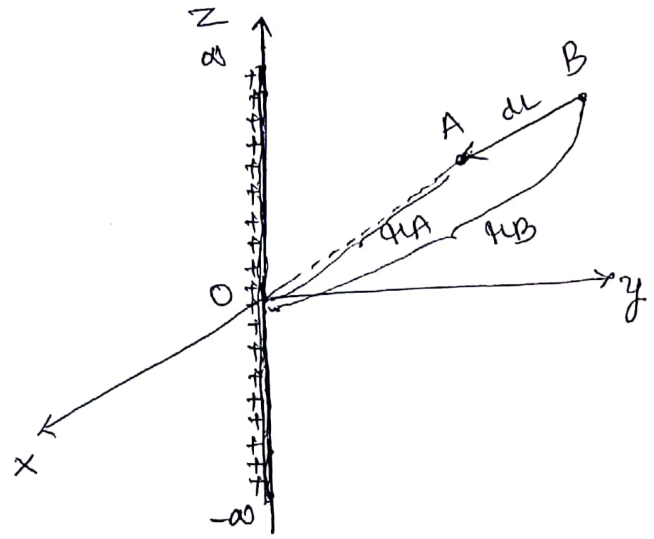
But $\rho_v = \frac{dQ}{dV} \Rightarrow dQ = \rho_v \, dV$

Sub in (2)

$$dV_A = \frac{\rho_v \, dV}{4\pi\varepsilon_0 R}$$

$$\boxed{V_A = \int_V \frac{\rho_v \, dV}{4\pi\varepsilon_0 R}}$$

Potential due to infinite line charge



- Consider an infinite line charge along z-axis having uniform charge density ρ_L C/m

$$V = -\int_B^A \bar{E} \cdot \bar{dl} \quad —(1)$$

The $\bar{E}$ due to infinite line charge along z-axis is $\bar{E} = \frac{\rho_L}{2\pi\varepsilon_0 r}\bar{a}_r$ —(2)

$$\bar{dl} = dr\,\bar{a}_r + r\,d\phi\,\bar{a}_\phi + dz\,\bar{a}_z \quad —(3)$$

(cylindrical)

sub (2) (3) in (1)

$$V = -\int_B^A \frac{\rho_L}{2\pi\varepsilon_0 r}\bar{a}_r \cdot [dr\,\bar{a}_r + r\,d\phi\,\bar{a}_\phi + dz\,\bar{a}_z]$$

$$= -\frac{\rho_L}{2\pi\varepsilon_0}\int_{r_B}^{r_A}\frac{1}{r}\,dr$$

$$= -\frac{\rho_L}{2\pi\varepsilon_0}[\log r]_{r_B}^{r_A}$$

$$= -\frac{\rho_L}{2\pi\varepsilon_0}[\log r_A - \log r_B]$$

$$\boxed{V = \frac{\rho_L}{2\pi\varepsilon_0}\log\frac{r_B}{r_A}}$$

NOTE: Workdone in moving a charge around any closed path in a static electric field is zero.

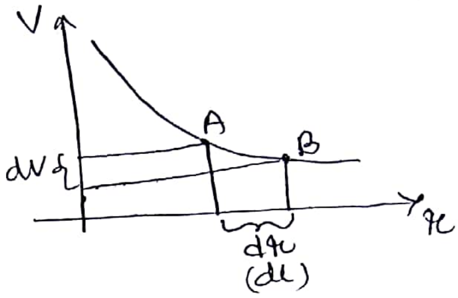
$$\boxed{V = -\oint_{\text{closed path}} \bar{E}\cdot\bar{dl} = 0}$$

Potential around a closed path is zero is called "Conservative Field".

Potential Gradient

(42)

$$V = -\int \bar{E} \cdot \bar{dL} = \frac{Q}{4\pi\varepsilon_0 r}$$



As $r \uparrow$ $V \downarrow$

$$dV = -\bar{E} \cdot \bar{dL}$$

$$dV = -|\bar{E}||\bar{dL}| \cos\theta$$

$$\frac{dV}{dL} = -|E| \cos\theta$$

Change of potential along the elementary length dl.

as $dl \to 0$

$$\lim_{dl \to 0} \frac{dV}{dl} = \frac{dV}{dL} = \text{potential gradient}$$

$$\boxed{\left.\frac{dV}{dL}\right|_{max} = E} \quad (\theta = 0)$$

"The maximum value of rate of change of potential w.r.t. the distance $\left(\frac{dV}{dL}\right)$ is called gradient of V (or) potential gradient."

Relationship between $\bar{E}$ and V :

$$\left.\frac{dV}{dL}\right|_{max} = E$$

↑

Gradient of V (or) grad V (or) ∇V

$$\therefore \nabla V = -\bar{E}$$

$$\boxed{\bar{E} = -\nabla V}$$

OR

$$dV = \frac{dV}{dx} dx + \frac{dV}{dy} dy + \frac{dV}{dz} dz \quad \text{—(1)}$$

$$\bar{E} = E_x \bar{a}_x + E_y \bar{a}_y + E_z \bar{a}_z$$
$$dL = dx\, \bar{a}_x + dy\, \bar{a}_y + dz\, \bar{a}_z$$

But $dV = -\bar{E} \cdot \bar{dL}$

$$dV = -[E_x dx + E_y dy + E_z dz] \quad \text{—(2)}$$

Compare ① ②

$$E_x = -\frac{dV}{dx}, \quad E_y = -\frac{dV}{dy}, \quad E_z = -\frac{dV}{dz}$$

But $\bar{E} = E_x \bar{a_x} + E_y \bar{a_y} + E_z \bar{a_z}$

$$= -\left[\frac{dV}{dx}\bar{a_x} + \frac{dV}{dy}\bar{a_y} + \frac{dV}{dz}\bar{a_z}\right]$$

$$\bar{E} = -\left[\frac{\partial}{\partial x}\bar{a_x} + \frac{\partial}{\partial y}\bar{a_y} + \frac{\partial}{\partial z}\bar{a_z}\right]V$$

$$\boxed{\bar{E} = -\nabla V}$$

$\bar{E} \Rightarrow$ vector

$V \rightarrow$ Scalar

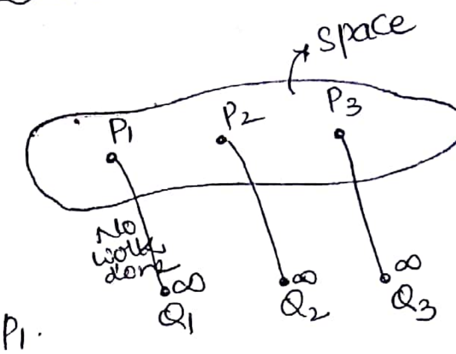
∴ gradient of a scalar is vector.

Cartesian	$\nabla V = \frac{\partial V}{\partial x}\bar{a_x} + \frac{\partial V}{\partial y}\bar{a_y} + \frac{\partial V}{\partial z}\bar{a_z}$
Cylindrical	$\nabla V = \frac{\partial V}{\partial r}\bar{a_r} + \frac{1}{r}\frac{\partial V}{\partial \phi}\bar{a_\phi} + \frac{\partial V}{\partial z}\bar{a_z}$
Spherical	$\nabla V = \frac{\partial V}{\partial r}\bar{a_r} + \frac{1}{r}\frac{\partial V}{\partial \theta}\bar{a_\theta} + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\bar{a_\phi}$

Properties of Gradient :

1. The gradient $\nabla \alpha$ gives the maximum rate of change α per unit distance.
2. $\nabla \alpha$ always indicates the direction of maximum rate of change of α.
3. $\nabla(\alpha + \beta) = \nabla\alpha + \nabla\beta$
4. $\nabla(\alpha\beta) = \alpha(\nabla\beta) + \beta(\nabla\alpha)$
5. $\nabla\left(\frac{\alpha}{\beta}\right) = \frac{\beta\nabla\alpha - \alpha\nabla\beta}{\beta^2}$

Energy Density in the Electrostatic Fields.



Consider an empty space where there is no electric field at all.
The charge Q_1 is moved from infinity to a point in the space at P_1.
This requires no work as there is no $\bar{E}$ present.
Now the charge Q_2 is to be placed at P_2. But there is an electric field due to Q_1 & Q_2 is required to be moved against the field of Q_1. Hence work is required to be done.

$$\text{Potential} = \text{work done per unit charge}$$
$$= \frac{W}{Q}$$

$\therefore$ work done = potential (V) × charge (Q)

$\therefore$ work done to position Q_1 at $P_1 = 0$ ($\because$ no field).

$\therefore$ work done to position Q_2 at $P_2 = V_{2,1}\, Q_2$ —— ①

where $V_{2,1}$ = potential at P_2 due to P_1

Now ~~work~~ Q_3 is moved to P_3. There are electric fields due to Q_1 and Q_2.

$\therefore$ work done to position Q_3 at $P_3 = V_{3,1} Q_3 + V_{3,2} Q_3$ —— ②

lly for charge Q_n to be placed at P_n then

$\therefore$ work done to position Q_n at $P_n = V_{n,1} Q_n + V_{n,2} Q_n + V_{n,3} Q_n + \cdots$ —— ③

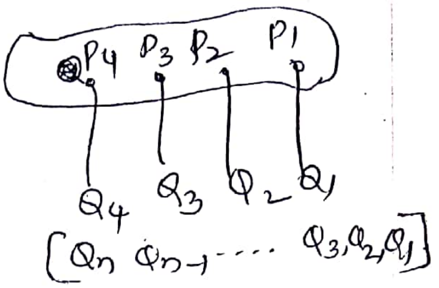
$\therefore$ total work done

$$W_E = Q_2 V_{2,1} + Q_3 V_{3,1} + Q_3 V_{3,2} + \cdots \cdots \quad \text{—— ④}$$

Total work done is nothing but the potential energy in the system of charges hence denoted as W_E.

If charges are placed in reverse order, then

$$W_E = Q_3 V_{3,4} + Q_2 V_{2,4} + Q_2 V_{2,3} + Q_1 V_{1,4} + Q_1 V_{1,3} + Q_1 V_{1,2} + \cdots \quad \text{—— (5)}$$



Add (4) & (5)

$$2W_E = Q_1(V_{1,2}+V_{1,3}+V_{1,4}+\cdots+V_{1,n}) + Q_2(V_{2,1}+V_{2,3}+V_{2,4}+\cdots+V_{2,n}) + Q_3(V_{3,1}+V_{3,2}+V_{3,4}+\cdots+V_{3,n}) + \cdots$$

Let $V_{1,2}+V_{1,3}+V_{1,4}+\cdots+V_{1,n} = V_1$

$V_{2,1}+V_{2,3}+V_{2,4}+\cdots+V_{2,n} = V_2$

above eqn becomes

$\therefore \quad 2W_E = Q_1V_1 + Q_2V_2 + Q_3V_3 + \cdots$

$$\boxed{W_E = \frac{1}{2}\sum_{m=1}^{n} Q_m V_m} \quad \text{Joules.}$$

This is the potential energy stored in the system.

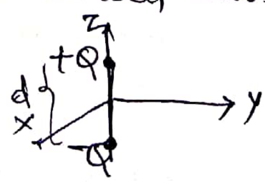
For line charge $\rho_L \Rightarrow W_E = \frac{1}{2}\int \rho_L\, dl\, V$ Joules.

For Surface charge $\rho_s \Rightarrow W_E = \frac{1}{2}\int \rho_s\, ds\, V$ Joules

For Volume charge $\rho_v \Rightarrow W_E = \frac{1}{2}\int \rho_v\, dv\, V$ Joules.

Electric Dipole:

The two point charges of equal magnitude but opposite sign, seperated by a very small distance give rise to an electric dipole.



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Energy stored interms of $\bar{D}$ and $\bar{E}$:

Total energy stored with charge density ρ_v C/m³

$$W_E = \frac{1}{2}\int_{vol} \rho_v \, dv \, V \qquad \text{—— (1)}$$

From maxwell eqn $\nabla \cdot \bar{D} = \rho_v$

$$\therefore W_E = \frac{1}{2}\int_{vol} (\nabla \cdot \bar{D}) \, dv \, V$$

vector identity

$$\nabla \cdot V\bar{A} = \bar{A} \cdot \nabla V + V(\nabla \cdot \bar{A})$$

$$(\nabla \cdot \bar{A})V = \nabla \cdot V\bar{A} - \bar{A} \cdot \nabla V$$

$$= \frac{1}{2}\int_{vol} (\nabla \cdot V\bar{D} - \bar{D} \cdot \nabla V) \, dv$$

$$W_E = \frac{1}{2}\int_{vol} (\nabla \cdot V\bar{D}) \, dv - \frac{1}{2}\int_{vol} (\bar{D} \cdot \nabla V) \, dv$$

using divergence theorem for first term

$$W_E = \frac{1}{2}\oint_S (V\bar{D}) \cdot \bar{ds} - \frac{1}{2}\int (\bar{D} \cdot \nabla V) \, dv \qquad \text{—— (2)}$$

we know that $V \propto \frac{1}{r}$ and $\bar{D} \propto \frac{1}{r^2}$ for point charge,

$V \propto \frac{1}{r^2}$ and $\bar{D} \propto \frac{1}{r^3}$ for dipoles

So, $V\bar{D}$ is $\propto \frac{1}{r} \cdot \frac{1}{r^2} \Rightarrow V\bar{D}$ is $\propto \frac{1}{r^3}$

$\bar{ds}$ is varies as r^2.

So, total integral varies $\frac{1}{r^3} \cdot r^2$ i.e. $\frac{1}{r}$. As surface becomes very large i.e. $r = \infty$ ~~and~~ then $\frac{1}{r} \to 0$. Hence closed surface integral is zero.

$\therefore$ eqn (2) becomes $W_E = -\frac{1}{2}\int (\bar{D} \cdot \nabla V) \, dv$

but $\bar{E} = -\nabla V$

$$\therefore W_E = -\frac{1}{2}\int_{vol} \bar{D} \cdot (-\bar{E}) \, dv \Rightarrow \boxed{W_E = \frac{1}{2}\int_{vol} \bar{D} \cdot \bar{E} \, dv}$$

but $\bar{D} = \varepsilon_0 \bar{E}$ $\quad \therefore W_E = \frac{1}{2}\int \varepsilon_0 \bar{E} \cdot \bar{E} \, dv$

$$\boxed{W_E = \frac{1}{2}\int_{vol} \varepsilon_0 \bar{E}^2 \, dv} \text{ Joule.}$$

$$\boxed{W_E = \frac{1}{2}\int_{vol} \frac{D^2}{\varepsilon_0} \, dv} \text{ Joules.}$$

In differential form

$$dW_E = \frac{1}{2} \bar{D} \cdot \bar{E} \, dv$$

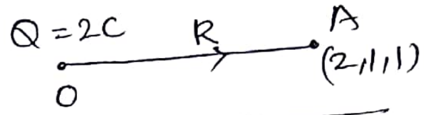
$$\boxed{\frac{dW_E}{dv} = \frac{1}{2} \bar{D} \cdot \bar{E}} \text{ J/m}^3$$

This is called Energy density.

1) a) A point charge $Q = 2C$ is located at the origin. obtain the absolute potential of $A(2,1,1)$

b) If same charge i.e. $Q = 2C$ is located at $P(2,2,3)$ then obtain absolute potential of $A(2,1,1)$.

Sol a)



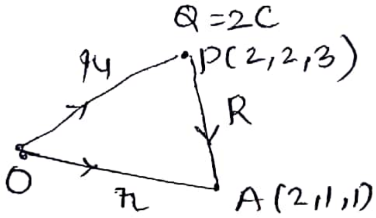
$$V_A = \frac{Q}{4\pi\varepsilon_0 R}$$

$$R = \sqrt{(2)^2+(1)^2+(1)^2} = \sqrt{6}$$

$$= \frac{2}{4\pi \times 8.854 \times 10^{-12} \times \sqrt{6}}$$

$$= 7.34 \times 10^{9}\ V$$

b)



$$V_A = \frac{Q}{4\pi\varepsilon_0 R}$$

$$R = \sqrt{(0)^2+(1)^2+(2)^2} = \sqrt{5}$$

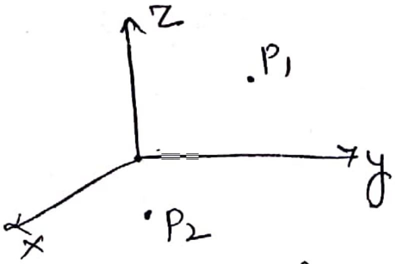
$$= \frac{2}{4\pi \times 8.854 \times 10^{-12} \times \sqrt{5}}$$

$$= 8.04 \times 10^{9}\ V.$$

2) An electric field is given by $\bar{E} = 6y^2z\,\bar{a}_x + 12xyz\,\bar{a}_y + 6xy^2\,\bar{a}_z$ V/m and $\bar{\Delta L} = -3\bar{a}_x + 5\bar{a}_y - 2\bar{a}_z$ μm. Find the workdone in moving a $2\mu C$ ~~charging~~ charge along this path location ~~of~~ at i) $P_1(0,3,5)$ ii) $P_2(1,1,0)$

Sol

$$W = -Q_t \int_B^A \bar{E}\cdot \bar{dL}$$



$$dW = -Q_t\, \bar{E}\cdot\bar{dL}$$

$$= -2\times10^{-6}\left[(6y^2z\,\bar{a}_x + 12xyz\,\bar{a}_y + 6xy^2\bar{a}_z)\cdot(-3\bar{a}_x + 5\bar{a}_y - 2\bar{a}_z)\right]10^{-6}$$

$$dW = -2\times10^{-6}\left[-18y^2z + 60xyz - 12xy^2\right]10^{-6}$$

i) At $P_1(0,3,5) \Rightarrow dW = -2\times10^{-6}\left[-18(3)^2 5 + 60(0) - 12(0)\right]10^{-6}$

$$= -2\times10^{-12}\left[-18(9)5\right] = 1620\ P\ \text{Joule}.$$

ii) At $P_2(1,1,0) \Rightarrow dW = -2\times10^{-12}\left[-18(0) + 60(0) - 12(1)(1)\right]$

$$= -2\times10^{-12}\left[-12\right] = 24\ P\ \text{Joules}$$

...he potential distribution is given by $V = \frac{4}{x^2+y+z^2}$

Find the expression for $\bar{E}$, $\bar{D}$ & ρ_v at $P(1,1,2)$. (45)

Sol:

$$\bar{E} = -\nabla V$$

$$= -\left[\frac{\partial V}{\partial x}\bar{a}_x + \frac{\partial V}{\partial y}\bar{a}_y + \frac{\partial V}{\partial z}\bar{a}_z\right]$$

$$= -\frac{\partial}{\partial x}\left[\frac{4}{x^2+y+z^2}\right]\bar{a}_x + \frac{\partial}{\partial y}\left[\frac{4}{x^2+y+z^2}\right]\bar{a}_y + \frac{\partial}{\partial z}\left[\frac{4}{x^2+y+z^2}\right]\bar{a}_z$$

$$= -\left[\frac{-4}{(x^2+y+z^2)^2}\cdot 2x\,\bar{a}_x + \frac{-4}{(x^2+y+z^2)^2}\cdot 1\,\bar{a}_y + \frac{-4\cdot 2z\bar{a}_z}{(x^2+y+z^2)^2}\right]$$

$$\bar{E} = \frac{4}{(x^2+y+z^2)^2}\left(2x\,\bar{a}_x + \bar{a}_y + 2z\,\bar{a}_z\right) \qquad \left[\because \frac{d}{dx}\left(\frac{1}{x}\right) = \frac{-1}{x^2}\right]$$

$\bar{E}$ at $P(1,1,2) \Rightarrow$ $\bar{E} = \frac{4}{(1+1+4)^2}\left(2\bar{a}_x + \bar{a}_y + 4\bar{a}_z\right)$

$$= 0.22\,\bar{a}_x + 0.11\,\bar{a}_y + 0.44\,\bar{a}_z \ \ V/m$$

$$\bar{D} = \varepsilon_0 \bar{E}$$

$$= 8.854\times10^{-12}\left[0.22\bar{a}_x + 0.11\bar{a}_y + 0.44\bar{a}_z\right]$$

$$\bar{D} = 1.95\,\bar{a}_x + 0.974\,\bar{a}_y + 3.9\,\bar{a}_z \ \ PC/m^2$$

$$\rho_v = \nabla\cdot\bar{D}$$

$$\cancel{= \left[\frac{\partial}{\partial x}\bar{a}_x + \frac{\partial}{\partial y}\bar{a}_y + \frac{\partial}{\partial z}\bar{a}_z\right]\cdot\left[1.95\bar{a}_x + 0.974\bar{a}_y + 3.9\bar{a}_z\right]10^{-12}}$$

$$= \nabla\cdot\varepsilon_0\bar{E}$$

$$\nabla\cdot\bar{E} = \left[\frac{\partial}{\partial x}\bar{a}_x + \frac{\partial}{\partial y}\bar{a}_y + \frac{\partial}{\partial z}\bar{a}_z\right]\cdot\left[\frac{8x}{(x^2+y+z^2)^2}\bar{a}_x + \frac{4}{(x^2+y+z^2)^2}\bar{a}_y + \frac{8z\,\bar{a}_z}{(x^2+y+z^2)}\right]$$

$$= \frac{\partial}{\partial x}\left[\frac{8x}{(x^2+y+z^2)^2}\right] + \frac{\partial}{\partial y}\left[\frac{4}{(x^2+y+z^2)^2}\right] + \frac{\partial}{\partial z}\left[\frac{8z}{(x^2+y+z^2)^2}\right]$$

$$= \left\{\frac{(x^2+y+z^2)^2 8 - 8x(2(x^2+y+z^2))2x}{(x^2+y+z^2)^4}\right\} + \left\{\frac{(x^2+y+z^2)^2 0 - 4\cdot 2(x^2+y+z^2)}{(x^2+y+z^2)^4}\right\}$$

$$+ \left\{\frac{(x^2+y+z^2)^2 8 - 8z\cdot 2(x^2+y+z^2)2z}{(x^2+y+z^2)^4}\right\}$$

$\nabla\cdot\bar{E}$ at $P(1,1,2) \Rightarrow \nabla\cdot\bar{E} = \frac{\left[(6)^2 8 - 8\cdot 2(6)2 + 0 - 4\cdot 2\cdot 6 + (6)^2 8 - 8\cdot 2\cdot 2(6)2(2)\right]}{(1+1+4)^4}$

$$= \frac{288 - 192 - 48 + 288 - 768}{1296} = -0.33$$

$$\rho_v = \varepsilon_0\left[\nabla\cdot\bar{E}\right]$$

$$= 2.95\ PC/m^3$$

4) A non uniform field $\bar{E}$ is given by $\bar{E} = y\bar{a}_x + x\bar{a}_y + z\bar{a}_z$ and determine the work experienced in carrying 2C charge from B(0.1,0,1) to A(0.8,0.6,1) along the shorter arc of the circle $x^2+y^2=1$ and $z=1$.

Sol

$$W = -Q\int_B^A \bar{E}\cdot d\bar{l}$$

$$= -2\int_B^A (y\bar{a}_x + x\bar{a}_y + z\bar{a}_z)\cdot(dx\bar{a}_x + dy\,\bar{a}_y + dz\bar{a}_z)$$

$$= -2\int_B^A y\,dx + x\,dy + z\,dz$$

$$= -2\int_{1}^{0.8} y\,dx - 2\int_0^{0.6} x\,dy - 2\int_1^1 z\,dz$$

$$= -2\int_1^{0.8}\sqrt{1-x^2}\,dx - 2\int_0^{0.6}\sqrt{1-y^2}\,dy - 2\int_1^1 z\,dz$$

$$= -2\left[\frac{1}{\cancel{2}\sqrt{1-x^2}}(-\cancel{2}x)\right]_{0.1}^{0.8} - 2\left[\frac{1}{\cancel{2}\sqrt{1-y^2}}(-\cancel{2}y)\right]_0^{0.6} - 2\left[\frac{z^2}{2}\right]_1^1$$

$$= 2\left[\frac{0.8}{\sqrt{1-(0.8)^2}} - \frac{0.1}{\sqrt{1-(0.1)^2}}\right] + \left[\frac{0.6}{\sqrt{1-(0.6)^2}} - 0\right] - 2[0]$$

$$= 2[1.33 - 0.10] + 0.75$$

$$W = 3.22 \text{ Joules}$$

5) If $V = x - y + xy + 2z$ V, find $\bar{E}$ at (1,2,3) and the energy stored in a cube of side 2m centred at the origin.

Sol.

$$\bar{E} = -\nabla V$$

$$= -\left[\frac{\partial V}{\partial x}\bar{a}_x + \frac{\partial V}{\partial y}\bar{a}_y + \frac{\partial V}{\partial z}\bar{a}_z\right]$$

$$\frac{\partial V}{\partial x} = 1+y, \quad \frac{\partial V}{\partial y} = -1+x, \quad \frac{\partial V}{\partial z} = 2$$

$$\therefore \bar{E} = -[(1+y)\bar{a}_x + (x-1)\bar{a}_y + 2\bar{a}_z]$$

At (1,2,3) $\Rightarrow$ $\bar{E} = -[(1+2)\bar{a}_x + (1-1)\bar{a}_y + 2\bar{a}_z]$

$$\bar{E} = -3\bar{a}_x + 2\bar{a}_z \text{ V/m.}$$

$$W_E = \frac{1}{2}\int_{\text{volume}} \varepsilon_0 |E|^2\,dV$$

$$|\bar{E}| = \sqrt{(1+y)^2 + (x-1)^2 + 4}$$

$$dv = dx\,dy\,dz$$

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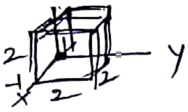
Cube is centred at origin 

(46)

$$\therefore W_E = \frac{\varepsilon_0}{2}\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}\left(\sqrt{(1+y)^2+(x-1)^2+4}\right)^2 dx\,dy\,dz.$$

$$= \frac{\varepsilon_0}{2}\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}(1+y^2+2y+x^2+1-2x+4)\,dx\,dy\,dz$$

$$= 2\times2\times\cancel{2}\times\frac{\varepsilon_0}{\cancel{2}}\int_{0}^{1}\int_{0}^{1}\int_{0}^{1}(6+x^2-2x+y^2+2y)\,dx\,dy\,dz$$

$$= 4\frac{\varepsilon_0}{2}\int_{y=0}^{1}\int_{z=0}^{1}\left(6x+\frac{x^3}{3}-\frac{2x^2}{2}+xy^2+2xy\right)dy\,dz$$

$$= 4\frac{\varepsilon_0}{2}\int_{z=0}^{1}\left(6xy+\frac{x^3}{3}y-x^2y+\frac{xy^3}{3}+\frac{2xy^2}{2}\right)dz.$$

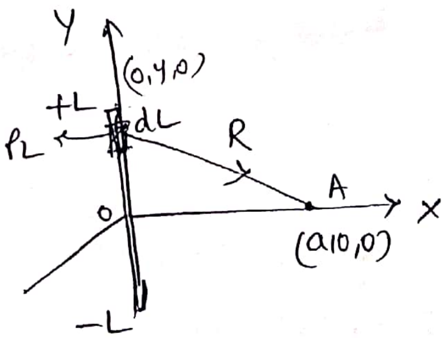
$$= 4\frac{\varepsilon_0}{2}\left[6xyz+\frac{x^3}{3}yz-x^2yz+\frac{xy^3z}{3}+xy^2z\right]_0^1$$

$$= 4\varepsilon_0\left[6+\frac{1}{3}-1+\frac{1}{3}+1\right] = 4\times8.854\times10^{-12}\times(6.67)$$
$$= 0.236\ \text{nJ}$$

6) A uniform line charge density ρ_L C/m is existing from $-L$ to $+L$ on y-axis. Find potential at $A(a,0,0)$.

Sol

$$dV_A = \frac{dQ}{4\pi\varepsilon_0 R} \quad \text{—①}$$



$$dQ = \rho_L\,dl = \rho_L\,dy$$
$$R = \sqrt{a^2+y^2}$$

Sub in ①

$$dV_A = \frac{\rho_L\,dl}{4\pi\varepsilon_0\sqrt{a^2+y^2}} = \frac{\rho_L\,dy}{4\pi\varepsilon_0\sqrt{a^2+y^2}}$$

$$V = \int_{y=-L}^{L}\frac{\rho_L\,dy}{4\pi\varepsilon_0\sqrt{a^2+y^2}} = 2\int_{y=0}^{L}\frac{\rho_L\,dy}{4\pi\varepsilon_0\sqrt{a^2+y^2}}$$

$$\int\frac{dx}{\sqrt{a^2+x^2}} = \ln\left[x+\sqrt{x^2+a^2}\right]$$
Std Result

$$= \frac{2\rho_L}{4\pi\varepsilon_0}\left[\ln\left[y+\sqrt{a^2+y^2}\right]\right]_{y=0}^{L}$$

$$= \frac{2\rho_L}{4\pi\varepsilon_0}\left[\ln\left(L+\sqrt{a^2+L^2}\right)-\ln a\right]$$

$$V = \frac{\rho_L}{2\pi\varepsilon_0}\ln\left[\frac{L+\sqrt{L^2+a^2}}{a}\right] \text{ volts}$$

7) An electric field is given by $E = 4x\,a_x + 2\,a_y$. Find the workdone to move a unit positive charge along the curve $xy = 4$ from $(2,2)$ to $(4,1)$.

Sol:

$$W = -Q\int_B^A \bar{E}\cdot \bar{dl}$$

$$= -Q\int_B^A (4x\bar{a}_x + 2\bar{a}_y)\cdot(dx\,\bar{a}_x + dy\,\bar{a}_y + dz\,\bar{a}_z)$$

$$= -Q\int_B^A 4x\,dx + 2\,dy$$

$$= -Q\left[\int_{x=2}^{4} 4x\,dx + \int_2^1 2\,dy\right]$$

$$= -Q\left[\left[\frac{4x^2}{2}\right]_2^4 + 2[y]_2^1\right] = -Q\left[(32-8) + 2(1-2)\right]$$

$$= -1.[22] = -22\,J$$

OR

$$xy = 4$$

$$y = \frac{4}{x}$$

$$dy = -\frac{4}{x^2}\,dx$$

$$W = -Q\left[\int_B^A 4x\,dx + 2\,dy\right]$$

$$= -Q\left[\int_2^4 4x\,dx + \int_2^4 2\cdot\left(-\frac{4}{x^2}\right)dx\right]$$

$$= -1\left[4\left(\frac{x^2}{2}\right)_2^4 + (-8)\left[-\frac{1}{x}\right]_2^4\right]$$

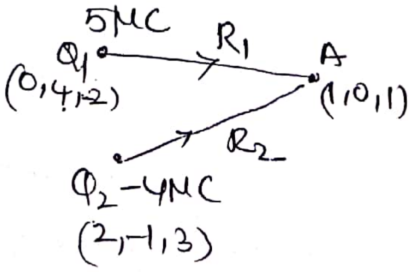
$$= -1\left[4(8-2) + (-8)\left(-\frac{1}{4} + \frac{1}{2}\right)\right]$$

$$= -1[24-2] = -22\,J$$

8) Two point charges $-4\mu C$ and $5\mu C$ are located at $(2,-1,3)$ and $(0,4,-2)$. Find the potential at $(1,0,1)$ assuming zero potential at infinity.

Sol

$$V = \frac{Q_1}{4\pi\varepsilon_0 R_1} + \frac{Q_2}{4\pi\varepsilon_0 R_2}$$



$$= \frac{1}{4\pi\varepsilon_0}\left[\frac{Q_1}{R_1} + \frac{Q_2}{R_2}\right]$$

$$= \frac{1}{4\pi\varepsilon_0}\left[\frac{5}{\sqrt{26}} + \frac{-4}{\sqrt{6}}\right]10^{-6}$$

$$V = 5.86\ \text{K volts}$$

$$R_1 = \sqrt{1^2 + 4^2 + 3^2} = \sqrt{26}$$

$$R_2 = \sqrt{1^2 + 1^2 + 2^2} = \sqrt{6}$$

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Boundary Conditions in Electrostatic Fields

When Electric field passes from one medium to other medium, it is necessary to ~~study~~ study the conditions at boundary b/w two medias.

The conditions existing at boundary when field passes from one media to other media are called boundary conditions.

NOTE: To determine the boundary conditions let us use Gaussian Surface and closed path.

To determine the boundary conditions, we use maxwell's eqns for electrostatic field.

i.e.

$$\boxed{\begin{array}{l} 1)\ \oint_S \bar{D}\cdot\bar{ds} = Q \\ 2)\ \oint_L \bar{E}\cdot\bar{dL} = 0 \end{array}}$$

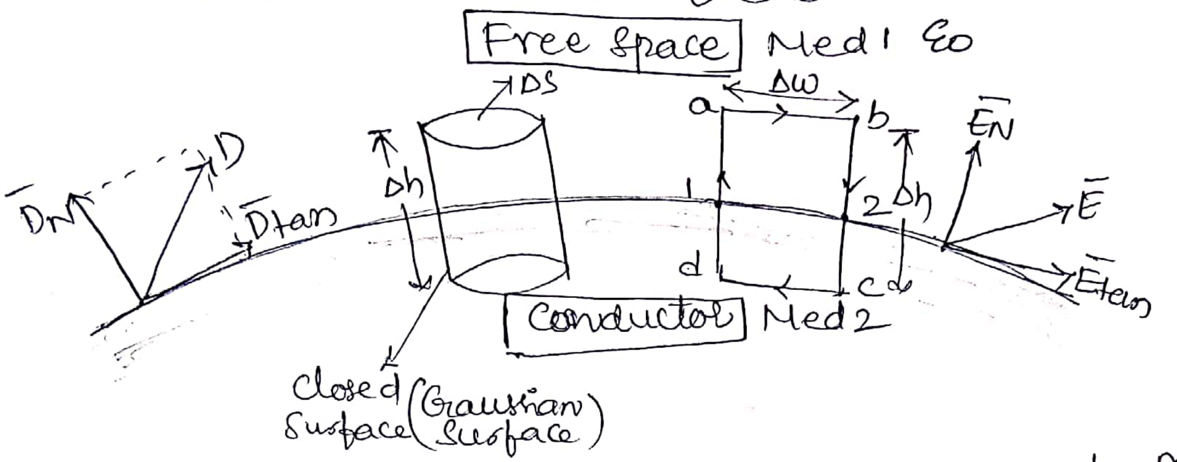
Total $\bar{E}$ & $\bar{D}$ at boundary is calculated using its components.

$$\boxed{\begin{array}{l} \bar{E} = \bar{E}_{tan} + \bar{E}_N \\ \bar{D} = \bar{D}_{tan} + \bar{D}_N \end{array}}$$

For Ideal conductors

1) The $\bar{E}$ and $\bar{D}$ inside a conductor is zero.
2) No charge can exist within a conductor, the charge appears on the surface in the form of surface charge density. ρ_s.
3) The charge density within the conductor is zero.

1) Boundary Conditions between Conductor and Free Space.



Assuming field is passing from medium 1 to med 2.

i) $\bar{E}$ at boundary (Tangential Component)

$$\oint \bar{E}\cdot \bar{dl} = 0$$

$$\int_a^b \bar{E}\cdot\bar{dl} + \int_b^c \bar{E}\cdot\bar{dl} + \int_c^d \bar{E}\cdot\bar{dl} + \int_d^a \bar{E}\cdot\bar{dl} = 0 \quad \text{--- (1)}$$

within conductor $\bar{E} = 0$.

$$\therefore \int_c^d \bar{E}\cdot\bar{dl} = 0$$

$\therefore$ eqn (1) becomes

$$\int_a^b \bar{E}\cdot\bar{dl} + \int_b^c \bar{E}\cdot\bar{dl} + \int_d^a \bar{E}\cdot\bar{dl} = 0 \quad \text{--- (2)}.$$

$$\int_a^b \bar{E}\cdot\bar{dl} = \int_a^b E_{tan}\, \bar{a}_{tan} \cdot \Delta w\, \bar{a}_{tan} = E_{tan_1}(\Delta w) \quad \text{--- (3)}$$

$$\int_b^c \bar{E}\cdot\bar{dl} = \int_b^2 \bar{E}\cdot\bar{dl} + \int_2^c \bar{E}\cdot\bar{dl}$$

$$= \int_b^2 E_{N_1}\, \bar{a}_N \cdot \frac{\Delta h}{2}\, \bar{a}_N + 0 = E_{N_1}\frac{\Delta h}{2} \quad \text{--- (4)}$$

$$\int_d^a \bar{E}\cdot\bar{dl} = \int_d^1 \bar{E}\cdot\bar{dl} + \int_1^a \bar{E}\cdot\bar{dl}$$

$$= 0 + \int_1^a E_{N_1}\, \bar{a}_N \cdot \frac{\Delta h}{2}\, \bar{a}_N = -E_{N_1}\frac{\Delta h}{2} \quad \text{--- (5)}$$

Sub ③④⑤ in ②

$$E_{tan1}\,\Delta w + E_{N1}\,\cancel{\frac{\Delta h}{2}} - E_{N1}\cancel{\frac{\Delta h}{2}} = 0$$

$$E_{tan1}\,\Delta w = 0$$

$$\Delta w \neq 0$$

$$\boxed{E_{tan1} = 0}$$

The tangential component of Electric field Intensity is zero at the boundary.

$$\bar{D} = \varepsilon_0 \bar{E}$$

$$\bar{D}_{tan1} = \varepsilon_0 \bar{E}_{tan1}$$

$$\boxed{D_{tan1} = 0}$$

The tangential component of $\bar{D}$ at the boundary is zero.

ii) D_N at the boundary (Normal component).

From Gauss's law $\oint \bar{D}\cdot\bar{ds} = Q$

$$\oint_{top} \bar{D}\cdot\bar{ds} + \oint_{bottom} \bar{D}\cdot\bar{ds} + \oint_{lateral} \bar{D}\cdot\bar{ds} = Q \quad \text{——— ①}$$

$$\oint_{top} \bar{D}\cdot\bar{ds} = D_{N1}\int_{top} \bar{ds} = D_{N1}\,\Delta S \quad \text{——— ②}$$

$$\oint_{bottom} \bar{D}\cdot\bar{ds} = 0 \quad \text{——— ③}$$

$$\oint_{lateral} \bar{D}\cdot\bar{ds} = 0 \quad \text{——— ④}$$

[∵ As $\Delta h \to 0$ surface comes towards at boundary. So, there is no lateral surface]

Sub ② ③ ④ in ①

$$D_{N1}\,\Delta S + 0 + 0 = Q \Rightarrow D_{N1} = \frac{Q}{\Delta S}$$

$$\boxed{D_{N1} = \rho_s}$$

The normal component of $\bar{D}$ at the boundary is equal to the surface charge density ρ_s.

$$D_{N1} = \varepsilon_0 E_{N1} \Rightarrow E_{N1} = \frac{D_{N1}}{\varepsilon_0} \Rightarrow \boxed{E_{N1} = \frac{\rho_s}{\varepsilon_0}}$$

$\therefore \bar{E} = \bar{E}_{tan} + \bar{E}_N$ $\qquad\qquad$ $\therefore \bar{D} = \bar{D}_{tan} + \bar{D}_N$

$= 0 + \frac{\rho_s}{\varepsilon_0}$ $\qquad\qquad$ $= 0 + \rho_s$

$$\boxed{\bar{E} = \frac{\rho_s}{\varepsilon_0}} \qquad \boxed{\bar{D} = \rho_s}$$

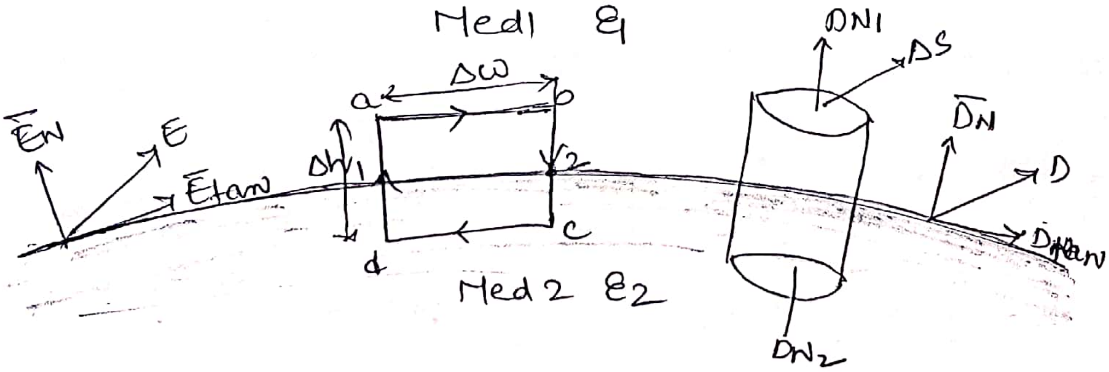
2) Boundary conditions between conductor and dielectric.

$E_{tan} = D_{tan} = 0$

$D_N = \rho_s$

$$\boxed{\bar{E} = \frac{\rho_s}{\varepsilon} = \frac{\rho_s}{\varepsilon_0 \varepsilon_r}} \qquad \boxed{\bar{D} = \rho_s}$$

3) Boundary conditions between two perfect Dielectrics



Assume field is entering from med1 and leaving from medium 2.

(i) Tangential component at boundary

$$\oint \bar{E} \cdot \bar{dL} = 0$$

$$\int_a^b \bar{E} \cdot \bar{dL} + \int_b^c \bar{E} \cdot \bar{dL} + \int_c^d \bar{E} \cdot \bar{dL} + \int_d^a \bar{E} \cdot \bar{dL} = 0 \quad \text{--- (1)}$$

$$E_{tan1} \int_a^b dL + \int_b^2 \bar{E} \cdot \bar{dL} + \int_2^c \bar{E} \cdot \bar{dL} + (-E_{tan2}) \int_c^d dL + \int_d^1 \bar{E} \cdot \bar{dL} + \int_1^a \bar{E} \cdot \bar{dL} = 0$$

$$E_{tan1} \Delta W + E_{N1} \frac{\Delta h}{2} + E_{N2} \frac{\Delta h}{2} - E_{tan2} \Delta W - E_{N2} \frac{\Delta h}{2} - E_{N1} \frac{\Delta h}{2} = 0$$

$$E_{tan1}\,\Delta w - E_{tan2}\,\Delta w = 0$$

$$(E_{tan1} - E_{tan2})\,\Delta w = 0$$

$$\Delta w \neq 0$$

$$E_{tan1} - E_{tan2} = 0$$

$$\boxed{E_{tan1} = E_{tan2}}$$

∴ Tangential component of $\bar{E}$ at the boundary is same in both dielectrics i.e. $\bar{E}$ is continuous across the boundary.

$$D = \varepsilon E$$

$$D_{tan1} = \varepsilon_1 E_{tan1} \quad \& \quad D_{tan2} = \varepsilon_2 E_{tan2}$$

$$\frac{D_{tan1}}{\varepsilon_1} = \frac{D_{tan2}}{\varepsilon_2}$$

$$\boxed{\frac{D_{tan1}}{D_{tan2}} = \frac{\varepsilon_1}{\varepsilon_2}}$$

∴ $\bar{D}$ is discontinuous at the boundary.

(ii) <u>Normal component</u>

$$\oint \bar{D}\cdot\bar{ds} = Q$$

$$\oint_{top} \bar{D}\cdot\bar{ds} + \oint_{bottom} \bar{D}\cdot\bar{ds} + \int_{lateral} \bar{D}\cdot\bar{ds} = Q \quad \text{——} (i)$$

$$D_{N1}\,\Delta S - D_{N2}\,\Delta S + 0 = Q. \qquad \left(\text{As } \Delta h \to 0 \;\; \oint_{lateral} \bar{D}\cdot\bar{ds} = 0\right).$$

$$(D_{N1} - D_{N2})\,\Delta S = Q$$

$$D_{N1} - D_{N2} = \frac{Q}{\Delta S} = \rho_s \Rightarrow \boxed{D_{N1} - D_{N2} = \rho_s}$$

For ideal dielectric med. $\rho_s = 0$

$$D_{N1} - D_{N2} = 0 \Rightarrow \boxed{D_{N1} = D_{N2}}$$

∴ Normal component is continuous at boundary

$$\varepsilon_1 E_{N1} = \varepsilon_2 E_{N2}$$

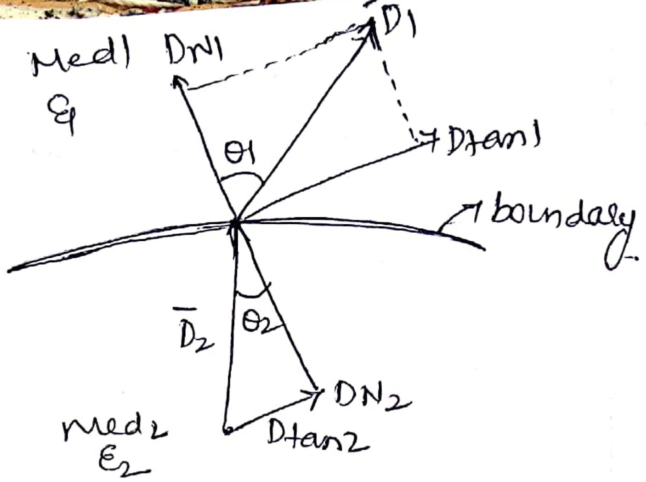
$$\boxed{\frac{E_{N1}}{E_{N2}} = \frac{\varepsilon_2}{\varepsilon_1}}$$

∴ Normal component of $\bar{E}$ is discontinuous

$\bar{E} = \bar{E}_{tan} + \bar{E}_N$

$\bar{E} = \bar{E}_1 + \bar{E}_2 \quad E_1 = E_{tan1} + E_{N1} \;\&\; E_2 = E_{tan2} + E_{N2}$ //

Law of refraction



$$\cos\theta_1 = \frac{D_{N1}}{D_1} \Rightarrow D_{N1} = D_1\cos\theta_1$$

$$\cos\theta_2 = \frac{D_{N2}}{D_2} \Rightarrow D_{N2} = D_2\cos\theta_2$$

But at boundary

$$D_{N1} = D_{N2}$$

$$\therefore D_1\cos\theta_1 = D_2\cos\theta_2 \quad \text{--- (1)}$$

Tangential component at boundary

$$\frac{D_{tan1}}{D_{tan2}} = \frac{\varepsilon_1}{\varepsilon_2} \quad \text{--- (2)}$$

From fig $\sin\theta_1 = \frac{D_{tan1}}{D_1}$ & $\sin\theta_2 = \frac{D_{tan2}}{D_2}$

$\therefore$ eqn (2) becomes

$$\frac{D_1\sin\theta_1}{D_2\sin\theta_2} = \frac{\varepsilon_1}{\varepsilon_2}$$

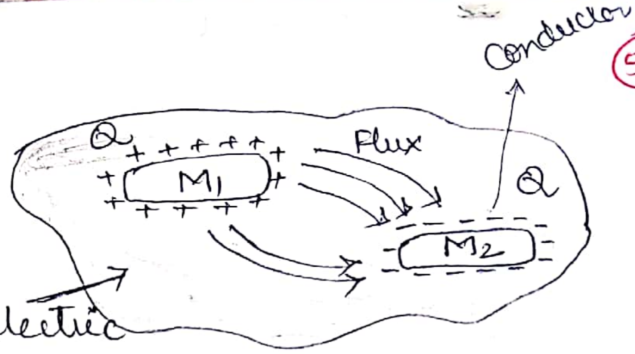
From fig $\tan\theta_1 = \frac{D_{tan1}}{D_{N1}}$ and $\tan\theta_2 = \frac{D_{tan2}}{D_{N2}}$

$$\frac{\tan\theta_1}{\tan\theta_2} = \frac{D_{tan1}/D_{N1}}{D_{tan2}/D_{N2}}$$

$$= \frac{D_{tan1}}{D_{tan2}} \cdot \frac{\cancel{D_{N2}}}{\cancel{D_{N1}}} \quad (\text{from (1)})$$

$$\boxed{\frac{\tan\theta_1}{\tan\theta_2} = \frac{\varepsilon_1}{\varepsilon_2}} \quad (\text{from (2)})$$

Capacitance :



Conducting materials M_1 and M_2 Carries +ve and -ve charge, equal magnitude as Q.

Total charge of the system is zero.

In conductors charge can not reside within the conductor and it resides only one the surface.

Potential difference between M_1 and M_2 is V_{12}

The ratio of the magnitude of the total charge to the potential difference is known as capacitance.

$$\boxed{C = \frac{Q}{V_{12}}}$$

in general $C = \frac{Q}{V}$

From Gauss's law

$$\oint \bar{D} \cdot \bar{ds} = Q$$

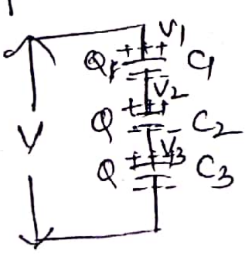
$$Q = \int_S \varepsilon \bar{E} \cdot \bar{ds} \quad \text{—— (1)}$$

But $V = -\int_L \bar{E} \cdot \bar{dL}$ —— (2)

$$\therefore C = \frac{Q}{V}$$

$$\boxed{C = \frac{\oint \varepsilon \bar{E} \cdot \bar{ds}}{-\int_{-}^{+} \bar{E} \cdot \bar{dL}}}$$

Capacitance are in Series

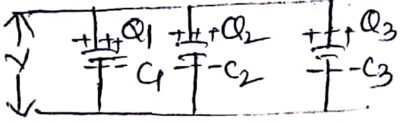


$$V = V_1 + V_2 + V_3$$

$$\frac{Q}{C_{eq}} = \frac{Q}{C_1} + \frac{Q}{C_2} + \frac{Q}{C_3}$$

$$\boxed{\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}}$$

Capacitance in parallel.



$$Q = Q_1 + Q_2 + Q_3$$

$$C_{eq} V = C_1 V + C_2 V + C_3 V$$

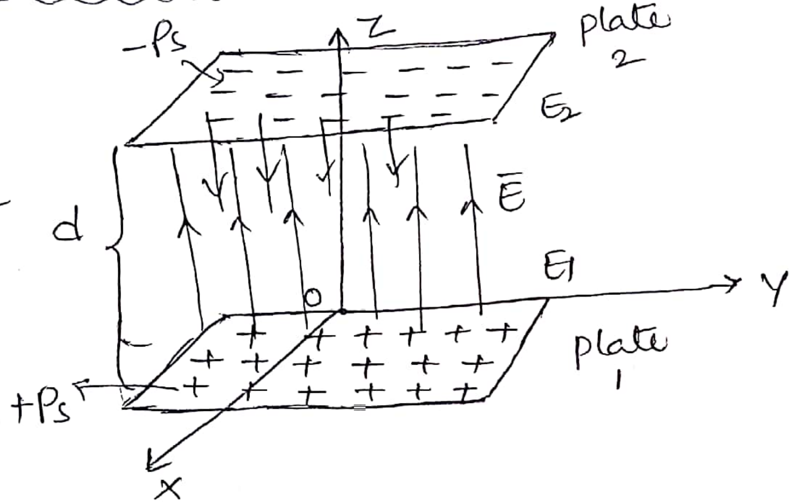
$$\boxed{C_{eq} = C_1 + C_2 + C_3}$$

Capacitance Calculations in

1) Between two parallel plates
2) Coaxial Cable
3) Spherical Capacitor
4) Two wire line.

1) capacitance between two parallel plates

Plate 1 is at $z=0$ having charge density $+\rho_s$; plate 2 is having charge density $-\rho_s$ kept at $z=d$.



$$C=\frac{Q}{V} \quad \text{—} \ (1)$$

$$Q=\rho_s A \quad \text{—} \ (2)$$

$\bar{E}$ due to plate 1

$$\bar{E}_1=\frac{+\rho_s}{2\varepsilon}\bar{a}_N=\frac{\rho_s}{2\varepsilon}\bar{a}_z \quad \text{—} \ (3)$$

$\bar{E}$ due to plate 2

$$\bar{E}_2=\frac{-\rho_s}{2\varepsilon}\bar{a}_N=-\frac{\rho_s}{2\varepsilon}(-\bar{a}_z) \quad \text{—} \ (4)$$

Total $\bar{E}$ between plates

$$\bar{E}=E_1+E_2$$

$$=\frac{\rho_s}{2\varepsilon}\bar{a}_z+\frac{\rho_s}{2\varepsilon}\bar{a}_z=\frac{\rho_s}{\varepsilon}\bar{a}_z \quad \text{—} \ (5)$$

$$V=-\int_{-}^{+}\bar{E}\cdot\bar{dL}=-\int_{-}^{+}\frac{\rho_s}{\varepsilon}\bar{a}_z\cdot\bar{dL}$$

$$=-\int_{z=d}^{0}\frac{\rho_s}{\varepsilon}\bar{a}_z\cdot(dx\,\bar{a}_x+dy\,\bar{a}_y+dz\,\bar{a}_z)$$

$$=-\int_{d}^{0}\frac{\rho_s}{\varepsilon}dz=-\frac{\rho_s}{\varepsilon}[z]_d^0=\frac{\rho_s}{\varepsilon}d$$

But $C=\dfrac{Q}{V}$

$$=\frac{\rho_s A}{\frac{\rho_s}{\varepsilon}d}=\frac{\varepsilon A}{d}$$

$$\boxed{C=\frac{\varepsilon A}{d}}$$

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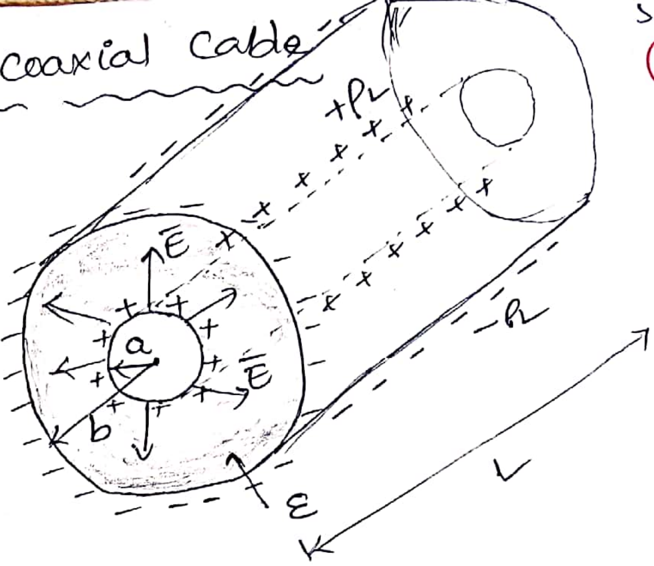
2) Capacitance of a coaxial cable



$$C = \frac{Q}{V} \quad \text{—(1)}$$

$$Q = \int \rho_L \, dL = \cancel{\rho_L L} \quad \text{—(2)}$$

$$V = -\int_{-}^{+} \bar{E} \cdot \bar{dL} \quad \text{—(3)}$$

But $\bar{E}$ due inner conductor

$$\bar{E} = \frac{\rho_L}{2\pi\varepsilon r} \bar{a_r}$$

Sub in (3)

$$V = -\int_{-}^{+} \frac{\rho_L}{2\pi\varepsilon r} \bar{a_r} \cdot dr \, \bar{a_r}$$

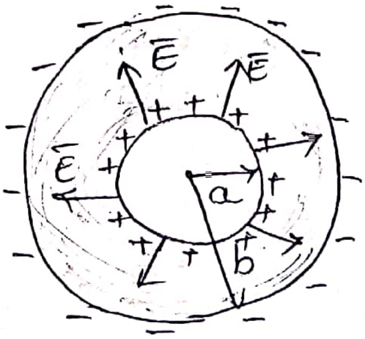
$$= -\frac{\rho_L}{2\pi\varepsilon} \int_{r=b}^{a} \frac{1}{r} dr = \frac{-\rho_L}{2\pi\varepsilon} \left[\log r\right]_b^a = \frac{-\rho_L}{2\pi\varepsilon} \left[\log a - \log b\right]$$

$$= \frac{\rho_L}{2\pi\varepsilon} \log b/a \quad \text{—(4)}$$

Sub (2) (4) in (1)

$$C = \frac{\cancel{\rho_L} L}{\frac{\cancel{\rho_L}}{2\pi\varepsilon} \log b/a} \implies \boxed{C = \frac{2\pi\varepsilon L}{\log b/a}} \text{ Farads.}$$

3) Spherical Capacitor



potential is nothing but workdone in moving charge from outer conductor to inner conductor. $C = \frac{Q}{V}$ —(1).

$$V = -\int_{-}^{+} \bar{E} \cdot \bar{dL}$$

$$= -\int_b^a \frac{Q}{4\pi\varepsilon r^2} \bar{a_r} \cdot dr \, \bar{a_r}$$

$$= -\frac{Q}{4\pi\varepsilon} \int_b^a \frac{1}{r^2} dr$$

$$V = -\frac{Q}{4\pi\varepsilon} \left[-\frac{1}{r}\right]_b^a = \frac{+Q}{4\pi\varepsilon} \left[\frac{1}{a} - \frac{1}{b}\right]$$

$$\therefore C = \frac{Q}{V} = \cancel{Q} / \frac{\cancel{Q}}{4\pi\varepsilon}\left[\frac{1}{a} - \frac{1}{b}\right] \implies \boxed{C = \frac{4\pi\varepsilon}{\left[\frac{1}{a} - \frac{1}{b}\right]}} \text{ Farads.}$$

Capacitance of isolated sphere

outer plate is at infinity (∞)

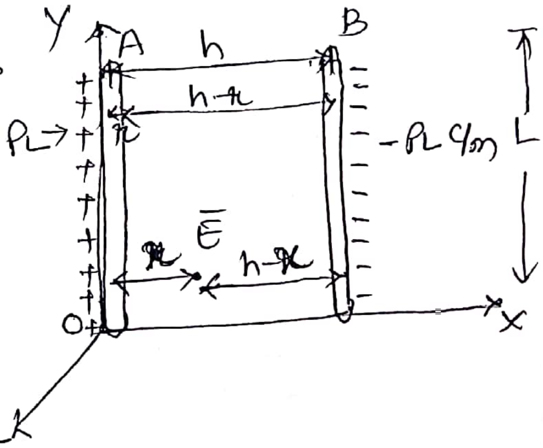
i.e. $b = \infty$

$$\therefore C = \frac{4\pi\varepsilon}{\frac{1}{a} - \frac{1}{b}} = \frac{4\pi\varepsilon}{\frac{1}{a} - \frac{1}{\infty}}$$

$$C = \frac{4\pi\varepsilon}{1/a} \Rightarrow \boxed{C = 4\pi\varepsilon a}$$

4) Capacitance of two wire line

The thickness of the wire is very small.



$$C = \frac{Q}{V_+} \quad \text{—— ①}$$

$$V = -\int \bar{E} \cdot \bar{dl} \quad \text{—— ②}$$

but $\bar{E} = \bar{E_1} + \bar{E_2}$

$$= \frac{\rho_L}{2\pi\varepsilon r}\bar{a_r} + \frac{\rho_L}{2\pi\varepsilon r}\bar{a_r}$$

$$= \frac{+\rho_L}{2\pi\varepsilon x}\bar{a_x} + \frac{-\rho_L}{2\pi\varepsilon(h-x)}(-\bar{a_x})$$

$$\bar{E} = \frac{\rho_L}{2\pi\varepsilon}\left[\frac{1}{x} + \frac{1}{h-x}\right]\bar{a_x}$$

Sub in ②

$$V = -\int_{h-r}^{r} \frac{\rho_L}{2\pi\varepsilon}\left[\frac{1}{x} + \frac{1}{h-x}\right]\bar{a_x} \cdot (dx\,\bar{a_x} + dy\,\bar{a_y} + dz\,\bar{a_z})$$

$$= -\int_{h-r}^{r} \frac{\rho_L}{2\pi\varepsilon}\left[\frac{1}{x} + \frac{1}{h-x}\right] dx$$

$$= -\frac{\rho_L}{2\pi\varepsilon}\Big[\log x + \log(h-x)\,(-1)\Big]_{h-r}^{r}$$

$$= -\frac{\rho_L}{2\pi\varepsilon}\left[\log\frac{x}{h-x}\right]_{h-r}^{r}$$

$$= \frac{-\rho_L}{2\pi\varepsilon}\left[\log\frac{r}{h-r} - \log\frac{h-r}{r}\right]$$

$$= \frac{-\rho_L}{2\pi\varepsilon}\left[\log r - \log(h-r) - \log(h-r) + \log r\right]$$

$$V = +\frac{\rho_L}{\cancel{2}\pi\varepsilon}\cdot\cancel{2}\log\frac{h-r}{r} \quad — (3)$$

Sub (3) in (1)

$$\therefore C = \frac{\cancel{\rho_L}\, L}{\frac{\cancel{\rho_L}}{\pi\varepsilon}\log\frac{h-r}{r}}$$

$$C = \frac{\pi\varepsilon L}{\log\left(\frac{h-r}{r}\right)}$$

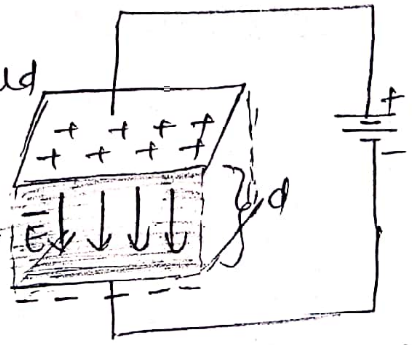
In practical

$h >> r$

$h - r \simeq h$

$$\boxed{\therefore C = \frac{\pi\varepsilon L}{\log\frac{h}{r}}} \quad \text{Farads}$$

Energy stored in a capacitor:



$E = \frac{V}{d}\,\bar{a}_N$ — (1)

energy stored in electrostatic field

$$W_E = \frac{1}{2}\int_{vol} \bar{D}\cdot\bar{E}\, dv$$

$$= \frac{1}{2}\int \varepsilon\, \bar{E}\cdot\bar{E}\, dv$$

$$= \frac{1}{2}\int \varepsilon |E|^2\, dv$$

$$= \frac{1}{2}\varepsilon|E|^2\int_{vol} dv = \frac{1}{2}\varepsilon\frac{V^2}{d^{\cancel{2}}}A\cancel{d}$$ ($\because \int dv = \text{Volume} = A \times d$)

$$= \frac{1}{2}\frac{\varepsilon A}{d}V^2 = \frac{1}{2}CV^2$$

$$\boxed{W_E = \frac{1}{2}CV^2} \text{ Joules.}$$

Energy density

$$W_E = \frac{1}{2}\varepsilon\int|E|^2 dv$$

$$\boxed{\frac{W_E}{dv} = \frac{1}{2}\varepsilon|E|^2} \; J/m^3.$$

energy density

or energy density $= \frac{1}{2}\varepsilon\frac{|D|^2}{\varepsilon^2} \Rightarrow \boxed{\frac{W_E}{V} = \frac{1}{2}\frac{|D|^2}{\varepsilon}}$

$$\boxed{\frac{W_E}{V} = \frac{1}{2}|D||E|}$$

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Current and current Density

"Current is defined as rate of flow of charge and is measured in amperes".

$$\boxed{i = \frac{dQ}{dt}}$$ C/s (or) Amperes.

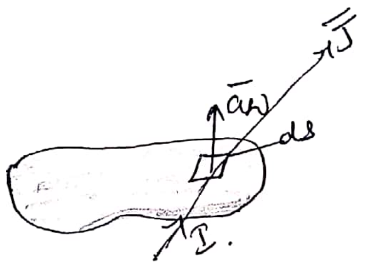
A current of 1 Ampere is said to be flowing across the surface when a charge of one Coulomb is passing across the ~~across~~ ~~the~~ surface in one second.

Current Density (J)

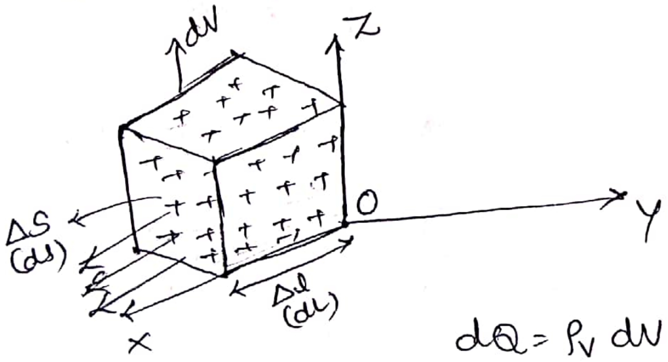
"It is defined as the current passing through the unit surface area, when the surface is held normal to the direction of current."

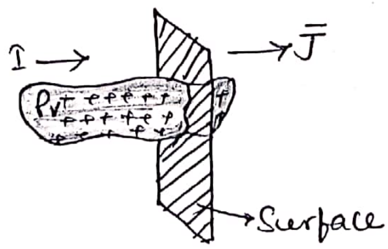
Units → A/m^2

$$\boxed{I = \int_S \bar{J} \cdot \bar{ds}}$$



Relationship between $\bar{J}$ and ρ_v





$$dQ = \rho_v \, dV \quad \text{—} \; ①$$

$$dV = ds \, dl \quad \text{—} \; ②$$

$$\therefore dQ = \rho_v \, ds \, dl$$

Let charge is moving in x-direction with velocity v.

$$dI = \frac{dQ}{dt}$$

$$= \frac{\rho_v \, ds \, dl}{dt}$$

$$= \frac{\rho_v \, ds \, dx}{dt}$$

$$dI = \rho_v \, ds \, v_x \quad \text{—} \; ③$$

But $I = \int \bar{J} \cdot \bar{ds}$

$$dI = \bar{J} \cdot \bar{ds}$$
$$dI = |\bar{J}||\bar{ds}| \quad \text{—} \; ④$$

Compare ④ ③

$$\boxed{\bar{J} = \rho_v \, \bar{v_x}}$$

Continuity equation

"It is based on the principle of conservation of charge".

It states that the charges can neither be created nor be destroyed.

$$I = \oint_S \bar{J} \cdot \bar{ds} \quad \text{——— (1)}$$

The current flows outwards from the closed surface. It has been mentioned that the current means the flow of positive charges.

– According to the principle of conservation of charge, there must be decrease of an equal amount of positive charge inside the closed surface.

Hence the outward rate of flow of positive charge gets balanced by the rate of decrease of charge inside the closed surface.

Let Q_i = charge within the closed surface

$-\frac{dQ_i}{dt}$ = rate of decrease of charge inside the closed surface.

(–ve sign indicates decrease in charge).

Due to principle of conservation of charge

rate of decrease of inside closed surface = current flows outward from the surface

$$-\frac{dQ_i}{dt} = I = \oint_S \bar{J} \cdot \bar{ds} \quad \text{——— (2)}$$

This is Integral form of continuity eqn.
(current is leaving from the surface)

If current is entering the volume then

$$\oint \bar{J} \cdot \bar{ds} = -I = +\frac{dQ_i}{dt} \quad \text{——— (3)}$$

using divergence theorem

$$\oint_S \bar{J}.\bar{ds} = \int_V (\nabla.\bar{J})\,dv$$

$\therefore$ eqn (2) becomes

$$-\frac{dQ_i}{dt} = \int_{vol} (\nabla.\bar{J})\,dv \longrightarrow (4)$$

but $Q_i = \int_{vol} \rho_v\,dv$

$\therefore$ eqn (4) becomes

$$\int_{vol} (\nabla.\bar{J})\,dv = -\frac{d}{dt}\left[\int \rho_v\,dv\right]$$

$$\int_{vol} (\nabla.\bar{J})\,\cancel{dv} = -\int_{vol} \frac{\partial \rho_v}{\partial t}\,\cancel{dv}$$

[$\because$ for a constant surface, the derivative become partial derivative]

$$\boxed{\nabla.\bar{J} = -\frac{\partial \rho_v}{\partial t}}$$

This is point form or differential form of continuity eqn.

For steady state currents which are not functions of time $\frac{\partial \rho_v}{\partial t} = 0$

$$\therefore \boxed{\nabla.\bar{J} = 0}$$

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Poisson's and Laplace's Equation

- $\bar{E}$ and $\bar{D}$ in the given region are obtained using coulomb's law and Gauss's law.
- Using these laws is easy, if the charge distribution or potential throughout the region is known.
- practically it is not possible in many situations, to know the charge distribution or potential variation throughout the region.
- Practically charge and potential may be known at some boundaries of the region only.
- From those values it is necessary to obtain potential and $\bar{E}$ throughout the region. Such electro-static problems are called boundary value problems.
- To solve such problems, poissons and laplace's eqns must be known.

From Gauss's law

$$\nabla \cdot \bar{D} = \rho_v$$

$$\nabla \cdot \varepsilon \bar{E} = \rho_v$$

$$\nabla \cdot \bar{E} = \frac{\rho_v}{\varepsilon} \quad \text{——— (1)}$$

We know that $\bar{E} = -\nabla V$

$$\therefore \quad \nabla \cdot (-\nabla V) = \frac{\rho_v}{\varepsilon}$$

$$\boxed{\nabla^2 V = -\frac{\rho_v}{\varepsilon}} \longrightarrow \text{This is called poisson's eqn.}$$

If $\rho_v = 0$ (for dielectric medium (or) charge free region)

$$\boxed{\nabla^2 V = 0} \longrightarrow \text{Laplace's eqn.}$$

This is special case of poisson's eqn & is called Laplace's equation.

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 \quad \text{(cartesian)}$$

$$\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\left(\frac{\partial^2 V}{\partial \phi^2}\right) + \frac{\partial^2 V}{\partial z^2} = 0 \quad \text{(cylindrical)}$$

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial \phi^2} = 0$$

(spherical)

Uniqueness Theorem:

Solution of Laplaces eqn solved by a method is the uniqueness theorem.

Assume that Laplace's eqn has two solutions say V_1 and V_2. These solutions must satisfy Laplace's eqn.

$$\therefore \nabla^2 V_1 = 0 \text{ and } \nabla^2 V_2 = 0 \quad \text{——} (1)$$

Both the solutions must satisfy the boundary conditions. At the boundary, the potentials at different points are same due to equipotential surface. Then

$$V_1 = V_2 \quad \text{——} (2)$$

Let the difference between the two solutions

$$V_d = V_2 - V_1 \quad \text{——} (3)$$

From Laplace eqn

$$\nabla^2 V_d = 0 \Rightarrow \nabla^2 (V_2 - V_1) = 0$$

$$\therefore \nabla^2 V_d = \nabla^2 V_2 - \nabla^2 V_1 = 0 \quad \text{——} (4)$$

At boundary $V_d = 0$.

From divergence theorem

$$\oint_S \bar{A}\cdot \bar{ds} = \int_{vol} (\nabla\cdot\bar{A})\, dv \quad \text{——} (5)$$

Let $\bar{A} = \underset{\alpha}{V_d}\underset{\beta}{(\nabla V_d)}$

$$\nabla\cdot(\alpha\bar{B}) = \alpha(\nabla\cdot\bar{B}) + \bar{B}\cdot(\nabla\alpha)$$

$\therefore \nabla \cdot (\underbrace{V_d}_{\alpha} \underbrace{\nabla V_d}_{\beta}) = V_d (\nabla \cdot \nabla V_d) + \nabla V_d \cdot \nabla V_d$

$= V_d (\nabla^2 V_d) + \nabla V_d \cdot \nabla V_d$

From (4)

$\nabla \cdot \underbrace{V_d \nabla V_d} = 0 + \nabla V_d \cdot \nabla V_d$

$\nabla \cdot \bar{A} = \nabla V_d \cdot \nabla V_d$

Sub this in (5)

$\int_{vol} (\nabla \cdot V_d \nabla V_d)\, dv = \oint_S V_d \nabla V_d \cdot \bar{ds}$

But $V_d = 0$ at boundary

$\int_{vol} (\nabla \cdot V_d \nabla V_d) dv = 0$

$\int_{vol} (\nabla V_d \cdot \nabla V_d) dv = 0$

$\int_{vol} |\nabla V_d|^2 dv = 0$

$dv \neq 0$, $|\nabla V_d|^2 = 0$

$\nabla V_d = 0$

$\nabla (V_2 - V_1) = 0$

$\nabla \neq 0$ & $V_2 - V_1 = 0$

$\boxed{V_2 = V_1}$

This proves that both solutions are equal.

Uniqueness Theorem states that "If the solution of Laplace's eqn satisfies the boundary conditions then that solution is unique, by whatever method it is obtained."

"The solution of Laplace's eqn gives the field (ie. V). which is unique, satisfying the same boundary conditions in a given region."

Problems :

1) Given that $\bar{J} = 10^3 \sin\theta \, \bar{a}_r$ A/m². Find the current passing through a spherical shell of $r = 0.2$ m

Sol $I = \oint \bar{J} \cdot \bar{ds}$ $\left.\begin{matrix} \cancel{dr} \\ r\,d\theta \\ r\sin\theta\, d\phi \end{matrix}\right\}$.

$= \oint 10^3 \sin\theta \, \bar{a}_r \cdot r^2 \sin\theta \, d\theta \, d\phi \, \bar{a}_r$

$= \int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} 10^3 r^2 \sin^2\theta \, d\theta \, d\phi$

$= 10^3 (0.2)^2 \left[\int_0^{\pi} \sin^2\theta \, d\theta \int_0^{2\pi} d\phi \right]$

$= 10^3 (0.2)^2 \left[\int_0^{\pi} \frac{1-\cos 2\theta}{2} d\theta \right] [\phi]_0^{2\pi}$

$= \frac{10^3 (0.2)^2}{\cancel{2}} \left[\theta - \frac{\sin 2\theta}{2} \right]_0^{\pi} [\cancel{2}\pi]$

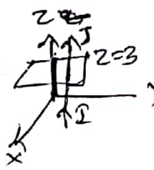
$= 1000 (0.008) [\pi][\pi]$

$= 394.38$ A.

2) i) Find the total current crossing the surface $z = 3$, $\rho < 6$ (i.e $r < 6$) in the $\bar{a}_z$ direction, if the current density in that region is given by

$$J = \left(\frac{100}{\rho^2}\right) \bar{a}_\rho + \left(\frac{10}{\rho^2+1}\right) \bar{a}_z \text{ A/m}^2$$

ii) Also find $\frac{\partial \rho_v}{\partial t}$.



Sol: i) $I = \iint \bar{J} \cdot \bar{ds}$ $\left.\begin{matrix} dr \\ r\,d\phi \\ \cancel{dz} \end{matrix}\right\}$

$= \int_{\rho=0}^{6}\int_{\phi=0}^{2\pi} \frac{100}{\rho^2+1} \bar{a}_z \cdot \rho \, d\rho \, d\phi \, \bar{a}_z$

$= \frac{10}{2} \int_{\rho=0}^{6}\int_0^{2\pi} \frac{2\rho}{\rho^2+1} d\rho \, d\phi$

$= 5 \left[\log_e(\rho^2+1) \right]_0^6 [\phi]_0^{2\pi}$

$= 5 \left[\log_e(37) - \log 1 \right] 2\pi$

$= 113.4$ A

ii) Continuity eqn

$\nabla \cdot \bar{J} = -\frac{\partial \rho_v}{\partial t}$

$\frac{\partial \rho_v}{\partial t} = -\nabla \cdot \bar{J} = -\left[\frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho J_\rho) + \frac{1}{\rho}\frac{\partial J_\phi}{\partial \phi} + \frac{\partial J_z}{\partial z} \right]$

$= -\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\frac{100}{\rho}\right) + \frac{\partial}{\partial z}\left(\frac{10}{\rho^2+1}\right)$ $= 0$

$= -\frac{1}{\rho} 100 \left(-\frac{1}{\rho^2}\right)$

$= \frac{100}{\rho^3}$ C/m³

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3) A potential field is given by $V = x^2 + y^2 + z^2$ volts. Let $P(1,1,1)$ located at a conductor – free space boundary. At point 'P', find the magnitude of a) V b) E c) E_n d) E_t and e) ρ_s. (56)

Sol: a) $V = x^2 + y^2 + z^2$

V at $P(1,1,1) = 1+1+1 = 3$ volts

b) $\bar{E} = -\nabla V$

$= -\left[\frac{\partial V}{\partial x}\bar{a}_x + \frac{\partial V}{\partial y}\bar{a}_y + \frac{\partial V}{\partial z}\bar{a}_z\right]$

$= -\left[2x\bar{a}_x + 2y\bar{a}_y + 2z\bar{a}_z\right]$

E at $(1,1,1)$ $= -\left[2\bar{a}_x + 2\bar{a}_y + 2\bar{a}_z\right]$

$|E| = \sqrt{2^2 + 2^2 + 2^2} = 3.46$ Volt/meter.

c) At boundary (conductor – freespace boundary).

$E_{tan} = 0$.

E_N exists.

$E_n = 3.46$ Volt/meter.

d) $E_t = 0$

e) $D_N = \rho_s$

$\rho_s = \varepsilon_0 E_N$

$= 8.854 \times 10^{-12} (3.46) = 30.67$ pC/m^2

4) A parallel plate capacitor has circular plates of 8 cm radius and 1 mm seperation. If the potential difference across the plates is 100V. Find the charge across the plates. $\varepsilon_0 = 8.854 \times 10^{-12}$ $C^2/N\text{-}m^2$.

Sol $C = \frac{\varepsilon_0 A}{d}$ $\quad (\because C = Q/V)$

$\frac{Q}{V} = \frac{\varepsilon_0 A}{d} \Rightarrow Q = \frac{\varepsilon_0 A}{d} \cdot V$

$A = \pi r^2$

$= \pi (0.08)^2$

$V = 100V$

$d = 1mm$

$= \frac{1}{1000} = 0.001m$

$Q = \frac{8.854 \times 10^{-12} \times \pi (0.08)^2 \times 100}{0.001}$

$= 1.79 \times 10^{-8}$ coulomb

5) Calculate the energy stored in a spherical capacitor of 10cm radius charged to a potential of 200Volt.

Sol:

$C = 4\pi\varepsilon_0 a$ $\qquad a = 10\,cm = 10/100\,m = 0.1\,m$.

$= 4\pi(8.854\times10^{-12})(0.1)$

$= 11.13\times10^{-12}$ Farads

$W_E = \frac{1}{2}CV^2$

$= \frac{1}{2}\times11.13\times10^{-12}\times(200)^2 = 0.222\times10^{-6}$ Joules.

6) A pair of 200mm long concentric cylindrical conductors of radii 50mm and 100mm is filled with a dielectric with $\varepsilon = 10\varepsilon_0$. A Voltage is applied between the conductors which establishes $\bar{E} = \frac{10^6}{r}\bar{a}_r$. Calculate a) Capacitance b) voltage applied c) Energy stored

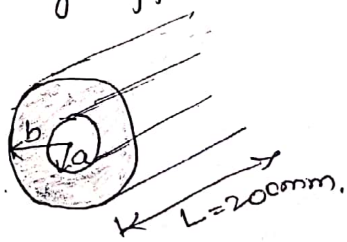
Sol:

a) $L = 200mm$

$= \frac{\cancel{200}}{\cancel{1000}}m = 0.2m$.

$a = 50mm = \frac{\cancel{50}}{\cancel{1000}}m = 0.05m$.

$b = 100mm = \frac{\cancel{100}}{\cancel{1000}}m = 0.1m$.



$C = \frac{2\pi\varepsilon L}{\ln\left[\frac{b}{a}\right]} = \frac{2\pi(10\varepsilon_0)\times0.2}{\ln\left[\frac{0.1}{0.05}\right]} = 160.5$ pf.

b) $V = -\int \bar{E}\cdot d\bar{L}$

$= -\int_{r=b}^{a} \frac{10^6}{r}\bar{a}_r\cdot dr\,\bar{a}_r = -10^6\int_{0.1}^{0.05}\frac{1}{r}dr$

$= -10^6\left[\ln r\right]_{0.1}^{0.5} = 693.15$ KV.

c) $W_E = \frac{1}{2}CV^2$

$= \frac{1}{2}\times160.5\times10^{-12}\times(693.15\times10^3)^2 = 38.56$ Joules

OR $W_E = \int_{vol}\frac{1}{2}\varepsilon|E|^2 dv$

$= \frac{\varepsilon}{2}\iint\left|\frac{10^6}{r}\right|^2 r\,dr\,d\phi\,dz$

$= \frac{10^{12}\varepsilon}{2}\int_{z=0}^{L}\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\frac{1}{r}dr\,d\phi\,dz = 38.56$ Joules.

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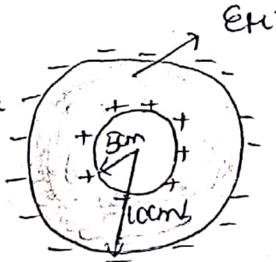
7) A conducting sphere of radius 5 cm has a total charge of $1\mu C$. The sphere is surrounded by an inhomogeneous dielectric sphere $5 \le r \le 10$ cm in which relative permittivity varies as $\epsilon_r = \frac{0.1}{r}$. A second conducting spherical surface is at $r = 10$ cm. Calculate the potential difference and capacitance between the conductors.



Sol $\epsilon_r \propto \frac{1}{r}$, the standard formula for spherical capacitor cannot be used.

$V = ?$
$C = ?$

$a = \frac{5}{100} = 0.05\,m$

$b = \frac{10}{100} = 0.1\,m$

$$V = -\int_{-}^{+} \bar{E} \cdot \bar{dL}$$

$$= -\int_{r=0.1}^{0.05} \frac{Q}{4\pi\epsilon r^2}\, \bar{a_r} \cdot dr\bar{a_r}$$

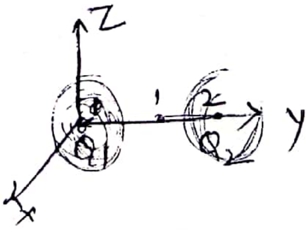
$$= -\int_{0.1}^{0.05} \frac{1\times10^{-6}}{4\pi\times\epsilon_0 \cdot \frac{0.1}{r} \cdot r^2}\, dr$$

$$= -\frac{10^{-6}\times10}{4\pi\times8.854\times10^{-12}} \left[\ln r\right]_{0.1}^{0.05} = \underline{62.3\,kV}$$

$$C = \frac{Q}{V}$$

$$= \frac{1\times10^{-6}}{62.3\times10^{3}} = \underline{16.05\,pf}$$

8) Find the energy stored in a system of two charges $Q_1 = 4nc$ & $Q_2 = -4nc$ seperated by 0.2m.



Sol $W = ?$ $\quad V = \frac{W}{Q}$

$W = Q_1 V_2 + Q_2 V_1$

potential at 1 due to Q_2

$V_{12} = \frac{Q_1}{4\pi\epsilon R} = \frac{4\times10^{-9}}{4\pi\epsilon_0(0.2)} = 0.179\times10^{3} = 179V$

$V_2 = \frac{Q_2}{4\pi\epsilon_0 R} = \frac{-4\times10^{-9}}{4\pi\epsilon_0(0.2)} = -179V$

$V_1 = V_{12} = \frac{W}{Q_2}$

$V_2 = V_{21} = \frac{W}{Q_1}$

$\therefore W = \underline{-1432\ \text{Joules}}$

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9) Using Poissons eqn obtain the volume charge density ρ_v inside a sphere of radius 'a' if the field Intensity is $E_r = Ar^4$ for $r<a$

$= -Ar^{-2}$ for $r>a$

Sol

$$\nabla^2 V = -\frac{\rho_v}{\varepsilon}$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) = -\frac{\rho_v}{\varepsilon} \quad \text{—①} \quad (\bar{E} \text{ is a function of } r \text{ only not of } \theta \text{ and } \phi).$$

but $\bar{E} = -\nabla V$

$$E_r \bar{a}_r = -\frac{\partial V}{\partial r}\bar{a}_r$$

$$E_r = -\frac{\partial V}{\partial r}$$

Case i) For $r<a$

$$E_r = Ar^4$$

$$E_r = -\frac{\partial V}{\partial r}$$

$$Ar^4 = -\frac{\partial V}{\partial r}$$

Sub this in ①

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left[\cancel{-}r^2 Ar^4\right] = \cancel{-}\frac{\rho_v}{\varepsilon}$$

$$\frac{1}{\cancel{r^2}}\left[A.6r^{\cancel{5}\,3}\right] = \frac{\rho_v}{\varepsilon}$$

$$\rho_v = 6A\varepsilon r^3 \ C/m^3.$$

Case ii) For $r>a$

$$E_r = -Ar^{-2}$$

$$E_r = -\frac{\partial V}{\partial r}$$

$$\therefore \frac{\partial V}{\partial r} = +Ar^{-2}$$

Sub in ①

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left[+r^2 Ar^{-2}\right] = -\frac{\rho_v}{\varepsilon}$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}[A] = -\frac{\rho_v}{\varepsilon}$$

$$\rho_v = 0 \ C/m^3.$$

10) Find potential and volume charge density at P(0.5, 1.5, 1) in free space of potential field is given by $V = x^2 - y^2 - z^2$ volts.

Sol. a) $V = x^2 - y^2 - z^2$

$V_{at\,P} = (0.5)^2 - (1.5)^2 - (1)^2 = -3V$

b) $\nabla^2 V = -\frac{\rho_v}{\varepsilon_0}$

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = -\frac{\rho_v}{\varepsilon_0}$$

$$\frac{\partial}{\partial x}(2x) + \frac{\partial}{\partial y}(-2y) + \frac{\partial}{\partial z}(-2z) = -\frac{\rho_v}{8.854\times10^{-12}}$$

$$\cancel{2} + 2 - 2 = -\frac{\rho_v}{8.854\times10^{-12}} \implies \rho_v = 17.6\times10^{-12} \ C/m^3.$$

11) Given the volume charge density $\rho_v = -2\times10^{7}\varepsilon_0\sqrt{x}\ C/m^3$ in free space. Let $V=0$ at $x=0$ and $V=2V$ at $x=2.5mm$. Find V at $x=1mm$.

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Sol

$$\nabla^2 V = -\frac{\rho_v}{\varepsilon}$$

$$\nabla^2 V = +\frac{2\times10^{7}\varepsilon_0\sqrt{x}}{\varepsilon_0}$$

$$\frac{\partial^2 V}{\partial x^2} = 2\times10^{7}x^{1/2}$$

Integrating on both sides

$$\frac{\partial V}{\partial x} = 2\times10^{7}\frac{x^{3/2}}{3/2} + C_1$$

again integrate

$$V = 2\times10^{7}\cdot\frac{2}{3}\cdot\frac{x^{5/2}}{5/2} + C_1x + C_2$$

$$V = 5.33\times10^{6}x^{5/2} + C_1x + C_2$$

At $x=0 \Rightarrow V=0$

$$\therefore 0 = 0 + 0 + C_2 \Rightarrow C_2 = 0.$$

At $x = 2.5mm = \frac{2.5}{1000}m = 2.5\times10^{-3}m. \Rightarrow V = 2V$

$$\therefore 2V = 5.33\times10^{6}(2.5\times10^{-3})^{5/2} + C_1(2.5\times10^{-3}) + 0.$$

$$C_1 = 133.75$$

$$\therefore V = 5.33\times10^{6}\times x^{5/2} + 133.75x$$

At $x = 1mm = 1\times10^{-3}m$

$$V = 5.33\times10^{6}\times(10^{-3})^{5/2} + 133.75(10^{-3})$$

$$= 0.302\ V$$

12) Let $V = 2xy^2z^3$ and $\varepsilon = \varepsilon_0$. Given point $P(1,3,-1)$

Find i) $\bar{E}$ at point 'p' ~~ii) find V also~~. (iii) find out V if satisfies Laplace's eqn.

Sol i) $\bar{E} = -\nabla V$

$$= -\left[\frac{\partial V}{\partial x}\bar{a_x} + \frac{\partial V}{\partial y}\bar{a_y} + \frac{\partial V}{\partial z}\bar{a_z}\right]$$

Given $V = 2xy^2z^3$

$$\frac{\partial V}{\partial x} = 2y^2z^3, \quad \frac{\partial V}{\partial y} = 4xyz^3, \quad \frac{\partial V}{\partial z} = 6xy^2z^2$$

$$\bar{E} = -\left[2y^2z^3\bar{a_x} + 4xyz^3\bar{a_y} + 6xy^2z^2\bar{a_z}\right]$$

$$\bar{E}_{at\,p} = -\left[2(3)^2(-1)^3\bar{a_x} + 4(1)(3)(-1)^3\bar{a_y} + 6(1)(3)^2(-1)^2\bar{a_z}\right]$$

$$\bar{E} = 18\bar{a_x} + 12\bar{a_y} - 54\bar{a_z} \text{ V/m.}$$

ii) $\nabla^2 V = 0$ $\qquad V = 2xy^2z^3.$

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

$$LHS = \nabla^2 V = 0 + 4xz^3 + 12xy^2z$$

$$\text{at } P(1,3,-1) = 4(1)(-1)^3 + 12(1)(3)^2(-1)$$

$$= -4 - 108 \neq 0$$

∴ Laplace eqn is not satisfying.

13) Determine whether following potential satisfies Laplace eqns. $V = cr\theta\phi$ where c = constant

Sol $\nabla^2 V = 0$

$$LHS\; \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial V}{\partial r}\right] + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\frac{\partial V}{\partial\theta}\right] + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2}$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2 c\theta\phi\right] + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\, cr\phi\right] + \frac{1}{r^2\sin^2\theta}[0]$$

$$= \frac{1}{r^{\cancel{2}}}\cdot 2\cancel{r}c\theta\phi + \frac{1}{r^{\cancel{2}}\sin\theta}c\cancel{r}\phi\cos\theta$$

$$= \frac{2c\theta\phi}{r} + \frac{c\phi\cos\theta}{r\sin\theta}$$

$$\neq 0$$

∴ It is not satisfying Laplace's eqn.

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14) Given $V = \frac{\cos\phi}{\rho}$ in free space.

i) Determine the volume charge density at $P(0.5, 60°, 1)$

ii) Also find electric field Intensity at $P(0.5, 60°, 1)$

Sol

i) $\nabla^2 V = \cancel{0} - \frac{\rho_v}{\varepsilon_0}$ cylindrical coordinate system

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} = \cancel{0} - \frac{\rho_v}{\varepsilon_0}$$

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\cancel{\rho}\frac{\cos\phi}{\rho^{\cancel{2}}}\right) + \frac{1}{\rho^2}\left(-\frac{\cos\phi}{\rho}\right) + 0 = -\frac{\rho_v}{\varepsilon_0}$$

$$+\frac{1}{\rho}\frac{\cancel{\cos\phi}}{\cancel{\rho^2}} - \frac{\cancel{\cos\phi}}{\cancel{\rho^3}} = -\frac{\rho_v}{\varepsilon_0}$$

$$\rho_v = 0$$

ii) $\bar{E} = -\nabla V$

$$= -\left[\frac{\partial V}{\partial \rho}\bar{a}_\rho + \frac{1}{\rho}\frac{\partial V}{\partial \phi}\bar{a}_\phi + \frac{\partial V}{\partial z}\bar{a}_z\right]$$

$$= -\left[-\frac{\cos\phi}{\rho^2}\bar{a}_\rho + \frac{1}{\rho}\left(-\frac{\sin\phi}{\rho}\right)\bar{a}_\phi + 0\right]$$

$$\bar{E} = \frac{\cos\phi}{\rho^2}\bar{a}_\rho + \frac{\sin\phi}{\rho^2}\bar{a}_\phi$$

$$E_{\text{at } P(0.5, 60°, 1)} = \frac{\cos 60°}{(0.5)^2}\bar{a}_\rho + \frac{\sin 60°}{(0.5)^2}\bar{a}_\phi$$

$$= \cancel{2+3.46}\ 2\bar{a}_\rho + 3.46\bar{a}_\phi \ \text{V/m}.$$

15) Find the permittivity of dielectric material present in parallel plate capacitor $A = 0.12\,m^2$ i) $d = 80\,\mu m$, $V = 12V$ and capacitor contains $1\,\mu$ Joule of energy?

ii) Stored energy density is $100\,J/m^3$, $V = 200V$ and $d = 45\,\mu m$.

Sol $W_E = \frac{1}{2}CV^2$

$$W_E = \frac{1}{2}\cdot\frac{\varepsilon A}{d}V^2 = \frac{1}{2}\frac{\varepsilon_0\varepsilon_r A V^2}{d}$$

i) $$1\times10^{-6} = \frac{1}{2}\,\frac{8.854\times10^{-12}\times0.12\times(12)^2\,\varepsilon_r}{80\times10^{-6}}$$

$$\varepsilon_r = 1.04$$

ii) energy density $= \frac{\text{energy stored}}{\text{Volume}}$

$$100 = \frac{\frac{1}{2}CV^2}{\text{Volume}}$$

$$100\times\text{Volume} = \frac{1}{2}CV^2$$

$$100\times A\times d = \tfrac{1}{2}\varepsilon_0\varepsilon_r \cancel{A} V^2 / \cancel{d}$$

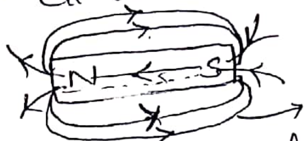
$$100\times0.\cancel{1}2\times45\times10^{-6} = \frac{1}{2}\,\frac{8.854\times10^{-12}\,\varepsilon_r(200)^2}{45\times10^{-6}}$$

$$\varepsilon_r = 1.14$$

Magnetic Fields

The electrostatic field exists due to the static charges i.e. charges at rest.

The magnetic field exist due to permanent magnet



magnetic line of force

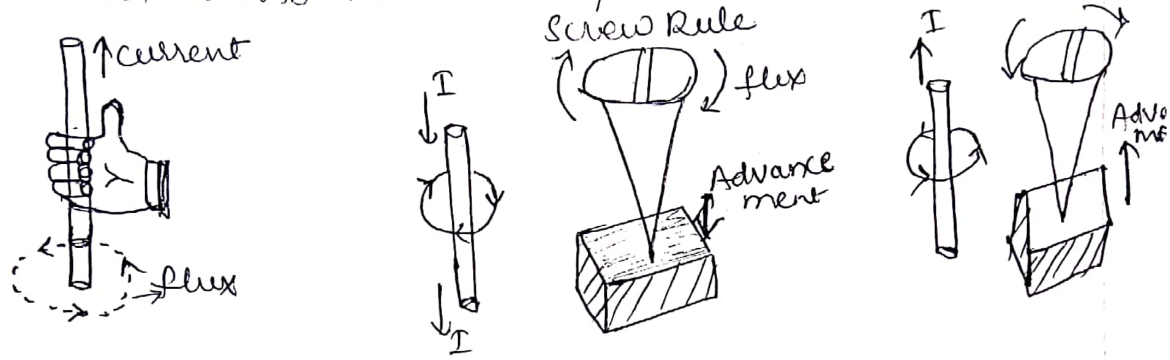
- But in electromagnetic engg. a link between electric and magnetic field is required to be studied. Such a link is absent with magnetic field due to a natural magnet.
- Scientist Oersted has discovered the relation between electric and magnetic fields. He stated that when the charges are in motion, they are surrounded by a magnetic field.
- The charges in motion i.e. flow of charges constitutes an electric current. Thus a current carrying conductor is always surrounded by a magnetic fields.
- If such a current flow is steady i.e. time invariant then the magnetic field produced is a steady magnetic field. which is also time invariant (dc current)

"The study of steady magnetic field, existing in a given space produced due to the flow of direct current through a conductor is called magnetostatics".

- The lines of force are also called magnetic lines of flux (or) magnetic flux lines.

Properties:
- An isolated magnetic pole can not exist.
- The magnetic flux lines exist in the form of closed loop.

Right hand thumb rule — Right hand Screw Rule



Magnetic Field Intensity ($\bar{H}$)

It is quantitative measure of strongness or weakness of the magnetic field.

It is defined as the force experienced by a unit north pole of one weber strength.

magnetic flux lines Units → webers (Wb)

magnetic Field Intensity ($\bar{H}$) Units → N/wb or A/m or Ampere-turn/meter

Magnetic Flux Density ($\bar{B}$)

magnetic flux lines passing through the unit surface area is known as magnetic Flux Density

Units → wb/m^2 (or) Tesla

Relationship between $\bar{B}$ and $\bar{H}$

$$\boxed{\bar{B} = \mu \bar{H}}$$

Where μ = permeability

$\mu = \mu_0 \mu_r$

μ_0 = permeability in free space

$= 4\pi \times 10^{-7}$ Henry/meter

μ_r = relative permeability.

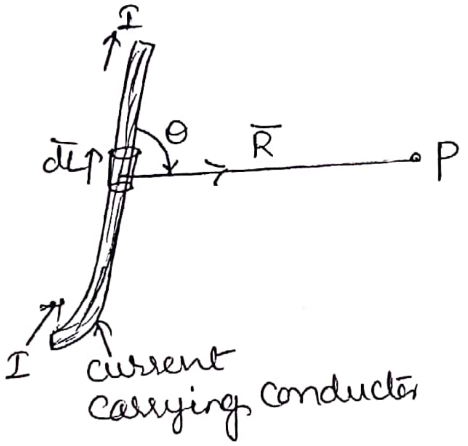
For magnetic materials $\mu_r > 1$

For Non-magnetic " $\mu_r = 1$

Biot-Savarts law

Biot-Savart law states that the magnetic field Intensity $\bar{dH}$ produced at a point 'P' due to differential current element Idl is

1) proportional to the product of current I and differential length dL.
2) The sine of the angle between the element and the line jointing point 'p' to the element.
3) Inversely proportional to the square of the distance 'R' between point 'p' and the element.



$$\bar{dH} \propto I\bar{dL}$$
$$\propto \sin\theta$$
$$\propto \frac{1}{R^2}$$

$$\therefore \bar{dH} \propto \frac{I dL \sin\theta}{R^2}$$

$$\boxed{\bar{dH} = \frac{K\, I dL \sin\theta}{R^2}}$$

K = constant

$$\boxed{\bar{dH} = \frac{I dL \sin\theta}{4\pi R^2}}$$

$$\bar{dL} \times \bar{a_R} = |\bar{dL}|\,|\bar{a_R}| \sin\theta\, \bar{a_n} = dL \cdot 1 \sin\theta\, \bar{a_n}$$

$$\therefore \boxed{\bar{dH} = \frac{I\bar{dL} \times \bar{a_R}}{4\pi R^2}}$$

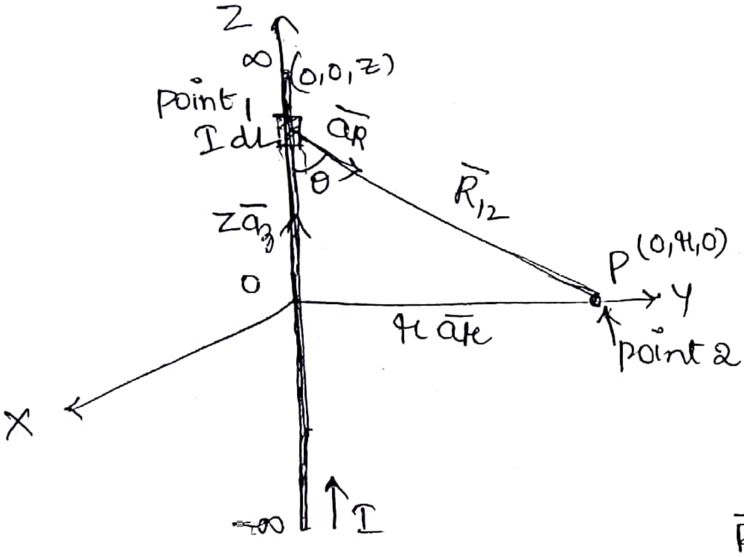
but $\bar{a_R} = \frac{\bar{R}}{|R|}$

$$\therefore \bar{dH} = \frac{I\bar{dL} \times \frac{\bar{R}}{|\bar{R}|}}{4\pi R^2}\, \bar{a_n}$$

$$\boxed{\bar{dH} = \frac{I\bar{dL} \times \bar{R}}{4\pi R^3}} \quad A/m$$

$$\boxed{\begin{aligned} H &= \int_L \frac{I\bar{dL} \times \bar{a_R}}{4\pi R^2} \\ H &= \int_S \frac{\bar{K}\, \bar{ds} \times \bar{a_R}}{4\pi R^2} \\ H &= \int_V \frac{\bar{J}\, dv \times \bar{a_R}}{4\pi R^2} \end{aligned}}$$

1) Magnetic Field Intensity ($\bar{H}$) due to infinitely long straight conductor.



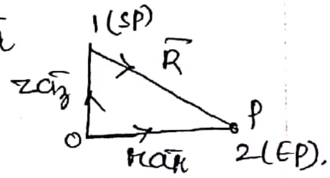
– Infinitely long conductor along z-axis

$$I\bar{dL} = I\,dz\,\bar{a}_z \quad \text{—(1)}$$

$$d\bar{H} = \frac{I\bar{dL} \times \bar{a}_R}{4\pi R^2} \quad \text{—(2)}$$

$$\bar{a}_R = \frac{\bar{R}}{|\bar{R}|}$$

$$\bar{R} = -z\bar{a}_z + r\bar{a}_r$$



$$\bar{a}_R = \frac{-z\bar{a}_z + r\bar{a}_r}{\sqrt{r^2+z^2}}$$

∴ ~~eqn (2) becomes~~

$$\bar{dL} \times \bar{a}_R = \begin{vmatrix} \bar{a}_r & \bar{a}_\phi & \bar{a}_z \\ 0 & 0 & dz \\ \frac{r}{\sqrt{r^2+z^2}} & 0 & \frac{-z}{\sqrt{r^2+z^2}} \end{vmatrix}$$

$$= \bar{a}_r(0) - \bar{a}_\phi\left(-\frac{r}{\sqrt{r^2+z^2}}dz\right) + \bar{a}_z(0)$$

$$= \frac{r}{\sqrt{r^2+z^2}}\,dz\,\bar{a}_\phi$$

∴ eqn (2) becomes.

$$d\bar{H} = \frac{I\,\frac{r}{\sqrt{r^2+z^2}}\,dz\,\bar{a}_\phi}{4\pi(r^2+z^2)}$$

$$\bar{H} = \int_{z=-\infty}^{\infty} \frac{I\,r\,dz\,\bar{a}_\phi}{4\pi(r^2+z^2)^{3/2}}$$

Let $\tan\theta = \frac{r}{z} \Rightarrow z = r\cot\theta$

$dz = -r\,\mathrm{cosec}^2\theta\, d\theta$

$z = -\infty \rightarrow \theta = \pi$

$z = \infty \rightarrow \theta = 0°$

$$\therefore \bar{H} = \int_{\theta=\pi}^{0} \frac{-I r \cdot r \operatorname{cosec}^2\theta \, d\theta \, a_\phi}{4\pi (r^2 + r^2\cot^2\theta)^{3/2}}$$

$$= -\frac{I r^2 \bar{a}_\phi}{4\pi r^{\cancel{2}\cdot\frac{3}{\cancel{2}}}} \int_{\pi}^{0} \frac{\operatorname{cosec}^2\theta \, d\theta}{(1+\cot^2\theta)^{3/2}}$$

$$= -\frac{I \cancel{r^2} \bar{a}_\phi}{4\pi r^{\cancel{3}}} \int_{\pi}^{0} \frac{\cancel{\operatorname{cosec}^2\theta} \, d\theta}{(\operatorname{cosec}^{\cancel{2}}\theta)^{3/\cancel{2}}}$$

$$= -\frac{I \bar{a}_\phi}{4\pi r} \left[\int_{\pi}^{0} \sin\theta \, d\theta \right]$$

$$= -\frac{I}{4\pi r} \bar{a}_\phi \left[-\cos\theta \right]_{\pi}^{0}$$

$$\bar{H} = -\frac{I}{\cancel{4}_{2}\pi r} \bar{a}_\phi \left[-(\cancel{2}) \right]$$

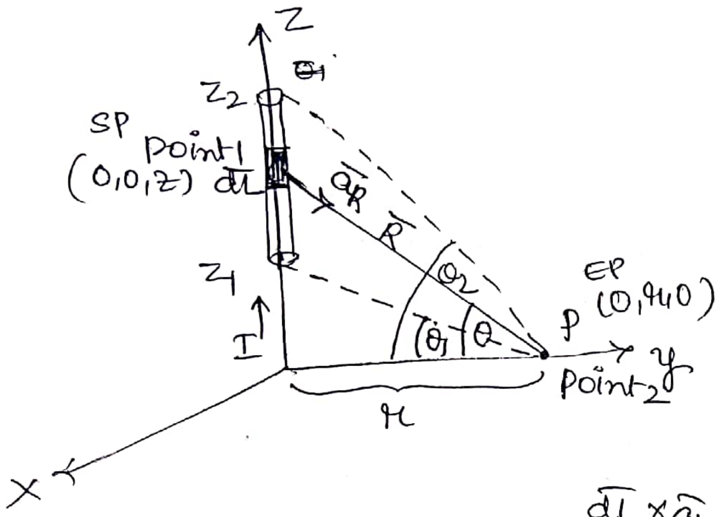
$$\therefore \boxed{\bar{H} = \frac{I}{2\pi r} \bar{a}_\phi}$$ N/wb (or) A/m.

Here r = perpendicular distance of point 'p' from current elem

$$\bar{B} = \mu \bar{H}$$

$$\boxed{\bar{B} = \frac{\mu I}{2\pi r} \bar{a}_\phi}$$ wb/m^2 (or) Tesla

2) $\bar{H}$ due to finite straight conductor :



$$\bar{dH} = \frac{I \bar{dL} \times \bar{a_R}}{4\pi R^2} \quad —(1)$$

$$\bar{dL} = dz\, \bar{a_z} \quad —(2)$$

$$\bar{a_R} = \frac{\bar{R}}{|\bar{R}|}$$

$$\bar{a_R} = \frac{r\bar{a_r} - z\bar{a_z}}{\sqrt{r^2+z^2}} \quad —(3)$$

$$|\bar{R}| = \sqrt{r^2+z^2}$$

$$\bar{dL} \times \bar{a_R} = \begin{vmatrix} \bar{a_r} & \bar{a_\phi} & \bar{a_z} \\ 0 & 0 & dz \\ \frac{r}{\sqrt{r^2+z^2}} & 0 & \frac{-z}{\sqrt{r^2+z^2}} \end{vmatrix}$$

$$= \bar{a_r}(0) - \bar{a_\phi}\left(-\frac{r}{\sqrt{r^2+z^2}}dz\right) + \bar{a_z}(0)$$

$$= \frac{r}{\sqrt{r^2+z^2}} dz\, \bar{a_\phi} \quad —(4)$$

sub (4) in (1)

$$\therefore \bar{dH} = \frac{I \frac{r}{\sqrt{r^2+z^2}} dz\, \bar{a_\phi}}{4\pi (r^2+z^2)} = \frac{I r\, dz\, \bar{a_\phi}}{4\pi (r^2+z^2)^{3/2}}$$

$$\bar{H} = \int_{z_1}^{z_2} \frac{I r\, \bar{a_\phi}}{4\pi (r^2+z^2)^{3/2}} dz$$

$$\tan\theta = \frac{z}{r} \Rightarrow z = r\tan\theta$$

$$dz = r\sec^2\theta\, d\theta$$

When $z = z_1 \Rightarrow z_1 = r\tan\theta_1$

$$\theta_1 = \tan^{-1}\left(\frac{z_1}{r}\right)$$

$z = z_2 \Rightarrow z_2 = r\tan\theta_2$

$$\theta_2 = \tan^{-1}\left(\frac{z_2}{r}\right)$$

$$\therefore \bar{H} = \int_{\theta_1}^{\theta_2} \frac{I r \bar{a_\phi}\, r\sec^2\theta\, d\theta}{4\pi (r^2 + r^2\tan^2\theta)^{3/2}}$$

$$= \frac{I \cancel{r^2} \bar{a_\phi}}{4\pi (r^2)^{3/2}} \int_{\theta_1}^{\theta_2} \frac{\cancel{\sec^2\theta}\, d\theta}{(\sec^2\theta)^{3/2}}$$

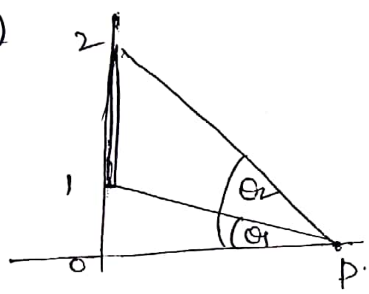
$$\bar{H} = \frac{I\,\bar{a}_\phi}{4\pi r}\int_{\theta_1}^{\theta_2} \cos\theta \, d\theta$$

$$= \frac{I\,\bar{a}_\phi}{4\pi r}\left[\sin\theta\right]_{\theta_1}^{\theta_2}$$

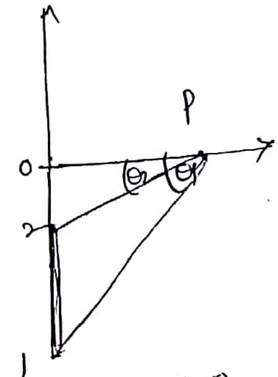
$$\boxed{\bar{H} = \frac{I}{4\pi r}\left[\sin\theta_2 - \sin\theta_1\right]\bar{a}_\phi} \quad A/m$$

$\bar{B} = \mu\bar{H}$

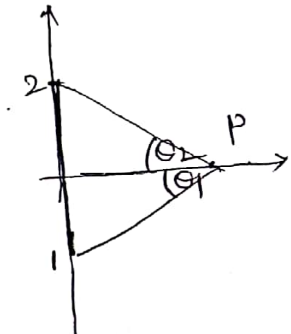
$$\therefore \boxed{\bar{B} = \frac{\mu I}{4\pi r}\left[\sin\theta_2 - \sin\theta_1\right]\bar{a}_\phi} \quad Wb/m^2 \text{ (or) Tesla}$$



i) both θ_1 & θ_2 Positive

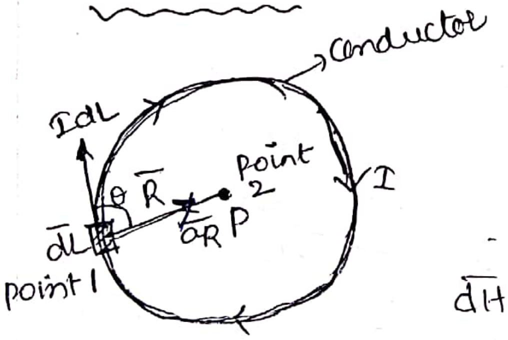


ii) both θ_1 & θ_2 Negative (−ve)



iii) θ_1 is −ve
θ_2 is +ve

3) Magnetic field Intensity ($\bar{H}$) at the centre of circular conductor



$$d\bar{H} = \frac{I\,d\bar{L}\times\bar{a}_R}{4\pi R^2} \quad \text{——} (1)$$

$$d\bar{L}\times\bar{a}_R = |d\bar{L}|\,|\bar{a}_R|\,\sin\theta\,\bar{a}_n$$
$$= dL \cdot 1 \cdot \sin\theta\,\bar{a}_n$$

$\therefore$ eqn (1) becomes,

$$d\bar{H} = \frac{I\,dL\sin\theta\,\bar{a}_n}{4\pi R^2}$$

$$\bar{H} = \frac{I\sin\theta\,\bar{a}_n}{4\pi R^2}\int dL$$

[$\bar{a}_n = \bar{a}_z$ if the circular loop is placed in XY plane]

but $\int dL = 2\pi R$ = circumference of the circle

$$\bar{H} = \frac{I\sin\theta\,\bar{a}_n}{\cancel{4\pi R^2}_{\,2}}\cdot\cancel{2\pi R}$$

$$\bar{H} = \frac{I\sin 90^\circ\,\bar{a}_n}{2R} \Rightarrow \bar{H} = \frac{I}{2R}\bar{a}_n \quad A/m$$

$$\boxed{\bar{H} = \frac{I}{2R}\bar{a}_z \quad A/m}$$

$$\bar{B} = \mu\bar{H} \Rightarrow \boxed{\bar{B} = \frac{\mu I}{2R}\bar{a}_z} \text{ Tesla}$$

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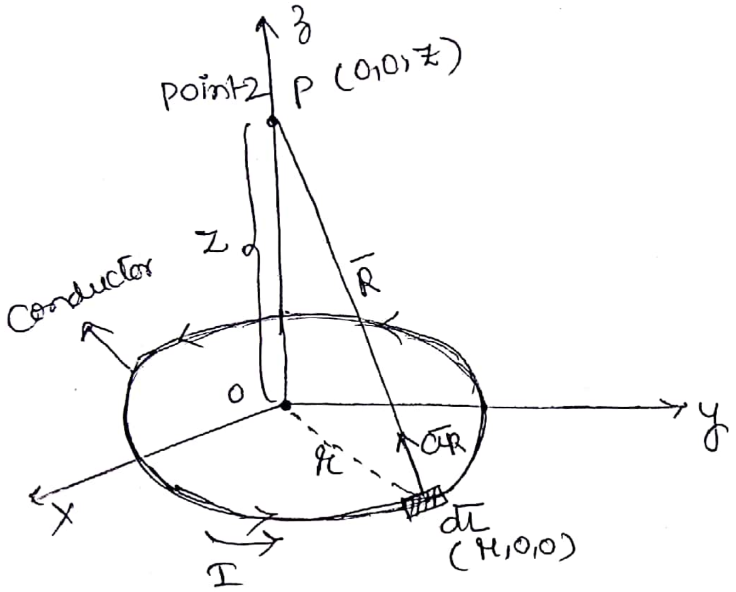
4) $\bar{H}$ on the axis of the circular conductor



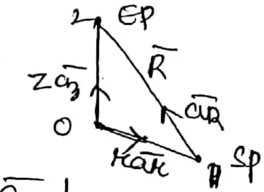
$$d\bar{H} = \frac{I\,\bar{dl}\times\bar{a_R}}{4\pi R^2} \quad \text{--- (1)}$$

$$\bar{dl} = dr\,\bar{a_r} + r\,d\phi\,\bar{a_\phi} + dz\,\bar{a_z}$$

But r = constant & $z=0$ plane.

$$\therefore \bar{dl} = r\,d\phi\,\bar{a_\phi} \quad \text{--- (2)}$$

$$\bar{a_R} = \frac{\bar{R}}{|\bar{R}|} = \frac{-r\bar{a_r} + z\bar{a_z}}{\sqrt{r^2+z^2}}$$



$$\bar{dl}\times\bar{a_R} = \begin{vmatrix} \bar{a_r} & \bar{a_\phi} & \bar{a_z} \\ 0 & r\,d\phi & 0 \\ \frac{-r}{\sqrt{r^2+z^2}} & 0 & \frac{z}{\sqrt{r^2+z^2}} \end{vmatrix}$$

$$= +\frac{z\,r\,d\phi}{\sqrt{r^2+z^2}}\,\bar{a_r} + \frac{r^2\,d\phi}{\sqrt{r^2+z^2}}\,\bar{a_z}$$

$\therefore$ eqn (1) becomes

$$d\bar{H} = \frac{I\left[\frac{z\,r\,d\phi}{\sqrt{r^2+z^2}}\,\bar{a_r} + \frac{r^2\,d\phi}{\sqrt{r^2+z^2}}\,\bar{a_z}\right]}{4\pi(r^2+z^2)}$$

$$d\bar{H} = \frac{I\,z\,r\,d\phi}{4\pi(r^2+z^2)^{3/2}}\,\bar{a_r} + \frac{I\,r^2\,d\phi}{4\pi(r^2+z^2)^{3/2}}\,\bar{a_z}$$

$$\bar{H} = \frac{I}{4\pi}\left[\cancel{\int_{\phi=0}^{2\pi}\frac{z\,r\,d\phi}{(r^2+z^2)^{3/2}}\,\bar{a_r}}_{=0} + \int_{\phi=0}^{2\pi}\frac{r^2\,d\phi}{(r^2+z^2)^{3/2}}\,\bar{a_z}\right]$$

Here circular conductor is radially symmetric so there is no $\bar{a_r}$ component. So, first term $=0$.

$$\therefore \bar{H} = \frac{I}{4\pi}\int_0^{2\pi}\frac{r^2\,d\phi}{(r^2+z^2)^{3/2}}\,\bar{a_z}$$

$$\bar{H} = \frac{I r^2 \, a_z}{4\pi (r^2+z^2)^{3/2}} \int_0^{2\pi} d\phi$$

$$= \frac{I r^2 \bar{a_z}}{\cancel{4\pi}_{2} (r^2+z^2)^{3/2}} [\cancel{2\pi}]$$

$$\boxed{\bar{H} = \frac{I r^2}{2(r^2+z^2)^{3/2}} \bar{a_z}}$$ A/m

$\bar{B} = \mu \bar{H}$

$$\boxed{\bar{B} = \frac{\mu I r^2}{2(r^2+z^2)^{3/2}} \bar{a_z}}$$ wb/m² (or) Tesla

NOTE:

If point 'p' is shifted to the origin (i.e. at the centre) of the circular loop i.e. z=0

$$\therefore \bar{H} = \frac{I r^2}{2(r^2+0)^{3/2}} \bar{a_z}$$

$$\bar{H} = \frac{I \cancel{r^2}}{2(\cancel{r^2})^{3/2}} \bar{a_z}$$

$$\boxed{\bar{H} = \frac{I}{2r} \bar{a_z}}$$ A/m.

$$\boxed{\bar{B} = \frac{\mu I}{2r} \bar{a_z}}$$ Tesla

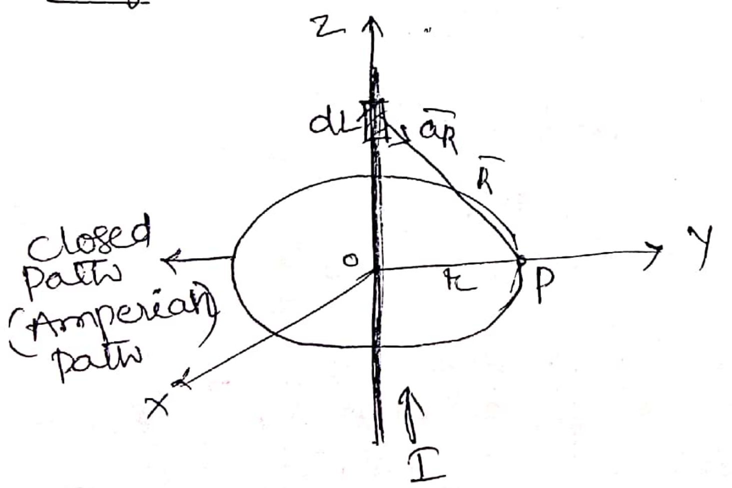
Ampere's circuital law (or Ampere's work law)

In electrostatics Gauss's law is useful to obtain $\bar{E}$ & $\bar{D}$ in case of complex problems.
In magnetostatics Ampere's law is used to obtain $\bar{H}$ and $\bar{B}$ in case of complex problems.

This law states that "the line integral of magnetic field Intensity $\bar{H}$ around a closed path is exactly equal to the direct current enclosed by that path".

$$\boxed{\oint \bar{H} \cdot \bar{dL} = I}$$

Proof:



$\bar{H} = \frac{I}{2\pi r} \bar{a_\phi}$ (magnetic field Intensity due to infinite conductor)

$\bar{dL} = r\,d\phi\,\bar{a_\phi}$ (r = const. & z = 0 plane)

$$L.H.S = \oint \bar{H} \cdot \bar{dL}$$

$$= \oint \frac{I}{2\pi r} \bar{a_\phi} \cdot r\,d\phi\,\bar{a_\phi}$$

$$= \frac{I}{2\pi} \int_{\phi=0}^{2\pi} d\phi$$

$$= \frac{I}{\cancel{2\pi}} \left[\cancel{2\pi}\right]$$

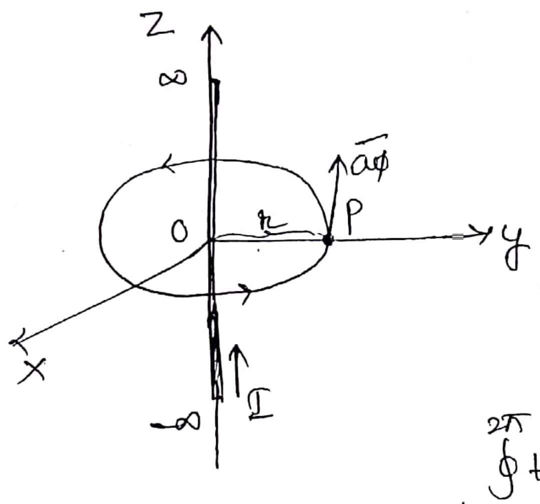
$$= I$$

$$= RHS$$

Applications of Amperes circuital law:

It is used to find $\bar{H}$ and $\bar{B}$ in various cases.

① $\bar{H}$ due to infinite long straight conductor



$$\oint \bar{H}\cdot\bar{dL} = I \quad \text{—— ①}$$

$\bar{H}$ is always tangential to the closed path i.e $\bar{a_\phi}$. So, $\bar{H}$ has only H_ϕ.

$$\bar{H} = H_\phi \bar{a_\phi}$$

$$\bar{dL} = r\,d\phi\,\bar{a_\phi}$$

Sub in ①

$$\oint_{\phi=0}^{2\pi} H_\phi \bar{a_\phi} \cdot r\,d\phi\,\bar{a_\phi} = I$$

$$H_\phi r \int_0^{2\pi} d\phi = I$$

$$H_\phi r [\phi]_0^{2\pi} = I$$

$$H_\phi r [2\pi] = I$$

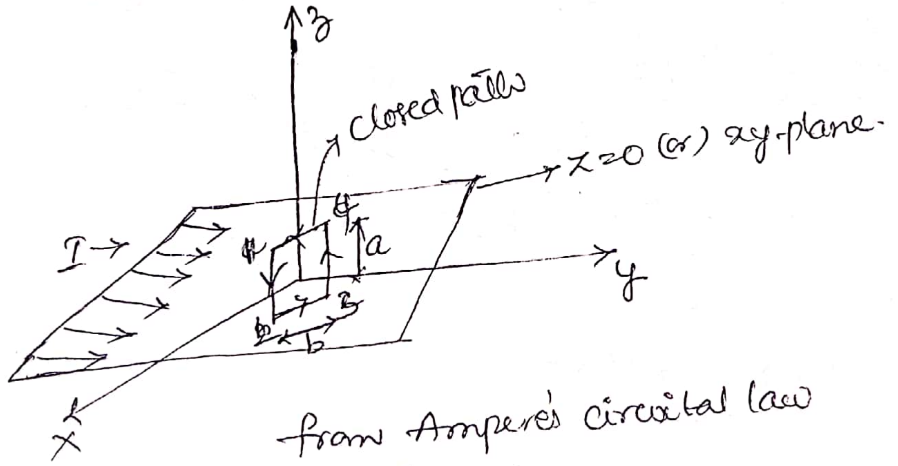
$$H_\phi = \frac{I}{2\pi r}$$

Hence $\bar{H}$ at 'p'

$$\boxed{\bar{H} = H_\phi \bar{a_\phi} = \frac{I}{2\pi r}\bar{a_\phi}} \quad A/m$$

$$\boxed{B = \mu \bar{H} = \frac{\mu I}{2\pi r}\bar{a_\phi}} \quad \text{Tesla}$$

2) $\bar{H}$ due to Infinite Sheet of current



from Ampere's circuital law

$$\oint \bar{H}\cdot\bar{dL} = I$$

$$\oint_1^2 \cancel{\bar{H}\cdot\bar{dL}} + \oint_2^3 \bar{H}\cdot\bar{dL} + \oint_3^4 \cancel{\bar{H}\cdot\bar{dL}} + \oint_4^1 \bar{H}\cdot\bar{dL} = I \quad \text{—— ①}$$

As current flowing in y-direction, $\bar{H}$ can not have component in y-direction. and also no field component along z-direction.

$$\therefore \int_2^3 \bar{H}\cdot \bar{dL} + \int_4^1 \bar{H}\cdot \bar{dL} = I$$

$$\int_2^3 -H_x\,\bar{a}_x \cdot dx(-\bar{a}_x) + \int_4^1 H_x\,\bar{a}_x \cdot dx\,\bar{a}_x = I$$

$$H_x \int_2^3 dx + H_x \int_4^1 dx = I$$

$$H_x\left[b+b\right] = I$$

$$\therefore H_x = \frac{I}{2b} \quad \text{—— (2)}$$

Surface current density J is flowing in the y-direction. Hence current density

$$J = J_y\,\bar{a}_y$$

The current flowing across the distance 'b' is

$$I = J_y\, b \quad \text{—— (3)}$$

Sub (3) in (1)

$$\therefore H_x = \frac{1}{2} J_y$$

$$\bar{H_x} = H_x\,\bar{a}_x \quad \text{for } z > 0$$

$$= -H_x\,\bar{a}_x \quad \text{for } z < 0$$

$$\boxed{\begin{aligned} \bar{H} &= \tfrac{1}{2} J_y\,\bar{a}_x \quad \text{for } z > 0 \\ &= -\tfrac{1}{2} J_y\,\bar{a}_x \quad \text{for } z < 0 \end{aligned}}$$

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3) Ampere's circuital law applied to surface element

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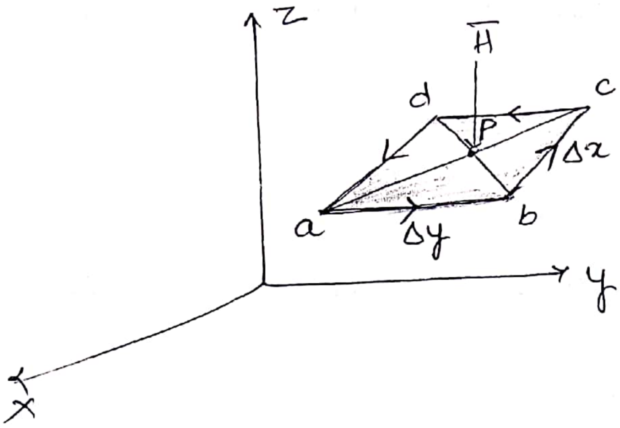


- In electrostatics Gauss's law applied to differential volume element to develop the concept of divergence
- Ily in magnetostatic Ampere's law applied to differential surface element to develop the concept of curl.

At 'p'

$\bar{H} = H_{x0}\bar{a_x} + H_{y0}\bar{a_y} + H_{z0}\bar{a_z}$ —— ①

Total current density

$\bar{J} = J_x\bar{a_x} + J_y\bar{a_y} + J_z\bar{a_z} \Rightarrow I_{enc} = J_z \Delta x \Delta y$ —— ②

From Ampere's ckt law

$\oint \bar{H}\cdot \bar{dL} = I$

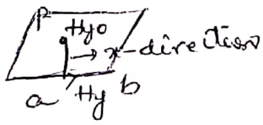
$\oint_a^b \bar{H}\cdot\bar{dL} + \oint_b^c \bar{H}\cdot\bar{dL} + \oint_c^d \bar{H}\cdot\bar{dL} + \oint_d^a \bar{H}\cdot\bar{dL} = I$ —— ③

For a-b path

$\int_a^b \bar{H}\cdot\bar{dL} = \int_a^b H_y\bar{a_y}\cdot\Delta y\bar{a_y}$

$= \int_a^b H_y\,\Delta y$

H_y can be expressed interms of H_{y0} existing at 'p' and the rate of change of H_y in x-direction.



$(\bar{H}\cdot\bar{dL})_{a-b} = \left[H_{y0} + \left(\frac{\partial H_y}{\partial x}\right)\frac{\Delta x}{2}\right]\Delta y$ —— ④

Ily For b-c path

$(\bar{H}\cdot\bar{dL})_{b-c} = -H_x\bar{a_x}\cdot\Delta x\bar{a_x}$

$= -H_x\,\Delta x$

$(\bar{H}\cdot\bar{dL})_{b-c} = -\left[H_{x0} + \left(\frac{\partial H_x}{\partial y}\right)\frac{\Delta y}{2}\right]\Delta x$ —— ⑤

For c-d path

$(\bar{H}\cdot\bar{dL})_{c-d} = -H_y\bar{a_y}\cdot\Delta y\bar{a_y}$

$= -H_y\,\Delta y$

As current flowing in y-direction, $\bar{H}$ can not have component in y-direction. and also no field component along z-direction.

$$\therefore \int_2^3 \bar{H}\cdot \bar{dl} + \int_4^1 \bar{H}\cdot \bar{dL} = I$$

$$\int_2^3 -H_x \bar{a}_x \cdot \bar{dx}(-\bar{a}_x) + \int_4^1 H_x \bar{a}_x \cdot dx \bar{a}_x = I$$

$$H_x \int_2^3 dx + H_x \int_4^1 dx = I$$

$$H_x [b+b] = I$$

$$\therefore H_x = \frac{I}{2b} \quad \text{--- (2)}$$

Surface current density J is flowing in the y-direction. Hence current density

$$J = J_y \bar{a}_y$$

The current flowing across the distance 'b' is

$$I = J_y b \quad \text{--- (3)}$$

Sub (3) in (1)

$$\therefore H_x = \frac{1}{2} J_y$$

$$\bar{H}_x = H_x \bar{a}_x \quad \text{for } z > 0$$

$$= -H_x \bar{a}_x \quad \text{for } z < 0$$

$$\boxed{\begin{aligned} \bar{H} &= \frac{1}{2} J_y \bar{a}_x \quad \text{for } z > 0 \\ &= -\frac{1}{2} J_y \bar{a}_x \quad \text{for } z < 0 \end{aligned}}$$

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1. Cartesian
$$\nabla \times \bar{H} = \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_x & H_y & H_z \end{vmatrix}$$

2. Cylindrical
$$\nabla \times \bar{H} = \frac{1}{r}\begin{vmatrix} \bar{a}_r & r\bar{a}_\phi & \bar{a}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ H_r & rH_\phi & H_z \end{vmatrix}$$

3. Spherical
$$\nabla \times \bar{H} = \frac{1}{r^2 \sin\theta}\begin{vmatrix} \bar{a}_r & r\bar{a}_\theta & r\sin\theta\, \bar{a}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ H_r & rH_\theta & r\sin\theta\, H_\phi \end{vmatrix}$$

Properties of curl :

1) The curl of a vector quantity is a vector.

2) $\nabla \times (\bar{A}+\bar{B}) = \nabla \times \bar{A} + \nabla \times \bar{B}$

3) $\nabla \times \nabla \times \bar{A} = \nabla(\nabla \cdot \bar{A}) - \nabla^2 \bar{A}$

4) The divergence of a curl is zero.

$$\nabla \cdot (\nabla \times \bar{A}) = 0$$

5) The curl of a scalar makes no meaning.

$$\nabla \times \alpha = \text{No meaning}$$

6) The curl of gradient ~~of a vector~~ is zero.

$$\nabla \times \nabla V = 0$$

7) $\nabla \times \bar{A} \times \bar{B} = \bar{A}(\nabla \cdot \bar{B}) - \bar{B}(\nabla \cdot \bar{A}) + (\bar{B}\cdot\nabla)\bar{A} - (\bar{A}\cdot\nabla)\bar{B}$

Stoke's Theorem :

Stoke's theorem in magnetostatics is analogous to the divergence theorem in electrostatics.

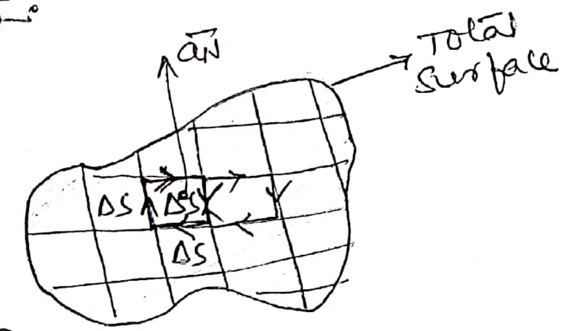
Stokes theorem converts line integral to a surface integral.

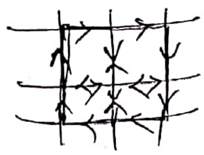
It states that

"the line integral of a vector $\bar{A}$ around a closed path is equal to the integral of curl of $\bar{A}$ over the open surface S enclosed by the closed path L".

$$\boxed{\oint_L \bar{A}\cdot \bar{dL} = \int_S (\nabla \times \bar{A})\cdot \bar{ds}}$$

Proof :





From Ampere's ckt law

$$\oint \bar{H}\cdot \bar{dL} = I$$

devide with ΔS

$$\frac{\oint \bar{H}\cdot \bar{dL}}{\Delta S} = \frac{I}{\Delta S} = \bar{J} \quad \text{—— (1)}$$

from curl $\nabla \times \bar{H} = \bar{J}$ —— (2)

Sub (2) in (1)

$$\frac{\oint \bar{H}\cdot \bar{dL}}{\Delta S} = \nabla \times \bar{H}$$

$$\oint \bar{H}\cdot \bar{dL} = (\nabla \times \bar{H})\,\Delta S$$

$$= (\nabla \times \bar{H})\cdot \bar{a_N}\,\Delta S$$

$$= (\nabla \times \bar{H})\cdot \bar{\Delta S}$$

(∵ curl of $\bar{H}$ in the normal direction is the dot product of curl of $\bar{H}$ with $\bar{a_N}$) i.e. $\nabla \times \bar{H} = (\nabla \times \bar{H})\cdot \bar{a_N}$

interior line integral gets cancelled, only out side boundary surface exist.

$$\therefore \boxed{\oint \bar{H}\cdot \bar{dL} = \int_S (\nabla \times \bar{H})\cdot \bar{ds}}$$

1) If a particular field is given by $\bar{E} = (x + 2y + az)\,\bar{a}_x + (bx - 3y - z)\,\bar{a}_y + (4x + cy + 2z)\,\bar{a}_z$. Then find the constant a, b, c such that the field is irrotational.

Sol

$\nabla \times \bar{E} = 0$ for $\bar{E}$ to be irrotational.

$$\nabla \times \bar{E} = \begin{vmatrix} a_x & \bar{a}_y & \bar{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix}$$

$$= \left[\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right]\bar{a}_x + \left[\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right]\bar{a}_y + \left[\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right]\bar{a}_z = 0$$

$$\left[\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right] = \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = 0.$$

$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = 0$$

$$c - 1 = 0 \Rightarrow c = 1$$

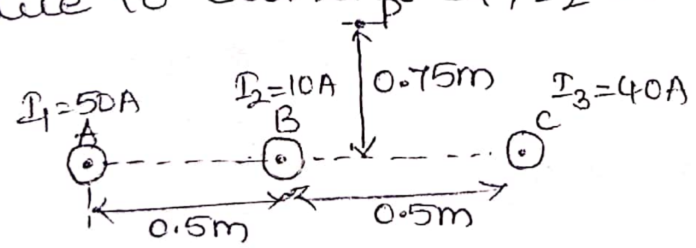
$$\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = 0.$$

$$a - 4 = 0 \Rightarrow a = 4$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = 0$$

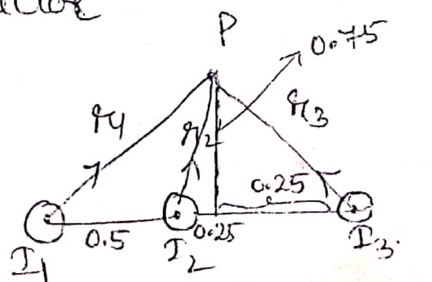
$$b - 2 = 0 \Rightarrow b = 2$$

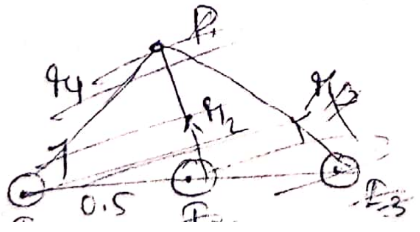
2) Find the magnetic flux density at point P due to currents I_1, I_2 and I_3 shown in fig.



Sol $\bar{H}$ due to straight conductor

$$\bar{H} = \frac{I}{2\pi r}\,\bar{a}_\phi$$





$$r_4 = \sqrt{(0.75)^2 + (0.75)^2} = 1.06$$

$$r_2 = \sqrt{(0.25)^2 + (0.75)^2} = 0.79$$

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$$\bar{H}_{at\,'p'} = \bar{H}_1 + \bar{H}_2 + \bar{H}_3$$

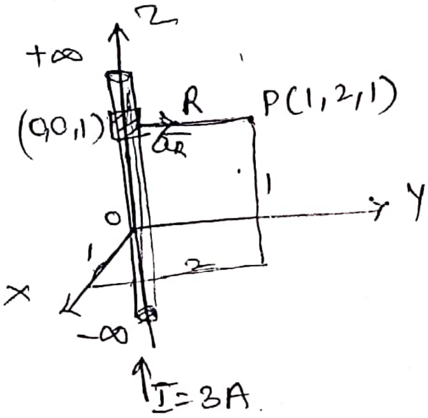
$$= \frac{I_1}{2\pi r_1}\bar{a}_\phi + \frac{I_2}{2\pi r_2}\bar{a}_\phi + \frac{I_3}{2\pi r_3}\bar{a}_\phi$$

$$= \left[\frac{50}{2\pi \times 1.06} + \frac{10}{2\pi \times 0.79} + \frac{40}{2\pi \times 0.79}\right]\bar{a}_\phi$$

$$\bar{B} = \mu \bar{H} = 2.208\,\bar{a}_\phi \times 10^{-5}\ \text{Tesla.}$$

$$\bar{H} = 17.57\,\bar{a}_\phi\ \text{A/m}$$

3) An infinite long straight conductor carrying a current of 3A is placed along z-axis & differential element kept at (0,0,1). Calculate the magnetic field Intensity at P(1,2,1).



Sol

$$\bar{H} = \frac{I}{2\pi R}\bar{a}_\phi$$

$$|\bar{R}| = \sqrt{(1)^2+(2)^2} = \sqrt{5}$$

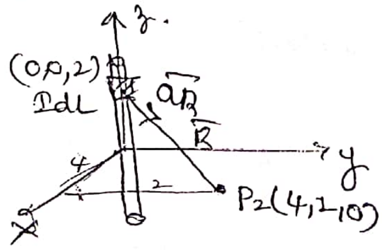
$$\bar{H}\ \text{at 'p'} = \frac{I}{2\pi r}\bar{a}_\phi$$

$$= \frac{3}{2\pi(\sqrt{5})}\bar{a}_\phi$$

$$= \frac{3}{2\pi\sqrt{5}}\left[\frac{\bar{a}_x + 2\bar{a}_y}{\sqrt{5}}\right]$$

$$= 0.954\,\bar{a}_x + 0.19\,\bar{a}_y\ \text{N/wb.}$$

4) Find $\bar{H}$ at point $P_2(4,2,0)$ due to carrying conductor $Idl = 2\pi \bar{a}_z\ \mu\,\text{A/m}$ placed at $P_1(0,0,2)$.



Sol:

$$d\bar{H} = \frac{Idl \times \bar{a}_R}{4\pi R^2}$$

$$|\bar{R}| = \sqrt{(4)^2+(2)^2+(2)^2} = \sqrt{24}$$

$$\bar{a}_R = \frac{\bar{R}}{|\bar{R}|} = \frac{4\bar{a}_x + 2\bar{a}_y - 2\bar{a}_z}{\sqrt{24}}$$

$$\therefore d\bar{H} = \frac{\cancel{2\pi}\,\bar{a}_z \times [4\bar{a}_x + 2\bar{a}_y - 2\bar{a}_z]\,10^{-6}}{\frac{\cancel{4\pi}}{2}(24)\sqrt{24}}$$

$$= -8.5\,\bar{a}_x + 17.01\,\bar{a}_y\ \text{nA/m}$$

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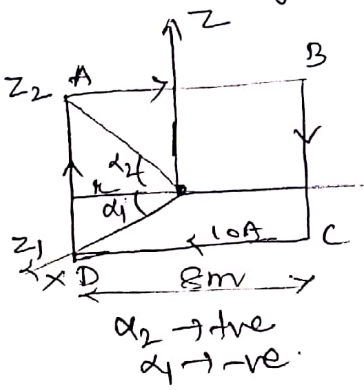
5) Find $\bar{H}$ at the centre of a square current loop of side 8m, if a current of 10A is passing through it.



Sol

$$\bar{H}_1 = \frac{I}{4\pi r}\left[\sin\alpha_2 - \sin\alpha_1\right]\bar{a}_\phi$$

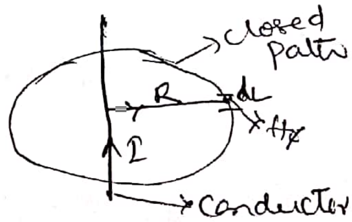
$$= \frac{I}{4\pi r}\left[\sin\alpha_2 - \sin(-\alpha_1)\right]\bar{a}_\phi$$

$$= \frac{10}{4\pi(4)}\left[\sin 45^\circ + \sin 45^\circ\right]\bar{a}_\phi$$

$$= 0.28\,\bar{a}_\phi \text{ A/m.}$$

From symmetry $\bar{H} = 4\bar{H}_1 = 4(0.28) = 1.13\,\bar{a}_\phi$ A/m.

6) Find the magnetic field at a distance R from a long straight wire carrying a steady current using Ampere's law.



Sol

$$\oint \bar{H}\cdot d\bar{L} = I$$

$$\oint H_\phi \bar{a}_\phi \cdot dL\,\bar{a}_\phi = I$$

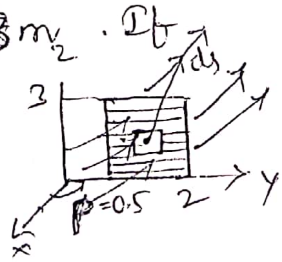
$$H_\phi \oint dL = I$$

$$H_\phi[2\pi R] = I \Rightarrow H_\phi = \frac{I}{2\pi R}$$

$$\therefore \bar{H} = H_\phi \bar{a}_\phi$$

$$\boxed{\bar{H} = \frac{I}{2\pi R}\bar{a}_\phi}$$

7) Find the flux passing through the plane surface defined by $0.5 \le \rho \le 2$ meter and $0 \le z \le 3m$. If $\bar{B} = \frac{4}{\rho}\bar{a}_\phi$ Tesla.



Sol

$$\phi = \oint_S \bar{B}\cdot d\bar{s}$$

$$= \iint_{\rho=0.5\,z=0}^{2\;\;3} \frac{4}{\rho}\bar{a}_\phi \cdot d\rho\, dz\,\bar{a}_\phi$$

$$= 4\int_{0.5}^{2}\frac{1}{\rho}d\rho \int_0^3 dz$$

$$= 4\left[\log_e \rho\right]_{0.5}^{2}\left[z\right]_0^3$$

$$= 16.64 \text{ weber.}$$

8) A Z-directed current distribution is given by $\bar{J} = (r^2 + ur)$ for $r \le a$. Find $\bar{B}$ at any point $r \ge a$ using Ampere's ckt law.

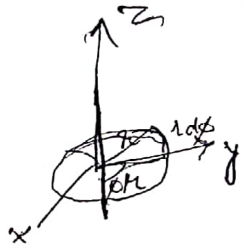
Sol:

$$\oint_L \bar{H} \cdot \bar{dl} = I \quad \text{—— (1)}$$

$$I = \oint \bar{J} \cdot \bar{ds} \qquad \bar{ds} = r\,dr\,d\phi\,\bar{a_z}$$

$$= \int_{\phi=0}^{2\pi}\int_{r=0}^{a} (r^2 + ur)\,\bar{a_z} \cdot r\,dr\,d\phi\,\bar{a_z}$$

$$I = \int_0^{2\pi}\int_0^{a} (r^2 + ur)\,r\,dr\,d\phi \quad \text{—— (2)}$$



$$\oint \bar{H} \cdot \bar{dL} = \int_{\phi=0}^{2\pi} H_\phi \bar{a_\phi} \cdot r\,d\phi\,\bar{a_\phi}$$

$$= \int_0^{2\pi} H_\phi\, r\,d\phi \quad \text{—— (3)}$$

Sub (2) (3) in (1)

$$\int_0^{2\pi} H_\phi\, r\,d\phi = \int_{\phi=0}^{2\pi}\int_0^{a} (r^2 + ur)\,r\,dr\,d\phi$$

$$H_\phi\, r[\phi]_0^{2\pi} = \int_0^{a} (r^3 + ur^2)\,dr\,[\phi]_0^{2\pi}$$

$$H_\phi\, \cancel{2\pi} = \frac{1}{r}\left[\frac{r^4}{4} + \frac{ur^3}{3}\right]_0^{a} \cancel{2\pi}$$

$$H_\phi = \frac{1}{r}\left[\frac{a^4}{4} + \frac{ua^3}{3}\right]$$

$$\bar{H} = H_\phi \bar{a_\phi}$$

$$\bar{H} = \frac{1}{12r}\left[3a^4 + 4ua^3\right]\bar{a_\phi} \quad \text{A/m}$$

$$B = \mu_0 \bar{H}$$

$$= \frac{\mu_0}{12r}\left[3a^4 + 4ua^3\right]\bar{a_\phi} \quad \text{Tesla.}$$

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9) The current density in an electron beam is given by $J = J_0\left[1-\frac{\rho^2}{b^2}\right]\bar{a}_z$ A/m² $(\rho<b)$ where J_0 is constant and b is the beam radius. At $\rho=\frac{b}{3}$, the field H is given by $H = KbJ_0\bar{a}_\phi$ A/m. Find K.

(10)

Sol

$$\oint \bar{H}\cdot \bar{dL} = I = \oint \bar{J}\cdot\bar{ds}$$

$$\oint KbJ_0\bar{a}_\phi \cdot dL\,\bar{a}_\phi = \int_{\rho=0}^{b/3}\int_{\phi=0}^{2\pi} J_0\left[1-\frac{\rho^2}{b^2}\right]\bar{a}_z \cdot \rho\, d\rho\, d\phi\, \bar{a}_z$$

$\cancel{KbJ_0}$

$$KbJ_0\oint dL = J_0\int_{\rho=0}^{b/3}\left[1-\frac{\rho^2}{b^2}\right]\rho\, d\rho\int_{\phi=0}^{2\pi} d\phi$$

$$KbJ_0(2\pi\rho) = J_0\left[\int_0^{b/3}\left(\rho-\frac{\rho^3}{b^2}\right)d\rho\right]\left[\phi\right]_0^{2\pi}$$

Consider. (∵ circular path)

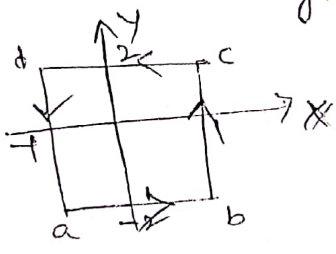
$$Kb\cancel{J_0}(\cancel{2\pi}\rho) = \cancel{J_0}\left[\frac{\rho^2}{2}-\frac{\rho^4}{4b^2}\right]_0^{b/3}(\cancel{2\pi}).$$

$$Kb\frac{b}{3} = \frac{b^2}{18} - \frac{b^{4}}{324 b^2}$$

$$K = 3\left[\frac{1}{18}-\frac{1}{324}\right]$$

$$K = \frac{306(3)}{18(324)} \Rightarrow K = 0.157$$

10) The magnetic field Intensity in the $z=0$ plane is given by $\bar{H} = -y(x^2+y^2)\bar{a}_x + x(x^2+y^2)\bar{a}_y$ A/m. Find the total current passing through $z=0$ plane in the $\bar{a}_z$ direction inside the rectangle bounded by $-1\le x<1$ and $-2<y<2$.



Sol.

$$\oint \bar{H}\cdot\bar{dL} = I$$

$$\left\{\int_a^b + \int_b^c + \int_c^d + \int_d^a\right\}\bar{H}\cdot\bar{dL} = I.$$

$$\int_a^b H_x\,dx\Big|_{y=-2} + \int_b^c H_y\,dy\Big|_{x=1} + \int_c^d H_x\,dx\Big|_{y=2} + \int_d^a H_y\,dy\Big|_{x=-1} = I.$$

$$\int_{-1}^{1} -y(x^2+y^2)\,dx\Big|_{y=-2} + \int_{-2}^{2} x(x^2+y^2)\,dy\Big|_{x=1} + \int_{1}^{-1} -y(x^2+y^2)\,dx\Big|_{y=2} + \int_{2}^{-2} x(x^2+y^2)\,dy\Big|_{x=-1} = I.$$

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$$\Rightarrow \int_{-1}^{1} 2(x^2+4)\,dx + \int_{-2}^{2} 1(1+y^2)\,dy + \int_{1}^{-1} -2(x^2+4)\,dx + \int_{2}^{-2} -1(1+y^2)\,dy = I$$

$$\Rightarrow \left[\frac{2x^3}{3} + 8x\right]_{-1}^{1} + \left[y + \frac{y^3}{3}\right]_{-2}^{2} - \left[\frac{2x^3}{3} + 8x\right]_{1}^{-1} - \left[y + \frac{y^3}{3}\right]_{2}^{-2} = I$$

$$\Rightarrow \left[\frac{2}{3} + 8 + \frac{2}{3} + 8\right] + \left[\cancel{2} + \cancel{\frac{8}{3}} + \cancel{2} + \cancel{\frac{8}{3}}\right] - \left[-\frac{2}{3} - 8 - \frac{2}{3} - 8\right] + \left[\cancel{-2} - \cancel{\frac{8}{3}} - \cancel{2} - \cancel{\frac{8}{3}}\right] = I$$

$$\Rightarrow \frac{4}{3} + 16 + \frac{4}{3} + 16 \cancel{-4} = I$$

$$\therefore I = 34.66\,A$$

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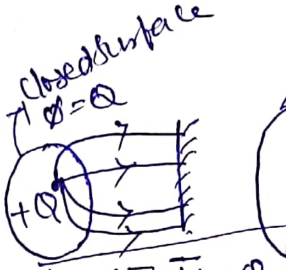
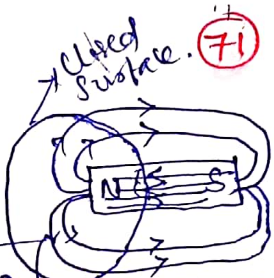
Magnetic Flux Density ($\bar{B}$)

$$B = \frac{d\phi}{ds}$$

$$\boxed{\phi = \int_S \bar{B} \cdot \bar{ds}}$$



$$\boxed{\phi = \oint \bar{D} \cdot \bar{ds} = Q}$$



$$\boxed{\phi = \oint \bar{B} \cdot \bar{dS} = 0}$$

applying divergence theorem

$$\oint_S \bar{B} \cdot \bar{ds} = \int_V (\nabla \cdot \bar{B})\, dV$$

$$\boxed{\oint \bar{B} \cdot \bar{ds} = 0}$$

($\because$ magnetic flux lines always exist in the form of closed loop.
$\therefore$ flux inside (no. of lines entering into surface) = flux coming out of surface)

This is maxwell's eqn in ~~IE~~ integral form (or) law of conservation of magnetic flux (or) Gauss's law in integral form for magnetic fields.

$$\oint_S \bar{B} \cdot \bar{ds} = \int_V (\nabla \cdot \bar{B})\, dV = 0$$

$$\therefore \boxed{\nabla \cdot \bar{B} = 0} \rightarrow$$ maxwell eqn in differential form (or) point-form.

Maxwell's eqns for static electric and magnetic fields.

Differential form		Integral form.
1) $\nabla \cdot \bar{D} = \rho_V$	Gauss's law	1) $\oint_S \bar{D} \cdot \bar{ds} = \int_V \rho_V\, dV = Q$
2) $\nabla \times \bar{E} = 0$	conservation of electric field	2) $\oint \bar{E} \cdot \bar{dl} = 0$
3) $\nabla \times \bar{H} = \bar{J}$	Ampere's ckt law	3) $\oint_L \bar{H} \cdot \bar{dL} = I = \int_S \bar{J} \cdot ds$
4) $\nabla \cdot \bar{B} = 0$	Gauss's law for magnetic field	4) $\oint \bar{B} \cdot \bar{ds} = 0$

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In electrostatics there exist a scalar potential V which is related to $\bar{E}$. i.e. $\bar{E} = -\nabla V$

In case of magnetic fields there are two types of potentials defined

1) Scalar magnetic potential V_m
2) vector " " $\bar{A}$

To define scalar & vector magnetic potentials, let us use two vector identities (i.e. properties of curl)

(1) $\nabla \times \nabla V = 0$

(2) $\nabla \cdot (\nabla \times \bar{A}) = 0$

1) Magnetic Scalar potential

$V_m \rightarrow$ scalar potential

$\nabla \times \nabla V_m = 0$ —— (1)

But $\bar{H} = -\nabla V_m$ —— (2)

sub (2) in (1)

$\nabla \times (-\bar{H}) = 0$

$\nabla \times \bar{H} = 0$ —— (3)

From the concept of curl

$\nabla \times \bar{H} = \bar{J}$ —— (4)

Compare (3) (4)

$\bar{J} = 0$

magnetic scalar potential V_m defined for source free region where $\bar{J} = 0$

$\therefore \boxed{\bar{H} = -\nabla V_m}$ only for $\bar{J} = 0$

Similar to the relation between $\bar{E}$ and V, magnetic scalar potential can be expressed in terms of $\bar{H}$

$$\boxed{V_{m\,a,b} = -\int_b^a \bar{H} \cdot d\bar{L}}$$

magnetic scalar potential satisfies Laplace eqn.

$$\nabla \cdot \bar{B} = 0$$
$$\nabla \cdot \mu \bar{H} = 0$$
$$\nabla \cdot \bar{H} = 0$$
$$\nabla \cdot (-\nabla V_m) = 0$$
$$\boxed{\nabla^2 V_m = 0} \longrightarrow \text{laplace eqn. for } \bar{J} = 0$$

2) magnetic Vector potential $(\bar{A})$

using vector identity

$$\nabla \cdot (\nabla \times \bar{A}) = 0 \quad \text{—— (1)}$$

but $\nabla \cdot \bar{B} = 0$ —— (2)

Compare (1) (2)

$$\boxed{\bar{B} = \nabla \times \bar{A}}$$

Curl on magnetic vector potential is nothing but magnetic flux density.

From curl on $\bar{H}$ $\quad \nabla \times \bar{H} = \bar{J}$

$$\nabla \times \frac{\bar{B}}{\mu_0} = \bar{J}$$
$$\nabla \times \bar{B} = \mu_0 \bar{J}$$
$$\nabla \times \nabla \times \bar{A} = \mu_0 \bar{J}$$
$$\nabla(\nabla \cdot \bar{A}) - \nabla^2 \bar{A} = \mu_0 \bar{J}$$
$$J = \frac{1}{\mu_0}\left[\nabla(\nabla \cdot \bar{A}) - \nabla^2 \bar{A}\right]$$

magnetic vector potential satisfies poisson's eqn.

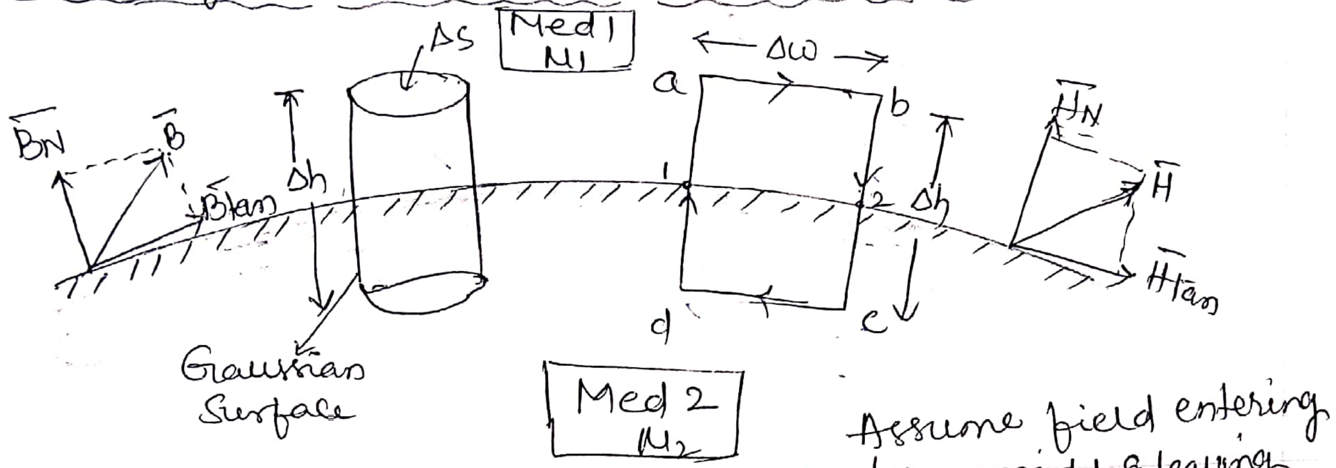
Completely define $\bar{A}$ its divergence must be known.

Assum $\nabla \cdot \bar{A} = 0$

$$\therefore \bar{J} = \frac{1}{\mu_0}\left[-\nabla^2 A\right]$$

$$\boxed{\nabla^2 A = -\mu_0 \bar{J}} \longrightarrow \text{poissons eqn.}$$

Boundary Conditions b/w two dielectrics (or medium)



Assume field entering from med 1 & leaving from med 2.

i) Tangential Component

$$\oint \overline{H} \cdot \overline{dL} = I \quad \text{(from Amperes Ckt law)}$$

$$\int_a^b \overline{H} \cdot \overline{dL} + \int_b^c \overline{H} \cdot \overline{dL} + \int_c^d \overline{H} \cdot \overline{dL} + \int_d^a \overline{H} \cdot \overline{dL} = I \quad \text{——} (1)$$

$$\int_a^b \overline{H} \cdot \overline{dL} = \int_a^b H_{tan1}\, \overline{a_L} \cdot dL\, \overline{a_L} = H_{tan1}\, \Delta w \quad \text{——} (2)$$

$$\int_b^c \overline{H} \cdot \overline{dL} = \int_b^2 \overline{H} \cdot \overline{dL} + \int_2^c \overline{H} \cdot \overline{dL}$$

$$= H_{N1} \frac{\Delta h}{2} + H_{N2} \frac{\Delta h}{2} \quad \text{——} (3)$$

$$\int_c^d \overline{H} \cdot \overline{dL} = -H_{tan2}\, \Delta w \quad \text{——} (4)$$

$$\int_d^a \overline{H} \cdot \overline{dL} = \int_d^1 \overline{H} \cdot \overline{dL} + \int_1^a \overline{H} \cdot \overline{dL}$$

$$= -H_{N2} \frac{\Delta h}{2} - H_{N1} \frac{\Delta h}{2} \quad \text{——} (5)$$

Sub (2)(3)(4)(5) in (1)

$$H_{tan1}\, \Delta w + \cancel{H_{N1} \frac{\Delta h}{2}} + \cancel{H_{N2} \frac{\Delta h}{2}} - H_{tan2}\, \Delta w - \cancel{H_{N2} \frac{\Delta h}{2}} - \cancel{H_{N1} \frac{\Delta h}{2}} = I$$

$$(H_{tan1} - H_{tan2})\, \Delta w = I$$

$$H_{tan1} - H_{tan2} = \frac{I}{\Delta w}$$

$$H_{tan1} - H_{tan2} = \overline{J}$$

$$\text{if } \overline{J} = 0 \implies \boxed{\overline{H}_{tan1} = \overline{H}_{tan2}}$$

Tangential Component of $\overline{H}$ is continuous at boundary.

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$\bar{B} = \mu H$

$\bar{B}_{tan1} = \mu_1 \bar{H}_{tan1}$

$\bar{B}_{tan2} = \mu_2 \bar{H}_{tan2}$

$\therefore \dfrac{\bar{B}_{tan1}}{\mu_1} = \dfrac{\bar{B}_{tan2}}{\mu_2}$

$\boxed{\dfrac{\bar{B}_{tan1}}{\bar{B}_{tan2}} = \dfrac{\mu_1}{\mu_2}}$

$\therefore$ Tangential component of $\bar{B}$ are discontinuous at the boundary, with the condition that boundary is current free.

ii) <u>Normal Component</u>

$\oint \bar{B}\cdot\bar{ds} = 0$ ($\because$ From Gauss's law)

$\oint_{top} \bar{B}\cdot\bar{ds} + \oint_{bottom} \bar{B}\cdot\bar{ds} + \oint_{lateral} \bar{B}\cdot\bar{ds} = 0$ —— (1)

$\oint_{top} \bar{B}\cdot\bar{ds} = B_{N1}\,\Delta S$ —— (2)

$\oint_{bottom} \bar{B}\cdot\bar{ds} = -B_{N2}\,\Delta S$ —— (3) (field is leaving from bottom surface)

$\oint_{lateral} \bar{B}\cdot\bar{ds} = 0$ —— (4) ($\because$ Since lateral surface moves towards the ~~surface~~ boundary).

Sub (2)(3)(4) in (1)

$B_{N1}\,\Delta S - B_{N2}\,\Delta S = 0$

$\boxed{B_{N1} = B_{N2}}$

$\therefore$ Normal component of $\bar{B}$ is continuous at boundary.

$\bar{B} = \mu \bar{H}$

$\bar{B}_{N1} = \mu_1 \bar{H}_{N1}$

$\bar{B}_{N2} = \mu_2 H_{N1}$

$\therefore \mu_1 H_{N1} = \mu_2 H_{N2}$

$\boxed{\dfrac{H_{N1}}{H_{N2}} = \dfrac{\mu_2}{\mu_1}}$

$\therefore$ Normal component of $\bar{H}$ is discontinuous at boundary.

$\therefore \bar{H} = \bar{H}_{tan} + \bar{H}_N$

$\bar{B} = \bar{B}_{tan} + \bar{B}_N$

$\left.\begin{array}{l} H_{tan} = H_{tan1} + H_{tan2} \\ H_N = H_{N1} + H_{N2}. \end{array}\right\}$

Illy $B_{tan} = B_{tan1} + B_{tan2}$

Magnetomotive Force

$\bar{F}_e = Q\bar{E}$ —(1) (From Coulomb's law)

magnetic force $\bar{F}_m = Q\,\bar{v}\times\bar{B}$ —(2)

$\therefore\ \bar{F} = \bar{F}_e + \bar{F}_m$

$$\boxed{\bar{F} = Q[\bar{E} + \bar{v}\times\bar{B}]}$$

This eqn is called Lorentz Force eqn.

$$\boxed{\bar{F} = m\bar{a} = m\frac{dv}{dt} = Q[\bar{E} + \bar{v}\times\bar{B}]}$$

force acting on differential current element

$$d\bar{F} = I\,d\bar{L}\times\bar{B}$$

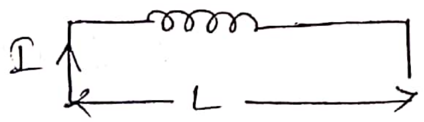
$$\bar{F} = \int I\,d\bar{L}\times\bar{B}$$

$$= I\bar{L}\times\bar{B}$$

$$= ILB\sin\theta$$

$$\boxed{\bar{F} = BIL\sin\theta}$$

Inductance (or) Self Inductance



A wire or conductor of certain length, when twisted into coil becomes Inductor.

The flux linkage is defined as "the product of number of turns N and the total flux ϕ linking each of the turn".

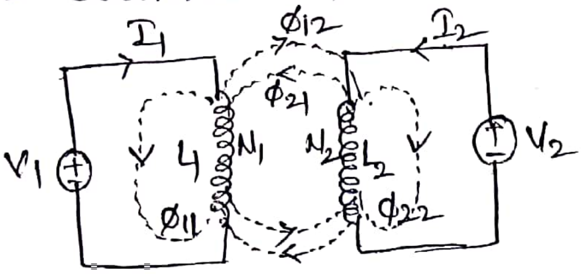
Flux linkage = $\boxed{\lambda = N\phi}$ weber-turn.

"The ratio of total flux linkage to the current flow through the circuit is called Inductance (or) Self Inductance"

$$\boxed{L = \frac{N\phi}{I}}$$ Henry (or) wb-turn/A

Mutual Inductance

"It is defined as the flux linkage of one circuit to the current in other circuit".



$$M_{12} = \frac{\text{flux linkage of circuit 1}}{\text{current in ckt 2}} = \frac{N_1 \phi_{21}}{I_2}$$

$$M_{21} = \frac{\text{flux linkage of ckt 2}}{\text{current in ckt 1}} = \frac{N_2 \phi_{12}}{I_1}$$

For a linear medium $M_{12} = M_{21}$

Derive an expression for coefficient of coupling between two circuits:

When two magnetic circuits kept closed to each other interacts with each other magnetically through the flux linkages in the ckt due to current in other ckt then the ckts are called magnetically coupled circuits.

$$L_1 = \frac{N_1 \phi_{11}}{I_1} = \frac{\lambda_{11}}{I_1} \quad \text{—— (1)}$$

$$L_2 = \frac{N_2 \phi_{22}}{I_2} = \frac{\lambda_{22}}{I_2} \quad \text{—— (2)}$$

The flux linking with ckt 2 due to current in ckt 1 is denoted by ϕ_{12}. This flux is part of total flux ϕ_{11}

$$\phi_{12} = K_1 \phi_{11} \quad \text{—— (3)}$$

lly $\phi_{21} = K_2 \phi_{22}$ —— (4)

Mutual inductance

$$M_{12} = \frac{N_1 \phi_{21}}{I_2} = \frac{N_1 (K_2 \phi_{22})}{I_2}$$

and $$M_{21} = \frac{N_2 \phi_{12}}{I_1} = \frac{N_2 (K_1 \phi_{11})}{I_1}$$

For linear medium $M_{12} = M_{21}$

$$\therefore M^2 = M_{12} \cdot M_{21}$$

$$= \frac{N_1 K_2 \phi_{22}}{I_2} \cdot \frac{N_2 K_1 \phi_{11}}{I_1}$$

$$= K_1 K_2 \frac{N_1 \phi_{11}}{I_1} \frac{N_2 \phi_{22}}{I_2}$$

$$M^2 = K_1 K_2 L_1 L_2$$

$$M = \sqrt{K_1 K_2 L_1 L_2}$$

let $K = \sqrt{K_1 K_2}$

$$\therefore M = K\sqrt{L_1 L_2} \Rightarrow \boxed{K = \frac{M}{\sqrt{L_1 L_2}}}$$

$K \rightarrow$ coupling coefficient

i) when two magnetic ckts are coupled together in series aiding

$$L_{eq} = L_1 + L_2 + 2M$$

ii) series opposing

$$L_{eq} = L_1 + L_2 - 2M$$

iii) parallel aiding

$$L_{eq} = \frac{L_1 L_2 - M^2}{L_1 + L_2 - 2M}$$

iv) parallel opposing

$$L_{eq} = \frac{L_1 L_2 - M^2}{L_1 + L_2 + 2M}$$

Inductance of a

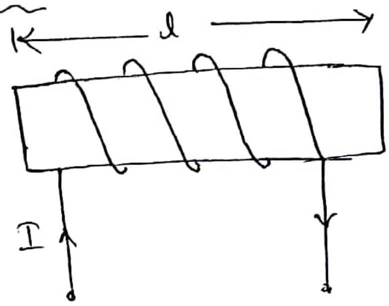
a) Solenoid

b) Toroid

c) coaxial cable

d) Two wire transmission line

1) Inductance of a Solenoid :



The magnetic field Intensity inside a solenoid

$$H = \frac{NI}{l} \text{ A/m} \quad \text{—①}$$

$$\text{Inductance} = L = \frac{\text{Total flux linkage}}{\text{current}} = \frac{N\phi}{I} \quad \text{—②}$$

$$\left[\because \phi = \int \bar{B}\cdot d\bar{s}\right]$$

$$\text{Total flux linkage} = N\phi = N(B)(A)$$

$$= N\mu H A$$

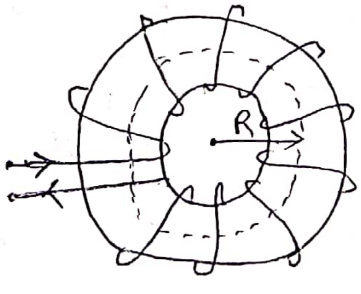
$$= N\mu\left[\frac{NI}{l}\right]A$$

$$= \frac{\mu N^2 I A}{l} \quad \text{—(3)}$$

Sub ③ in ②

$$\therefore L = \frac{\mu N^2 \cancel{I} A}{\cancel{I} l} \Rightarrow \boxed{L = \frac{\mu N^2 A}{l}}$$

2) Inductance of a Toroid :



The magnetic flux density inside a toroidal ring

$$B = \frac{\mu N I}{2\pi R} \quad \text{—①}$$

$$\text{Total flux linkage} = N\phi = NBA$$

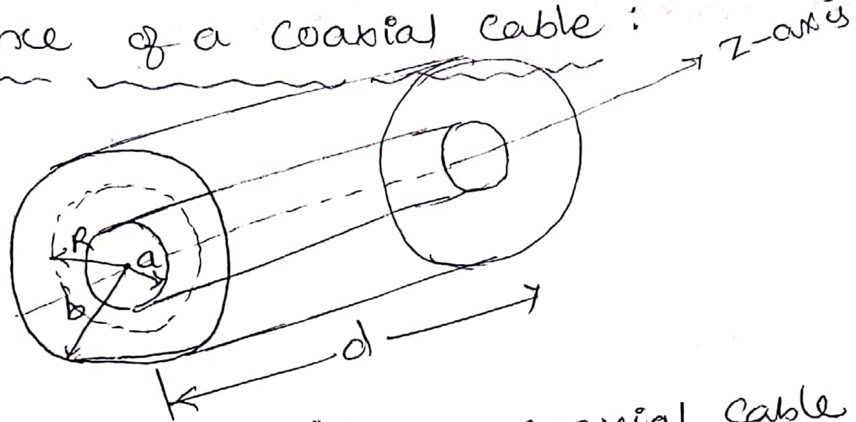
$$= N\frac{\mu N I}{2\pi R}A = \frac{\mu N^2 I A}{2\pi R}$$

$$\therefore \text{Inductance} = L = \frac{N\phi}{I}$$

$$= \frac{\mu N^2 \cancel{I} A}{\cancel{I}\, 2\pi R}$$

$$\boxed{L = \frac{\mu N^2 A}{\ldots}}$$

③ Inductance of a coaxial cable :



magnetic field Intensity in coaxial cable

$$\vec{H} = \frac{I}{2\pi R} \quad \text{when } a<R<b \qquad \text{— (1)}$$

$$\bar{B} = \mu \bar{H} = \frac{\mu I}{2\pi R}\,\bar{a_\phi}$$

Flux linkage $= N\phi$

$= \cancel{N B A}$

The magnetic flux density will be on radial plane extending from $r=a$ to $r=b$ and $z=0$ to $z=d$.

$= \cancel{N \frac{\mu I}{2\pi R} A}$

$\therefore$ Total magnetic flux $= \phi = \int_S \bar{B}\cdot\bar{ds}$

$$= \int_{z=0}^{d}\int_{R=a}^{b} \frac{\mu I}{2\pi R}\,\bar{a_\phi}\cdot dR\,dz\,\bar{a_\phi}$$

$$= \frac{\mu I}{2\pi}\left[\log R\right]_a^b \left[z\right]_0^d$$

$$= \frac{\mu I}{2\pi}\left(\log\frac{b}{a}\right)d$$

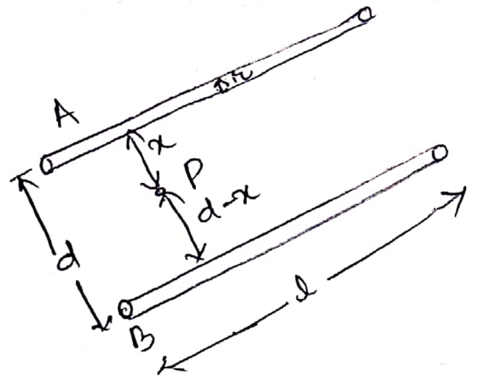
$\therefore$ Total inductance $= L = \frac{N\phi}{I}$

$$= \frac{N}{\cancel{I}}\cdot\frac{\mu \cancel{I} d}{2\pi}\left(\log\frac{b}{a}\right)$$

$$\boxed{L = \frac{\mu N d}{2\pi}\log\left(\frac{b}{a}\right)}$$

H-bys.

(4) Inductance of two wire transmission line



r = radius of wire

The magnetic field Intensity at 'p' because of conductor.

$$\overline{H_A} = \frac{I}{2\pi x}\,\overline{a_\phi} \quad \text{——— (1)}$$

The magnetic field Intensity due to conductor B

$$\overline{H_B} = \frac{I}{2\pi(d-x)}\,\overline{a_\phi} \quad \text{——— (2)}$$

Total magnetic field Intensity at 'p'

$$\overline{H} = \overline{H_A} + \overline{H_B}$$

$$= \frac{I}{2\pi x}\,\overline{a_\phi} + \frac{I}{2\pi(d-x)}\,\overline{a_\phi}$$

$$\overline{H} = \frac{I}{2\pi}\left[\frac{1}{x} + \frac{1}{d-x}\right]\overline{a_\phi} \quad \text{——— (3)}$$

Flux linkage $= N\phi$ $\quad [\phi = \int \overline{B}\cdot\overline{ds} = B\,dx]$

$= NBA$

$= N\mu \frac{I}{2\pi}\left[\frac{1}{x} + \frac{1}{d-x}\right] A$

Total flux linkage $= \int_{x=r}^{d-r} \frac{N\mu I \cancel{A}}{2\pi}\left[\frac{1}{x} + \frac{1}{d-x}\right] dx$

$$= \frac{N\mu I \cancel{A}}{2\pi}\Big[\log x + \log(d-x)\,(-1)\Big]_r^{d-r}$$

$$= \frac{N\mu I \cancel{A}}{2\pi}\Big[\log(d-r) - \log(\cancel{d}-\cancel{d}+r) - \log r + \log(d-r)\Big]$$

Flux linkage due to one line $= N\phi/2$

$$= \frac{N\mu I \cancel{A}}{2\pi}\left[\cancel{2}\log\frac{d-r}{r}\right]$$

$\therefore$ Inductance $= L = \frac{N\phi/2}{I}$

$$L = \frac{\mu N}{2\pi}\log\left[\frac{d-r}{r}\right] \quad \text{Henries}$$

Energy in Inductor:

When an electric current is flowing in an inductor there is energy stored in the magnetic field.

$$P = IV$$
$$= I L \frac{di}{dt}$$

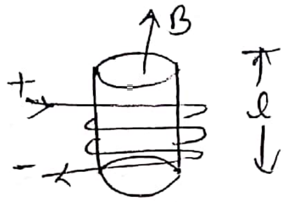
energy stored $= \int_0^t P\,dt = \int_0^t I L \frac{di}{dt}\,dt$

$$= \frac{L \cdot I^2}{2}$$

$$\boxed{\text{energy stored} = \frac{1}{2} L I^2}$$

Energy in magnetic field:

Solenoid $= B = \frac{\mu N I}{l}$



Energy stored $= \frac{1}{2} L I^2$

energy density $= \frac{\text{energy}}{\text{volume}}$

$$= \frac{\frac{1}{2} L I^2}{A l}$$

$$= \frac{\frac{1}{2} \frac{\mu N^2 A}{l} \left(\frac{B l}{\mu N}\right)^2}{A l}$$

$$= \frac{1}{2} \frac{\cancel{\mu} \cancel{N^2} \cancel{A}}{\cancel{A} \cancel{l^2}} \frac{B^2 \cancel{l^2}}{\mu^{\cancel{2}} \cancel{N^2}}$$

$$\boxed{W_E = \frac{1}{2} \frac{B^2}{\mu}}$$

or $W_E = \frac{1}{2} \frac{\mu^{\cancel{2}} H^2}{\cancel{\mu}}$ $(\because B = \mu H)$

$$\boxed{W_E = \frac{1}{2} \mu H^2}$$

or $W_E = \frac{1}{2} \frac{B}{\cancel{H}} H^{\cancel{2}}$

$$\boxed{W_E = \frac{1}{2} B H}$$

Problems

1) Calculate the inductance of solenoid of 4000 turns wound uniformly over a length of 600mm on a cylindrica paper tube 50mm diameter. The medium is air.

Sol:

$$L = \frac{\mu N^2 A}{l}$$

$N = 4000$ turns

$l = \frac{600}{1000} m = 0.6m.$

$d = \frac{50}{1000} = 0.05m$

$r = \frac{0.05}{2} = 0.025m.$

$$\therefore L = \frac{4\pi \times 10^{-7} \times (4000)^2 \times 0.1242}{0.6}$$

$A = 2\pi r^2 h$

$= 2\pi (0.025)^2 \times 0.6$

$= 0.1242 m^2$

$$= 4.15 \text{ H}$$

2) A very long 1m solenoid with 2x2 cm cross section has an iron core ($\mu r = 1000$) and 4000 turns/m. If it carries a current of 500mA. Find its self inductance per meter.

Sol

$$L = \frac{\mu N^2 A}{l}$$

$A = \frac{2}{100} \times \frac{2}{100} = 4 \times 10^{-4} m.$

$N = 4000$ turns.

$\mu = \mu_0 \mu_r$

$= 4\pi \times 10^{-7} \times 1000$

$= 4\pi \times 10^{-4}$

$$\frac{L}{A} = \frac{4\pi \times 10^{-4} \times (4000)^2 \times 4 \times 10^{-4}}{1}$$

$$= 256\pi \times 10^{-2} \text{ H}$$

3) An air cored toroidal coil has 500 turns and a mean diameter of 32cm with a cross section area of $2 cm^2$. Calculate the inductance of the coil.

Sol

$$L = \frac{\mu_0 N^2 A}{2\pi R}$$

$A = 2 cm^2$

$= \frac{2}{(100)^2} = 2 \times 10^{-4} m$

$d = \frac{32}{100} = 32 \times 10^{-2}$

$R = \frac{32 \times 10^{-2}}{2}$

$= 16 \times 10^{-2}$

$$= \frac{4\pi \times 10^{-7} \times (500)^2 \times 2 \times 10^{-4}}{2\pi \times 16 \times 10^{-2}}$$

$$= 6.25 \times 10^{-5} \text{ Henry}$$

4) A charge of 10 nC is moving with velocity of $10^{7}[-0.5\bar{a}_x + \bar{a}_y + \cancel{0.71}\,\bar{a}_z]$ m/s. Determine the force exerted on the charge when

i) A magnetic induction $\bar{B} = \bar{a}_x + 2\bar{a}_y + 3\bar{a}_z$ milli Wb/m² applied

ii) An electric induction $\bar{E} = 3\bar{a}_x + 2\bar{a}_y + \bar{a}_z$ kV/m applied

iii) When $\bar{B}$ and $\bar{E}$ are applied simultaneously.

Sol. i) $F_m = q(\bar{v} \times \bar{B})$ (due to magnetic field)

$$\bar{v} \times \bar{B} = [10^{7}(-0.5\bar{a}_x + \bar{a}_y - 0.71\bar{a}_z] \times [\bar{a}_x + 2\bar{a}_y + 3\bar{a}_z]10^{-3}$$

$$\bar{v} \times \bar{B} = 10^{4}\begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ -0.5 & 1 & +\cancel{0.71} \\ 1 & 2 & 3 \end{vmatrix}$$

$$\bar{v} \times \bar{B} = 10^{4}\left[4.42\bar{a}_x - 2.5\bar{a}_y - 2\bar{a}_z\right]$$

$$\therefore F = q(\bar{v} \times \bar{B})$$

$$= 10 \times 10^{-9} \times 10^{4}[4.42\bar{a}_x - 2.5\bar{a}_y - 2\bar{a}_z].$$

$$F = 10^{-4}[4.42\bar{a}_x - 2.5\bar{a}_y - 2\bar{a}_z] \text{ newton}$$

ii) $F_e = q\bar{E}$

$$= 10 \times 10^{-9}[3\bar{a}_x + 2\bar{a}_y + \bar{a}_z]10^{3}$$

$$= 10^{-5}[3\bar{a}_x + 2\bar{a}_y + \bar{a}_z] \text{ newton}$$

iii) Total Force $= F = \bar{F}_e + \bar{F}_m$

$$= 10^{-4}(4.42\bar{a}_x - 2.5\bar{a}_y - 2\bar{a}_z) + 10^{-5}(3\bar{a}_x + 2\bar{a}_y + \bar{a}_z)$$

$$= 10^{-4}[4.72\bar{a}_x - 2.7\bar{a}_y - 2.1\bar{a}_z] \text{ newton}$$

Time Varying Fields
(EM fields)

In static electromagnetic fields, electric and magnetic fields are independant of each other.

In time varying (or) dynamic fields, the electric and magnetic fields are interdependent.

Static electric field is produced due to stationary charge. static magnetic fields are produced due to the motion of charges.

The time varying fields are produced due to time varying currents.

Static magnetic field cannot produce any current but in time varying fields emf (electromotive force) induces.

According to Faraday's law "emf in a closed path is equal to the rate of change of magnetic flux enclosed by a closed path".

$$emf \propto -\frac{d\phi}{dt}$$

$$e = -N\frac{d\phi}{dt}$$

assume $N=1$

$$\boxed{e = -\frac{d\phi}{dt}} \quad \text{——— (1)}$$

Lenz's Law:

"The induced emf acts to produce an opposing flux (or) "The direction of induced emf is such that it opposes the cause producing it i.e. change in magnetic flux"

$$e = +\frac{d\phi}{..}$$

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The induced emf is a scalar quantity and measured in volts.

$$e = \oint \bar{E} \cdot \bar{dL} \quad \text{---} \quad (2)$$

but $\phi = \oint_S \bar{B} \cdot \bar{ds} \quad \text{---} \quad (3)$

Sub (3) in (1)

$$e = -\frac{d}{dt} \oint_S \bar{B} \cdot \bar{ds}$$

$$e = -\oint_S \frac{\partial \bar{B}}{\partial t} \cdot \bar{ds}$$

Sub this in ②

$$-\oint_S \frac{\partial \bar{B}}{\partial t} \cdot \bar{ds} = \oint_L \bar{E} \cdot \bar{dL}$$

applying Stoke's theorem for RHS.

$$-\oint \frac{\partial \bar{B}}{\partial t} \cdot \cancel{\bar{ds}} = \oint_S (\nabla \times \bar{E}) \cdot \cancel{\bar{ds}}$$

Comparing on both sides

$$\boxed{\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t}}$$

if $\bar{B} = 0$ ($\because$ if $\bar{B}$ is not varying with time)

$\nabla \times \bar{E} = 0$ (Static electric field)

Modified Ampere's Circuital Law :

From Ampere's law

$$\oint \bar{H} \cdot d\bar{L} = I$$

$$\nabla \times \bar{H} = \bar{J} \quad \text{——— (1)}$$

applying divergence on both sides

$$\nabla \cdot (\nabla \times \bar{H}) = \nabla \cdot \bar{J}$$

$$0 = \nabla \cdot \bar{J}$$

(Divergence on curl is zero) $\nabla \cdot (\nabla \times \bar{H}) = 0$.

$$\therefore \nabla \cdot \bar{J} = 0 \quad \text{——— (2)}$$

From continuity eqn

$$\nabla \cdot \bar{J} = -\frac{\partial \rho_v}{\partial t} \quad \text{——— (3)}$$

$$\frac{\partial \rho_v}{\partial t} = 0 \Rightarrow \rho_v = 0.$$

$\frac{\partial \rho_v}{\partial t} = 0$, then eqn (2) becomes true.

eqn (2) & (3) are not compatable for time varying fields.

Therefore unknown parameter 'N' can be added to eqn (1)

$$\nabla \times \bar{H} = \bar{J} + \bar{N} \quad \text{——— (I)}$$

apply divergence on both sides

$$\nabla \cdot (\nabla \times \bar{H}) = \nabla \cdot (\bar{J} + \bar{N})$$

$$0 = \nabla \cdot \bar{J} + \nabla \cdot \bar{N}$$

$$0 = -\frac{\partial \rho_v}{\partial t} + \nabla \cdot \bar{N} \Rightarrow \nabla \cdot \bar{N} = \frac{\partial \rho_v}{\partial t} \quad \text{——— (4)}$$

From Gauss's law $\nabla \cdot \bar{D} = \rho_v$ — (5)

Sub (5) in (4)

$$\nabla \cdot \bar{N} = \frac{\partial}{\partial t}(\nabla \cdot \bar{D})$$

$$\nabla \cdot \bar{N} = \nabla \cdot \frac{\partial \bar{D}}{\partial t}$$

Comparing we get $\Rightarrow \bar{N} = \frac{\partial \bar{D}}{\partial t}$

$\therefore$ eqn (I) becomes.

$$\boxed{\nabla \times \bar{H} = \bar{J} + \frac{\partial \bar{D}}{\partial t}} \quad \text{or} \quad \boxed{\nabla \times \bar{H} = \bar{J}_c + \bar{J}_D}$$

or

$$\boxed{\nabla \times \bar{H} = \sigma \bar{E} + \frac{\partial \bar{D}}{\partial t}}$$

where $\bar{J}_c$ = conduction current density (drift current)
$\bar{J}_D$ = Displacement " " (~~diffusion~~ current)

Maxwell's eqn for Non-Varying Fields

1) Maxwell's eqn from Faraday's law for electric field

$$\boxed{\oint_L \bar{E}\cdot \bar{dL} = 0}$$ → Integral form

apply ~~stokes~~ Theorem

$$\oint_S (\nabla \times \bar{E})\cdot \bar{ds} = 0$$

$$\bar{ds} \neq 0$$

$$\therefore \boxed{\nabla \times \bar{E} = 0}$$ → differential form.

2) Maxwell's eqn from Ampere's ckt law for magnetic field.

$$\boxed{\oint_L \bar{H}\cdot \bar{dL} = I = \oint_S \bar{J}\cdot \bar{ds}}$$

apply stokes theorem

$$\oint_S (\nabla \times \bar{H})\cdot \bar{ds} = \oint_S \bar{J}\cdot \bar{ds}$$

$$\therefore \boxed{\nabla \times \bar{H} = \bar{J}}$$

3) Maxwell's eqn from Gauss's law for electric field.

$$\boxed{\oint_S \bar{D}\cdot \bar{ds} = Q = \int_V \rho_v \, dv}$$

apply divergence Theorem for L.H.S.

$$\int_V (\nabla\cdot \bar{D})\, dv = \int_V \rho_v \, dv$$

$$\therefore \boxed{\nabla\cdot \bar{D} = \rho_v}$$

4) Maxwell eqn from Gauss's law for magnetic fields.

$$\boxed{\oint_S \bar{B}\cdot \bar{ds} = 0}$$

apply divergence theorem

$$\int_V (\nabla\cdot \bar{B})\, dv = 0$$

$$\therefore dv \neq 0$$

$$\boxed{\nabla\cdot \bar{B} = 0}$$

Maxwells eqn for time-varying fields (EM fields)

1) Maxwell's eqn from Faraday's law

$$\boxed{\oint_L \bar{E}\cdot d\bar{L} = \oint_S -\frac{\partial \bar{B}}{\partial t}\cdot d\bar{s}}$$

apply stokes theorem for L.H.S

$$\oint_S (\nabla\times\bar{E})\cdot d\bar{s} = \oint_S -\frac{\partial \bar{B}}{\partial t}\cdot d\bar{s}$$

$$\therefore \boxed{\nabla\times\bar{E} = -\frac{\partial \bar{B}}{\partial t}}$$

2) Maxwell's eqn from Ampere's circuit law

$$\oint \bar{H}\cdot d\bar{L} = I = \oint_S \bar{J}\cdot d\bar{s}$$

$$\oint_L \bar{H}\cdot d\bar{L} = \oint_S (\bar{J}_C+\bar{J}_D)\cdot d\bar{s}$$

$$\boxed{\oint_L \bar{H}\cdot d\bar{L} = \oint_S \left(\bar{J}_C+\frac{\partial \bar{D}}{\partial t}\right)\cdot d\bar{s}}$$

applying Stokes theorem for LHS.

$$\oint_S (\nabla\times\bar{H})\cdot d\bar{s} = \oint_S \left(\bar{J}_C+\frac{\partial \bar{D}}{\partial t}\right)\cdot d\bar{s}$$

$$\therefore \boxed{\nabla\times\bar{H} = \bar{J}_C+\frac{\partial \bar{D}}{\partial t}}$$

3) Maxwell's eqn from Gauss's law

$$\boxed{\oint_S \bar{D}\cdot d\bar{s} = Q = \int_V \rho_v\, dV}$$

apply divergence theorem

$$\int_V (\nabla\cdot\bar{D})\, dV = \int_V \rho_v\, dV$$

$$\therefore \boxed{\nabla\cdot\bar{D} = \rho_v}$$

4) Maxwell's eqn from Gauss's law

$$\boxed{\oint_S \bar{B}\cdot d\bar{s} = 0}$$

apply divergence theorem

$$\int_V (\nabla\cdot\bar{B})\, dV = 0$$

$$dV \neq 0$$

$$\therefore \boxed{\nabla\cdot\bar{B} = 0}$$

Significance	Non-varying fields		Time varying fields (Em field)	
	Differential form	Integral form	Differential form	Integral form
1) Faraday's law	$\nabla \times \bar{E} = 0$	$\oint \bar{E} \cdot \bar{dL} = 0$	$\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t}$	$\oint \bar{E} \cdot \bar{dL} = -\oint_S \frac{\partial \bar{B}}{\partial t} \cdot \bar{ds}$
2) Gauss's law	$\nabla \cdot \bar{D} = \rho_v$	$\oint_S \bar{D} \cdot \bar{ds} = Q = \int_V \rho_v dv$	$\nabla \cdot \bar{D} = \rho_v$	$\oint_S \bar{D} \cdot \bar{ds} = Q = \int_V \rho_v dV$
3) Ampere's circuital law	$\nabla \times \bar{H} = \bar{J}$	$\oint_L \bar{H} \cdot \bar{dL} = I = \oint_S \bar{J} \cdot \bar{ds}$	$\nabla \times \bar{H} = \bar{J}_c + \bar{J}_D$ or $\nabla \times \bar{H} = \bar{J}_c + \frac{\partial \bar{D}}{\partial t}$	$\oint_L \bar{H} \cdot \bar{dL} = \oint_S \left(\bar{J}_c + \frac{\partial \bar{D}}{\partial t}\right) \cdot \bar{ds}$
4) Gauss's law (No isolated magnetic charge)	$\nabla \cdot \bar{B} = 0$	$\oint_S \bar{B} \cdot \bar{ds} = 0$	$\nabla \cdot \bar{B} = 0$	$\oint_S \bar{B} \cdot \bar{ds} = 0$

Continuity eqn. (Refer page No. 53)

From $\nabla \times \bar{H} = \bar{J}_c + \frac{\partial \bar{D}}{\partial t}$.

divergence on curl is zero. (From curl property)

$$\nabla \cdot \nabla \times \bar{H} = 0.$$

$$\nabla \cdot \left(\bar{J}_c + \frac{\partial \bar{D}}{\partial t} \right) = 0.$$

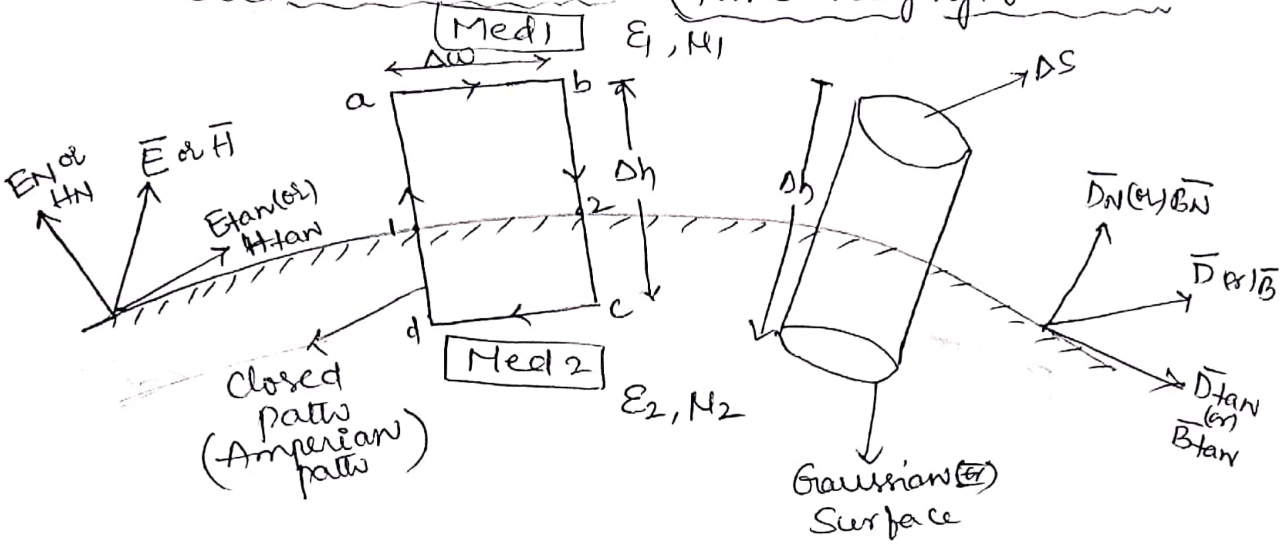
$$\nabla \cdot \bar{J}_c + \frac{\partial}{\partial t} (\nabla \cdot \bar{D}) = 0$$

$$\nabla \cdot J_c + \frac{\partial}{\partial t} (-\rho_v) = 0.$$

$$\boxed{\nabla \cdot \bar{J} = -\frac{\partial \rho_v}{\partial t}}$$

Charges have to move away from a closed surface (based on principle of conservation of charge)

Boundary conditions in Electromagnetic field (Time varying field)



Total $\bar{E} = \bar{E}_{tan} + \bar{E}_N$

$\bar{H} = \bar{H}_{tan} + \bar{H}_N$

$\bar{D} = \bar{D}_{tan} + \bar{D}_N$

$\bar{B} = \bar{B}_{tan} + \bar{B}_N$

Assume field entering from Med 1 & leaving from Med 2.

(i) Tangential Component

$$\oint_L \bar{E}\cdot \bar{dL} = \int_S -\frac{\partial \bar{B}}{\partial t}\cdot \bar{ds}$$

$$\left\{\int_a^b + \int_b^c + \int_c^d + \int_d^a\right\} \bar{E}\cdot\bar{dL} = \int_S -\frac{\partial \bar{B}}{\partial t}\cdot \bar{ds} \quad \text{——— (1)}$$

$$\int_a^b \bar{E}\cdot\bar{dL} = \bar{E}_{tan1}\,\Delta w \quad \text{—— (2)}$$

As $\Delta h \to 0$, the closed path comes towards boundary.

$$\int_b^c \bar{E}\cdot\bar{dL} + \int_d^a \bar{E}\cdot\bar{dL} = 0 \quad \text{—— (3)}$$

$$\int_c^d \bar{E}\cdot\bar{dL} = -\bar{E}_{tan2}\,\Delta w \quad \text{——— (4)}$$

Sub (2) (3) (4) in (1)

$$\bar{E}_{tan1}\,\Delta w - \bar{E}_{tan2}\,\Delta w = -\int \frac{\partial \bar{B}}{\partial t}\cdot \bar{ds}$$

$$(\bar{E}_{tan1} - \bar{E}_{tan2})\,\Delta w = -\int \frac{\partial \bar{B}}{\partial t}\cdot \bar{ds}$$

if $\bar{B} = 0$ (for non varying fields)

$\bar{E}_{tan1} = \bar{E}_{tan2}$

$\bar{D} = \varepsilon\bar{E}$

$$\frac{D_{tan1}}{D_{tan2}} = \frac{\varepsilon_1}{\varepsilon_2}$$

$$\oint \bar{H} \cdot \bar{dL} = \oint (\bar{J_c} + \bar{J_D}) \cdot \bar{ds}$$

$$\oint \bar{H} \cdot \bar{dL} = \oint \left(J_c + \frac{\partial \bar{D}}{\partial t}\right) \cdot \bar{ds}$$

$$\oint \bar{H} \cdot \bar{dL} = I + \oint_s \frac{\partial \bar{D}}{\partial t} \cdot \bar{ds}$$

$$\left\{ \oint_a^b + \oint_b^c + \int_c^d + \int_d^a \right\} \bar{H} \cdot \bar{dL} = I + \oint_s \frac{\partial \bar{D}}{\partial t} \cdot \bar{ds} \quad \text{—— (5)}$$

$$\int_a^b \bar{H} \cdot \bar{dL} = \bar{H}_{tan1}\, \Delta w \quad \text{—— (6)}$$

As $\Delta h \to 0$ closed path at boundary

$$\oint_b^c \bar{H} \cdot \bar{dL} + \int_d^a \bar{H} \cdot \bar{dL} = 0 \quad \text{—— (7)}$$

$$\int_c^d \bar{H} \cdot \bar{dL} = -\bar{H}_{tan2}\, \Delta w \quad \text{—— (8)}$$

Sub (6) (7) (8) in (5)

$$\boxed{\bar{H}_{tan1}\, \Delta w - \bar{H}_{tan2}\, \Delta w = I + \oint \frac{\partial \bar{D}}{\partial t} \cdot \bar{ds}}$$

if $\dot{\bar{D}} = 0$ for non varying fields.

$$\boxed{\bar{H}_{tan1} = \bar{H}_{tan2}}$$

$$\bar{B} = \mu \bar{H}$$

$$\therefore \frac{\bar{B}_{tan1}}{\bar{B}_{tan2}} = \frac{\mu_1}{\mu_2}$$

(ii) Normal Component

From Gauss's law

$$\oint \bar{D} \cdot \bar{ds} = Q$$

$$\left[\oint_{top} + \oint_{bottom} + \oint_{lateral} \right] \bar{D} \cdot \bar{ds} = Q \quad \text{—— (9)}$$

$$\oint_{top} \bar{D} \cdot \bar{ds} = \bar{D}_{N1}\, \Delta S \quad \text{—— (10)}$$

$$\oint \bar{D} \cdot \bar{ds} = -\bar{D}_{N2}\, \Delta S \quad \text{—— (11)}$$ (flux leaving from media)

As $\Delta h \to 0$ Surface comes towards boundary.

$$\oint_{lateral} \bar{D} \cdot \bar{ds} = 0 \text{ ——— } (12)$$

Sub (10)(11)(12) in (9)

$$\bar{D}_{N1} \Delta S - \bar{D}_{N2} \Delta S = Q$$

$$\bar{D}_{N1} - \bar{D}_{N2} = \frac{Q}{\Delta S}$$

$$\bar{D}_{N1} - \bar{D}_{N2} = \rho_S$$

if $\rho_S = 0$ (Charge free Medium)

$$\boxed{\bar{D}_{N1} = D_{N2}}$$

$$\bar{D} = \varepsilon \bar{E}$$

$$\boxed{\frac{\bar{E}_{N1}}{\bar{E}_{N2}} = \frac{\varepsilon_2}{\varepsilon_1}}$$

From Gauss law for magnetic field

$$\oint \bar{B} \cdot \bar{ds} = 0$$

$$\left\{ \oint_{top} + \oint_{bottom} + \oint_{lateral} \right\} \bar{B} \cdot \bar{ds} = 0 \text{ ——— } (13)$$

$$\oint_{top} \bar{B} \cdot \bar{ds} = \bar{B}_{N1} \Delta S \text{ ——— } (14)$$

$$\oint_{bottom} \bar{B} \cdot \bar{ds} = -\bar{B}_{N2} \Delta S \text{ — } (15)$$

$$\oint_{lateral} \bar{B} \cdot \bar{ds} = 0 \text{ ——— } (16)$$

Sub (14)(15)(16) in (13)

$$\bar{B}_{N1} \Delta S - \bar{B}_{N2} \Delta S = 0$$

$$\boxed{\bar{B}_{N1} = \bar{B}_{N2}}$$

$$\bar{B} = \mu \bar{H}$$

$$\boxed{\frac{\bar{H}_{N1}}{\bar{H}_{N2}} = \frac{\mu_2}{\mu_1}}$$

Electromagnetic Wave equation (or) EM wave in Conductor

- Waves are means of transporting energy (or) information
- A wave is a function of both space & time.

From maxwell's eqn

$$\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t} \quad \text{——— (1)}$$

$$\nabla \times \bar{H} = \bar{J}_c + \frac{\partial \bar{D}}{\partial t} \quad \text{——— (2)}$$

$$\nabla \cdot \bar{D} = \rho_v \quad \text{——— (3)}$$

$$\nabla \cdot \bar{B} = 0 \quad \text{——— (4)}$$

i) wave eqn interms of $\bar{E}$

applying curl for eqn (1)

$$\nabla \times \nabla \times \bar{E} = \nabla \times \left[-\frac{\partial \bar{B}}{\partial t} \right]$$

$$= -\nabla \times \mu \frac{\partial \bar{H}}{\partial t}$$

$$= -\mu \frac{\partial}{\partial t} \left[\nabla \times \bar{H} \right]$$

$$= -\mu \frac{\partial}{\partial t} \left[\bar{J}_c + \frac{\partial \bar{D}}{\partial t} \right]$$

$$= -\mu \frac{\partial}{\partial t} \left[\sigma \bar{E} + \varepsilon \frac{\partial \bar{E}}{\partial t} \right]$$

$$\nabla(\nabla \cdot \bar{E}) - \nabla^2 \bar{E} = -\left[\mu\sigma \frac{\partial \bar{E}}{\partial t} + \varepsilon \frac{\partial^2 \bar{E}}{\partial t^2} \right]$$

From eqn (3) $\nabla \cdot \bar{E} = 0$ (for Conductor).

$$0 \cancel{-} \nabla^2 \bar{E} = \cancel{-} \left[\mu\sigma \frac{\partial \bar{E}}{\partial t} + \mu\varepsilon \frac{\partial^2 \bar{E}}{\partial t^2} \right]$$

$$\boxed{\nabla^2 \bar{E} = \mu\sigma \frac{\partial \bar{E}}{\partial t} + \mu\varepsilon \frac{\partial^2 \bar{E}}{\partial t^2}}$$

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ii) wave eqn interms of H

applying curl for eqn (2)

$$\nabla\times\nabla\times\bar{H} = \nabla\times\left[\bar{J}_c+\frac{\partial\bar{D}}{\partial t}\right]$$

$$= \nabla\times\left[\sigma\bar{E}+\varepsilon\frac{\partial\bar{E}}{\partial t}\right]$$

$$= \sigma(\nabla\times\bar{E})+\varepsilon\left(\nabla\times\frac{\partial\bar{E}}{\partial t}\right)$$

$$= \sigma(\nabla\times\bar{E})+\varepsilon\frac{\partial}{\partial t}(\nabla\times\bar{E})$$

$$= \sigma\left(-\frac{\partial\bar{B}}{\partial t}\right)+\varepsilon\frac{\partial}{\partial t}\left(-\frac{\partial B}{\partial t}\right)$$

$$= -\sigma\mu\frac{\partial\bar{H}}{\partial t}-\mu\varepsilon\frac{\partial^2\bar{H}}{\partial t^2}$$

$$\nabla(\nabla\cdot\bar{H})-\nabla^2H = -\left[\mu\sigma\frac{\partial\bar{H}}{\partial t}+\mu\varepsilon\frac{\partial^2\bar{H}}{\partial t^2}\right]$$

$$0\cancel{-}\nabla^2H = \cancel{-}\left[\mu\sigma\frac{\partial\bar{H}}{\partial t}+\mu\varepsilon\frac{\partial^2\bar{H}}{\partial t^2}\right]$$

$$\boxed{\nabla^2\bar{H} = \mu\sigma\frac{\partial\bar{H}}{\partial t}+\mu\varepsilon\frac{\partial^2\bar{H}}{\partial t^2}}$$

Standard EM wave eqn

$$\nabla^2\begin{bmatrix}\bar{E}\\\bar{H}\\\bar{D}\\\bar{B}\end{bmatrix} = \mu\sigma\frac{\partial}{\partial t}\begin{bmatrix}\bar{E}\\\bar{H}\\\bar{D}\\\bar{B}\end{bmatrix}+\mu\varepsilon\frac{\partial^2}{\partial t^2}\begin{bmatrix}\bar{E}\\\bar{H}\\\bar{D}\\\bar{B}\end{bmatrix}$$

This is also called Helmholtz's eqn (or) wave eqn.

EM wave eqn in free space :

In free space $\sigma = 0$, $\varepsilon = \varepsilon_0$, $\mu = \mu_0$

$$\therefore \nabla^2 \bar{E} = \mu_0 \varepsilon_0 \frac{\partial^2 \bar{E}}{\partial t^2}$$

Ily $$\nabla^2 \bar{H} = \mu_0 \varepsilon_0 \frac{\partial^2 \bar{H}}{\partial t^2}$$

$$\nabla^2 \begin{bmatrix} \bar{E} \\ \bar{H} \\ \bar{D} \\ \bar{B} \end{bmatrix} = \mu_0 \varepsilon_0 \frac{\partial^2}{\partial t^2} \begin{bmatrix} \bar{E} \\ \bar{H} \\ \bar{D} \\ \bar{B} \end{bmatrix}$$

From wave eqn

$$\nabla^2 \bar{E} = \mu \sigma \frac{\partial \bar{E}}{\partial t} + \mu \varepsilon \frac{\partial^2 \bar{E}}{\partial t^2}$$

Let $\frac{\partial}{\partial t} = j\omega$

$$\nabla^2 \bar{E} = \mu \sigma j\omega \bar{E} + \mu \varepsilon (j\omega)^2 \bar{E}$$

$$\boxed{\nabla^2 \bar{E} = \underbrace{j\omega\mu [\sigma + j\omega\varepsilon]}_{\gamma^2} \bar{E}}$$

$$\therefore \boxed{\nabla^2 \bar{E} = \gamma^2 \bar{E}}$$

where γ = propagation constant

$\gamma = \alpha + j\beta$

α = attenuation constant

β = phase shift constant

EM wave propagation (motion) :

1) Free space

$\sigma = 0$, $\varepsilon = \varepsilon_0$ and $\mu = \mu_0$

2.) Lossless dielectric (or perfect dielectric or Good dielectric).

$\sigma = 0$, $\varepsilon = \varepsilon_0 \varepsilon_r$, and $\mu = \mu_0 \mu_r$ (or) $\sigma \ll \omega\varepsilon$ (or) $\frac{\sigma}{\omega\varepsilon} \ll 1$

3) Lossy dielectric (or conducting medium)

$\sigma \neq 0$, $\varepsilon = \varepsilon_0 \varepsilon_r$ and $\mu = \mu_0 \mu_r$

4) Good conductor (or perfect conductor)

$\sigma = \infty$, $\varepsilon = \varepsilon_0 \varepsilon_r$ and $\mu = \mu_0 \mu_r$ (or) $\sigma \gg \omega\varepsilon$

When wave is propagated in different medium the following parameters to be calculated.

i) Propagation constant (γ)
ii) Attenuation constant (α)
iii) phase shift constant (β)
iv) phase velocity (v_p)
v) Wave length (λ)
vi) characteristic impedance (η)

1) Electromagnetic wave propagation in free space.

From Em wave eqn

$$\nabla^2 \bar{E} = \mu\sigma \frac{\partial \bar{E}}{\partial t} + \mu\varepsilon \frac{\partial^2 \bar{E}}{\partial t^2}$$ Let $\frac{\partial}{\partial t} = j\omega$

or
$$\nabla^2 \bar{E} = j\omega\mu\,[\sigma + j\omega\varepsilon]\,\bar{E}$$

For free space $\sigma = 0$, $\varepsilon = \varepsilon_0$, $\mu = \mu_0$

$$\therefore \nabla^2 \bar{E} = \underbrace{j\omega\mu_0\,[0 + j\omega\varepsilon_0]}_{\gamma^2}\,\bar{E}$$

$$\gamma^2 = j\omega\mu_0\,(j\omega\varepsilon_0)$$

$$\gamma^2 = (j\omega)^2 \mu_0\varepsilon_0$$

i) $$\gamma = \alpha + j\beta = j\omega\sqrt{\mu_0\varepsilon_0}$$

ii) Attenuation constant (α) = 0.

iii) $$\beta = \omega\sqrt{\mu_0\varepsilon_0}$$

iv) phase velocity (v_p) $= \frac{\omega}{\beta} = \frac{\cancel{\omega}}{\cancel{\omega}\sqrt{\mu_0\varepsilon_0}}$

$$= \frac{1}{\sqrt{4\pi\times10^{-7}\times 8.854\times10^{-12}}} = 3\times10^8\, m$$

v) wavelength $= \lambda = \frac{2\pi}{\beta}$

$$= \frac{2\pi}{\omega\sqrt{\mu_0\varepsilon_0}} = \frac{\cancel{2\pi}}{\cancel{2\pi} f\sqrt{\mu_0\varepsilon_0}} = \frac{3\times10^8}{f}\ \text{mel}$$

vi) Characteristic impedance (η)

$$\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}}$$

$$= \sqrt{\frac{j\omega\mu_0}{0 + j\omega\varepsilon_0}} = \sqrt{\frac{4\pi\times10^{-7}}{8.854\times10^{-12}}} = 377\Omega \text{ (or) } 120\pi.$$

(2) EM wave propagation in lossless dielectric (or perfect dielectric)

From EM wave eqn

$$\nabla^2\bar{E} = \mu\sigma\frac{\partial\bar{E}}{\partial t} + \mu\varepsilon\frac{\partial^2\bar{E}}{\partial t^2}$$

(or)

$$\nabla^2 E = j\omega\mu\left[\sigma + j\omega\varepsilon\right]\bar{E}$$

In lossless dielectric

$$\sigma = 0,\ \varepsilon = \varepsilon_0\varepsilon_r,\ \mu = \mu_0\mu_r$$

$$\therefore \nabla^2\bar{E} = \underbrace{j\omega\mu_0\mu_r\left[0 + j\omega\,\varepsilon_0\varepsilon_r\right]}_{\gamma^2}\bar{E}$$

$$\gamma^2 = (j\omega)^2\,\mu_0\mu_r\,\varepsilon_0\,\varepsilon_r \qquad \gamma = \alpha + j\beta.$$

i) Attenuation constant $= \gamma = \sqrt{(j\omega)^2\mu_0\mu_r\varepsilon_0\varepsilon_r}$

$$\boxed{\gamma = j\omega\sqrt{\mu_0\mu_r\varepsilon_0\varepsilon_r}}$$

ii) Propagation constant $(\alpha) = 0$

iii) Phase shift constant $= \beta = \omega\sqrt{\mu_0\mu_r\varepsilon_0\varepsilon_r}$

iv) phase velocity $= v_p = \frac{\omega}{\beta} = \frac{\cancel{\omega}}{\cancel{\omega}\sqrt{\mu_0\mu_r\varepsilon_0\varepsilon_r}}$

$$= \frac{3\times10^8}{\sqrt{\mu_r\varepsilon_r}}\ /.$$

v) wave length $= \lambda = \frac{2\pi}{\beta}$

$$= \frac{2\pi}{\omega\sqrt{\mu_0\mu_r\varepsilon_0\varepsilon_r}} = \frac{\cancel{2\pi}}{\cancel{2\pi}f\sqrt{\mu_0\varepsilon_0}\sqrt{\mu_r\varepsilon_r}} = \frac{3\times10^8}{f\sqrt{\mu_r\varepsilon_r}}$$

vi) Characteristic impedance (η)

$$\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}} = \sqrt{\frac{j\omega\mu}{0 + j\omega\varepsilon}} = \sqrt{\frac{\mu}{\varepsilon}} = 377\sqrt{\frac{\mu_r}{\varepsilon_r}}$$

3) EM wave propagation in Lossy dielectric :

$$\nabla^2 \bar{E} = \underbrace{j\omega\mu\,[\sigma + j\omega\varepsilon]}_{\gamma^2}\,\bar{E}$$

$$\gamma^2 = j\omega\mu\,[\sigma + j\omega\varepsilon] \quad \text{——} (1)$$

for lossy dielectric

$\sigma \neq 0$, $\varepsilon = \varepsilon_0 \varepsilon_r$, $\mu = \mu_0 \mu_r$

we know $\gamma = \alpha + j\beta$

$$\gamma^2 = (\alpha + j\beta)^2 = \alpha^2 - \beta^2 + j\,2\alpha\beta$$

Sub this in ①

$$\alpha^2 - \beta^2 + j\,2\alpha\beta = j\omega\mu\sigma - \omega^2\mu\varepsilon$$

Comparing on both sides

$$\left.\begin{aligned} \alpha^2 - \beta^2 &= -\omega^2\mu\varepsilon \\ 2\alpha\beta &= \omega\mu\sigma \end{aligned}\right\} \quad \text{——} ②$$

$$\alpha^2 + \beta^2 = \sqrt{(\alpha^2 - \beta^2)^2 + 4\alpha^2\beta^2}$$

$$= \sqrt{(-\omega^2\mu\varepsilon)^2 + (\omega\mu\sigma)^2}$$

$$= \sqrt{(\omega^2\mu\varepsilon)^2\left[1 + \frac{\omega^2\mu^2\sigma^2}{\omega^4\mu^2\varepsilon^2}\right]}$$

$$= \sqrt{(\omega^2\mu\varepsilon)^2\left[1 + \frac{\sigma^2}{\omega^2\varepsilon^2}\right]}$$

$$\alpha^2 + \beta^2 = \omega^2\mu\varepsilon\sqrt{1 + \left(\frac{\sigma}{\omega\varepsilon}\right)^2} \quad \text{——} ③$$

i) Add (2)+(3)

$$\alpha^2 + \cancel{\beta^2} + \alpha^2 - \cancel{\beta^2} = \omega^2\mu\varepsilon\sqrt{1 + \left(\frac{\sigma}{\omega\varepsilon}\right)^2} - \omega^2\mu\varepsilon$$

$$\alpha^2 = \frac{\omega^2\mu\varepsilon}{2}\left[\sqrt{1 + \left(\frac{\sigma}{\omega\varepsilon}\right)^2} - 1\right]$$

$$\boxed{\alpha = \sqrt{\frac{\omega^2\mu\varepsilon}{2}\left[\sqrt{1 + \left(\frac{\sigma}{\omega\varepsilon}\right)^2} - 1\right]}}$$

ii) (3) − (2)

$$\cancel{\alpha^2}+\beta^2-\cancel{\alpha^2}+\beta^2 = \omega^2\mu\varepsilon\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}+\omega^2\mu\varepsilon$$

$$2\beta^2 = \omega^2\mu\varepsilon\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}+1\right]$$

$$\boxed{\beta = \sqrt{\frac{\omega^2\mu\varepsilon}{2}\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}+1\right]}}$$

iii) $\gamma = \alpha + j\beta$

iv) phase velocity $= v_p = \frac{\omega}{\beta} = \frac{\omega}{\sqrt{\frac{\omega^2\mu\varepsilon}{2}\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}+1\right]}}$

v) wavelength $= \lambda = \frac{2\pi}{\beta}$

$$= \frac{2\pi}{\sqrt{\frac{\omega^2\mu\varepsilon}{2}\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}+1\right]}}$$

vi) Intrinsic Impedance (η)

$$\eta = \sqrt{\frac{j\omega\mu}{\sigma+j\omega\varepsilon}}$$

or Alternate method

$$\gamma^2 = j\omega\mu\,[\sigma + j\omega\varepsilon]$$

$$\gamma = \sqrt{j\omega\mu \cdot j\omega\varepsilon\left[1+\frac{\sigma}{j\omega\varepsilon}\right]}$$

$$\boxed{\gamma = j\omega\sqrt{\mu\varepsilon}\sqrt{1+\frac{\sigma}{j\omega\varepsilon}}}$$

By using this eqn we can calculate above parameters.

(4) EM wave propagation in good conductor

$\sigma \neq 0$, $\varepsilon = \varepsilon_0 \varepsilon_r$, $\mu = \mu_0 \mu_r$ (or) $\sigma >> \omega\varepsilon$

From wave eqn

$$\nabla^2 \bar{E} = \underbrace{j\omega\mu[\sigma + j\omega\varepsilon]}_{\gamma^2} \bar{E}$$

$$\gamma^2 = j\omega\mu[\sigma + j\omega\varepsilon]$$

$$\gamma^2 = j\omega\mu[\sigma]$$

(or)

$$\gamma^2 = j\omega\mu\, j\omega\varepsilon\left[1 + \frac{\sigma}{j\omega\varepsilon}\right]$$
$$= j\omega\mu j\omega\varepsilon\left[1 - j\frac{\sigma}{\omega\varepsilon}\right]$$

$\sigma >> \omega\varepsilon$

$$\therefore \gamma^2 = (j\omega)^2 \mu \cancel{\varepsilon}\left(-j\frac{\sigma}{\cancel{\omega\varepsilon}}\right)$$
$$\gamma^2 = +j\omega\mu\sigma$$

let $j = e^{j\pi/2} = \cos\pi/2 + j\sin\pi/2$
$= 0 + j = j$.

$$\sqrt{j} = (e^{j\pi/2})^{1/2} = \cos\pi/4 + j\sin\pi/4$$
$$= 1/\sqrt{2} + j\frac{1}{\sqrt{2}}$$

i) $\therefore \gamma = \sqrt{j\omega\mu\sigma}$

$$\gamma = \sqrt{\omega\mu\sigma}\left[\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}\right]$$

ii) $\gamma = \alpha + j\beta$.

$$\therefore \alpha = \sqrt{\frac{\omega\mu\sigma}{2}}$$

iii) $$\beta = \sqrt{\frac{\omega\mu\sigma}{2}}$$

iv) Phase velocity $= v_p = \frac{\omega}{\beta} = \frac{\omega}{\sqrt{\frac{\omega\mu\sigma}{2}}} = \sqrt{\frac{2\omega}{\mu\sigma}}$ //

v) wave length $= \lambda = \frac{2\pi}{\beta} = \frac{2\pi}{\sqrt{\frac{\omega\mu\sigma}{2}}}$

vi) Characteristic impedance (η)

$$\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}}$$

$$= \sqrt{\frac{j\omega\mu}{\sigma\left[1 + j\frac{\omega\varepsilon}{\sigma}\right]}}$$

but $\sigma >> \omega\varepsilon$

i.e. $\frac{\omega\varepsilon}{\sigma} << 1$

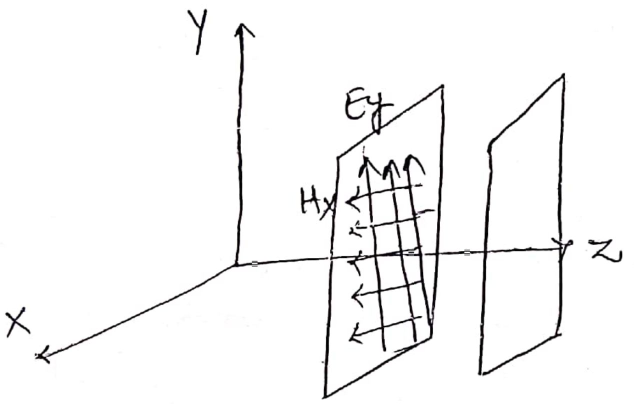
$$\therefore \eta = \sqrt{\frac{j\omega\mu}{\sigma}}$$

$$= \sqrt{\frac{\omega\mu}{\sigma}}\left[\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}\right]$$

$$\eta = \sqrt{\frac{\omega\mu}{\sigma}} \angle 45^\circ$$ //

Uniform plane wave :

Plane means that electric and magnetic field vectors both lie in a plane and all such planes are parallel.
The phase of the wave is constant over the plane.
Uniform means that the amplitude and phase of vectors $\bar{E}$ and $\bar{H}$ are constant over a plane.



- If the electric field is in X-direction and magnetic field is in y-direction, then the wave is travelling in the z-direction
- If the phase in a wave is same for all points on a plane surface it is called a plane wave.

- In addition if the amplitude is also constant over the plane surface it is called Uniform Plane Wave.
- Uniform plane waves do not exist in practice because they can not be produced by finite size antennas.

$v_x \rightarrow E_y, H_z$
$v_x \rightarrow E_z, H_y$

Em wave eqn

$$\nabla^2 E = \mu\sigma \frac{\partial E}{\partial t} + \mu\varepsilon \frac{\partial^2 E}{\partial t^2}$$

Solution of Em wave eqn.

$$\bar{E} = E_0 e^{j(\omega t - \beta x)}$$

$$\bar{E} = E_0 \left[\cos(\omega t - \beta x) + j \sin(\omega t - \beta x)\right]$$

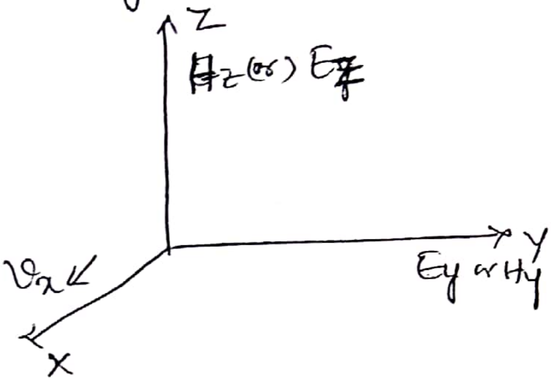
$$\bar{H} = H_0 e^{j(\omega t - \beta x)}$$

$$\bar{H} = H_0 \left[\cos(\omega t - \beta x) + j \sin(\omega t - \beta x)\right]$$

Relationship between $\bar{E}$ and $\bar{H}$

Assume that the wave is travelling along x-axis.

i.e. $\nabla \rightarrow \frac{\partial}{\partial x}\bar{a}_x + \cancel{\frac{\partial}{\partial y}\bar{a}_y}^{\to 0} + \cancel{\frac{\partial}{\partial y}\bar{a}_y}^{\to 0}$



$\bar{E} = 0 + E_y\bar{a}_y + E_z\bar{a}_z$

$\bar{H} = 0 + H_y\bar{a}_y + H_z\bar{a}_z$

$$\nabla\times\bar{E} = \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ \frac{\partial}{\partial x} & 0 & 0 \\ 0 & E_y & E_z \end{vmatrix} = -\bar{a}_y\left[\frac{\partial E_z}{\partial x}\right] + \bar{a}_z\left[\frac{\partial E_y}{\partial x}\right] \quad \text{—— (1)}$$

$$\nabla\times\bar{H} = \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ \frac{\partial}{\partial x} & 0 & 0 \\ 0 & H_y & H_z \end{vmatrix} = -\bar{a}_y\left[\frac{\partial}{\partial x}H_z\right] + \bar{a}_z\left[\frac{\partial H_y}{\partial x}\right] \quad \text{—— (2)}$$

From maxwell's eqn

$$\nabla\times\bar{E} = -\frac{\partial\bar{B}}{\partial t} = -\mu\frac{\partial\bar{H}}{\partial t} \quad \text{—— (3)}$$

$$\nabla\times\bar{H} = \underbrace{J_c}_{=0} + \frac{\partial\bar{D}}{\partial t} = \varepsilon\frac{\partial\bar{E}}{\partial t} \quad \text{—— (4)}$$

Sub (1) in (3)

$$-\bar{a}_y\frac{\partial E_z}{\partial x} + \bar{a}_z\frac{\partial E_y}{\partial x} = -\mu\frac{\partial}{\partial t}\left[H_y\bar{a}_y + H_z\bar{a}_z\right]$$

$$-\frac{\partial E_z}{\partial x}\bar{a}_y + \frac{\partial E_y}{\partial x}\bar{a}_z = -\mu\frac{\partial H_y}{\partial t}\bar{a}_y - \mu\frac{\partial H_z}{\partial t}\bar{a}_z$$

Comparing

$$+\frac{\partial E_z}{\partial x}\cancel{\bar{a}_y} = +\mu\frac{\partial H_y}{\partial t}\cancel{\bar{a}_y}$$

$$\frac{\partial E_z}{\partial x} = \mu\frac{\partial H_y}{\partial t} \quad \text{—— (5)}$$

lly $$\frac{\partial E_y}{\partial x} = -\mu\frac{\partial H_z}{\partial t} \quad \text{—— (6)}$$

We know that $H_y = H_{yo}\, e^{j(\omega t - \beta x)}$

Differentiate w.r.t. 't'

$$\frac{\partial H_y}{\partial t} = H_{yo}\, e^{j(\omega t-\beta x)} \cdot j\omega$$

Sub this in ⑤

$$\frac{\partial E_z}{\partial x} = \mu\left[j\omega \underbrace{H_{yo}\, e^{j(\omega t-\beta x)}}_{H_y}\right]$$

$$\partial E_z = j\omega\mu\, H_{yo}\, e^{j(\omega t-\beta x)}\, dx$$

$$E_z = \int j\omega\mu\, H_{yo}\, e^{j(\omega t-\beta x)}\, dx$$

$$E_z = j\omega\mu\, H_{yo}\, \frac{e^{j(\omega t-\beta x)}}{-j\beta}$$

$$E_z = \frac{\omega\mu H_y}{\beta}$$

$$\frac{E_z}{H_y} = \frac{-\omega\mu}{\beta} \qquad \left[\because v_p = \frac{\omega}{\beta}\right]$$

$$= -v_p\,\mu$$

$$= -\frac{1}{\sqrt{\mu\varepsilon}}\cdot\mu$$

$$\frac{E_z}{H_y} = -\sqrt{\mu/\varepsilon}$$

in free space

$$\boxed{\left|\frac{E_z}{H_y}\right| = \left|\sqrt{\frac{\mu_0}{\varepsilon_0}}\right| = 377\,\Omega \text{ (or) } 120\pi}$$

Ily from ⑥

$$\boxed{\left|\frac{E_y}{H_z}\right| = \sqrt{\frac{\mu_0}{\varepsilon_0}} = 377\,\Omega}$$

Depth of penetration (or) Skin Depth (δ)

When the wave is penetrating through the medium of finite conductivity, it gets attenuated. This rate of attenuation is very high and the wave is penetrates into small distance.

- The depth of penetration is defined as the depth in which the wave has got attenuated to $\frac{1}{e}$ or 37% of its initial value.

Let x is depth of penetration, then attenuation factor is $e^{-\alpha x}$.

$$e^{-\alpha x} = e^{-1} = \frac{1}{e}$$

$$\therefore \alpha x = 1 \Rightarrow \underbrace{x = \frac{1}{\alpha}}_{\text{skin depth}}$$

$$\boxed{\delta = \frac{1}{\alpha}}$$

For Good conductor

$$\alpha = \sqrt{\frac{\omega\mu\sigma}{2}}$$

$$\therefore \text{Skin depth} = \delta = \frac{1}{\alpha}$$

$$= \frac{1}{\sqrt{\frac{\omega\mu\sigma}{2}}}$$

$$= \sqrt{\frac{2}{2\pi f\mu\sigma}}$$

$$\delta = \frac{1}{\sqrt{\pi f\mu\sigma}}$$

Phase velocity (v_p) $v_p = \frac{\omega}{\beta}$

The velocity with which constant phase point travels is called phase velocity.

$$\boxed{v_p v_g = c^2}$$

c = velocity of light

Group velocity (v_g) $v_g = \frac{\Delta\omega}{\Delta\beta}$

$$v_g = \frac{2\pi\Delta f}{2\pi/\Delta\lambda} = \Delta f\Delta\lambda$$

Polarization:

In optics polarization interms of plane of polarization
In electromagnetic field theory we only talk of a direction of polarization.

Def: " Orientation of electromagnetic wave at a given instant of time in the space.

(or)

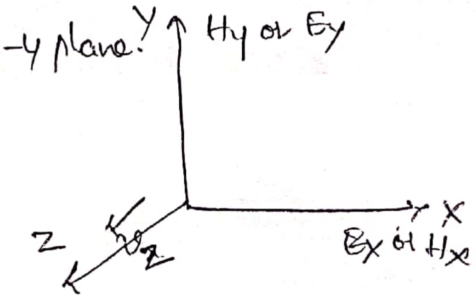
"orientation of electric field vector $\bar{E}$ at a given instant of time in space.

3 types of polarization:

1) Linear polarization
2) Elliptical "
3) circular "

① Linear polarization:

Let wave travelling along Z-direction.
i.e. $\bar{E}$ & $\bar{H}$ fields lie on x-y plane.



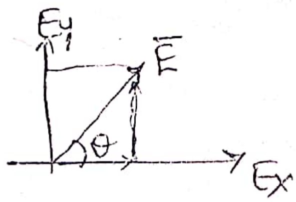
If $E_y = 0$ and $E_x \neq 0$, then the wave is said to be linearly polarized in X-direction (LPx)

If $E_x = 0$ and $E_y \neq 0$ then wave is said to be linearly polarized in y-direction. (LPy)

$$\bar{E} = \sqrt{E_x^2 + E_y^2}$$

$$\bar{H} = \sqrt{H_x^2 + H_y^2}$$

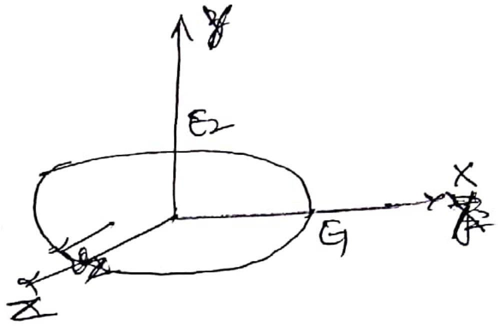
$$\theta = \tan^{-1} \frac{E_y}{E_x}$$



E_x, E_y both are in phase.

(2) Elliptical polarization

"The magnitude of $\bar{E}$ components along x and y-axis are not equal. Then wave is said to be elliptically polarized."



i.e. $E_1 \neq E_2$

$E_x \neq E_y$

$E_x = E_{x0}\, e^{j(\omega t - \beta z)}$

$Re\{E_x\} = E_{x0} \cos(\omega t - \beta z)$

$= E_1 \cos(\omega t - \beta z)$

$E_y = E_{y0}\, e^{j(\omega t - \beta z)}$

$Re\{E_y\} = E_{y0} \cos(\omega t - \beta z)$

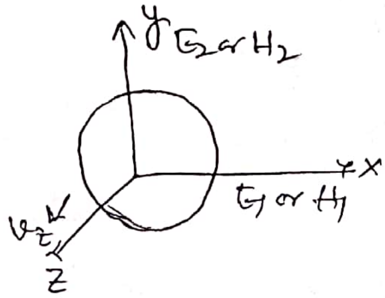
$= E_2 \cos(\omega t - \beta z)$

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (Ellipse eqn)

$\therefore \frac{x^2}{E_1^2} + \frac{y^2}{E_2^2} = 1$

(3) Circular polarization

The magnitude of electric field component along x and y-axis are equal, then the wave is said to be circularly polarized.



$E_1 = E_2$

$E_x = E_{x0}\, e^{j(\omega t - \beta z)}$

$Re\{E_x\} = E_1 \cos(\omega t - \beta z)$

lly $E_y = E_{y0}\, e^{j(\omega t - \beta z)}$

$Re\{E_y\} = E_2 \cos(\omega t - \beta z)$

$= E_1 \cos(\omega t - \beta z)$ $(\because E_1 = E_2)$

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$\frac{x^2}{E_1^2} + \frac{y^2}{E_1^2} = 1 \Rightarrow x^2 + y^2 = E_1^2$

Problems :

1) A uniform plane wave in medium having $\sigma = 10^{-3}$ S/m, $\varepsilon = 80\varepsilon_0$ and $\mu = \mu_0$ is having a frequency of 10 kHz. Calculate the different parameters of the wave?

Sol

$$\frac{\sigma}{\omega\varepsilon} = \frac{10^{-3}}{2\pi(10\times10^{3})\,80\times8.854\times10^{-12}}$$

$$= 22.4 >> 1$$

$\frac{\sigma}{\omega\varepsilon} >> 1$ for Good conductor.

i) $$\gamma = \sqrt{\frac{\omega\mu\sigma}{2}} + j\sqrt{\frac{\omega\mu\sigma}{2}}$$

$$= 6.28\times10^{-3} + j\,6.28\times10^{-3}$$

ii) $$\alpha = \sqrt{\frac{\omega\mu\sigma}{2}} = 6.28\times10^{-3} \text{ Nepers/m.}$$

iii) $$\beta = \sqrt{\frac{\omega\mu\sigma}{2}} = 6.28\times10^{-3} \text{ NP/m.}$$

iv) $$v_p = \frac{\omega}{\beta} = \frac{2\pi(10\times10^{3})}{6.28\times10^{-3}} = 10\times10^{6} \text{ m/s.}$$

v) $$\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{6.28\times10^{-3}} = 1000.5 \text{ meter}$$

vi) $$\eta = \sqrt{\frac{\omega\mu}{\sigma}}\left[\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}\right] = \angle 45^\circ$$

$$= 2\pi(1+j)\,\Omega$$

2) Calculate the refractive index of copper at 10 MHz assuming that conductivity of copper is 5.8×10^{7} mho/m and its relative permeability and permittivity is unity.

Sol Copper is Good Conductor

$$\therefore v_p = \sqrt{\frac{2\omega}{\mu\sigma}}$$

$$= \sqrt{\frac{2\times2\pi(10\times10^{6})}{4\pi\times10^{-7}\times5.8\times10^{7}}}$$

$$= 0.131\times10^{4} \text{ m/s}$$

Refractive index

$$n = \frac{c}{v_p}$$

$$= \frac{3\times10^{8}}{0.131\times10^{4}}$$

$$= 0.0229\times10^{7}$$

90

3) For copper $\sigma = 58\,MS\text{-}m^{-1}$, for Teflon $\sigma = 30\,nS\text{-}m^{-1}$ and $\varepsilon = 2.1\varepsilon_0$. Verify that at 1 MHz, copper is a good conductor and Teflon is good dielectric.

Sol For Copper

$$\frac{\sigma}{\omega\varepsilon} = \frac{58\times10^{6}}{2\pi(10^{6})\,2.1\times8.854\times10^{-12}}$$

$$= 0.496\times10^{12} >> 1$$

$\therefore$ copper is a good conductor.

For Teflon

$$\frac{\sigma}{\omega\varepsilon} = \frac{30\times10^{-9}}{2\pi(10^{6})\,2.1\times8.854\times10^{-12}}$$

$$= 2.57\times10^{-4} << 1$$

$\therefore$ Teflon is a good dielectric.

4) After which frequency, the earth may be considered as perfect dielectric? Assume $\frac{\sigma}{\omega\varepsilon_r} = \frac{1}{100}$.

Given $\sigma = 5\times10^{-3}\,S/m$, $\mu_r = 10$ and $\varepsilon_r = 8$.

Sol In case of perfect dielectric

$$\frac{\sigma}{\omega\varepsilon} << 1$$

$$\frac{\sigma}{\omega\varepsilon} = \frac{1}{100} << 1$$

$$\omega = \frac{100\sigma}{\varepsilon} \Rightarrow 2\pi f = \frac{100\sigma}{\varepsilon_0\varepsilon_r}$$

$$f = \frac{100\times5\times10^{-3}}{8.854\times10^{-12}\times8\times2\pi} = 1.12\,GHz.$$

So, after 1.12 GHz, the earth may be considered as perfect dielectric.

5) Consider $B = B_0 e^{kt}\,\bar{a}_z$ for cylindrical region $\rho < b$. Find electric field intensity E using Faraday's law.

Sol

$$\oint \bar{E}\cdot\bar{dl} = -\frac{\partial}{\partial t}\oint_s \bar{B}\cdot\bar{ds}$$

$$\int_0^{2\pi} E_\phi\, r\,d\phi = -\frac{\partial}{\partial t}\int_0^{2\pi}\int_0^{r} B_0 e^{kt}\, r\,dr\,d\phi$$

$$E_\phi\, r\, 2\pi = -\frac{\partial}{\partial t}\left[B_0\, e^{kt}\left(\frac{r^2}{2}\right)_0^r (\phi)_0^{2\pi}\right]$$

$$E_\phi\, \cancel{r}\, \cancel{2\pi} = -B_0\, e^{kt}(k)\,\frac{r^{\cancel{2}}}{2}\cdot \cancel{2\pi}$$

$$E_\phi = -\frac{1}{2} B_0 k\, e^{kt}\, r \qquad \text{or.} \quad E = -\frac{1}{2}B_0 k e^{kt}\, r\, a_\phi$$

6) Show that for a sinusoidally varying fields, the conduction current and displacement currents are always displaced by 90° in phase.

Sol Let $E = E_m \cos\omega t$ $\qquad (\because J_c = \frac{I}{Area})$

Conduction current $= I_c = J_c A$

$= \sigma E A = \sigma A E_m \cos\omega t$ ——(1)

Displacement current $= I_D = J_D A$

$= \frac{\partial D}{\partial t} A = A \varepsilon \frac{\partial E}{\partial t}$

$= A \varepsilon \frac{\partial}{\partial t}[E_m \cos\omega t]$

$= - A \varepsilon E_m \sin\omega t \; \omega$

$= A \varepsilon E_m \omega \cos(\omega t + 90°)$ —(2)

$\therefore$ So, I_c and I_D are displaced in phase by 90°.

7) Use maxwell's eqn curl $E = -\frac{\partial B}{\partial t}$. Prove that div. $B = 0$

Sol

$$\nabla \times \bar{E} = -\frac{\partial B}{\partial t}$$

Taking divergence on both sides

$$\nabla \cdot (\nabla \times \bar{E}) = \nabla \cdot \left(-\frac{\partial \bar{B}}{\partial t}\right)$$

$$0 = -\frac{\partial}{\partial t}[\nabla \cdot \bar{B}] \qquad (\because \text{divergence on curl is zero})$$

$$\therefore \boxed{\nabla \cdot \bar{B} = 0}$$

8) Using maxwell eqn curl $B = \mu_0 \left(J + \frac{\partial D}{\partial t}\right)$. Prove that div. $D = \rho$.

Sol

$$\text{curl} B = \mu_0 \left(J + \frac{\partial D}{\partial t}\right)$$

$$\nabla \times B = \mu_0 \left[J + \frac{\partial D}{\partial t}\right]$$

Taking divergence

$$\nabla \cdot (\nabla \times \bar{B}) = \nabla \cdot \left[\mu_0 \bar{J} + \mu_0 \frac{\partial D}{\partial t}\right]$$

$$0 = \mu_0 \nabla \cdot \bar{J} + \mu_0 \frac{\partial}{\partial t}(\nabla \cdot \bar{D})$$

$$\cancel{\mu_0} \nabla \cdot \bar{J} = -\cancel{\mu_0} \frac{\partial}{\partial t}(\nabla \cdot \bar{D})$$

$$\cancel{+} \frac{\partial \rho_v}{\partial t} = \cancel{+} \frac{\partial}{\partial t}(\nabla \cdot \bar{D}) \qquad (\text{from Continuity eqn})$$

$$\therefore \nabla \cdot \bar{D} = \rho_v$$

Poynting Vector & poynting Theorem (Energy Theorem)

In general electric circuits, power can be expressed interms of voltage and current.

In case of Em waves, the power and energy relationship can be explained interms of amplitudes of electric and magnetic fields.

$P = V I$ for electric ckts

$P = \bar{E} \times \bar{H}$ for Electromagnetic fields. (EM)

The resulting theorem is the most fundamental relationships of the electromagnetic theory which is known as Poynting Theorem.

By means of Em wave an energy can be transfered from transmitter to receiver.

$\bar{E} \rightarrow V/m$

$\bar{H} \rightarrow A/m$

$\bar{E}\,\bar{H} \rightarrow \frac{V}{m} \cdot \frac{A}{m}$ (or) $Watt/m^2$

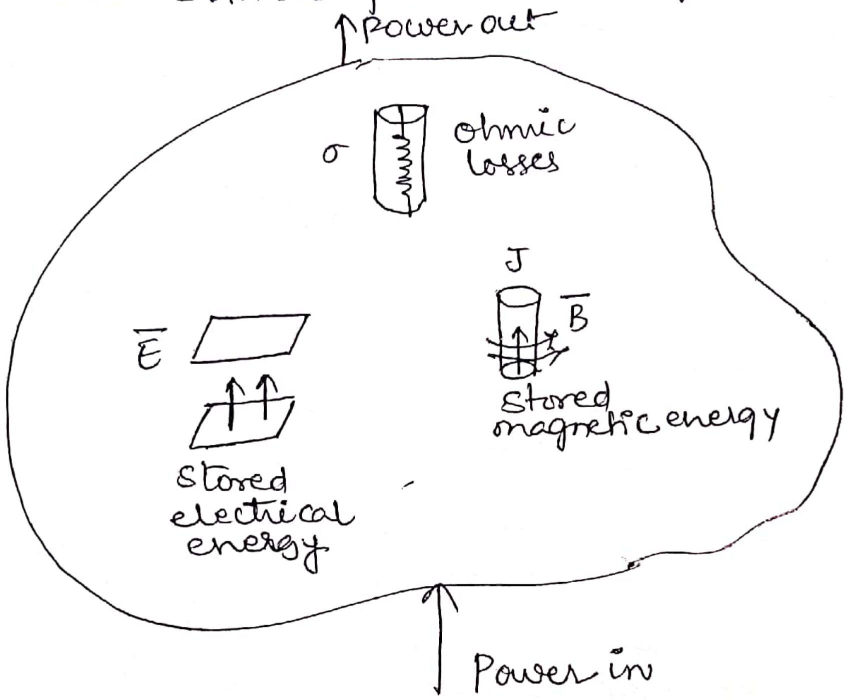
- The product of $\bar{E}$ and $\bar{H}$ gives a new quantity is called Power density and expressed in $Watt/m^2$.
- The power radiated from antenna has a particular direction. Therefore power density is expressed as

$$\boxed{\bar{P} = \bar{E} \times \bar{H}}$$

where $\bar{P}$ is called Poynting vector.

Poynting Theorem is based on law of conservation of energy in electromagnetism.

Poynting Theorem states " The net power flowing out of a given volume V is equal to the time rate of decrease in the energy stored within volume (V) minus the ohmic power dissipated."



From maxwell eqn

$$\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t} = -\mu \frac{\partial \bar{H}}{\partial t} \quad \text{---(1)}$$

$$\nabla \times \bar{H} = \bar{J}_C + \bar{J}_D = \sigma \bar{E} + \varepsilon \frac{\partial \bar{E}}{\partial t} \quad \text{--- (2)}$$

Taking dot product with $\bar{E}$ of eqn (2)

$$\bar{E} \cdot (\nabla \times \bar{H}) = \bar{E} \cdot (\sigma \bar{E}) + \bar{E} \cdot \left(\varepsilon \frac{\partial \bar{E}}{\partial t}\right) \quad \text{---(3)}$$

we know vector identity

$$\nabla \cdot (\bar{A} \times \bar{B}) = \bar{B} \cdot (\nabla \times \bar{A}) - \bar{A} \cdot (\nabla \times \bar{B})$$

let $\bar{A} = \bar{E}$
$\bar{B} = \bar{H}$

$$\nabla \cdot (\bar{E} \times \bar{H}) = \bar{H} \cdot (\nabla \times \bar{E}) - \bar{E} \cdot (\nabla \times \bar{H}) \quad \text{---(4)}$$

Sub (4) in (3)

$$\bar{H} \cdot (\nabla \times \bar{E}) - \nabla \cdot (\bar{E} \times \bar{H}) = \bar{E} \cdot (\sigma \bar{E}) + \bar{E} \cdot \left(\varepsilon \frac{\partial \bar{E}}{\partial t}\right)$$

$$= \sigma E^2 + \bar{E}\left(\varepsilon \frac{\partial \bar{E}}{\partial t}\right)$$

Sub (1) in above

$$\bar{H} \cdot \left(-\mu \frac{\partial \bar{H}}{\partial t}\right) - \nabla \cdot (\bar{E} \times \bar{H}) = \sigma E^2 + \varepsilon \left(\bar{E} \cdot \frac{\partial \bar{E}}{\partial t}\right) \quad \text{---(5)}$$

Now consider term

$$\frac{\partial}{\partial t}(\underset{u}{\bar{H}} \cdot \underset{v}{\bar{H}}) = \bar{H} \cdot \frac{\partial \bar{H}}{\partial t} + \bar{H} \cdot \frac{\partial \bar{H}}{\partial t}$$

$$\therefore \frac{\partial}{\partial t} H^2 = 2 \bar{H} \cdot \frac{\partial \bar{H}}{\partial t} \Rightarrow \frac{1}{2} \frac{\partial}{\partial t}(H^2) = \bar{H} \cdot \frac{\partial \bar{H}}{\partial t}$$

||ly $\frac{1}{2} \frac{\partial}{\partial t}(E^2) = \bar{E} \cdot \frac{\partial \bar{E}}{\partial t}$

Sub these values in eqn (5)

$$-\mu\left(\frac{1}{2} \frac{\partial}{\partial t} H^2\right) - \nabla \cdot (\bar{E} \times \bar{H}) = \sigma E^2 + \frac{\varepsilon}{2} \frac{\partial}{\partial t} E^2$$

$$-\nabla \cdot (\bar{E} \times \bar{H}) = \sigma E^2 + \frac{1}{2} \frac{\partial}{\partial t}\left[\mu H^2 + \varepsilon E^2\right]$$

$$\boxed{-\nabla \cdot P = +\sigma E^2 + \frac{1}{2} \frac{\partial}{\partial t}\left[\mu H^2 + \varepsilon E^2\right]}$$

Above eqn is Poynting theorem in point form.

Integrate above eqn w.r.t. volume

$$-\int_V \nabla \cdot \bar{P} \, dv = +\int_V \sigma E^2 dv + \int_V \frac{1}{2} \frac{\partial}{\partial t}\left[\mu H^2 + \varepsilon E^2\right] dv$$

$$-\int_V (\nabla \cdot \bar{P}) dv = +\int_V \sigma E^2 dv + \int_V \frac{\partial}{\partial t}\left[\frac{\varepsilon E^2}{2} + \frac{\mu H^2}{2}\right] dv$$

using divergence Theorem

$$\boxed{+\oint_S \bar{P} \cdot \bar{ds} = \cancel{+}\int_V \sigma E^2 dv + \int_V \frac{\partial}{\partial t}\left[\frac{\varepsilon E^2}{2} + \frac{\mu H^2}{2}\right] dv}$$

Above eqn is Poynting theorem in integral form.

* -ve sign on LHS of above eqn indicates that the power is flowing into the surface.

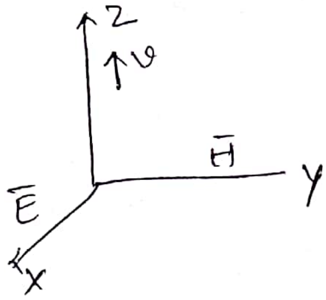
First terms gives the total ohmic power loss within the volume.

Second term gives time rate of increase of total energy stored in electric and magnetic fields.

Instantaneous, Average and Complex Power:

Let $\bar{E} = E_x \bar{a}_x$

$\bar{H} = H_y \bar{a}_y$



then $\bar{P} = \bar{E} \times \bar{H}$

$= (E_x \bar{a}_x) \times (H_y \bar{a}_y)$

$= E_x H_y \bar{a}_z$

$= P_z \bar{a}_z$

$\therefore$ $\bar{E}, \bar{H}, \bar{P}$ are mutually perpendicular to each.

In general $\boxed{\bar{P} = \bar{E} \times \bar{H}}$ → This is instantaneous power flow per unit area. Hence it is also called instantaneous poynting vector.

Complex poynting vector $\bar{P} = \frac{1}{2} \bar{E} \times \bar{H}^*$

Average power $= \bar{P}_{avg} = \frac{1}{2} \mathrm{Re}[\bar{E} \times \bar{H}^*]$

$\bar{P}_{react} = \frac{1}{2} \mathrm{Im}[\bar{E} \times \bar{H}^*]$

$\boxed{\bar{P}_{inst} = \bar{P}_{avg} + \bar{P}_{react}}$

Let $\bar{E} = E_m \cos(\omega t - \beta z)\, \bar{a}_x$

$\bar{H} = H_m \cos(\omega t - \beta z)\, \bar{a}_y$

$= \frac{E_m}{\eta_0} \cos(\omega t - \beta z)\, \bar{a}_y$

$\left(\because \frac{\bar{E}}{\bar{H}} = \eta_0 = \frac{E_m}{H_m} = 377\Omega\right)$.

According to poynting theorem

$\bar{P} = \bar{E} \times \bar{H}$

$= [E_m \cos(\omega t - \beta z)\, \bar{a}_x] \times \left[\frac{E_m}{\eta_0} \cos(\omega t - \beta z)\, \bar{a}_y\right]$

$$\boxed{\bar{P} = \frac{E_m^2}{\eta_0} \cos^2(\omega t - \beta z)\, \bar{a}_z} \quad W/m^2$$

↓ power density

The power passing particular area is given by

Power = power density × Area.

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Average power Density (Pavg)

$$P_{avg} = \frac{1}{T}\int_0^T \text{Power}$$

$$= \frac{1}{T}\int_0^T \frac{E_m^2}{\eta_0}\cos^2(\omega t - \beta z)\,\bar{a}_z\,dt$$

$$= \frac{E_m^2}{T\eta_0}\int_0^T \frac{1+\cos 2(\omega t - \beta z)}{2}\,dt$$

$$= \frac{E_m^2}{2T\eta_0}\left[t + \frac{\sin(2(\omega t - \beta z))}{2\omega}\right]_0^T$$

$$= \frac{E_m^2}{2T\eta_0}\left[T + \frac{\sin(2\omega T - 2\beta z)}{2\omega} + \frac{\sin 2\beta z}{2\omega}\right]$$

but $\omega T = 2\pi f T = \frac{2\pi}{\cancel{T}}\cdot\cancel{T} = 2\pi$

$$P_{avg} = \frac{E_m^2}{2T\eta}\left[T + \frac{\sin(4\pi - 2\beta z)}{2\omega} + \frac{\sin 2\beta z}{2\omega}\right]$$

$$= \frac{E_m^2}{2\cancel{T}\eta}\left[\cancel{T} - \frac{\sin 2\beta z}{\cancel{2\omega}} + \frac{\sin 2\beta z}{\cancel{2\omega}}\right]$$

$$\boxed{P_{avg} = \frac{1}{2}\frac{E_m^2}{\eta}} \quad W/m^2.$$

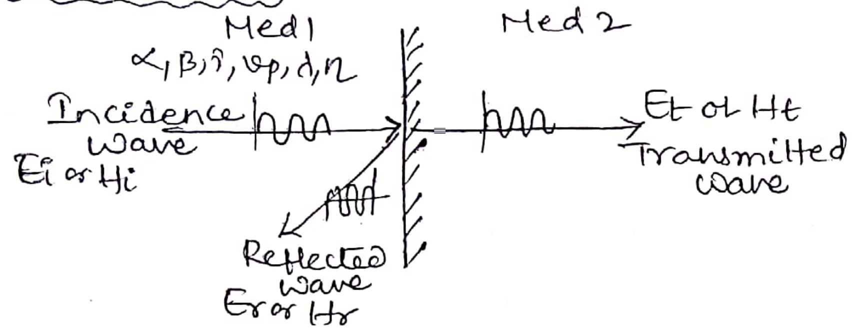
$$\bar{P} = \bar{E}\times\bar{H}$$

$$= E_m e^{j(\omega t - \beta z)}\,\bar{a}_x \times H_m e^{j(\omega t - \beta z)}\,\bar{a}_y$$

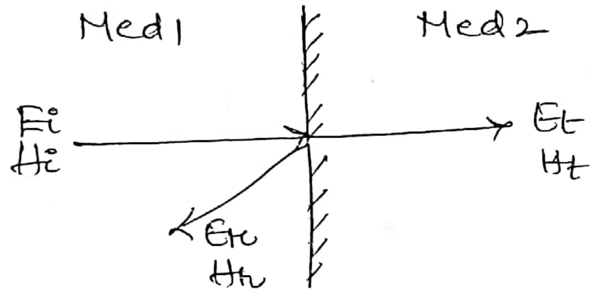
$$\bar{P} = E_m H_m\left[\cos(\omega t - \beta z) + j\sin(\omega t - \beta z)\right]\left[\cos(\omega t - \beta z) - j\sin(\omega t - \beta z)\right]\bar{a}_z$$

$$P_{inst} = P_{avg} + P_{react}$$

Reflection:



Reflection by a perfect Conductor – Normal incidence



$E_i = E_i e^{-j\beta x} = E_i e^{j(\omega t - \beta x)}$

$E_r = E_r e^{+j\beta x} = E_r e^{j(\omega t + \beta x)}$

Boundary is the surface of the conductor given by $x=0$.

The boundary conditions to be applied are:

1. The Tangential component of electric field is continuous across the boundary.
2. Electric field $\bar{E}$ inside the conductor is zero.

i.e. $E_i + E_r = 0$

$$\boxed{E_r = -E_i}$$

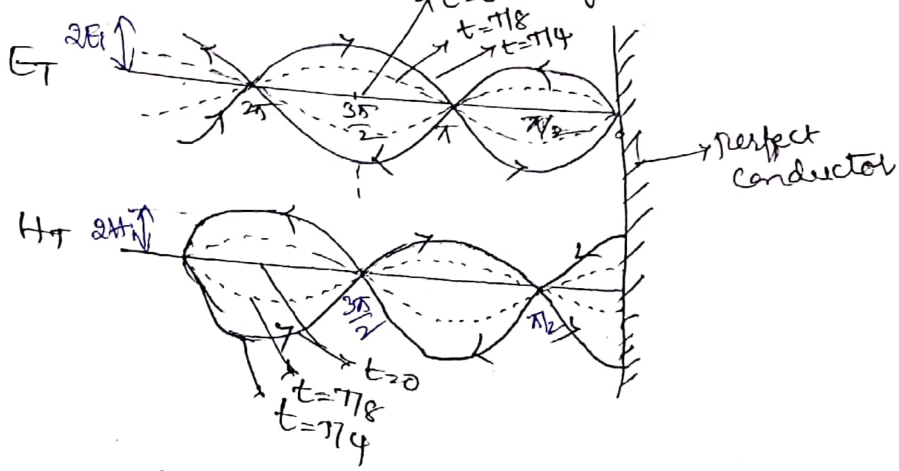
Assume the wave is travelling along x-axis

$$\begin{aligned}
E_T &= E_i + E_r \\
&= E_i e^{j(\omega t - \beta x)} + E_r e^{j(\omega t + \beta x)} \\
&= E_i \left[e^{j(\omega t - \beta x)} - e^{j(\omega t + \beta x)} \right] \\
&= E_i \left[e^{j\omega t} \left[e^{-j\beta x} - e^{+j\beta x} \right] \right] \\
&= E_i e^{j\omega t} (-2j \sin\beta x) \\
&= -E_i (\cos\omega t + j \sin\omega t)\, 2j \sin\beta x \\
&= \underbrace{2 E_i \sin\beta x \sin\omega t}_{\text{Real part}} - j\, \underbrace{2 E_i \sin\beta x \cos\omega t}_{\text{Imaginary part}}
\end{aligned}$$

for E_T to be real

$\therefore$ $\boxed{E_T = 2E_i \sin\beta x \sin\omega t}$

This eqn shows that incident wave and reflected wave combine to give a standing wave.



The above eqn is ~~for~~ time varying fields.
The magnitude of electric field varies sinusoidally with distance from the reflecting plane.

i) $E_T = 0$ when $x=0$ or multiples of $\lambda/2$

$$\beta x = \frac{2\pi}{\lambda} \times \frac{\lambda}{2} = \pi$$

ii) $E_T = 2E_i$ when $x =$ odd multiples of $\lambda/4$

$$\beta x = \frac{\cancel{2}\pi}{\cancel{\lambda}} \cdot \frac{\cancel{\lambda}}{\cancel{4}_2} = \pi/2$$

As the electric field is reflected with the phase reversal, then the magnetic field is reflected without the phase reversal.

i.e. $H_R = H_i$

$$H_T = H_i e^{j(\omega t - \beta x)} + H_R e^{j(\omega t + \beta x)}$$

$$= H_i e^{j\omega t} (e^{-j\beta x} + e^{j\beta x})$$

$$= H_i (\cos\omega t + j\sin\omega t)\, 2\cos\beta x$$

$$= 2H_i \cos\beta x \cos\omega t + j\, 2H_i \cos\beta x \sin\omega t$$

$H_T = 0$, $\beta x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2} \ldots$

$H_T = max$ when $\beta x = 0, \pi, 2\pi \ldots$

H_T to be real

$\therefore$ $\boxed{H_T = 2H_i \cos\beta x \cos\omega t}$

$\boxed{\begin{aligned} \bar{E} &= 2E_i \sin\beta x \sin\omega t \\ \bar{H} &= 2H_i \cos\beta x \cos\omega t \end{aligned}}$

H_T is min (≈ 0) when $x =$ odd multiples of $\lambda/4$

H_T is max when $x =$ multiples of $\lambda/2$

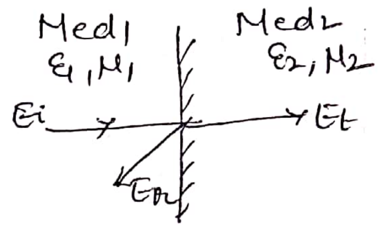
Reflection by a perfect dielectric – Normal Incidence (Insulator)

Let us consider two media with parameters ϵ_1, μ_1 and ϵ_2, μ_2. The boundary is parallel to $x=0$ plane

$E_i \rightarrow$ incident wave Electric field strength

$E_r \rightarrow$ Reflected wave " "

$E_T \rightarrow$ Transmitted wave " "



$$\eta_1 = \sqrt{\frac{\mu_1}{\epsilon_1}} \text{ and } \eta_2 = \sqrt{\frac{\mu_2}{\epsilon_2}}$$

$$\left.\begin{aligned} E_i &= \eta_1 H_i \\ E_r &= -\eta_1 H_r \\ E_t &= \eta_2 H_t \end{aligned}\right\} \text{ —— (1)}$$

At boundary $\quad E_{1tan} = E_{2tan}$ & $H_{1tan} = H_{2tan}$

$$\left.\begin{aligned} E_i + E_r &= E_T \\ H_i + H_r &= H_T \end{aligned}\right\} \text{ — (2)}$$

From (1)

$$H_t = \frac{E_t}{\eta_2} = \frac{1}{\eta_2}\left[E_i + E_r\right]$$

Sub this in (2)

$$H_i + H_r = H_T$$

$$\frac{E_i}{\eta_1} - \frac{1}{\eta_1}E_r = \frac{1}{\eta_2}\left[E_i + E_r\right]$$

$$E_r\left[\frac{1}{\eta_1} + \frac{1}{\eta_2}\right] = E_i\left[\frac{1}{\eta_1} - \frac{1}{\eta_2}\right]$$

$$\frac{E_r}{E_i} = \frac{1/\eta_1 - 1/\eta_2}{1/\eta_1 + \frac{1}{\eta_2}}$$

$$\boxed{\frac{E_r}{E_i} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}}$$

From (2) $\quad E_i + E_r = E_T$

$$1 + \frac{E_r}{E_i} = \frac{E_T}{E_i} \Rightarrow \therefore \frac{E_T}{E_i} = 1 + \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}$$

$$\boxed{\frac{E_T}{E_i} = \frac{2\eta_2}{\eta_1 + \eta_2}}$$

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From (1)

$$E_r = -\eta_1 H_r \quad \text{and} \quad E_i = \eta_1 H_i$$

divide these two eqn.

$$\frac{E_r}{E_i} = -\frac{\cancel{\eta_1} H_r}{\cancel{\eta_1} H_i} \Rightarrow \frac{H_r}{H_i} = -\frac{E_r}{E_i}$$

$$\frac{H_r}{H_i} = -\left[\frac{\eta_2-\eta_1}{\eta_2+\eta_1}\right]$$

$$\therefore \boxed{\frac{H_r}{H_i} = \frac{\eta_1-\eta_2}{\eta_1+\eta_2}}$$

From (2)

$$H_i + H_r = H_t$$

$$1 + \frac{H_r}{H_i} = \frac{H_t}{H_i}$$

$$\therefore \frac{H_t}{H_i} = 1 + \frac{\eta_1-\eta_2}{\eta_1+\eta_2}$$

$$\boxed{\frac{H_t}{H_i} = \frac{2\eta_1}{\eta_1+\eta_2}}$$

Assume $\mu_1 = \mu_2 = 1$

$$\eta_1 = \sqrt{\frac{1}{\epsilon_1}} \quad \text{and} \quad \eta_2 = \sqrt{\frac{1}{\epsilon_2}}$$

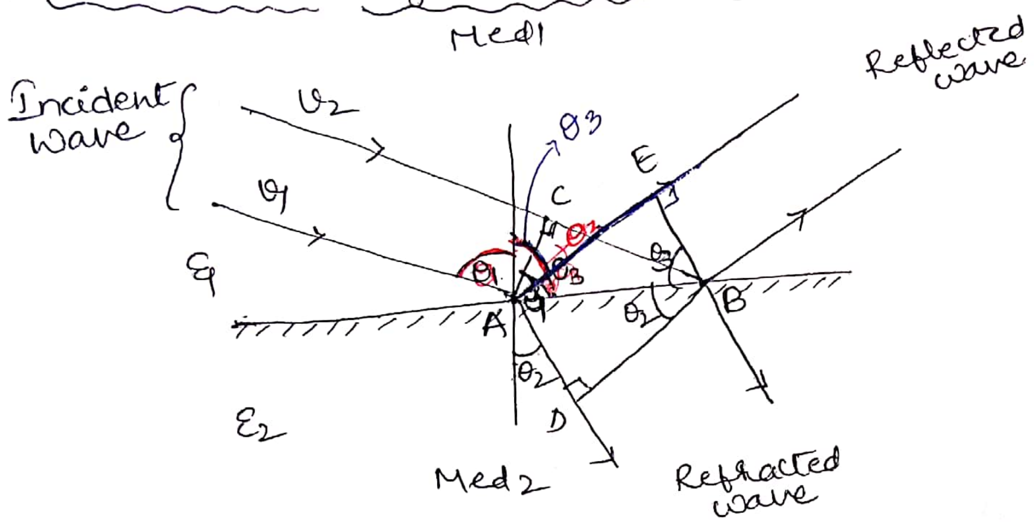
$$\therefore \frac{E_r}{E_i} = \frac{\eta_2-\eta_1}{\eta_2+\eta_1} = \frac{\frac{1}{\sqrt{\epsilon_2}} - \frac{1}{\sqrt{\epsilon_1}}}{\frac{1}{\sqrt{\epsilon_2}} + \frac{1}{\sqrt{\epsilon_1}}} = \frac{\sqrt{\epsilon_1}-\sqrt{\epsilon_2}}{\sqrt{\epsilon_1}+\sqrt{\epsilon_2}}$$

$$\frac{E_t}{E_i} = \frac{2\eta_2}{\eta_1+\eta_2} = \frac{2\,{}^1\!/\!\sqrt{\epsilon_2}}{{}^1\!/\!\sqrt{\epsilon_1} + {}^1\!/\!\sqrt{\epsilon_2}} = \frac{2\sqrt{\epsilon_1}}{\sqrt{\epsilon_1}+\sqrt{\epsilon_2}}$$

$$\frac{H_r}{H_i} = \frac{\eta_1-\eta_2}{\eta_1+\eta_2} = \frac{\frac{1}{\sqrt{\epsilon_1}} - \frac{1}{\sqrt{\epsilon_2}}}{\frac{1}{\sqrt{\epsilon_1}} + \frac{1}{\sqrt{\epsilon_2}}} = \frac{\sqrt{\epsilon_2}-\sqrt{\epsilon_1}}{\sqrt{\epsilon_2}+\sqrt{\epsilon_1}}$$

$$\frac{H_t}{H_i} = \frac{2\eta_1}{\eta_1+\eta_2} = \frac{2\,{}^1\!/\!\sqrt{\epsilon_1}}{\frac{1}{\sqrt{\epsilon_1}} + \frac{1}{\sqrt{\epsilon_2}}} = \frac{2\sqrt{\epsilon_2}}{\sqrt{\epsilon_1}+\sqrt{\epsilon_2}}$$

Reflection by a perfect dielectric – oblique incidence



$v_1 \rightarrow$ velocity of the incident wave (in medium)

$v_2 \rightarrow$ velocity of the medium

$\sin\theta = \frac{\text{opp. side}}{\text{hypotenuse}}$

From fig $\frac{CB}{AD} = \frac{v_1}{v_2}$

distance = vel × time

time = $\frac{\text{distance}}{\text{vel}}$

$= \frac{CB}{v_1} = \frac{AD}{v_2}$

In $\Delta ACB \Rightarrow \sin\theta_1 = \frac{BC}{AB} \Rightarrow BC = AB\sin\theta_1$

$\Delta ADB \Rightarrow \sin\theta_2 = \frac{AD}{AB} \Rightarrow AD = AB\sin\theta_2$

$$\therefore \frac{CB}{AD} = \frac{\cancel{AB}\sin\theta_1}{\cancel{AB}\sin\theta_2} = \frac{v_1}{v_2}$$

$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{1/\sqrt{\mu_1\varepsilon_1}}{1/\sqrt{\mu_2\varepsilon_2}}$$

$$\frac{\sin\theta_1}{\sin\theta_2} = \sqrt{\frac{\mu_2\varepsilon_2}{\mu_1\varepsilon_1}}$$

if $\mu_1 = \mu_2 = 1$

$$\boxed{\frac{\sin\theta_1}{\sin\theta_2} = \sqrt{\frac{\varepsilon_2}{\varepsilon_1}}}$$ Snell's law.

According to law of reflection

angle of incidence = angle of reflection

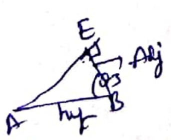
$\boxed{\theta_1 = \theta_3}$

In Δ CAB

$CB = AB\sin\theta_1$

In ABE $\Rightarrow$

$AE = AB\sin\theta_3$.



$CB = AE$

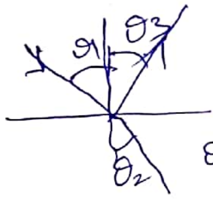
$\cancel{AB}\sin\theta_1 = \cancel{AB}\sin\theta_3$

$\boxed{\theta_1 = \theta_3}$

From law of conservation of energy

Incident energy = Reflected energy + transmitted energy

$$\frac{E_i^2 \cos\theta_1}{\eta_1} = \frac{E_r^2 \cos\theta_1}{\eta_1} + \frac{E_t^2 \cos\theta_2}{\eta_2}$$



$\theta_1 = \theta_3$

$$1 = \frac{\frac{E_r^2 \cancel{\cos\theta_1}}{\cancel{\eta_1}}}{\frac{E_i^2 \cancel{\cos\theta_1}}{\cancel{\eta_1}}} + \frac{\frac{E_t^2 \cos\theta_2}{\eta_2}}{\frac{E_i^2 \cos\theta_1}{\eta_1}}$$

$P_i \cos\theta_1 = P_r \cos\theta_3 + P_t \cos\theta_2$

$$\boxed{\frac{E_r^2}{E_i^2} = 1 - \frac{\eta_1}{\eta_2}\frac{E_t^2}{E_i^2}\frac{\cos\theta_2}{\cos\theta_1}} \quad \text{—— (1)}$$

Case(i) perpendicular polarization

$\eta_1 = \frac{1}{\sqrt{\varepsilon_1}}$

$\eta_2 = 1/\sqrt{\varepsilon_2}$

$$E_i + E_r = E_t$$

$$\frac{E_t}{E_i} = 1 + \frac{E_r}{E_i}$$

Sub this in above eqn (1)

$$\frac{E_r^2}{E_i^2} = 1 - \frac{\eta_1}{\eta_2}\left(1 + \frac{E_r}{E_i}\right)^2 \frac{\cos\theta_2}{\cos\theta_1}$$

$$\frac{E_r^2}{E_i^2} = 1 - \frac{\sqrt{\varepsilon_2}}{\sqrt{\varepsilon_1}}\left(1 + \frac{E_r}{E_i}\right)^2 \frac{\cos\theta_2}{\cos\theta_1}$$

$$\frac{\sqrt{\varepsilon_2}}{\sqrt{\varepsilon_1}}\left(1 + \frac{E_r}{E_i}\right)^2 \frac{\cos\theta_2}{\cos\theta_1} = 1 - \frac{E_r^2}{E_i^2}$$

$$\frac{\sqrt{\varepsilon_2}}{\sqrt{\varepsilon_1}}\left(1 + \frac{E_r}{E_i}\right)^{\cancel{2}} \frac{\cos\theta_2}{\cos\theta_1} = \cancel{\left(1 + \frac{E_r}{E_i}\right)}\left(1 - \frac{E_r}{E_i}\right)$$

$$\frac{E_r}{E_i}\left[\frac{\sqrt{\varepsilon_2}}{\sqrt{\varepsilon_1}}\frac{\cos\theta_2}{\cos\theta_1} + 1\right] = 1 - \frac{\sqrt{\varepsilon_2}}{\sqrt{\varepsilon_1}}\frac{\cos\theta_2}{\cos\theta_1}$$

$$\frac{E_r}{E_i}\left[\frac{\sqrt{\varepsilon_2}\cos\theta_2 + \sqrt{\varepsilon_1}\cos\theta_1}{\cancel{\sqrt{\varepsilon_1}\cos\theta_1}}\right] = \frac{\sqrt{\varepsilon_1}\cos\theta_1 - \sqrt{\varepsilon_2}\cos\theta_2}{\cancel{\sqrt{\varepsilon_1}\cos\theta_1}}$$

$$\boxed{\frac{E_r}{E_i} = \frac{\sqrt{\varepsilon_1}\cos\theta_1 - \sqrt{\varepsilon_2}\cos\theta_2}{\sqrt{\varepsilon_1}\cos\theta_1 + \sqrt{\varepsilon_2}\cos\theta_2}}$$

Case(ii) Parallel polarization

applying boundary condition that

$$(E_i - E_r)\cos\theta_1 = E_t \cos\theta_2$$

$$E_i\cos\theta_1 = E_r\cos\theta_1 + E_t\cos\theta_2$$

$$1 = \frac{E_r}{E_i} + \frac{E_t}{E_i}\frac{\cos\theta_2}{\cos\theta_1}$$

$$\frac{E_t}{E_i}\frac{\cos\theta_2}{\cos\theta_1} = 1 - \frac{E_r}{E_i}$$

$$\frac{E_t}{E_i} = \left[1 - \frac{E_r}{E_i}\right]\frac{\cos\theta_1}{\cos\theta_2}$$

Sub this value in eqn (1)

$$\therefore \frac{E_r^2}{E_i^2} = 1 - \frac{n_1}{n_2}\left[1-\frac{E_r}{E_i}\right]^2 \frac{\cos^{\cancel{2}}\theta_1}{\cos^{\cancel{2}}\theta_2}\,\frac{\cancel{\cos\theta_2}}{\cancel{\cos\theta_1}}$$

$$= 1 - \frac{\sqrt{\varepsilon_2}}{\sqrt{\varepsilon_1}}\left[1-\frac{E_r}{E_i}\right]^2\frac{\cos\theta_1}{\cos\theta_2}$$

$$\frac{\sqrt{\varepsilon_2}}{\sqrt{\varepsilon_1}}\left[\cancel{1+\frac{E_r}{E_i}}\right]\left[1-\frac{E_r}{E_i}\right]^2\frac{\cos\theta_1}{\cos\theta_2} = 1 - \frac{E_r^2}{E_i^2}$$

$$\frac{\sqrt{\varepsilon_2}}{\sqrt{\varepsilon_1}}\left[\cancel{1+\frac{E_r}{E_i}}\right]\left[1-\frac{E_r}{E_i}\right]^{\cancel{2}}\frac{\cos\theta_1}{\cos\theta_2} = \left(\cancel{1-\frac{E_r}{E_i}}\right)\left(1+\frac{E_r}{E_i}\right)$$

$$\frac{\sqrt{\varepsilon_2}}{\sqrt{\varepsilon_1}}\left[1-\frac{E_r}{E_i}\right]\frac{\cos\theta_1}{\cos\theta_2} = 1 + \frac{E_r}{E_i}$$

$$\frac{\sqrt{\varepsilon_2}}{\sqrt{\varepsilon_1}}\frac{\cos\theta_1}{\cos\theta_2} - 1 = \frac{E_r}{E_i}\left[1 + \frac{\sqrt{\varepsilon_2}}{\sqrt{\varepsilon_1}}\frac{\cos\theta_1}{\cos\theta_2}\right]$$

$$\frac{\sqrt{\varepsilon_2}\cos\theta_1 - \sqrt{\varepsilon_1}\cos\theta_2}{\cancel{\sqrt{\varepsilon_1}\cos\theta_2}} = \frac{E_r}{E_i}\left[\frac{\sqrt{\varepsilon_1}\cos\theta_2 + \sqrt{\varepsilon_2}\cos\theta_1}{\cancel{\sqrt{\varepsilon_1}\cos\theta_2}}\right]$$

$$\boxed{\frac{E_r}{E_i} = \frac{\sqrt{\varepsilon_2}\cos\theta_1 - \sqrt{\varepsilon_1}\cos\theta_2}{\sqrt{\varepsilon_2}\cos\theta_1 + \sqrt{\varepsilon_1}\cos\theta_2}}$$

Brewster angle :

This is angle of incidence for which the angle of reflection is zero.

i.e. $\dfrac{E_R}{E_i} = 0$

$$\frac{E_R}{E_i} = \frac{\sqrt{\varepsilon_2}\cos\theta_1 - \sqrt{\varepsilon_1}\cos\theta_2}{\sqrt{\varepsilon_2}\cos\theta_1 - \sqrt{\varepsilon_1}\cos\theta_2}$$

$$0 = \sqrt{\varepsilon_2}\cos\theta_1 - \sqrt{\varepsilon_1}\cos\theta_2$$

$$\sqrt{\varepsilon_2}\cos\theta_1 = \sqrt{\varepsilon_1}\cos\theta_2$$

$$\sqrt{\varepsilon_2}\cos\theta_1 = \sqrt{\varepsilon_1}\left[\sqrt{1-\sin^2\theta_2}\right]$$

$$\sqrt{\varepsilon_2}\cos\theta_1 = \sqrt{\varepsilon_1}\left[\sqrt{1-\frac{\varepsilon_1}{\varepsilon_2}\sin^2\theta_1}\right]$$

$\left(\because \text{From Snell's law } \dfrac{\sin\theta_1}{\sin\theta_2} = \sqrt{\dfrac{\varepsilon_2}{\varepsilon_1}}\right)$

Squaring on both sides

$$\varepsilon_2\cos^2\theta_1 = \varepsilon_1\left[1-\frac{\varepsilon_1}{\varepsilon_2}\sin^2\theta_1\right]$$

$$\varepsilon_2(1-\sin^2\theta_1) = \varepsilon_1 - \frac{\varepsilon_1^2}{\varepsilon_2}\sin^2\theta_1$$

$$\sin^2\theta_1\left[\frac{\varepsilon_1^2}{\varepsilon_2} - \varepsilon_2\right] = \varepsilon_1 - \varepsilon_2$$

$$\sin^2\theta_1\left[\frac{\varepsilon_1^2-\varepsilon_2^2}{\varepsilon_2}\right] = \varepsilon_1 - \varepsilon_2$$

$$\sin^2\theta_1\,\frac{(\varepsilon_1+\varepsilon_2)\cancel{(\varepsilon_1-\varepsilon_2)}}{\varepsilon_2} = \cancel{(\varepsilon_1-\varepsilon_2)}$$

$$\sin^2\theta_1 = \frac{\varepsilon_2}{\varepsilon_1+\varepsilon_2} \Rightarrow \sin\theta_1 = \sqrt{\frac{\varepsilon_2}{\varepsilon_1+\varepsilon_2}}$$

$$\cos^2\theta_1 = 1-\sin^2\theta_1$$

$$= 1 - \frac{\varepsilon_2}{\varepsilon_1+\varepsilon_2} = \frac{\varepsilon_1}{\varepsilon_1+\varepsilon_2}$$

$$\cos\theta_1 = \sqrt{\frac{\varepsilon_1}{\varepsilon_1+\varepsilon_2}}$$

$$\therefore \tan\theta_1 = \frac{\sin\theta_1}{\cos\theta_1} = \frac{\sqrt{\varepsilon_2/\varepsilon_1+\varepsilon_2}}{\sqrt{\varepsilon_1/\varepsilon_1+\varepsilon_2}}$$

$$\therefore \boxed{\tan\theta_1 = \sqrt{\frac{\varepsilon_2}{\varepsilon_1}}}$$

Reflection coefficient (K)

It is the ratio of magnitude of electric or magnetic field component of the reflected wave to the incident wave.

$$\boxed{K = \frac{E_r}{E_i}} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}$$

or

$$K = \frac{H_r}{H_i} = \frac{\eta_1 - \eta_2}{\eta_1 + \eta_2}$$

$$S = \frac{1+K}{1-K}$$

or $K = \frac{S-1}{S+1}$

Transmission coefficient (T)

It is the ratio of magnitude of electric or magnetic field component of the transmitted wave to the incident wave.

$$\boxed{T = \frac{E_t}{E_i}} = \frac{2\eta_2}{\eta_1 + \eta_2}$$

or

$$T = \frac{H_t}{H_i} = \frac{2\eta_1}{\eta_1 + \eta_2}$$

* Incase of parallel polarization prove that $\frac{E_r}{E_i} = \frac{\tan(\theta_1 - \theta_2)}{\tan(\theta_1 + \theta_2)}$ where θ_1 = angle of incidence and θ_2 = angle of reflection.

Sol For parallel polarization

$$\frac{E_r}{E_i} = \frac{\sqrt{\epsilon_2}\cos\theta_1 - \sqrt{\epsilon_1}\cos\theta_2}{\sqrt{\epsilon_2}\cos\theta_1 + \sqrt{\epsilon_1}\cos\theta_2} \quad \text{---(1)}$$

from Snell's law $\frac{\sin\theta_1}{\sin\theta_2} = \sqrt{\frac{\epsilon_2}{\epsilon_1}}$ ---(2)

From (1)

$$\frac{E_r}{E_i} = \frac{\cancel{\sqrt{\epsilon_1}}\left[\frac{\sqrt{\epsilon_2}}{\sqrt{\epsilon_1}}\cos\theta_1 - \cos\theta_2\right]}{\cancel{\sqrt{\epsilon_1}}\left[\frac{\sqrt{\epsilon_2}}{\sqrt{\epsilon_1}}\cos\theta_1 + \cos\theta_2\right]}$$

$$\frac{E_r}{E_i} = \frac{\frac{\sin\theta_1}{\sin\theta_2}\cos\theta_1 - \cos\theta_2}{\frac{\sin\theta_1}{\sin\theta_2}\cos\theta_1 + \cos\theta_2}$$

$$\frac{E_r}{E_i} = \frac{\sin\theta_1\cos\theta_1 - \frac{\sin\theta_2}{\cos\theta_2}}{\sin\theta_1\cos\theta_1 + \frac{\sin\theta_2}{\cos\theta_2}}$$

$$= \frac{2\sin\theta_1\cos\theta_1 - \frac{2\sin\theta_2}{\cos\theta_2}}{2\sin\theta_1\cos\theta_1 + \frac{2\sin\theta_1}{\cos\theta_1}}$$

$$= \frac{\sin 2\theta_1 - \sin 2\theta_2}{\sin 2\theta_1 + \sin 2\theta_2}$$

$$= \frac{\cancel{2}\cos(\theta_1+\theta_2)\sin(\theta_1-\theta_2)}{\cancel{2}\sin(\theta_1+\theta_2)\cos(\theta_1-\theta_2)}$$

$$\frac{E_r}{E_i} = \frac{\tan(\theta_1-\theta_2)}{\tan(\theta_1+\theta_2)} \;//.$$

Problems:

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1) A plane wave is reflected at normal incidence from boundary surface $E_{in} = 10\,mV/m$, $E_r = 5\,mV/meter$. Find VSWR. What should be the value of reflected wave to produce a pure standing wave ~~...~~ one reflection

Sol)

$$S = \frac{1+K}{1-K} \qquad K = \frac{E_r}{E_i} = \frac{5\times10^{-3}}{10\times10^{-3}} = 0.5$$

$$= \frac{1+0.5}{1-0.5} = \frac{1.5}{0.5} = 3$$

$$K = \frac{E_r}{E_i}$$

$$1 = \frac{E_r}{E_i} \Rightarrow E_r = E_i = 10\,mV/m$$

2) Given $E = Ae^{j\omega(t-\frac{x}{v})}$, $E_x = 0$, $E_z = 0$. Find other H-components?

Sol

$$\frac{E_y}{H_z} = \eta = \sqrt{\frac{\mu_0}{\varepsilon_0}} = 377.$$

$$H_z = \frac{1}{377} E_y = \frac{1}{377} A e^{j\omega(t-\frac{x}{v})}$$

$$H_x = 0$$

$$H_y = 0$$

3) Determine the amplitudes of the reflected and transmitted $\bar{E}$ and $\bar{H}$ at the interface. If $E_i = 1.5\times10^{-3}\,V$, $\varepsilon_{r1} = 8.5$, $\mu_{r1} = 1$, $\sigma = 0$. Region 2 is free space. Assume normal incidence.

Med1 | Med2

Sol

$$\eta_1 = \sqrt{\frac{\mu}{\varepsilon}} = \sqrt{\frac{\mu_0\mu_r}{\varepsilon_0\varepsilon_r}} = \sqrt{\frac{4\pi\times10^{-7}}{8.854\times10^{-12}\times8.5}} = 129\,\Omega$$

$$\eta_2 = \sqrt{\frac{\mu_0}{\varepsilon_0}} = 377\,\Omega$$

$$H_i = \frac{E_i}{\eta_1} = 11.6\times10^{-6}\,A/m$$

$$\frac{E_r}{E_i} = \frac{\eta_2-\eta_1}{\eta_2+\eta_1} = 7.55\times10^{-3}\,V/m.$$

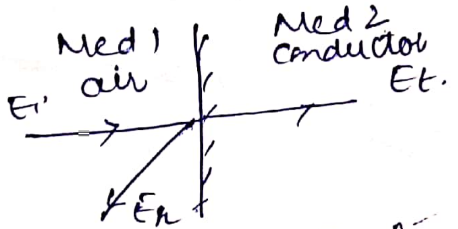
$$H_r = \frac{E_r}{\eta_1} = -5.69\times10^{-6}\,A/m$$

$$\frac{E_t}{E_i} = \frac{2\eta_2}{\eta_1+\eta_2} = 2.24\times10^{-3}\,V/m$$

$$H_t = \frac{E_t}{\eta_2} = 5.9\times10^{-6}\,A/m$$

(4) A wave is incident from air on to a perfect conductor then ~~what~~ find the reflection coefficient?

Sol



Med 1 is air

$\sigma = 0, \ \varepsilon = \varepsilon_0 \varepsilon_r, \ \mu = \mu_0 \mu_r$

$\eta_1 = \sqrt{\frac{\mu_0}{\varepsilon_0}} =$

$K = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}$

Med 2 is conductor (perfect conductor).

$\sigma = \infty, \ \varepsilon = \varepsilon_0 \varepsilon_r, \ \mu = \mu_0 \mu_r$

$$\eta_2 = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}} = \sqrt{\frac{j\omega\mu}{\sigma\left[1 + j\frac{1}{\sigma}\omega\varepsilon\right]}} = \sqrt{\frac{\omega\mu}{\sigma}}\sqrt{j}$$

$$\eta_2 = \sqrt{\frac{\omega\mu}{\sigma}}\left[\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}\right]$$

$$\eta_2 = 0$$

$$\therefore K = \frac{0 - \eta_1}{0 + \eta_1} = -1$$

(5) Determine the surface impedance of a conductor at 1 GHz? Given $\mu_r = 100$, $\sigma = 50 \times 10^6$ ℧/m.

Sol $\eta = Z_s = \sqrt{\frac{j\omega\mu}{\sigma}}$

$$= \sqrt{\frac{2\pi f \mu_0 \mu_r}{\sigma}}\left[\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}\right]$$

$$= \sqrt{\frac{2\pi(1\times10^9)\,4\pi\times10^{-7}\times100}{50\times10^6}}\left[\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}\right]$$

$$Z_s = 0.088 + j\,0.088 \ \Omega$$

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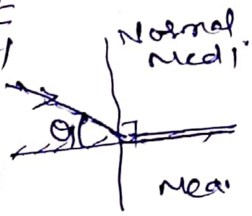
Critical angle

"angle of incidence for which total reflection occurs" (or) "angle of incidence for which angle of reflection is 90°".

From Snell's law

$$\frac{\sin\theta_1}{\sin\theta_2} = \sqrt{\frac{\varepsilon_2}{\varepsilon_1}} \qquad (\text{or}) \qquad \frac{\sin i}{\sin r} = \sqrt{\frac{\varepsilon_2}{\varepsilon_1}}$$



$$\frac{\sin\theta_1}{\sin 90^\circ} = \sqrt{\frac{\varepsilon_2}{\varepsilon_1}}$$

$$\sin\theta_c = \sqrt{\frac{\varepsilon_2}{\varepsilon_1}}$$

$$\boxed{\theta_c = \sin^{-1}\sqrt{\frac{\varepsilon_2}{\varepsilon_1}}}$$

Loss Tangent

$$\tan\theta = \left|\frac{J_c}{J_D}\right|$$

$$= \left|\frac{\sigma E}{j\omega\varepsilon E}\right|$$

$$\boxed{\tan\theta = \frac{\sigma}{\omega\varepsilon}}$$

$$J_D = \frac{\partial D}{\partial t}$$

$$= \varepsilon\frac{\partial E}{\partial t}$$

$$= \varepsilon j\omega E.$$

ELECTROMAGNETIC THEORY

UNIT-I

1. State coulombs law.

Coulombs law states that the force between any two point charges is directly proportional to the product of their magnitudes and inversely proportional to the square of the distance between them. It is directed along the line joining the two charges.

$F=Q_1Q_2 / 4\pi\varepsilon r^2$ ar

2. State Gauss law for electric fields

The total electric flux passing through any closed surface is equal to the total charge enclosed by that surface.

3. Define electric flux.

The lines of electric force are electric flux.

4. Define electric flux density.

Electric flux density is defined as electric flux per unit area.

5. Define electric field intensity.

Electric field intensity is defined as the electric force per unit positive charge.

$E = F/Q$

$= Q/4\pi\varepsilon r^2$ V/m

6. Name few applications of Gauss law in electrostatics.

Gauss law is applied to find the electric field intensity from a closed surface.e.g)Electric field can be determined for shell, two concentric shell or cylinders etc.

7. What is a point charge?

Point charge is one whose maximum dimension is very small in comparison with any other length.

8. Define linear charge density.

It is the charge per unit length.

9. Define surface charge density.

It is the charge per surface area.

10. State the principle of superposition of fields.

The total electric field at a point is the algebraic sum of the individual electric field at that point.

11. Explain the conservative property of electric field.

The work done in moving a point charge around a closed path in a electric field is zero.
Such a field is said to be conservative.

$\oint \bar{E}\cdot\bar{dl} = 0$

12. Define ohms law at a point

Ohms law at appoint states that the field strength within a conductor is proportional to current density.

13. Give the relation between electric field intensity and electric flux density.

$\bar{D}=\varepsilon\bar{E}$ C/m^2

14. What is the physical significance of div'D?

$\nabla\bullet D=\rho_v$

The divergence of a vector flux density is electric flux per unit volume leaving a small volume. This is equal to the volume charge density.

15. What is the effect of permittivity on the force between two charges?

Increase in permittivity of the medium tends to decrease the force between two charges and decrease in permittivity of the medium tends to increase the force between two charges.

16. State electric displacement.

The electric flux or electric displacement through a closed surface is equal to the charge enclosed by the surface.

17. What is displacement flux density?

The electric displacement per unit area is known as electric displacement density or electric flux density.

18. State Divergence Theorem.

The integral of the divergence of a vector over a volume v is equal to the surface integral o f the normal component of the vector over the surface bounded by the volume.

19. Give the expression for electric field intensity due to a single shell of charge

$\bar{E} = Q / 4\pi\varepsilon r^2 \; \bar{a}_r$

20. What is electrostatic force?

The force between any two particles due to existing charges is known as electrostatic force, repulsive for like and attractive for unlike.

21. Define divergence.

The divergence of a vector F at any point is defined as the limit of its surface integral per unit volume as the volume enclosed by the surface around the point shrinks to zero.

22. Define dielectric strength.

The dielectric strength of a dielectric is defined as the maximum value of electric field that canbe applied to the dielectric without its electric breakdown.

UNIT-II

1. Write Poisson's and Laplace's equations.

Poisson's eqn:

$\nabla^2 V = -\rho_v / \varepsilon$

Laplace's eqn:

$\nabla^2 V = 0$

2. Define potential difference.

Potential difference is defined as the work done in moving a unit positive charge from one point to another point in an electric field.

3. Define potential.

Potential at any point is defined as the work done in moving a unit positive charge from infinity to that point in an electric field.

$V = Q / 4\pi\varepsilon r$

4. Give the relationship between potential gradiant and electric field.

$E = -\nabla V$

5. Write the expression for energy density in electrostatic field.

$W = 1/2\, \varepsilon E^2$

6. Write the boundary conditions at the interface between two perfect dielectrics.

i)The tangential component of electric field is continuous i.e $E_{t1} = E_{t2}$

ii) The normal component of electric flux density is continuous i.e $D_{n1} = D_{n2}$

7. Write down the expression for capacitance between two parallel plates.

$C = \varepsilon A / d$

8. Give the expression for potential between two spherical shells

$V = 1/4\pi\varepsilon (Q_1/a - Q_2/b)$

9. Define electric dipole.

Electric dipole is nothing but two equal and opposite point charges separated by a finite distance.

10. Give significant physical difference between poisons and laplaces equations.

When the region contains charges poisons equation is used and when there is no charges laplaces equation is applied.

11. Define Potential gradient?

It is the maximum rate of change of potential w.r.t distance i.e. $|dv/dl|_{max}$

12. Describe what are the sources of electric field and magnetic field?

Stationary charges produce electric field that are constant in time, hence the term electrostatics. Moving charges produce magnetic fields hence the term magnetostatics.

13. How is electric energy stored in a capacitor?

In a capacitor, the work done in charging a capacitor is stored in the form of electric energy.

14. What meaning would you give to the capacitance of a single conductor?

A single conductor also possesses capacitance. It is a capacitor whose one plate is at infinity.

15. Why water has much greater dielectric constant than mica?

Water has a much greater dielectric constant than mica .because water has a permanent dipole moment, while mica does not have.

16. What is Lorentz force?

Lorentz force is the force experienced by the test charge .It is maximum if the direction of movement of charge is perpendicular to the orientation of field lines.

17. What are dielectrics?

Dielectrics are materials that may not conduct electricity through it but on applying electric field induced charges are produced on its faces .The valence electron in atoms of a dielectric are tightly bound to their nucleus.

18. What is a capacitor?

A capacitor is an electrical device composed of two conductors which are separated through a dielectric medium and which can store equal and opposite charges ,independent of whether other conductors in the system are charged or not.

19. What are the significant physical differences between Poisson 's and laplace 's equations.

Poisson's and Laplace's equations are useful for determining the electrostatic potential V in regions whose boundaries are known.

When the region of interest contains charges poissons equation can be used to find the potential. When the region is free from charge laplace equation is used to find the potential.

UNIT-III

1.State Biot –Savarts law.

It states that the magnetic flux density at any point due to current element is proportional to the current element and sine of the angle between the elemental length and inversely proportional to the square of the distance between them

$dB=\mu_0 Idl \sin\theta / 4\pi r^2$

2. State stokes theorem.

The line integral of a vector around a closed path is equal to the surface integral of the normal component of its curl over any surface bounded by the path

$\oint_L \bar{H}.dl = \int_S (\nabla \times \bar{H}) \cdot ds$

3. State the condition for the vector F to be solenoidal.

$\nabla \bullet F = 0$

4. State the condition for the vector F to be irrotational.

$\nabla \times F = 0$

5. Define current density.

Current density is defined as the current per unit area. $J= I/A$ Amp/m^2

6. Write the point form of continuity equation and explain its significance.

$\nabla \bullet J = -\partial \rho v / \partial t$

7. What is meant by displacement current?

Displacement current is nothing but the current flowing through capacitor. $J= D / t$

8. State point form of ohms law.

Point form of ohms law states that the field strength within a conductor is proportional to the current density. $J=\sigma E$

9. State amperes circuital law.

Magnetic field intensity around a closed path is equal to the current enclosed by the path.

$\oint_L \bar{H} \bullet d\bar{l} = I$

10. Define magnetic vector potential.

It is defined as that quantity whose curl gives the magnetic flux density.

$B=\nabla \times A$

$=\mu / 4\pi \; J/r \; dv$ web/m^2

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11. Write down the expression for magnetic field at the centre of the circular coil.

$H = I/2a$.

12. Give the relation between magnetic flux density and magnetic field intensity.

$B = \mu H$

13. Write down the magnetic boundary conditions.

i) The normal components of flux density B is continuous across the boundary.

ii) The tangential component of field intensity is continuous across the boundary.

14. Give the force on a current element.

$dF = BIdl\sin\theta$

15. Define magnetic moment.

Magnetic moment is defined as the maximum torque per magnetic induction of flux density. $m=IA$

16. State Gauss law for magnetic field.

The total magnetic flux passing through any closed surface is equal to zero. $B.ds = 0$

17. Define self inductance.

Self inductance is defined as the rate of total magnetic flux linkage to the current through the coil.

18. State Lenz law.

Lenz's law states that the induced emf in a circuit produces a current which opposes the change in magnetic flux producing it.

19. Define magnetic field strength.

The magnetic field strength (H) is a vector having the same direction as magnetic flux density.

$H = B/\mu$

20. Give the formula to find the force between two parallel current carrying conductors.

$F = \mu I\, I_1 / 2\pi R$

21. Give the expression for torque experienced by a current carrying loop situated in a magnetic field. $T = IAB\sin\theta$

22. What is torque on a solenoid?

$T = NIAB\sin\theta$

23. Write he expression for field intensity due to a toroid carrying a filamentary current I

$H = NI / 2\pi R$

24. What are equipotential surfaces?

An equipotential surface is a surface in which the potential energy at every point is of the same vale.

25. What is the expression for energy stored in a magnetic field?

$W = \frac{1}{2} LI^2$

26. What is energy density in magnetic field?

$W = \frac{1}{2} \mu H^2$

27. Distinguish between solenoid and toroid.

Solenoid is a cylindrically shaped coil consisting of a large number of closely spaced turns of insulated wire wound usually on a non magnetic frame.

If a long slender solenoid is bent into the form of a ring and there by closed on itself it becomes a toroid.

28. Define magnetic moment.

Magnetic moment is defined as the maximum torque on the loop per unit magnetic induction.

29. Define inductance.

The inductance of a conductor is defined as the ratio of the linking magnetic flux to the current producing the flux. $L = N\Phi / I$

30. What is main cause of eddy current?

The main cause of eddy current is that it produces ohmic power loss and causes local heating.

31. How can the eddy current losses be eliminated?

The eddy current losses can be eliminated by providing laminations. It can be proved that the total eddy current power loss decreases as the number of laminations increases.

32. What is the fundamental difference between static electric and magnetic field lines?

There is a fundamental difference between static electric and magnetic field lines .The tubes of electric flux originate and terminates on charges, whereas magnetic flux tubes are continuous.

UNIT-IV

1. State Maxwells fourth equation.

The net magnetic flux emerging through any closed surface is zero. $\oint \bar{B} \cdot \bar{ds} = 0$

2. State Maxwells Third equation

The total electric displacement through the surface enclosing a volume is equal to the total charge within the volume. $\oint \bar{D} \cdot \bar{ds} = Q = \int_v \rho_v dV$

3. What is the significance of displacement current?

The concept of displacement current was introduced to justify the production of magnetic field in empty space. It signifies that a changing electric field induces a magnetic field .In empty space the conduction current is zero and the magnetic fields are entirely due to displacement current.

4. Distinguish between conduction and displacement currents.

The current through a resistive element is termed as conduction current whereas the current through a capacitive element is termed as displacement current. $J_c = \sigma E$

5. Define a wave.

If a physical phenomenon that occurs at one place at a given time is reproduced at other places at later times, the time delay being proportional to the space separation from the first location then the group of phenomena constitutes a wave. Waves is nothing but it carries some information or energy.

6. Define faraday's law.

Emf in a closed path is proportional to the rate of change of magnetic flux enclosed by a closed path.

$$emf = -N\frac{d\phi}{dt}$$

7. Define Lenz's law

The direction of induced emf is such that it opposes the cause producing it i.e. changes in magnetic flux.

8. What are the conditions for different mediums?

i) Free space $\sigma = 0, \ \varepsilon = \varepsilon_0, \mu = \mu_0$

ii) Lossless (perfect dielectric) or Good dielectric

$\sigma = 0, \ \varepsilon = \varepsilon_0 \varepsilon_r, \mu = \mu_0 \mu_r$

(or) $\frac{\sigma}{\omega\varepsilon} \ll 1$

iii) Lossy dielectric (Conducting medium)

$\sigma \neq 0, \ \varepsilon = \varepsilon_0 \varepsilon_r, \ \mu = \mu_0 \mu_r$

iv) Good conductor (Perfect Conductor)

$\sigma = \infty, \ \varepsilon = \varepsilon_0 \varepsilon_r, \ \mu = \mu_0 \mu_r$ (or) $\frac{\sigma}{\omega\varepsilon} \gg 1$

9. Define phase velocity.

The velocity with which constant phase point travels is called phase velocity.

$$\boxed{V_p = \frac{\omega}{\beta}}$$

10. Group Velocity

The velocity with which change in constant phase points travels is called Group velocity

$$V_g = \frac{\Delta \omega}{\Delta \beta} = \frac{2\cancel{\pi}\Delta f}{2\cancel{\pi}/\Delta\lambda} = \Delta\lambda \cdot \Delta f.$$

11. Relationship between Group velocity and phase velou

$$\boxed{V_p \times V_g = c^2}$$ C = velocity of light.

12. What is polarization & what are the different type are there?

Orientation of EM wave at a given instant of time in the space.

1) Linear polaization 2) Elliptical 3) circulas

13. write down Relationship between $\bar{E}$ & $\bar{H}$

$$\boxed{\frac{E}{H} = \eta}$$ (or) $\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t} = -\mu \frac{\partial \bar{H}}{\partial t}$

14.

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UNIT-V

1. Mention the properties of uniform plane wave.

i) At every point in space, the electric field E and magnetic field H are perpendicular to each other.

ii) The fields vary harmonically with time and at the same frequency everywhere in space.

2. Write down the wave equation for E and H in free space.

$\nabla^2 H - \mu_0 \varepsilon_0 \partial^2 H / \partial t^2 = 0$ $\quad \nabla^2 \bar{H} = \mu_0 \varepsilon_0 \frac{\partial^2 H}{\partial t^2}$; $\nabla^2 \bar{E} = \mu_0 \varepsilon_0 \frac{\partial^2 E}{\partial t^2}$

3. Define intrinsic impedance or characteristic impedance.

It is the ratio of electric field to magnetic field.or It is the ratio of square root of permeability to permittivity of medium. $\eta = \sqrt{\mu/\varepsilon}$

4. Give the characteristic impedance of free space.

377ohms i.e. $\eta = \sqrt{\mu_0/\varepsilon_0}$

$\mu_0 = 4\pi \times 10^{-7}$ H/m

$\varepsilon_0 = 8.854 \times 10^{-12}$ F/m

5. Define propagation constant.

Propagation constant is a complex number

$$\gamma = \alpha + j\beta$$

where α is attenuation constant

β is phase constant

$\gamma = j\omega\mu\,(\sigma + j\omega\varepsilon)$

6. Define skin depth (δ) (or) Depth of penetration.

It is defined as that depth in which the wave has been attenuated to 1/e or approximately 37% of its original value. $= 1/\alpha = 2 / j\omega\sigma$

7. Define Poynting vector.

The pointing vector is defined as rate of flow of energy of a wave as it propagates.

$\bar{P} = \bar{E} \times \bar{H}$

8. Define pointing vector.

The vector product of electric field intensity and magnetic field intensity at a point is a measure of the rate of energy flow per unit area at that point.

9. State Poyntings Theorem.

The net power flowing out of a given volume is equal to the time rate of decrease of the the energy stored within the volume- conduction losses.

10. Explain the steps in finite element method.

i) Discretisation of the solution region into elements.

ii) Generation of equations for fields at each element

iii) Assembly of all elements

iv) Solution of the resulting system

11. Define loss tangent.

Loss tangent is the ratio of the magnitude of conduction current density to displacement cuurrent density of the medium.

$\boxed{\tan\delta = \sigma / \omega\varepsilon}$

12. Define reflection ~~and transmission~~ coefficients.

Reflection coefficient is defined as the ratio of the magnitude of the reflected field to that of the incident field. $K = \frac{E_r}{E_i}$ or $K = \frac{H_r}{H_i}$

13. Define transmission coefficients.

Transmission coefficient is defined as the ratio of the magnitude of the transmitted field to that of incident field. $T = \frac{E_t}{E_i}$ or $T = H_t / H_i$

14. What will happen when the wave is incident obliquely over dielectric –dielectric boundary?

When a plane wave is incident obliquely on the surface of a perfect dielectric part of the energy is transmitted and part of it is reflected .But in this case the transmitted wave will be refracted, that is the direction of propagation is altered.

15. What are uniform plane waves?

Electromagnetic waves which consist of electric and magnetic fields that are perpendicular to each other and to the direction of propagation and are uniform in plane perpendicular to the direction of propagation are known as uniform plane waves.

16. Write short notes on imperfect dielectrics.

A material is classified as imperfect dielectrics for $\sigma \ll \omega\varepsilon$ that is conduction current density is small in magnitude compared to the displacement current density.

17. What is the significant feature of wave propagation in an imperfect dielectric?

The only significant feature of wave propagation in an imperfect dielectric compared to that in a perfect dielectric is the attenuation undergone by the wave.

18. What is the major drawback of finite difference method?

The major drawback of finite difference method is its inability to handle curved boundaries accurately.

19. What is method of images?

The replacement of the actual problem with boundaries by an enlarged region or with image charges but no boundaries is called the method of images.

20. When is method of images used?

Method of images is used in solving problems of one or more point charges in the presence of boundary surfaces.

21 State the meaning of Brewster angle?

This is the angle of incidence for which the angle of reflection is zero.

i.e. $\frac{E_r}{E_i} = 0$.

$$\boxed{\tan\theta_i = \sqrt{\varepsilon_2/\varepsilon_1}}$$

22. Give snell's law.

$$\frac{\sin\theta_i}{\sin\theta_t} = \sqrt{\frac{\varepsilon_2}{\varepsilon_1}}$$

23. Define law of reflection.

It is defined as that angle of incidence is equal to the angle of reflection.

$$\boxed{\theta_i = \theta_r}$$

UNIT-IV

... electrical ...

... transmission lines

... information

... energy from source to load.

... transmission lines

... waveguides

... power distribution

... communications (at high frequencies).

... Optical fiber

... waveguide.

(1)

UNIT-IV

Transmission Lines:

Objectives:

1. Types of Transmission lines
2. Transmission line parameters, Line equations.
3. Characteristic impedance, propagation constant, α, β.
4. lossless and distortionless lines.

- Till now we have studied the propagation of waves through free space or materials.
- The conductors or waveguides used for transmitting electric or EM waves over long distances b/w the transmitter and the receiver are called transmission lines.
- The electrical lines which are used to transmit the electrical waves along them are called Transmission lines.
- There are other means of transmitting power or information by using guided structures.
- guided structures, guide the energy from source to load.

Examples: for guided structures:

- Transmission lines
- waveguides.

- Transmission lines are commonly used in power distribution (at low frequencies) and in communications (at high frequencies).

Examples:
- Open wires lines,
- coaxial cables,
- Optical Fiber waveguide.

- Generally, a transmission line is a distributed parameter n/w
- Generally, The transmission line parameters such as R, L, G are distributed uniformly along the transmission line.
- The distributed elements are measured per unit length.

Types of Transmission lines:

① open wire:

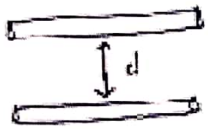


fig: open wire transmission line.

- Parallel conducting wires separated by a distance in free space and mounted on towers or posts.

Ex: Telephone lines
Telegraphy lines
power lines.

These are suited for

- Low frequency (LF) short and medium distance Transmission
- upto 100MHz as antenna feeders.

② coaxial cable:

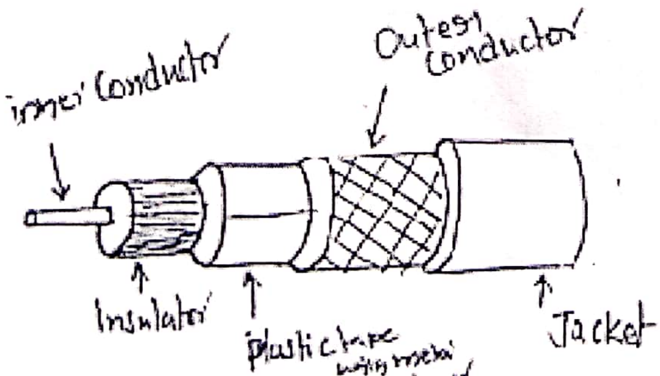


fig: coaxial cable.

- Two conductors are placed coaxially and filled with dielectric material.

Example: TV, and Telephone cables

frequency: 1GHz

③ **Waveguides:**

- Hollow/ dielectric filled conductors used to transmit EM waves at microwave frequency
- Energy is transmitted through reflections from the inner surface of the conductor.

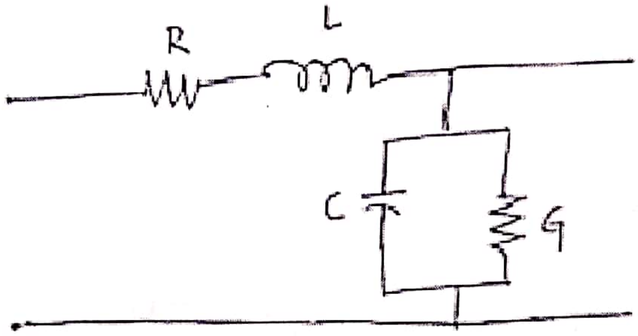
Ex: Communication network.

④ **Optical fibers:**

Transmission line parameters:

1) Primary constants.

- The equivalent electric circuit of a transmission line consists of series 'R', series 'L', shunt 'C' and shunt conductance G, along the length.



Resistance (R):

- loop resistance per unit length of the line (sum of resistance of both the wires).
- units: Ω/m.

Inductance (L):

- loop inductance per unit length of the line.

units: H/m.

Capacitance (C):

- shunt capacitance between two wires per unit line length.
- F/m.

Conductance (G):

- shunt conductance between two wires per unit line length.

-units: ℧/m, or S/m.

- The series impedance Z and shunt admittance Y of the line per unit length can be expressed as

$$Z = R + j\omega L$$

$$Y = G + j\omega C$$

- Primary constants (R, L, G, C) are independent of operating frequency

2) Secondary constants

i) Characteristic Impedance (Z_0)

$$\boxed{Z_0 = \sqrt{\frac{R+j\omega L}{G+j\omega C}}} \quad \text{(OR)} \quad \boxed{Z_0 = \sqrt{\frac{Z}{Y}}}$$

ii) Propagation constant (γ)

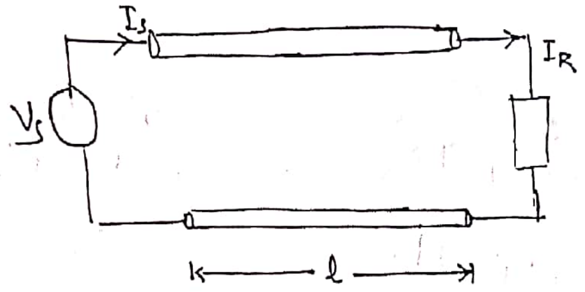
$$\boxed{\gamma = \sqrt{(R+j\omega L)(G+j\omega C)}} \quad \text{(OR)} \quad \boxed{\gamma = \sqrt{ZY}}$$

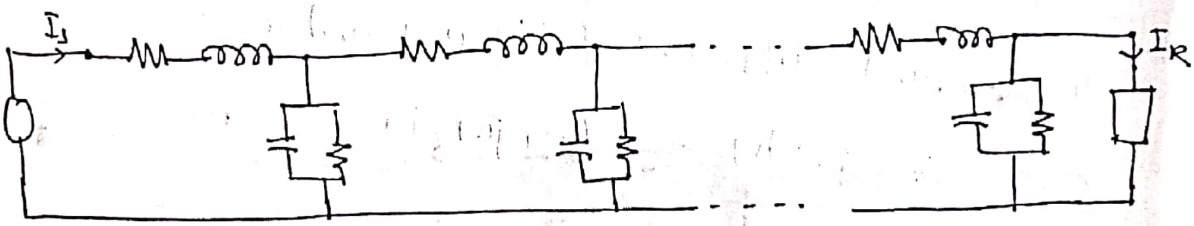
Skin effect:

- The electrical properties of a transmission line are determined by the primary constants of the line
- But, at a radio frequency L and R are controlled by the skin effect.
- When an alternating current flows in a conductor, the alternating magnetic flux within the conductor induces an emf.
- This emf causes current density to decrease in the interior of the wire and to increase towards the outer surface.
  This is known as skin effect.
- When the cross-sectional dimension of the conductor is much larger than effective thickness of the
  the current density varies exponentially inward from the surface.
- The distance at which the current density decreases to 1/e of its surface value is called skin depth (δ)

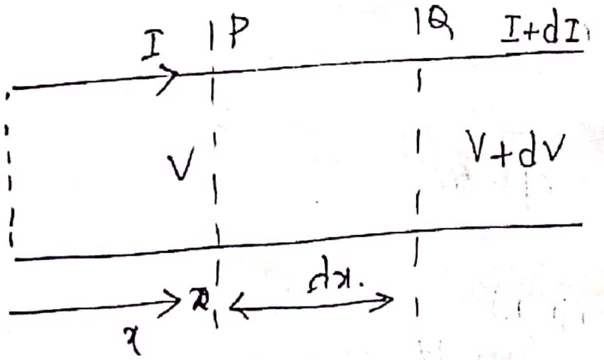
Transmission line equations:

Consider a transmission line with two parallel conductors.





- let R, L, C, and G be the primary constants.
- assuming these values donot vary with frequency.
- Consider a point 'P' on the line at adistance 'x' from the source.



- let Q be another point at a small distance dx from point P.
- let 'V' and 'I' be Voltage and current at point P.

as the voltages and currents are uniformly distributed along the line, at Q.

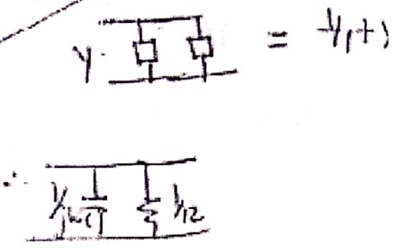
$$V = V + dV$$

$$I = I + dI.$$

for small length dx

The series impedance $= (R + j\omega L)\,dx$

Shunt admittance $= (G + j\omega C)\,dx$



$y = \frac{1}{\frac{1}{j\omega C}} + \frac{1}{R}$

$y = G + j\omega C$.

∴ The potential difference between P and Q is

$$V - (V + dV) = I\,(R + j\omega L)\,dx. \quad \text{—①}$$

Current difference between P and Q is.

$$I - (I + dI) = V\,(G + j\omega C)\,dx \quad \text{—②}$$

from eq① $\quad -\frac{dV}{dx} = I\,(R + j\omega L)\,. \quad \text{—③}$

from eq② $\quad -\frac{dI}{dx} = (G + j\omega C)\,V. \quad \text{—④}$

③ taking derivative of eq③

$$-\frac{d^2V}{dx^2} = (R + j\omega L)\,\frac{dI}{dx} \quad \text{—⑤}$$

Ily taking derivative of eq④

$$-\frac{d^2I}{dx^2} = (G + j\omega C)\,\frac{dV}{dx} \quad \text{—⑥}$$

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now $$-\frac{d^2V}{dx^2} = (R+j\omega L)[-(G+j\omega C)V]$$

$$\cancel{-}\frac{d^2V}{dx^2} = \cancel{-}(R+j\omega L)(G+j\omega C)V$$

$$\therefore \frac{d^2V}{dx^2} = (R+j\omega L)(G+j\omega C)V \quad \text{--- (7)}$$

Illy from eq (6) $$-\frac{d^2I}{dx^2} = (G+j\omega C)[-(R+j\omega L)I]$$

$$\cancel{-}\frac{d^2I}{dx^2} = \cancel{-}(G+j\omega C)(R+j\omega L)I$$

$$\therefore \frac{d^2I}{dx^2} = (R+j\omega L)(G+j\omega C)I. \quad \text{--- (8)}$$

let $(R+j\omega L)(G+j\omega C) = \gamma^2$

$$\gamma = \alpha + j\beta$$

$$= \sqrt{(R+j\omega L)(G+j\omega C)}$$

$$\gamma = \sqrt{ZY} \quad \text{--- (9)}$$

$$\therefore \frac{d^2V}{dx^2} = \gamma^2 V \quad \text{--- (10)}$$

and $$\frac{d^2I}{dx^2} = \gamma I. \quad \text{--- (11)}$$

eq (10) and (11) are differential equations of the transmission lines.

The standard solutions for the equations (10) and (11) are.

$$V = ae^{\gamma x} + be^{-\gamma x} \quad — (12)$$

and $$I = ce^{\gamma x} + de^{-\gamma x} \quad — (13)$$

Where a, b, c, d are constants.

Substituting eq (12) into eq (8).

$$-\frac{d}{dx}\left(ae^{\gamma x} + be^{-\gamma x}\right) = (R+j\omega L)\,I.$$

$$\cancel{\gamma(ae^{\gamma x}+}$$

$$-\gamma\left(ae^{\gamma x} - be^{-\gamma x}\right) = (R+j\omega L)I$$

or $$I = \frac{\left(be^{-\gamma x} - ae^{\gamma x}\right)\sqrt{(R+j\omega L)(G+j\omega C)}}{(R+j\omega L)}$$

$$\Rightarrow I = \frac{\left(be^{-\gamma x} - ae^{\gamma x}\right)\sqrt{(R+j\omega L)(G+j\omega C)}}{\sqrt{(R+j\omega L)^2}}$$

$$I = \left(be^{-\gamma x} - ae^{\gamma x}\right)\sqrt{\frac{G+j\omega C}{(R+j\omega L)}}$$

let another constant, Z_0, The characteristic impedance

$$Z_0 = \sqrt{\frac{(R+j\omega L)}{(G+j\omega C)}}$$

$$\therefore I = \frac{1}{Z_0}\left(be^{-\gamma x} - ae^{\gamma x}\right) \quad — (14)$$

eq 12 and 14 in terms of a and b

~~~~ using hyperbolic functions:

$$e^{\gamma x} = \cosh\gamma x + \sinh\gamma x \quad \text{— i)}$$

and $$e^{-\gamma x} = \cosh\gamma x - \sinh\gamma x \quad \text{— ii)}$$

Substituting ~~~~ i and ii into equation 12 and 14.

$$\therefore V = a\cosh\gamma x + a\sinh\gamma x + b\cosh\gamma x - b\sinh\gamma x$$

$$= \cosh\gamma x(a+b) + \sinh\gamma x(a-b)$$

$$= A\cosh\gamma x + B\sinh\gamma x \quad \text{— 15}$$

$$\therefore I = \frac{1}{Z_0}\left[b\cosh\gamma x - b\sinh\gamma x - a\cosh\gamma x - a\sinh\gamma x\right]$$

$$= \frac{1}{Z_0}\left[\cosh\gamma x(b-a) - (a+b)\sinh\gamma x\right]$$

$$= -\frac{1}{Z_0}\left[A\sinh\gamma x + B\cosh\gamma x\right] \quad \text{— 16}$$

here $A = a+b$
$B = a-b$.

at source:

let $V = V_s$, $I = I_s$
$x = 0$,
~~~~

Then eq (15)

$$V_s = A\cosh\gamma(0) + B\sinh\gamma(0)$$

$$V_s = A.$$

$$\therefore \boxed{A = V_s}$$

The from eq (16)

$$I_s = -\frac{1}{Z_0} B$$

$$\therefore \boxed{B = -Z_0 I_s}$$

now substituting A and B into eq (15) and (16) we obtain

$$V = V_s \cosh\gamma x - I_s Z_0 \sinh\gamma x \quad \text{--- (17)}$$

$$I = I_s \cosh\gamma x - \frac{V_s}{Z_0} \sinh\gamma x \quad \text{--- (18)}$$

→ Called as Transmission line equations.

eq (17) and eq (18) give voltage and current at a point of distance 'x'. from sending end interms of source voltage and current.

*** Note:** γ and Z_0 are called Secondary Constants of a Transmission line

at Receiving end:

· When the conditions at the receiving end are known,

$x = l,$

$V = V_R$

$I = I_R$

from eq (15) $V = V_R = A\cosh\gamma l + B\sinh\gamma l$ — (19)

from eq (16) $I = I_R = -\frac{1}{Z_0}\left[A\sinh\gamma l + B\cosh\gamma l\right]$ — (20)

now from eq (19)

$$V_R - A\cosh\gamma l = B\sinh\gamma l \quad \text{— (21)}$$

and from eq (20)

$$-Z_0 I_R - A\sinh\gamma l = B\cosh\gamma l \quad \text{— (22)}$$

now dividing equation (21) by

$$\frac{V_R - A\cosh\gamma l}{-I_R Z_0 - A\sinh\gamma l} = \frac{\cancel{B}\sinh\gamma l}{\cancel{B}\cosh\gamma l}$$

$$V_R\cosh\gamma l - A\cosh^2\gamma l = -I_R Z_0\sinh\gamma l - A\sinh^2\gamma l \quad \text{— (23)}$$

$\therefore$ $\cancel{V_R\cosh\gamma l} - \cancel{I_R Z_0}$

$\therefore\ V_R \cosh\gamma l - A\cosh^2\gamma l = -I_R Z_0 \sinh\gamma l - A\sinh^2\gamma l$

we know $\cosh^2 x - \sinh^2 x = 1$

now

$$V_R \cosh\gamma l + I_R Z_0 \sinh\gamma l = A\cosh^2\gamma l - A\sinh^2\gamma l$$

$$V_R \cosh\gamma l + I_R Z_0 \sinh\gamma l = A(\cosh^2\gamma l - \sinh^2\gamma l)$$

$$V_R \cosh\gamma l + I_R Z_0 \sinh\gamma l = A \quad \text{—— (23)}$$

from (15) $\quad V_R - B\sinh\gamma l = A\cosh\gamma l \quad$ — (24)

from (16) $\quad -I_R Z_0 - B\cosh\gamma l = A\sinh\gamma l \quad$ — (25)

dividing eq (24) by (25)

$$\frac{V_R - B\sinh\gamma l}{-I_R Z_0 - B\cosh\gamma l} = \frac{A\cosh\gamma l}{A\sinh\gamma l}$$

$$V_R \sinh\gamma l - B\sinh^2\gamma l = -I_R Z_0 \cosh\gamma l - B\cosh^2\gamma l$$

$$B = -\left[V_R \sinh\gamma l + I_R Z_0 \cosh\gamma l\right] \quad \text{—— (26)}$$

Substituting A and B in (15) and (16), we get.

$$V = \left[V_R \cosh\gamma l + I_R Z_0 \sinh\gamma l\right]\cosh\gamma x - (V_R \sinh\gamma l + I_R Z_0 \cosh\gamma l)\sinh\gamma x$$

$$= V_R \cosh\gamma l \cosh\gamma x + I_R Z_0 \sinh\gamma l \cosh\gamma x - V_R \sinh\gamma l \sinh\gamma x - I_R Z_0 \cosh\gamma l \sinh\gamma x.$$

$$= V_R \cosh\gamma(l-x) + I_R Z_0 \sinh\gamma(l-x) \quad \text{—— (27)}$$

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lly.

$$I = \frac{V_R}{Z_0}\sinh\gamma(l-x) + I_R\cosh\gamma(l-x) \quad \text{--- (28)}$$

Equations (27) and (28) are used to find voltage and currents at any point. If V_R and I_R are known.

Infinite line:

when $l = \infty$, the line is Infinite line.

from equation (12) and (13)

we know

$$V = ae^{\gamma x} + be^{-\gamma x}$$

$$I = ce^{\gamma x} + de^{-\gamma x}$$

$\left.\begin{matrix} ae^{\gamma x}, \\ ce^{\gamma x} \end{matrix}\right\} \rightarrow$ reverse wave

$\left.\begin{matrix} be^{-\gamma x} \\ de^{-\gamma x} \end{matrix}\right\} \rightarrow$ forward wave

at $x = 0$, $V = V_s$

from (12) $\Rightarrow V_s = a.1 + b.1$

$= a + b.$

at $x = 0$

$I = I_s,$

from (13)

$I_s = c + d.$

at $x = \infty$,

$V = 0$

$\therefore \boxed{a = 0}$

$\Rightarrow \boxed{b = V_s}$

$$\therefore \boxed{V = V_s e^{-\gamma x}} \quad \text{—(29)}$$

at $x = \infty$

$I_s = 0$

$\Rightarrow I = 0.$

$0 = c(\infty) + d(0)$

$\boxed{c = 0}$

$\boxed{d = I_s}$

$$\boxed{I = I_s e^{-\gamma x}} \quad \text{—(30)}$$

Note: $\therefore$ for infinite line, there is no reflected wave as $V_R = 0$, $I_R = 0$.

Telephone cables

The ordinary telephone cable is an underground cable which consists of conductors (wires) insulated with paper and twisted in pair.

The inductance (L) and conductance (G) of such a cable is negligibly small at the audio frequency range, and hence can be neglected.

For this cables, the series impedance and shunt admittance are given by

$$Z = R + j\omega L \quad (\because R \gg \omega L)$$

$$Y = G + j\omega C \quad (\because G \ll \omega C)$$

Propagation constant $= \gamma = \alpha + j\beta = \sqrt{ZY}$

$$= \sqrt{R \cdot j\omega C}$$

$$= \sqrt{\omega RC}\,\sqrt{j}$$

$$= \sqrt{\omega RC}\,\angle 45^\circ.$$

$$= \sqrt{\omega RC}\left(\cos\pi/4 + j\sin\pi/4\right)$$

$$= \sqrt{\omega RC}\left[\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}\right]$$

$j = e^{j\frac{\pi}{2}} = \cos\frac{\pi}{2} + j\sin\frac{\pi}{2}$

$\sqrt{j} = (j)^{1/2} = \left(e^{j\frac{\pi}{2}}\right)^{1/2}$

$= \cos\pi/4 + j\sin\frac{\pi}{4}$

$= 1/\sqrt{2} + j\,1/\sqrt{2}$

$$\alpha + j\beta = \sqrt{\frac{\omega RC}{2}}\,[1 + j]$$

$$\boxed{\alpha = \beta = \sqrt{\frac{\omega RC}{2}}}$$

In phasor form $\alpha = \sqrt{\omega CR}\,\angle 45^\circ$.

Characteristic Impedance $= Z_0 = \sqrt{\dfrac{Z}{Y}} = \sqrt{\dfrac{R}{j\omega C}}$

$$= \sqrt{\frac{R}{\omega C}(-j)} = \sqrt{\frac{R}{\omega C}}\,\angle -45^\circ$$

Velocity of propagation $= V_p = \dfrac{\omega}{\beta} = \dfrac{\omega}{\sqrt{\omega RC/2}}$

$$\boxed{V_p = \sqrt{\frac{2\omega}{RC}}}$$

α and V_p are functions of ω $\therefore$ at high frequencies, attenuation is more and also wave travels faster. It results phase & freq. distortion

Distortion in Transmission Line

~~EHD~~ A transmission line is said to be distorted when the received signal is not the exact replica of the transmitted signal.

Two types

1) Frequency distortion:

In this various frequency components of transmitting signal represents different attenuations.

$$\alpha = \sqrt{\frac{1}{2}\left[(RG - \omega^2 LC) + \sqrt{[R^2 + (\omega L)^2][G^2 + (\omega C)^2]}\right]}$$

Frequency distortion raises serious problem in audio signals but not much important for video signals. Hence in high frequency radio broad-casting such frequency distortion is eliminated by using equalizers.

2) Delay distortion: (or phase distortion)

$$\beta = \sqrt{\frac{1}{2}\left[(\omega^2 LC - RG) + \sqrt{(R^2 + (\omega L)^2)(G^2 + (\omega C)^2}\right]}$$

$$v_p = \frac{\omega}{\beta}.$$

Velocity of propagation of wave varies with frequency. Hence all waves cannot reach at receiver end in a same time. Thus o/p wave at the receiver end will not be exact replica of the i/p.

It is not much important for the audio signals due to the characteristic of human ears. But such a distortion is very serious in case of video and picture transmission.

The remedy for this is to use coaxial cables for the picture transmission of television and video signals.

Distortionless Transmission:

The transmission line in which the signal at the receiver end is an exact replica of signal at the sending end is called distortion less line or distortion free line.

"It is defined as a transmission line for which secondary line constants are independent of frequency".

Condition for Distortionless Transmission

$$\alpha = \sqrt{\frac{1}{2}\left\{\sqrt{(R^2+\omega^2L^2)(G^2+\omega^2C^2)} + (RG-\omega^2LC)\right\}}$$

The value of L for which attenuation is reduced to minimum is obtained by differentiating w.r.t. L and equating to zero.

$$\therefore \frac{d\alpha}{dL} = \frac{1}{2}\left[\frac{1}{2}\left\{\sqrt{(R^2+\omega^2L^2)(G^2+\omega^2C^2)} + (RG-\omega^2LC)\right\}\right]^{\frac{1}{2}-1} \cdot$$

$$\frac{1}{2}\left\{\frac{1}{2}\left[(R^2+\omega^2L^2)(G^2+\omega^2C^2)\right]^{-1/2} \cdot \omega^2 2L(G^2+\omega^2C^2) - \omega^2C\right\}$$

$$= \frac{1}{2}\,\frac{\frac{1}{2}\left[\frac{\omega^2L(G^2+\omega^2C^2)}{\sqrt{(R^2+\omega^2L^2)(G^2+\omega^2C^2)}} - \omega^2C\right]}{\sqrt{\frac{1}{2}\sqrt{(R^2+\omega^2L^2)(G^2+\omega^2C^2)} + (RG-\omega^2LC)}} = 0.$$

$$\frac{\omega^2L(G^2+\omega^2C^2)}{\sqrt{(R^2+\omega^2L^2)(G^2+\omega^2C^2)}} - \omega^2C = 0$$

$$\frac{\cancel{\omega^2}L(G^2+\cancel{\omega^2C^2})}{\sqrt{(R^2+\omega^2L^2)(G^2+\cancel{\omega^2C^2})}} = \cancel{\omega^2}C$$

$$L\sqrt{G^2+\omega^2C^2} = C\sqrt{R^2+\omega^2L^2}$$

$$L^2(G^2+\cancel{\omega^2C^2}) = C^2(R^2+\cancel{\omega^2L^2})$$

$$L^2G^2 = R^2C^2 \Rightarrow \boxed{LG = RC} \text{ (or) } \boxed{\frac{R}{L} = \frac{G}{C}}$$

i) Attenuation Constant (α) and phase Constant (β) for distortionless Transmission

$$\alpha = \left[\frac{1}{2}\left\{(RG-\omega^2LC)+\sqrt{(R^2+\omega^2L^2)(G^2+\omega^2C^2)}\right\}\right]^{1/2}$$

Sub $LG = RC \Rightarrow L = \frac{RC}{G}$

$$\alpha = \left[\frac{1}{2}\left\{\left(RG-\omega^2\frac{RC^2}{G}\right)+\sqrt{\left(R^2+\omega^2\frac{R^2C^2}{G^2}\right)(G^2+\omega^2C^2)}\right\}\right]^{1/2}$$

$$= \left[\frac{1}{2}\left\{\left(\frac{RG^2-\omega^2RC^2}{G}\right)+\sqrt{\left(\frac{R^2G^2+\omega^2R^2C^2}{G^2}\right)(G^2+\omega^2C^2)}\right\}\right]^{1/2}$$

$$= \left[\frac{1}{2}\left\{\frac{R}{G}(G^2-\omega^2C^2)+\sqrt{\frac{R^2}{G^2}(G^2+\omega^2C^2)(G^2+\omega^2C^2)}\right\}\right]^{1/2}$$

$$= \left[\frac{1}{2}\left\{\frac{R}{G}(G^2-\omega^2C^2)+\frac{R}{G}(G^2+\omega^2C^2)\right\}\right]^{1/2}$$

$$= \left[\frac{1}{2}\frac{R}{G}\left\{G^2-\cancel{\omega^2C^2}+G^2+\cancel{\omega^2C^2}\right\}\right]^{1/2}$$

$$\alpha = \left[\frac{1}{\cancel{2}}\frac{R}{\cancel{G}}\cancel{2}G^{\cancel{2}}\right]^{1/2} = \sqrt{RG} \Rightarrow \boxed{\alpha = \sqrt{RG}}$$

Since α is independent of frequency.

ii) Phase constant (β)

$$\beta = \left[\frac{1}{2}\left\{(\omega^2LC-RG)+\sqrt{(R^2+\omega^2L^2)(G^2+\omega^2C^2)}\right\}\right]^{1/2}$$

$$= \left[\frac{1}{2}\left\{\left(\omega^2\frac{RC^2}{G}-RG\right)+\sqrt{\left(R^2+\omega^2\frac{R^2C^2}{G^2}\right)(G^2+\omega^2C^2)}\right\}\right]^{1/2}$$

$$= \left[\frac{1}{2}\left\{\left(\frac{\omega^2RC^2-RG^2}{G}\right)+\sqrt{\left(\frac{R^2G^2+\omega^2R^2C^2}{G^2}\right)(G^2+\omega^2C^2)}\right\}\right]^{1/2}$$

$$= \left[\frac{1}{2}\left\{\frac{R}{G}(\omega^2C^2-G^2)+\sqrt{\frac{R^2}{G^2}(G^2+\omega^2C^2)(G^2+\omega^2C^2)}\right\}\right]^{1/2}$$

$$= \left[\frac{1}{2}\frac{R}{G}\left\{\omega^2C^2-\cancel{G^2}+\cancel{G^2}+\omega^2C^2\right\}\right]^{1/2} = \left[\frac{1}{\cancel{2}}\frac{R}{G}\cancel{2}\omega^2C^2\right]^{1/2}$$

$$= \omega\left(\frac{RC^2}{G}\right)^{1/2} = \omega(LC)^{1/2}$$

$$\therefore \boxed{\beta = \omega\sqrt{LC}}$$

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iii) Propagation constant (γ)

$$\gamma = \alpha + j\beta$$

$$\boxed{\gamma = \sqrt{RG} + j\omega\sqrt{LC}}$$

iv) Velocity of propagation (v_p)

$$v_p = \frac{\omega}{\beta}$$

$$= \frac{\cancel{\omega}}{\cancel{\omega}\sqrt{LC}} \Rightarrow \boxed{v_p = \frac{1}{\sqrt{LC}}}$$

v) Characteristic impedance (Z_0)

$$Z_0 = \sqrt{\frac{R+j\omega L}{G+j\omega C}}$$

$$= \sqrt{\frac{R+j\omega \frac{RC}{G}}{G+j\omega C}} = \sqrt{\frac{\frac{R}{G}\left[\cancel{G+j\omega C}\right]}{\left[\cancel{G+j\omega C}\right]}}$$

$$\therefore \boxed{Z_0 = \sqrt{\frac{R}{G}}} \text{ or } \boxed{Z_0 = \sqrt{\frac{L}{C}}}$$

Practical Methods to obtain Distortionless Transmission

In an actual transmission line

$$\frac{R}{L} \gg \frac{G}{C}$$

Hence to make a line distortionless either $\frac{R}{L}$ is to be decreased or $\frac{G}{C}$ is to be increased.

Three methods are there to achieve this.

1) Reducing R, $\therefore \frac{R}{L}$ is decreases.

2) Increasing L, $\therefore$ the value of $\frac{R}{L}$ decreases.

3) Decreasing C, $\therefore \frac{G}{C}$ is increased.

Loading of Lines

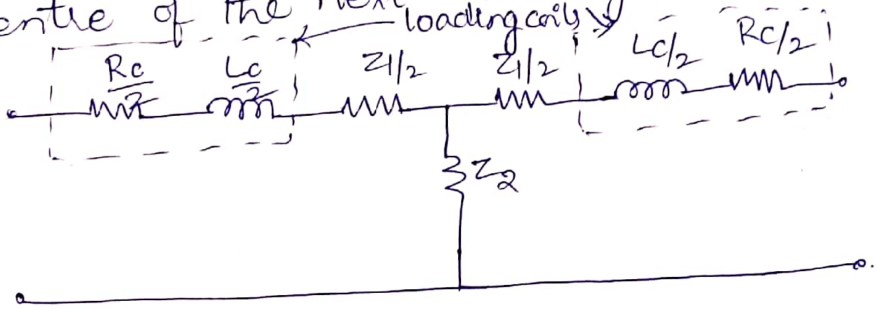
- The process of increasing the inductance (L) of a transmission line artificially inorder to obtain a distortionless transmission condition is called loading of lines. And such a line is called loaded line.

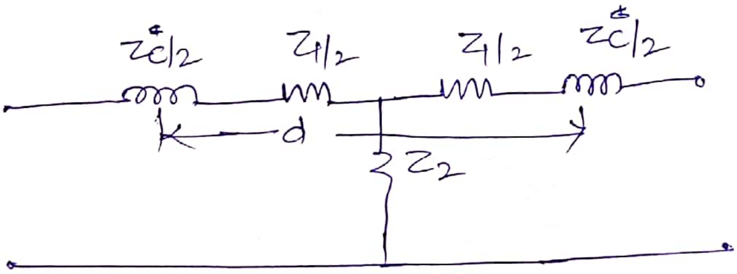
- There are various ways by which line loading can be increased such as
  - i) By installation of coaxial cable in series with the line such that line inductance increases.
  - ii) By adding induction in lumped form at specific locations in a particular form.

- There are two methods of loading a line
  1) Continuous loading (or Heavy side loading)
  2) Lumped loading (or coil loading)

Campbell's Formula:

Campbell's formula provides an analysis for the Performance of a loaded line at uniform intervals. The analysis can be done by considering a symmetrical section of line from the centre of one loading coil to the centre of the next loading coil.





The spacing between two loading coils is 'd' Kms.

The propagation constant for symmetric-T section

$$\left.\begin{array}{l} \sinh(d\gamma) = \dfrac{Z_0}{Z_2} \\ \text{(or)}\ \cosh(d\gamma) = 1+\dfrac{Z_1}{2Z_2} \end{array}\right\} \text{— ①}$$

when loaded section is added $\cosh(d\gamma') = 1+\frac{Z_1'}{2Z_2}$ — ②

$$\frac{Z_1'}{2} = \frac{Z_c}{2}+\frac{Z_1}{2} \quad \text{— ③}$$

From ①

$$\sinh(d\gamma) = \frac{Z_0}{Z_2}$$

$$Z_2 = \frac{Z_0}{\sinh(d\gamma)} \quad \text{— ④}$$

From ① $\frac{Z_1}{2Z_2} = \cosh(d\gamma) - 1$

$$\frac{Z_1}{2} = Z_2\left[\cosh(\gamma l) - 1\right]$$

$$\frac{Z_1}{2} = \frac{Z_0}{\sinh(\gamma l)}\left[\cosh(\gamma l) - 1\right] \quad — ⑤ \quad (\because \text{from } ④)$$

Sub ④ in ③

$$\frac{Z_1'}{2} = \frac{Z_c}{2} + \frac{Z_0}{\sinh(\gamma l)}\left[\cosh(\gamma l) - 1\right]$$

Sub this in ②

$$\cosh(\gamma l') = 1 + \frac{Z_c}{2Z_2} + \frac{\frac{Z_0}{\sinh(\gamma l)}\left[\cosh(\gamma l)-1\right]}{Z_2}$$

$$= 1 + \frac{Z_c}{2Z_2} + \frac{\frac{Z_0}{\sinh(\gamma l)}\left[\cosh(\gamma l)-1\right]}{\frac{Z_0}{\sinh(\gamma l)}}$$

$$= \cancel{1} + \frac{Z_c}{2Z_2} + \cosh(\gamma l) - \cancel{1}$$

$$= \cosh(\gamma l) + \frac{Z_c}{2\left(\frac{Z_0}{\sinh(\gamma l)}\right)}$$

$$\boxed{\cosh(\gamma l') = \cosh(\gamma l) + \frac{Z_c}{2Z_0}\sinh(\gamma l).}$$

Lossless Transmission Line

$R = 0 = G.$

i) propagation constant $(\gamma) = \sqrt{ZY}$

$$= \sqrt{(R+j\omega L)(G+j\omega C)}$$

$$= \sqrt{j\omega L \cdot j\omega C} = j\omega\sqrt{LC}.$$

ii) Attenuation constant (α)

$$\gamma = \alpha + j\beta = j\omega\sqrt{LC}$$

$$\boxed{\alpha = 0}$$

iii) phase shift constant (β)

$$\boxed{\beta = \omega\sqrt{LC}}$$

iv) phase velocity $= v_p = \frac{\omega}{\beta} = \frac{\cancel{\omega}}{\cancel{\omega}\sqrt{LC}}$

$$\boxed{v_p = \frac{1}{\sqrt{LC}}}$$

v) Char Impedance

$$Z_0 = \sqrt{\frac{R+j\omega L}{G+j\omega C}} = \sqrt{\frac{L}{C}}$$

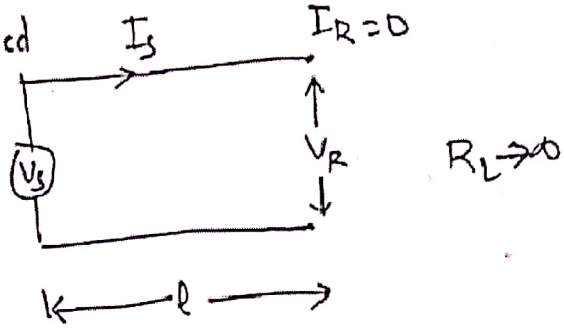
vi) Wavelength (λ)

$$\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{\omega\sqrt{LC}}$$

UNIT-V

Open Circuited and Short Circuited Impedances of a line:

Open Circuited Finite line:



assume line to be of finite length and open circuited at the receiving end.

i.e, $I_R = 0$

we know that when conditions at sending end are known

$$V = V_s \cosh \gamma x - I_s Z_0 \sinh \gamma x$$

$$I = I_s \cosh \gamma x - \frac{V_s}{Z_0} \sinh \gamma x$$

$\Rightarrow$ So for open circuited finite line ~~form~~.

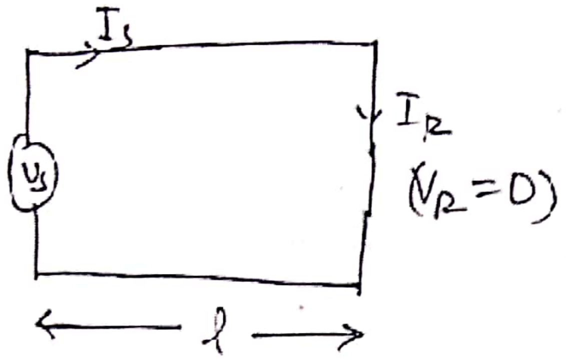
i.e, $0 = I_s \cosh \gamma l - \frac{V_s}{Z_0} \sinh \gamma l \qquad (\because x = l)$

$$\Rightarrow I_s \cosh \gamma l = \frac{V_s}{Z_0} \sinh \gamma l$$

$$\frac{V_s}{I_s} = Z_0 \frac{\cosh \gamma l}{\sinh \gamma l}$$

$$\therefore \boxed{Z_{oc} = \frac{V_s}{I_s} = Z_0 \coth \gamma l} \quad \text{——} \; \cancel{30} \; (31)$$

On the other hand, if the receiving end is short circuited,



from eq (17) $\quad V = V_s \cosh\gamma x - I_s Z_0 \sinh\gamma x$

$$0 = V_s \cosh\gamma l - I_s Z_0 \sinh\gamma l$$

$$\frac{V_s}{I_s} = Z_0 \tanh\gamma l$$

$$\therefore \boxed{Z_{sc} = \frac{V_s}{I_s} = Z_0 \tanh\gamma l} \quad \text{--- (32)}$$

- The product of eq (31) and eq (32) gives

$$Z_{oc} \cdot Z_{sc} = Z_0 \coth\gamma l \,.\, Z_0 \tanh\gamma l$$

$$Z_{oc} \cdot Z_{sc} = Z_0^2 .$$

$$\therefore \boxed{Z_0 = \sqrt{Z_{oc} \cdot Z_{sc}}} \quad \text{--- (33)}$$

$$\frac{Z_{sc}}{Z_{oc}} = \frac{Z_0 \tanh \gamma l}{Z_0 \coth \gamma l} = \tanh^2 \gamma l$$

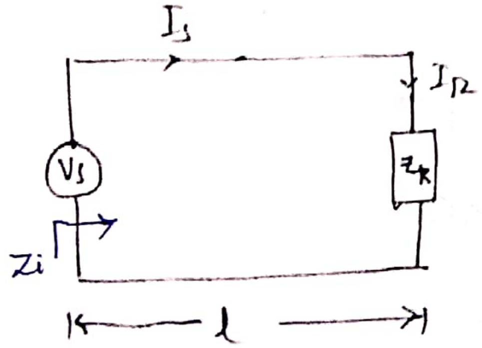
$$\therefore \tanh \gamma l = \sqrt{\frac{Z_{sc}}{Z_{oc}}}$$

$$\Rightarrow \gamma l = \tanh^{-1}\left(\sqrt{\frac{Z_{sc}}{Z_{oc}}}\right)$$

$$\Rightarrow \boxed{\gamma = \frac{1}{l} \tanh^{-1}\left(\sqrt{\frac{Z_{sc}}{Z_{oc}}}\right)} \quad \text{—— (34)}$$

Input Impedance when Receiver terminated with Load (Z_R)

In general, when the line is terminated with a finite impedance 'Z_R'.



at the receiving end

$x = l$,

$I = I_R$

$V = V_R$

we know $I_R = I_s \cosh \gamma l - \frac{V_s}{Z_0} \sinh \gamma l$ —— (35)

also $\quad V_R = V_s \cosh\gamma l - I_s Z_0 \sinh\gamma l \quad$ — (36).

~~dividing~~

divide eq (35) by (36).

$$\Rightarrow Z_R = \frac{V_R}{I_R}$$

$$Z_R = \frac{V_s \cosh\gamma l - I_s Z_0 \sinh\gamma l}{I_s \cosh\gamma l - \frac{V_s}{Z_0} \sinh\gamma l}.$$

$$Z_R = \frac{I_s \cosh\gamma l \left(\frac{V_s}{I_s} - Z_0 \frac{\sinh\gamma l}{\cosh\gamma l}\right)}{I_s \cosh\gamma l \left(1 - \frac{V_s}{I_s Z_0} \frac{\sinh\gamma l}{\cosh\gamma l}\right)}$$

$$Z_R = \frac{Z_{in} - Z_0 \tanh\gamma l}{1 - \frac{Z_{in}}{Z_0} \tanh\gamma l} \qquad \left(\because \frac{V_s}{I_s} = Z_{in}\right)$$

$$Z_R - Z_R \frac{Z_{in}}{Z_0} \tanh\gamma l = Z_{in} - Z_0 \tanh\gamma l$$

$$Z_{in} + Z_R \frac{Z_{in}}{Z_0} \tanh\gamma l = Z_R + Z_0 \tanh\gamma l$$

$$Z_{in}\left(1 + \frac{Z_R}{Z_0} \tanh\gamma l\right) = Z_R + Z_0 \tanh\gamma l$$

$$Z_{in}\left(1+\frac{Z_R}{Z_0}\tanh\gamma l\right) = Z_0\tanh\gamma l + Z_R$$

$$\therefore Z_{in} = \frac{Z_0\tanh\gamma l + Z_R}{\frac{Z_R}{Z_0}\tanh\gamma l + 1}$$

$$= \frac{Z_0\tanh\gamma l + Z_R}{\frac{Z_R\tanh\gamma l + Z_0}{Z_0}}$$

$$\boxed{Z_{in} = Z_0\left(\frac{Z_0\tanh\gamma l + Z_R}{Z_R\tanh\gamma l + Z_0}\right)} \quad \text{— (37)}$$

case i):

if $Z_R = Z_0$ (i.e, line is terminated by characteristic Impedance).

$\therefore$ eq (37) becomes

$$\boxed{Z_{in} = Z_0}$$

Case ii) if $Z_R = \infty$ (O/P open)

$$Z_{in} = Z_0\left(\frac{\cancel{Z_R}\left(\frac{Z_0}{Z_R}\tanh\gamma l + 1\right)}{\cancel{Z_R}\left(\tanh\gamma l + \frac{Z_0}{Z_R}\right)}\right)$$

$$\boxed{Z_{oc} = Z_0\coth\gamma l}$$

Case iii) if $Z_R = 0$ (Short ckt)

$$\boxed{Z_{sc} = Z_0\tanh\gamma l}$$

Thus, we can say the line is matched perfectly, when the termination impedance is equal to its characteristic Impedance.

Reflection and Reflection Coefficient:

- Fundamental equations for voltage and current at any point of transmission line are.

$$V = a e^{\gamma x} + b e^{-\gamma x}$$

$$I = -\frac{a}{Z_0} e^{\gamma x} + \frac{b}{Z_0} e^{-\gamma x}$$

b → incident
a → reflected

- when the parameters of a transmission line are not uniform along the line (or).
- The termination impedance is different from characteristic impedance. (i.e., $Z_R \neq Z_0$)
- The incident wave is reflected back at the load

* The phenomenon of a wave being reflected at the load due to improper termination is called reflection.

- Power loss occurs due to reflection.
- if the attenuation is less, the reflected wave again reflects back at the source.
- The wave travels back and forth on the line until it dissipated

The wave travels back and forth on the line until it disappears

· If 'y' is the distance measured from the termination Z_R, then by substituting '$-y$' in place of x into eq (38) we get

$$\left.\begin{aligned} V &= \underset{\text{reflected}}{a e^{-\gamma y}} + \underset{\text{incident}}{b e^{\gamma y}} \\ I &= -\frac{a}{Z_0} e^{-\gamma y} + \frac{b}{Z_0} e^{\gamma y} \end{aligned}\right\} \quad \text{--- (39)}$$

To find the constants,

The conditions at the load (i.e., at Z_R) are:

$y = 0, \quad V = V_R, \quad I = I_R.$

$\Rightarrow$ now eq (39) becomes

$$V_R = a e^{-\gamma(0)} + b e^{\gamma(0)}$$

and $$I_R = -\frac{a}{Z_0} e^{-\gamma(0)} + \frac{b}{Z_0} e^{\gamma(0)}$$

$$\Rightarrow \quad V_R = a + b \quad \text{--- (40)}$$

$$I_R = \frac{1}{Z_0}(b - a) \quad \text{--- (41)}$$

Solving (40) and (41), we get

$$a = \frac{V_R - I_R Z_0}{2} \quad \text{--- (42)}$$

$$b = \frac{V_R + I_R Z_0}{2} \quad \text{--- (43)}$$

Reflection coefficient:

Reflection coefficient is defined as the ratio of reflected voltage to the incident voltage (or) reflected current to the incident current.

$\therefore$ reflection coefficient $K = \frac{V_r}{V_i}$ (or) $K = \frac{-I_r}{I_i}$ (here negative sign indicates I_r is in reverse direction to I_i)

$$\therefore K = \frac{V_r}{V_i} = \frac{a e^{-\gamma y}}{b e^{\gamma y}} = \frac{a}{b} e^{-2\gamma y}$$

now at $y=0$.

$$K = \frac{a}{b}$$

Substituting a, b, from eq (42) and (43)

$$\therefore K = \frac{V_R - I_R Z_0/2}{V_R + I_R Z_0/2} = \frac{V_R - I_R Z_0}{V_R + I_R Z_0}$$

$$K = \frac{\frac{V_R}{I_R} - Z_0}{\frac{V_R}{I_R} + Z_0}$$

and now $K = \dfrac{\frac{V_R}{I_R} - Z_0}{\frac{V_R}{I_R} + Z_0}$

but we know $\dfrac{V_R}{I_R} = Z_R$

∴ The reflection coefficient $\boxed{K = \dfrac{Z_R - Z_0}{Z_R + Z_0}}$ —— (4.4)

- K completely depends only on load impedance and Z_0
- K is a complex quantity, and always less than or equal to ±1.

Cases:

① : for matched termination.

i.e., $Z_R = Z_0$

$\Rightarrow \boxed{K = 0}$

i.e., The reflected wave is zero.

② for short circuited termination. $(Z_R = 0)$

$\therefore \boxed{K = -1}$

i.e., entire incident wave reflects back using 180° phaseshift.

③ For open circuit termination i.e., $Z_R = \infty$.

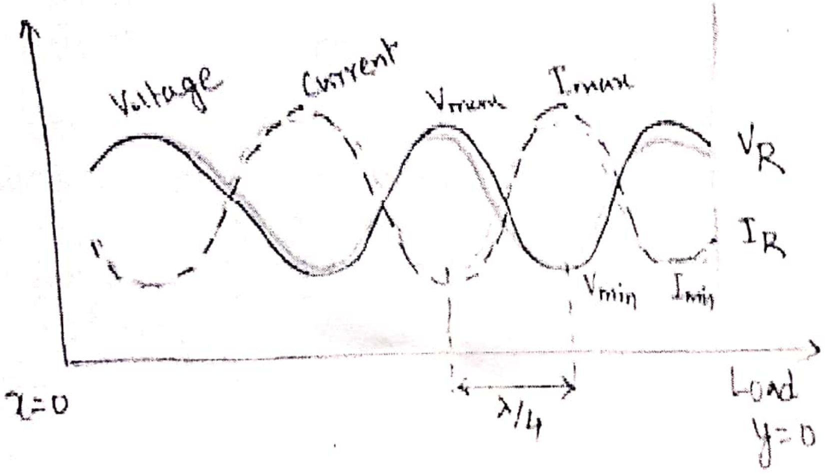
i.e., $K = \dfrac{Z_R - Z_0}{Z_R + Z_0} = \dfrac{1 - Z_0/Z_R}{1 + Z_0/Z_R} = \dfrac{1-0}{1+0} = 1$

$\therefore \boxed{K = 1}$

i.e, entire incident wave reflects back with same phase.

Standing wave Ratio:

when a line is not terminated with characteristic impedance (Z_0) the combination of incident and reflected waves gives rise to standing waves. (here the line is lossless)



Let V_{max} - Maximum voltage

V_{min} - minimum voltage

I_{max} - maximum current

I_{min} - minimum current.

- The distance between two maximum & minimum points is $\lambda/2$.

- The maximum values occur when the incident and reflected waves are added.

i.e., $|V_{max}| = |V_r| + |V_i|$

and $|I_{max}| = |I_r| + |I_i|$

The minimum values occur when the incident and reflected waves are subtracted.

i.e, $|V_{min}| = |V_i| - |V_r|$

and $|I_{min}| = |I_i| - |I_r|$

Voltage standing wave ratio: (VSWR)

- The ratio of the maximum magnitude of the voltage to the minimum magnitude of the voltage is called voltage standing wave ratio.

- denoted by S

Thus $VSWR = S = \dfrac{|V_{max}|}{|V_{min}|}$

$$\Rightarrow S = \frac{|V_{max}|}{|V_{min}|} = \frac{|V_r| + |V_i|}{|V_i| - |V_r|} = \frac{1 + \left|\frac{V_r}{V_i}\right|}{1 - \left|\frac{V_r}{V_i}\right|}$$

$$\Rightarrow \boxed{S = \frac{1+|K|}{1-|K|}} \quad \text{--- (4.5)}$$

eq (4.5) is VSWR in terms of reflection coefficient (K).

$$\text{or} \quad \boxed{|K| = \frac{S-1}{S+1}} \quad \text{--- (4.6)}$$

VSWR is a real Quantity. It is always greater than 1.

i.e., $\underline{S > 1}$

also we have

Current standing wave ratio: (ISWR)

The ratio of the maximum magnitude of the current to the minimum magnitude of the current is called current standing wave ratio.

$$ISWR = \frac{|I_{max}|}{|I_{min}|}$$

different conditions are:

① for a perfectly matched line

$Z_R = Z_0$

and $K = 0$.

$\therefore$ $\boxed{S = 1}$ and there is no reflection.

② For an open circuited or short circuited line,

$K = \pm 1$

and $\boxed{S = \infty}$

i.e, when the incident wave is completely reflected back, VSWR becomes infinite.

* ∴ The range of reflection coefficient is $-1 \leq K \leq 1$

* ∴ The range of VSWR is $0 \leq S \leq \infty$

Maximum and Minimum Impedance of Tx line

$V_{max} = |V_i| + |V_r|$ ||ly $I_{max} = |I_i| + |I_r|$

$V_{min} = |V_i| - |V_r|$ $\quad I_{min} = |I_i| - |I_r|$.

$$Z_{max} = \frac{V_{max}}{I_{min}}$$

$$= \frac{|V_i| + |V_r|}{|V_i| - |V_r| / Z_0} \quad (\text{from } 1)$$

$$Z_{max} = Z_0 \frac{|V_i| + |V_r|}{|V_i| - |V_r|}$$

$$= Z_0 \left[\frac{1 + \left|\frac{V_r}{V_i}\right|}{1 - \left|\frac{V_r}{V_i}\right|} \right]$$

$$Z_{max} = Z_0 \left[\frac{1 + |K|}{1 - |K|} \right]$$

$$\boxed{Z_{max} = Z_0 S}$$

where $S = VSWR$.

$$I_{min} = \frac{|V_{min}|}{Z_0}$$

$$= \frac{|V_i| - |V_r|}{Z_0} \quad \text{--- (1)}$$

$$I_{max} = \frac{|V_{max}|}{Z_0} = \frac{|V_i| + |V_r|}{Z_0}. \quad \text{--- (2)}$$

$$Z_{min} = \frac{V_{min}}{I_{max}}$$

$$= \frac{|V_i| - |V_r|}{\frac{|V_i| + |V_r|}{Z_0}} \quad (\text{from } 2)$$

$$Z_{min} = Z_0 \left[\frac{1 - \left|\frac{V_r}{V_i}\right|}{1 + \left|\frac{V_r}{V_i}\right|} \right]$$

$$= Z_0 \left[\frac{1 - |K|}{1 + |K|} \right]$$

$$\boxed{Z_{min} = \frac{Z_0}{S}}$$

Input Impedance in terms of reflection coefficient :

We know

$$Z_{in} = Z_0 \frac{Z_R \cosh\gamma l + Z_0 \sinh\gamma l}{Z_0 \cosh\gamma l + Z_R \sinh\gamma l} \quad \text{------} \quad (4.7)$$

or

$$Z_{in} = Z_0 \left(\frac{Z_0 \tanh\gamma l + Z_R}{Z_R \tanh\gamma l + Z_0} \right)$$

we also know

$$\cosh\gamma l = \frac{e^{\gamma l} + e^{-\gamma l}}{2} \quad \text{and} \quad \sinh\gamma l = \frac{e^{\gamma l} - e^{-\gamma l}}{2}$$

$$\therefore Z_{in} = Z_0 \frac{Z_R (e^{\gamma l} + e^{-\gamma l}) + Z_0 (e^{\gamma l} - e^{-\gamma l})}{Z_0 (e^{\gamma l} + e^{-\gamma l}) + Z_R (e^{\gamma l} - e^{-\gamma l})}$$

$$Z_{in} = Z_0 \cdot \frac{e^{\gamma l} (Z_R + Z_0) + e^{-\gamma l} (Z_R - Z_0)}{e^{\gamma l} (Z_R + Z_0) - e^{-\gamma l} (Z_R - Z_0)}$$

$$Z_{in} = Z_0 \frac{1 + e^{-2\gamma l} \left(\frac{Z_R - Z_0}{Z_R + Z_0} \right)}{1 - e^{-2\gamma l} \left(\frac{Z_R - Z_0}{Z_R + Z_0} \right)}$$

$$\boxed{Z_{in} = Z_0 \frac{(1 + K e^{-2\gamma l})}{(1 - K e^{-2\gamma l})}} \quad \text{------} \quad (4.8)$$

we know where $K = \dfrac{Z_R - Z_0}{Z_R + Z_0}$

Thus the input impedance of line depends on the reflection coefficient (K), characteristic impedance (Z_0) ~~and~~ propagation constant (γ) and length of line (l)

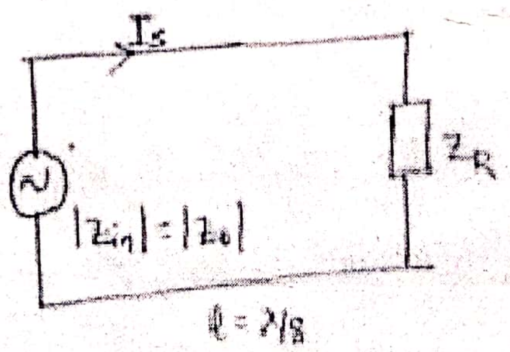
Impedance Transformation:

- Input impedance of a transmission line, depends on its length.

- The important short length transmission lines are:
  1. The eighth wave ($\lambda/8$ length) transmission line.
  2. The quarter wave ($\lambda/4$ length) transmission line.
  3. The half wave ($\lambda/2$ length) transmission line.

Eighth wave transmission line:

- The length of the eighth wave transmission line is $\lambda/8$,
- where λ is the wavelength.
- Consider a $\lambda/8$ length transmission line terminated with impedance Z_R and characteristic impedance Z_0.



- We know Z_{in} of a transmission line is

$$Z_{in} = Z_0 \left[\frac{Z_0 \tanh \gamma l + Z_R}{Z_R \tanh \gamma l + Z_0} \right]$$

for lossless line $\alpha = 0$ and $\gamma = j\beta$

$$Z_{in} = Z_0\left[\frac{Z_R + Z_0 \tanh j\beta l}{Z_0 + Z_R \tanh j\beta l}\right]$$

$$Z_{in} = Z_0\left[\frac{Z_R + jZ_0 \tan\beta l}{Z_0 + jZ_R \tan\beta l}\right]$$

now for length $l = \lambda/8$

($\because$ as $\beta = 2\pi/\lambda$)

$$\beta l = \frac{2\pi}{\lambda} \cdot \frac{\lambda}{8} = \frac{\pi}{4}$$

$$\therefore \quad Z_{in} = Z_0\left[\frac{Z_R + jZ_0 \tan(\pi/4)}{Z_0 + jZ_R \tan(\pi/4)}\right]$$

or $$Z_{in} = Z_0\left[\frac{Z_R + jZ_0}{Z_0 + jZ_R}\right]$$

or $$|Z_{in}| = |Z_0|\left|\frac{Z_R + jZ_0}{Z_0 + jZ_R}\right|$$

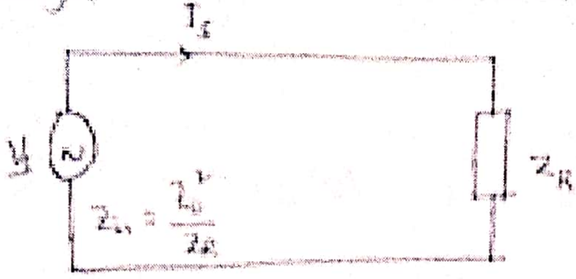
$$\left.\begin{array}{l} \text{but } |Z_R + jZ_0| = \sqrt{Z_R^2 + Z_0^2} \\ \text{also } |Z_0 + jZ_R| = \sqrt{Z_R^2 + Z_0^2} \end{array}\right\}$$

$$\therefore \quad \boxed{|Z_{in}| = |Z_0|}$$

Thus a $\lambda/8$ length transmission line is used to transform any impedance (Z_R) to a magnitude of Z_0.

Quarter wave ($\lambda/4$) transmission line:

- Consider a $\lambda/4$ length transmission line..



$l = \lambda/4$

- The Transmission line terminated with impedance Z_R. and characteristic impedance Z_0.

- $\lambda/4$ length transmission line is also called a quarter wave transformer.

- We know that Z_{in} of a distortionless transmission line is

$$Z_{in} = Z_0 \left(\frac{Z_R + jZ_0 \tan\beta l}{Z_0 + jZ_R \tan\beta l} \right)$$

for length $l = \lambda/4$

$$\beta l = \frac{2\pi}{\lambda} \cdot \frac{\lambda}{4} = \frac{\pi}{2}$$

$$\therefore Z_{in} = Z_0 \left[\frac{\frac{Z_R}{\tan\beta l} + jZ_0}{\frac{Z_0}{\tan\beta l} + jZ_R} \right]$$

Substituting, $\beta l = \frac{\pi}{2}$

$$Z_{in} = Z_0 \frac{jZ_0}{jZ_R} \quad \text{or}$$

$$Z_0^2 = Z_{in} Z_R \quad \text{or} \quad \boxed{Z_0 = \sqrt{Z_{in} Z_R}}$$

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Applications of a $\lambda/4$ line transformer:

1. To match the impedance between a transmission line and an antenna.

2. To step up or step down the characteristic impedance Z_0 of a transmission line.

if the load of a line is not resistive, the impedance of the line at node points is either SZ_0 or $\frac{Z_0}{S}$, irrespective of load impedance. 'S' is VSWR

$\therefore$ Z_0 of the line $\lambda/4$ is

$$Z_0' = \sqrt{Z_0 Z_0 S} \qquad (\because Z_0 = \sqrt{Z_{in} Z_R})$$

$$Z_0' = Z_0\sqrt{S}$$

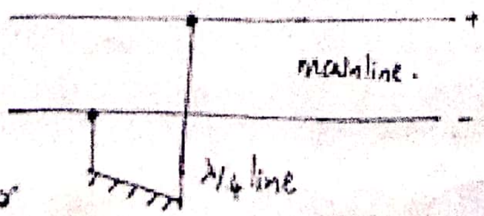
or

$$Z_0' = \sqrt{Z_0 \frac{Z_0}{S}}$$

$$Z_0' = \frac{Z_0}{\sqrt{S}}$$

Since $S > 1$, the impedance may be either step up or stepdown.

3. It can provide a mechanical support to the transmission line in addition to the impedance.



Quarter wave transformer as insulator.

- Half wave Transmission line:

- The length of the half wave transmission line is $\lambda/2$.

- $$Z_{in} = Z_0 \frac{Z_R + jZ_0 \tan\beta l}{Z_0 + jZ_R \tan\beta l}.$$

for length $l = \lambda/2$.

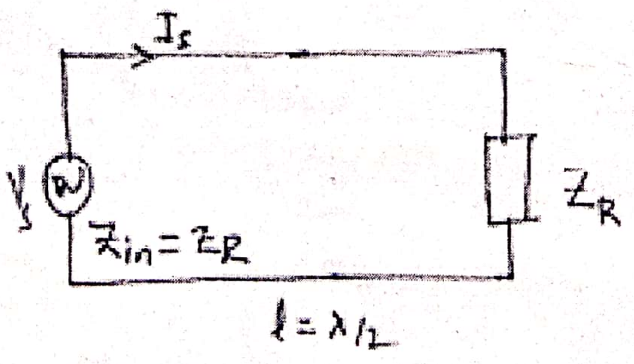
$$\beta l = \frac{2\pi}{\lambda} \times \frac{\lambda}{2} = \pi.$$

$$\therefore \; Z_{in} = Z_0 \frac{Z_R}{Z_0}.$$

or $\boxed{Z_{in} = Z_R.}$

$\therefore$ The input impedance of a half wave line is equal to its termination impedance.

- The termination impedance of the line repeats at every $\lambda/2$ distan[ce]



Application:

If the load and source cannot be made adjacent, a half wavelength line may be connected at the load point for accurate measurements.

Input Impedance of lossless SC and OC Transmission lines:

- we know the input impedance of short circuited and open circuited lines are

$$Z_{sc} = Z_0 \tanh \gamma l$$

and $Z_{oc} = Z_0 \coth \gamma l$

for a lossless transmission line

$$\gamma = \alpha + j\beta$$

$\alpha = 0$ for loss less line

$$\therefore \gamma = 0 + j\beta = j\beta$$

$$\Rightarrow \quad Z_{sc} = Z_0 \tanh j\beta l$$

$$Z_{sc} = jZ_0 \tan \beta l \quad \text{———}$$

and $Z_{oc} = Z_0 \coth j\beta l$

$$Z_{oc} = -jZ_0 \cot \beta l \quad \text{———}$$

- Since Z_0 is resistive, the Z_{in} for both S.C and O.C lossless lines is pure reactive.

- Depending on the length, the transmission line can provide either a capacitive or an inductive effect.

Case (a): for $l < \frac{\lambda}{4}$

let $l = \lambda/8$, Then $\beta l = \frac{2\pi}{\lambda} \cdot \frac{\lambda}{8} = \frac{\pi}{4}$.

$\therefore Z_{sc} = jZ_0 \tan(\pi/4)$

$\boxed{Z_{sc} = jZ_0}$ — (inductive) —

and $Z_{oc} = -jZ_0 \cot(\pi/4)$

$\boxed{Z_{oc} = -jZ_0}$ — (capacitive). —

$\therefore$ In first quarter wavelength (i.e., $0 < l < \lambda/4$), the SC line acts as Inductive.
while the OC line acts as Capacitive.

Case(b): for $\lambda/4 < l < \lambda/2$.

let $l = \lambda/3$, Then $\beta l = \frac{2\pi}{\lambda} \cdot \frac{\lambda}{3} = \frac{2\pi}{3}$.

$\therefore Z_{sc} = jZ_0 \tan\left(\frac{2\pi}{3}\right)$

$\boxed{Z_{sc} = -j\sqrt{3}\, Z_0}$ (capacitive) —

and $Z_{oc} = -jZ_0 \cot\left(\frac{2\pi}{3}\right)$

$\boxed{Z_{oc} = j\sqrt{3}\, Z_0}$ (inductive) —

$\therefore$ for $\lambda/4 < l < \lambda/2$

- SC line acts as Capacitive
- OC line acts as Inductive.

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Case (c): for $l = \lambda/4$

$$\beta l = \frac{2\pi}{\lambda} \cdot \frac{\lambda}{4} = \frac{\lambda}{2}.$$

$$\therefore Z_{sc} = jZ_0 \tan(\lambda/2) = \pm\infty$$

and $$Z_{oc} = -jZ_0 \cot(\lambda/2) = 0$$

Therefore, for $l = \lambda/4$

The SC line acts as open circuit

The OC line acts as short circuit

Case (d): for $l = \lambda/2$.

$$\beta l = \frac{2\pi}{\lambda} \cdot \frac{\lambda}{2} = \pi$$

$$\therefore Z_{sc} = jZ_0 \tan(\pi) = 0$$

$$Z_{oc} = -jZ_0 \cot(\pi) = \pm\infty$$

$\therefore$ for $l = \lambda/2$

S.C line acts as short circuit

OC line acts as open circuit.

Figure: Input Impedance variation

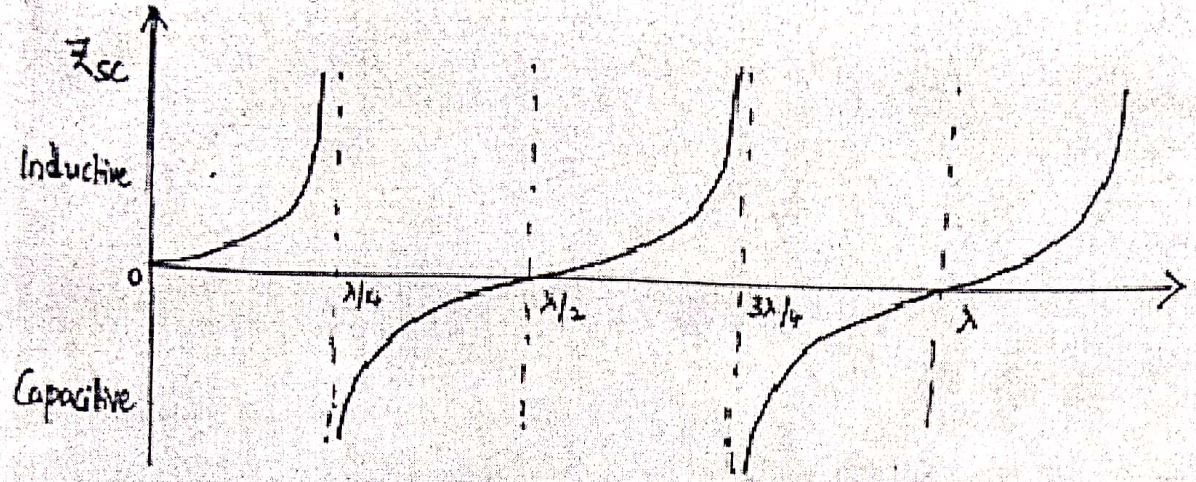

fig (a) SC line.

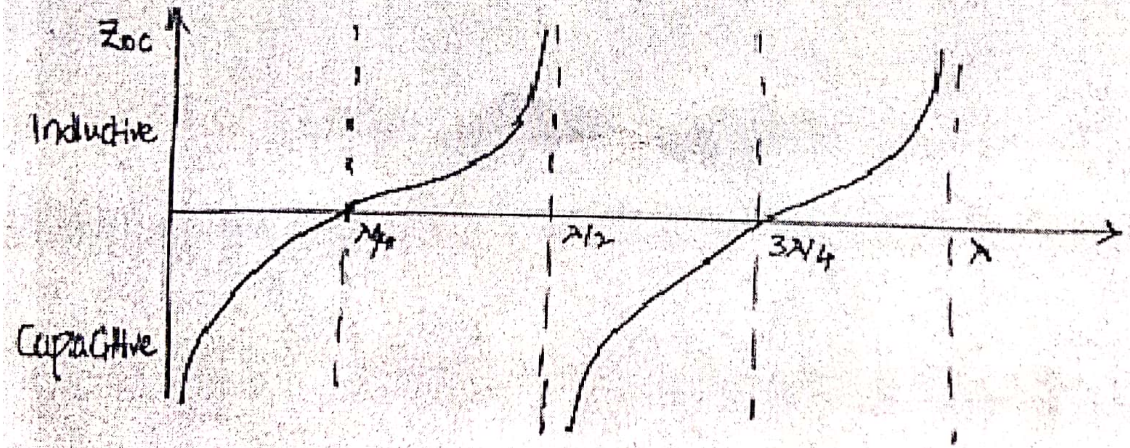

fig (b) OC line.

Note:

- It is observed that after each quarter wavelength of the line, the nature of reactance reverses.
- The same reactance value repeat every half wave length distance.

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UHF lines as circuit elements:

- At ultra high frequency, The transmission line becomes lossless.
- The short length SC and OC transmission lines can be used as circuit elements.

(a) for lengths $(0<l<\lambda/4)$ of an SC line:

SC line:

$$Z_{sc} = jZ_0 \tan\beta l$$

if L_{eq} = equivalent inductance,

Then $j\omega L_{eq} = jZ_0 \tan\beta l$

$$\therefore \boxed{L_{eq} = \frac{Z_0}{\omega}\tan\beta l}$$

∴ line acts as an inductor

OC line:

$$Z_{oc} = -jZ_0 \cot\beta l$$

If C_{eq} = equivalent Capacitance.

Then $\frac{1}{j\omega C_{eq}} = -jZ_0 \cot\beta l$

$$\therefore \boxed{C_{eq} = \frac{1}{\omega Z_0 \cot\beta l}}$$

∴ The line acts as a capacitor.

(b) for lengths $\lambda/4 < l < \lambda/2$.

for a SC line:

$$Z_{sc} = jZ_0 \tan\beta l.$$

$$\frac{1}{j\omega C_{eq}} = jZ_0 \tan\beta l$$

$$\boxed{C_{eq} = \frac{-1}{\omega Z_0 \tan\beta l}}$$

The line acts as a capacitor.

for an OC line:

$$j\omega L_{eq} = jZ_0 \cot\beta l$$

$$\boxed{L_{eq} = \frac{Z_0}{\omega} \cot\beta l}$$

The line acts as an Inductor.

(c) for $l = \lambda/4$.

- The SC line has infinite input impedance.
- It acts as a parallel or anti-resonant ckt at every odd multiples of $\lambda/4$ length.

- for $l = \lambda/2$;
  - The SC line has zero input impedance.
  - acts as a series resonant circuit at every even multiples of the $\lambda/4$ length.

(d): **for $l = \lambda/4$**

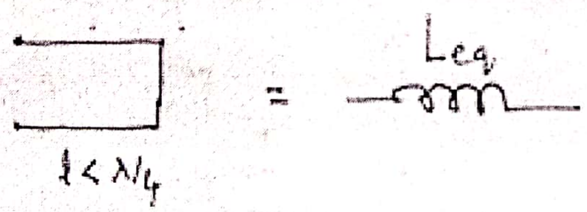
- The OC line has zero input Impedance.
- acts as Series resonant Circuit at every odd multiple of the $\lambda/4$ length.

for $l = \lambda/2$:

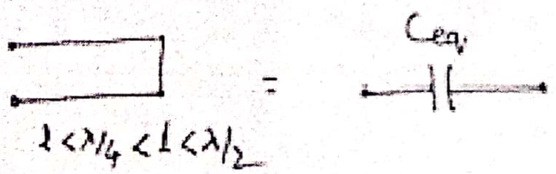
- The OC line has infinite input impedance.
- It acts as a parallel resonant & anti-resonant Circuit at every even multiple of $\lambda/4$ length.

* **SC lines at different lengths and equivalent Circuit elements.**

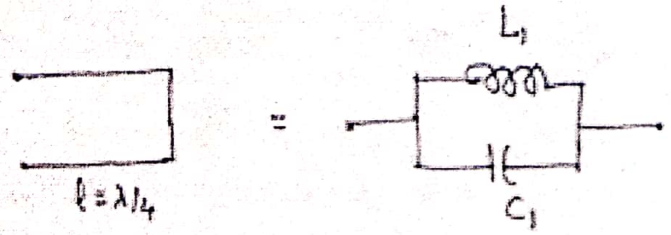
(a)



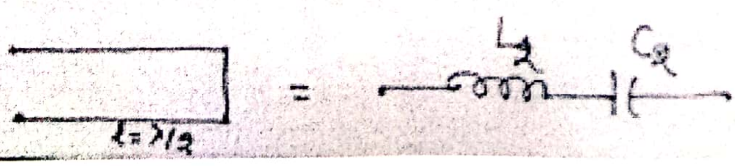
(b)



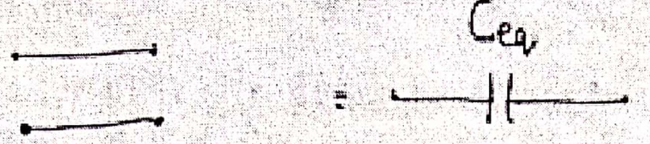
(c)



(d)



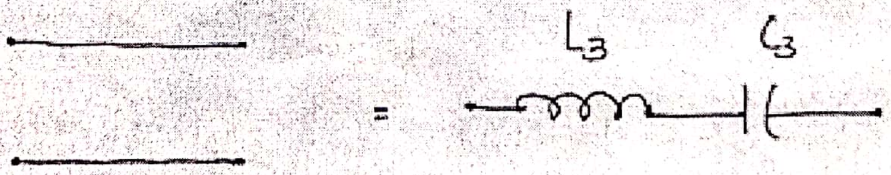
OC lines at different lengths with equivalent circuit elements:

ⓐ 

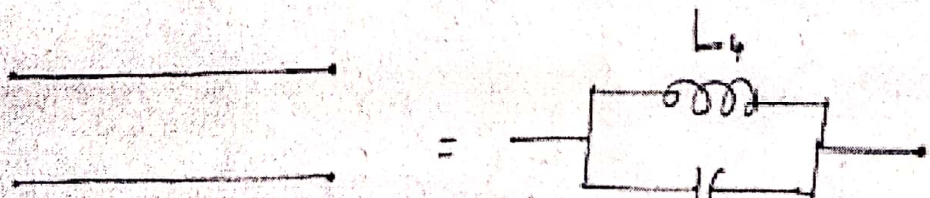
$l < \lambda/4$

ⓑ 

$\lambda/4 < l < \lambda/2$

ⓒ 

$l = \lambda/4$

ⓓ 

$l = \lambda/2$

Stub Matching:

- When a UHF line is terminated with a load impedance which is not equal to the characteristic impedance of the line, Mismatch occurs. (i.e., when $Z_R \neq Z_0$)
- To avoid mismatching, it is necessary to add impedance matching devices b/w load and the line.
- To achieve impedance matching we have to cut the line to insert a transformer (here quarter wave transformer i.e., $l = \lambda/4$) between the line and load.
- The other method is to use open or short circuited short length transmission lines as a matching device.
- which can be connected in parallel to the line at certain distance or distances from the load.
- This matching device is called Stub matching.