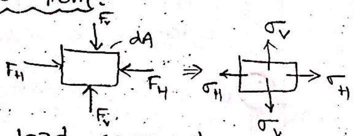
Stresses & equilibrium :-

When a mechanical component is subjected to a load, some stresses induced here depending on the nature of load, body force (self weight), friction forces etc.



In FEA technique, a body is to be considered under equilibrium the following condition are

$$\Sigma F_x = 0, \ \Sigma F_y = 0, \ \Sigma F_z = 0 \ ; \ \Sigma M = 0$$

$$\Rightarrow \sigma_x = 0 \ ; \ \sigma_y = 0 \ ; \ \sigma_z = 0$$

Boundary conditions:-

When analysing any mechanical component, the nature of load at various locations and different surface conditions, which are specified with certain limits are to be known, which is known as boundary conditions.

eg : no. of faces is limited to 3

deformation is limited to a certain value

Temp is limited to a certain limit (freezing, melting etc)

Strain-displacement:- $\left(e = \frac{\Delta l}{l} \right)$

When a ~~body~~ load is applied on a body, corresponding displacements takes place. ie, material is deformed corresponding to various stresses, strains are induced here. the no of stresses (σ) & strains are equal to '6'.

3-normal strains & 3-shear strains.

$$e = \{ e_x, e_y, e_z, \phi_{xy}, \phi_{yz}, \phi_{zx} \}$$

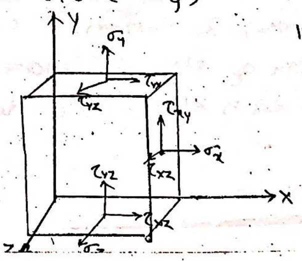
1) $e_x = \frac{\partial u}{\partial x}$; 4) $\phi_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}$

2) $e_y = \frac{\partial v}{\partial y}$; 5) $\phi_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}$

3) $e_z = \frac{\partial w}{\partial z}$; 6) $\phi_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}$

Stress-strain relations:-

For linear elastic materials, the stress-strain relation is given by Hookes law. ($\sigma \propto \varepsilon$)



If μ = Poisson ratio

E = Young's modulus

then

the stress-strain relations are → (3-D)

$e_x = \frac{P}{E}(1-2\mu)$

1) $e_x = \frac{\sigma_x}{E} - \mu\frac{\sigma_y}{E} - \mu\frac{\sigma_z}{E}$ (4) $\phi_{xy} = \frac{\tau_{xy}}{G}$

2) $e_y = \frac{\sigma_y}{E} - \mu\frac{\sigma_z}{E} - \mu\frac{\sigma_x}{E}$ (5) $\phi_{yz} = \frac{\tau_{yz}}{G}$

3) $e_z = \frac{\sigma_z}{E} - \mu\frac{\sigma_x}{E} - \mu\frac{\sigma_y}{E}$ (6) $\phi_{zx} = \frac{\tau_{zx}}{G}$

→ For a 2-D body & 2-directional loading.

$e = \frac{P}{E}(1-\mu)$

(1) $e_x = \frac{\sigma_x}{E} - \mu.\frac{\sigma_y}{E}$ (4) $\boxed{\phi_{xy} = \frac{\tau_{xy}}{G}}$

(2) $e_y = \frac{\sigma_y}{E} - \mu.\frac{\sigma_x}{E}$ $\Rightarrow \phi_{xy} = \frac{2(1+\mu).\tau_{xy}}{E}$

(3) $e_z = -\mu\frac{\sigma_x}{E} - \mu\frac{\sigma_y}{E}$ $\{ \because E = 2G(1+\mu)$ $\Rightarrow G = \frac{E}{2(1+\mu)} \}$

→ for 1-D body : $\boxed{\sigma = Ee}$ (one direction loading)

* one dimensional Problems:-

→ F.E.A. co-ordinates & shape functions:-

co-ordinate system is used to identify, no of elements, nodes, stiffness of each element, & displacements.

→ Two types of co-ordinate systems are:-

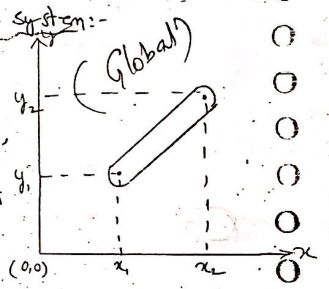
1. Global co-ordinate system
2. local co-ordinate system.

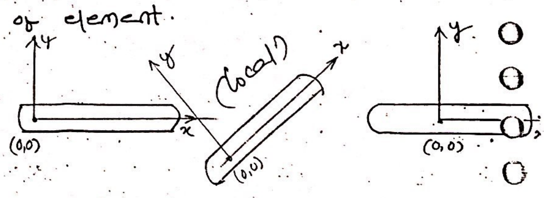
(*) Global co-ordinate system:- (complicated)

In Global system, the various elements & their nodes are specified by the common axis whose origin is mostly away from the elements.

(*) Local co-ordinate system:-

Also known as natural co-ordinate system. In this system, the co-ordinate axis is placed on element such that their origin is at on end of element.





Shape Functions:-

In F.E.A, the complex domain like a complicated structure is discretized into many no. of finite elements & the solution like nodal displacements, element stresses are evaluated & by combining the nodal displacements on element stresses in a proper format, the overall displacement or stress of the whole structure can be decided.

To achieve the displacements at nodal points & inside the element, we have to make use of two mathematical expressions

(1) Finite element equations

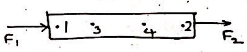
(2) shape functions.

→ (1) : The displacement at the primary nodes (at extreme ends of the elements) can be evaluated by finite element eqⁿ.

→ (2) : by using shape functions, the displacements at interior points of the elements can be known (interior nodes)

In any coordinate system, the shape functions are used to find variables like displacement in b/w the nodes (within the element).

→ eg: consider a bar (1-D) element :-



1, 2 → Primary nodes

3, 4 → interior node:

let F_1 & F_2 be applied forces at the primary nodes 1 & 2.

K be the stiffness of bar

U_1, U_2, U_3 & U_4 be the displacements of nodes 1, 2, 3 & 4 due to forces F_1 & F_2.

by Hookes Law,

stress ∝ strain

$\Rightarrow \sigma \propto \xi$

$\Rightarrow \sigma = E . \xi$

$\Rightarrow \frac{F}{A} = E . \frac{\Delta l}{l}$

$\Rightarrow \frac{F}{\Delta l} = \frac{AE}{l}$

$\Rightarrow \boxed{K = \frac{AE}{l}}$ Assembly of global stiffness matrix

where,

$K \longrightarrow$ force per unit deflection

* **Derivation of shape functions by direct Method:-**

Consider a one-dimensional bar element (Linear) of length 'l' with 2 nodes, one at each end. Let the nodes be denoted by as 1 & 2 & the nodal values (displacement) of field variable U as U_1 & U_2.

Let U be expressed in polynomial form with global-coordinate

$$U(x) = a_1 + a_2 x + a_3 x^2 + \dots$$

a_1, a_2 are unknown polynomial coeff

$$\Rightarrow U(x) = a_1 + a_2 x \quad \text{--- (1)}$$

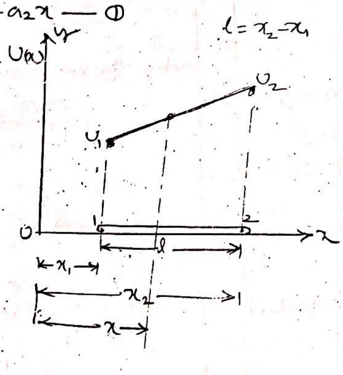
$l = x_2 - x_1$

The values of a_1 & a_2 can be determined by using nodal conditions

at

$U = U_1$; $x = x_1$

$U = U_2$; $x = x_2$.



x_1 & x_2 are global co-ordinates of nodes 1 & 2

$$\therefore U_1 = a_1 + a_2 x_1 \quad \text{--- (a)}$$

&

$$U_2 = a_1 + a_2 x_2 \quad \text{--- (b)}$$

now, eqn (b) − (a)

$$\Rightarrow U_2 - U_1 = a_2(x_2 - x_1)$$

$$\Rightarrow a_2 = \frac{U_2 - U_1}{x_2 - x_1} \quad \text{--- (x)}$$

substitute eqn (x) in (a)

$$\Rightarrow U_1 = a_1 + \left[\frac{U_2 - U_1}{x_2 - x_1}\right] x_1$$

$$\Rightarrow a_1 = U_1 - \left[\frac{U_2 - U_1}{x_2 - x_1}\right] x_1$$

$$\Rightarrow a_1 = \frac{U_1 x_2 - \cancel{U_1 x_1} - U_2 x_1 + \cancel{U_1 x_1}}{x_2 - x_1}$$

$$a_1 = \frac{U_1 x_2 - U_2 x_1}{x_2 - x_1} \quad \text{--- (y)}$$

sub... (x) & (y) in (1)

$$\Rightarrow U(x) = a_1 + a_2 x$$

$$= \left[\frac{U_1 x_2 - U_2 x_1}{x_2 - x_1}\right] + \left[\frac{U_2 - U_1}{x_2 - x_1}\right] x$$

$$= \frac{U_1 x_2 - U_2 x_1 + U_2 x - U_1 x}{x_2 - x_1}$$

$$=\frac{(x_2-x)U_1+(x-x_1)U_2}{x_2-x_1}$$

$$=\left(\frac{x_2-x}{x_2-x_1}\right)U_1+\left(\frac{x-x_1}{x_2-x_1}\right)U_2$$

$\therefore$ It is in the form of $N_1U_1+N_2U_2$

$$N_1=\frac{x_2-x}{l}\ ;\ N_2=\frac{x-x_1}{l} \quad \{\because l=x_2-x_1\}$$

N_1 & N_2 are shape functions.

Note:–

* $\boxed{U(x)=N_1U_1+N_2U_2}$

similar to displacements, any field variable like temperature at various locations can be evaluated by using shape function.

$$\boxed{T(x)=N_1T_1+N_2T_2}$$

* Calculation of inner point displacement stress & strain for the element:–

We know

$$N_1=\frac{x_2-x}{l}\ ;\ N_2=\frac{x-x_1}{l}$$

$$\because e=\frac{dU}{dx}=\frac{d}{dx}(N_1U_1+N_2U_2)$$

$[\because U=N_1U_1+N_2U_2]$

$$=\frac{d}{dx}(N_1U_1)+\frac{d}{dx}(N_2U_2)$$

$$=\frac{d}{dx}\left[\frac{x_2-x}{l}\right]U_1+\frac{d}{dx}\left[\frac{x-x_1}{l}\right]U_2$$

$$=\frac{-1}{l}U_1+\frac{1}{l}U_2$$

$$=\frac{1}{l}[-U_1+U_2]$$

$$\boxed{e=\frac{1}{l}[-1\ \ 1]\begin{Bmatrix}U_1\\U_2\end{Bmatrix}}$$

i.e. $\boxed{e=[B]\{U\}}$

$$\because E=\frac{\sigma}{e}\Rightarrow \sigma=E.e$$

$$=\frac{E}{l}[-1\ \ 1]\begin{Bmatrix}U_1\\U_2\end{Bmatrix}$$

Potential Energy Approach: (π)

In F.E.A, the total potential energy of elastic body is defined as sum of strain energy & work potential

$$\pi=\text{strain energy}(U)\pm\text{work potential (W.P)}$$

$$\boxed{\pi=U\pm W}$$

where, $U=\frac{1}{2}F\delta$

W = self weight + Traction force + point load.

$+W$ = Workdone due to external forces.

$-W$ = W.D. by the ext forces.

* **Assembly of Global Stiffness matrix & load vector :-**

We know

stiffness K = Force/deflection

$$\therefore K = \frac{F}{U} \quad (\text{or}) \quad F = KU \qquad (\Delta l = U)$$

by Hookes law,

$$\sigma \propto \xi$$

$$\sigma = E\xi$$

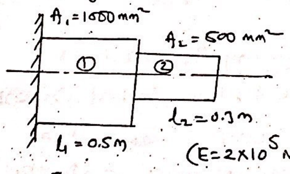
$$\Rightarrow \frac{F}{A} = E \cdot \frac{\Delta l}{l}$$

$$\Rightarrow \frac{F}{\Delta l} = \frac{AE}{l} \Rightarrow \boxed{K = \frac{AE}{l}}$$

2M

Prob:

① Find the stiffness of given element in N/mm ?



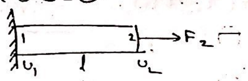
Sol:- we know

$$K = \frac{AE}{l} \qquad (E = 2\times10^5\ N/mm^2)$$

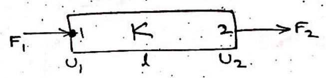
$$\rightarrow K_1 = \frac{A_1E}{l_1} = \frac{1000\times2\times10^5}{0.5\times1000} = 4\times10^5\ N/mm.$$

$$\rightarrow K_2 = \frac{A_2E}{l_2} = \frac{500\times2\times10^5}{0.3\times1000} = 3.3\times10^5\ N/mm.$$

* **Formulation of stiffness Matrix:-**



considering the displacement at node 1 as U_1, & at node 2 as U_2.



$(K = K_1)$

stiffness at node 1 : $F_1 = K_1U_1$

stiffness at node 2 : $F_2 = K_1U_2$

— considering the displacement at node 1 w.r.to node 2. The total displacement will be $(U_1 - U_2)$

$$\therefore F_1 = K_1(U_1 - U_2) = K_1U_1 - K_1U_2 = (KU_1 - KU_2)$$

— similarly the total displacement of node 2 w.r.to node 1

$$\therefore F_2 = K_1(U_2 - U_1) = -KU_1 + KU_2$$

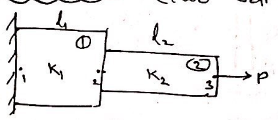
$\therefore$ Matrix form

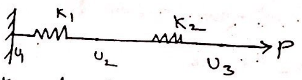
$$\begin{bmatrix} K & -K \\ -K & K \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

$$\Rightarrow \underbrace{K\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}}_{\text{stiffness matrix}} \underbrace{\begin{Bmatrix} U_1 \\ U_2 \end{Bmatrix}}_{\text{displacement vector}} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} \text{ — load vector.}$$

$\therefore$ stiffness matrix $= \frac{AE}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

* Stiffness matrix for a stepped bar:- (two bar element)

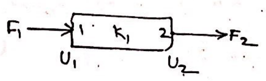




let the displacements at nodes 1, 2 & 3 be u_1, u_2 & u_3.

- Consider nodes ① & ② :-

element 1 :-



$F_1 = k_1(u_1 - u_2) \Rightarrow F_1 = k_1u_1 - k_1u_2$

$F_2 = k_1(u_2 - u_1) \Rightarrow F_2 = k_1u_2 - k_1u_1$

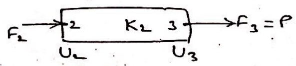
$= -k_1u_1 + k_1u_2$

$\therefore$ Matrix form

$$\begin{array}{c|c} & \begin{matrix} 1 & \quad 2 \end{matrix} \\ \begin{matrix} 1 \\ 2 \end{matrix} & \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix} \end{array} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$$

$\Rightarrow [K][U] = [F]$

element 2 :



$F_2 = k_2(u_2 - u_3) \Rightarrow F_2 = k_2u_2 - k_2u_3$

$F_3 = k_2(u_3 - u_2) \Rightarrow F_3 = -k_2u_2 + k_2u_3$

$\Rightarrow$ matrix form

$$\begin{array}{c|c} & \begin{matrix} 2 & \quad 3 \end{matrix} \\ \begin{matrix} 2 \\ 3 \end{matrix} & \begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \end{array} \begin{bmatrix} u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} F_2 \\ F_3 \end{bmatrix}$$

$\therefore$ combined global stiffness matrix

$$\begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1+k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix}$$

Here $u_1 = 0$ (no displacement)

$F_2 = 0$ (no load applied)

$F_1 = -P$ (opp to force applied)

$F_3 = P$

$\because U_1 = 0$ (we can neglect 1st row & coloumn in global matrix)

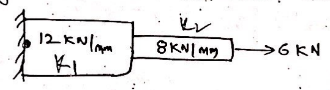
$$\begin{bmatrix} K_1+K_2 & -K_2 \\ -K_2 & K_2 \end{bmatrix} \begin{Bmatrix} U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ P \end{Bmatrix}$$

ie, $(K_1+K_2)U_2 - K_2U_3 = 0$

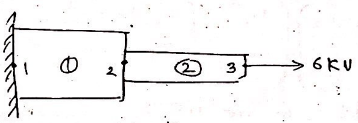
$-K_2U_2 + K_2U_3 = P$ (U_2 & U_3 can be determined)

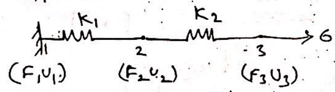
Problems:-

① Calculate the nodal displacements & forces for the bar loaded as shown



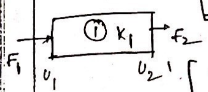
sol:





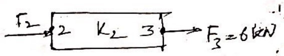
[spring element]

element ①: $\begin{bmatrix} K_1 & -K_1 \\ -K_1 & K_1 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$



$$\begin{array}{c|cc} & 1 & 2 \\ \hline 1 & 12 & -12 \\ 2 & -12 & 12 \end{array} \begin{Bmatrix} U_1 \\ U_2^0 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

lly element ②:



$$\begin{array}{c|cc} & 2 & 3 \\ \hline 2 & 8 & -8 \\ 3 & -8 & 8 \end{array} \begin{Bmatrix} U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} F_2 \\ F_3 \end{Bmatrix}$$

Combine matrix:

$$\begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 12 & -12 & 0 \\ 2 & -12 & 12+8 & -8 \\ 3 & 0 & -8 & 8 \end{array} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix}$$

$\because$ node 1 is fixed; $U_1 = 0$

(can eliminate 1st row & coloumn)

$\because$ no load at node 2; $F_2 = 0$

$F_3 = P = 6$ KN

$$\Rightarrow \begin{bmatrix} 12 & -12 & 0 \\ -12 & 20 & -8 \\ 0 & -8 & 8 \end{bmatrix} \begin{Bmatrix} 0 \\ U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ 0 \\ 6 \end{Bmatrix}$$

~~$\Rightarrow -12U_2 = F_1$~~ — ①

$\Rightarrow 20U_2 - 8U_3 = 0$ — ②

$-8U_2 + 8U_3 = 6$ — ③

solving ① & ②

$\Rightarrow U_2 = 0.5$ mm ; $U_3 = 1.25$ mm

Reaction force:

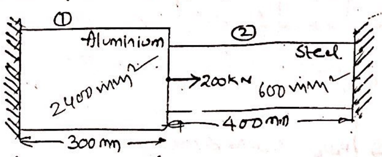
$F_1 = -12(0.5)$ $[-12U_2]$

$\therefore F_1 = -6$ KN

Results: $U_1 = 0$ mm , $U_2 = 0.5$ mm , $U_3 = 1.25$ mm

$F_1 = -6$ KN ; $F_2 = 0$ KN ; $F_3 = 6$ KN

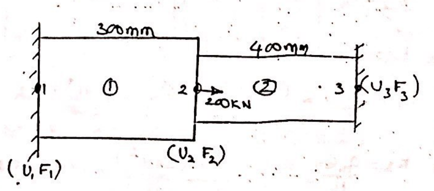
(2) A stepped bar is subjected to an axial load of 200 KN at the place of change of cross-section. Find
a) nodal displacement
b) reaction forces
c) Induced stresses in each element.



$A_1 = 2400 \, mm^2$; $E_1 = 70 \times 10^3 \, N/mm^2$

$A_2 = 600 \, mm^2$; $E_2 = 200 \times 10^3 \, N/mm^2$

Sol:



$$K_1 = \frac{A_1 E_1}{l_1} = \frac{2400 \times 70 \times 10^3}{300} = 56 \times 10^4 \, N/mm$$

$$K_2 = \frac{A_2 E_2}{l_2} = \frac{600 \times 200 \times 10^3}{400} = 30 \times 10^4 \, N/mm$$

→ element ① :

$$\begin{bmatrix} K_1 & -K_1 \\ -K_1 & K_1 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$$

$$10^4 \begin{bmatrix} 56 & -56 \\ -56 & 56 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$$

→ element ② :

$$10^4 \begin{bmatrix} 30 & -30 \\ -30 & 30 \end{bmatrix} \begin{bmatrix} U_2 \\ U_3 \end{bmatrix} = \begin{bmatrix} F_2 \\ F_3 \end{bmatrix}$$

$\because U_1 \,\&\, U_3 = 0$ (fixed)

can eliminate 1 & 3 row's & coloumn's.

→ combine matrix :

$$10^4 \begin{bmatrix} 56 & -56 & 0 \\ -56 & 86 & -30 \\ 0 & -30 & 30 \end{bmatrix} \begin{bmatrix} 0 \\ U_2 \\ 0 \end{bmatrix} = \begin{bmatrix} F_1 \\ 200 \times 10^3 \\ F_3 \end{bmatrix}$$

($\because$ 1 & 3 is fixed)

$\Rightarrow 10^4(-56)U_2 = F_1$ —— ①

$10^4 (86) U_2 = 200 \times 10^3 \Rightarrow U_2 = 0.233$ mm

$10^4(-30)U_2 = F_3$ —— ②

$\Rightarrow F_1 = -130$ KN ; $F_3 = -70$ KN

$$\xi = \frac{\Delta l}{l}$$

→ Induced stresses :-

$$\underline{\xi_1} = e_1 = \left(\frac{du}{dx}\right)_1 = \left[\frac{U_2 - U_1}{l_1}\right]$$

$e =$

$\epsilon = \frac{\sigma}{e}$

$\epsilon = \frac{\sigma}{\epsilon}$

$$= \frac{0.233 - 0}{300} = 0.0075$$

$$e_2 = \left[\frac{du}{dx}\right]_2 = \left(\frac{U_3 - U_2}{l_2}\right)$$

$$= \frac{0 - 0.233}{400} = 0.0005$$

$$\because E = \frac{\sigma}{e} \Rightarrow \sigma_1 = E \times e_1$$

$$= 70 \times 10^3 \times 0.0075 = 54.4 \ N/mm^2$$

(Tensile)

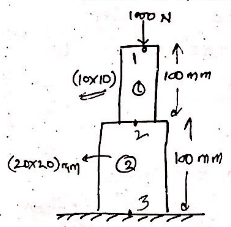
ly

$$\sigma_2 = E \times e_2$$

$$= 200 \times 10^3 \times (-0.0005)$$

$$= -116.5 \ N/mm^2 \ \text{(Compressive)}$$

③ For the column made of mild steel steel
find a) nodal displacement
b) reaction forces at supports
c) stresses & strains ?



Sol :-

(spring diagram)

we know

$$K_1 = \frac{A_1 E_1}{l_1} = \frac{(10 \times 10)(2 \times 10^5)}{100} = 2 \times 10^5 \ N/mm$$

$$K_2 = \frac{A_2 E_2}{l_2} = \frac{(20 \times 20)(2 \times 10^5)}{100} \simeq 4 \times 10^5 \ N/mm$$

→ Element stiffness matrix:-

element 1 :

$$2 \times 10^5 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \quad \text{— ①}$$

element 2:

$$\Rightarrow 4\times10^5 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_2 \\ F_3 \end{Bmatrix}$$

$$\Rightarrow 2\times10^5 \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_2 \\ F_3 \end{Bmatrix} \quad \text{—} ②$$

combine matrix:-

$$2\times10^5 \begin{bmatrix} 1 & -1+0 & 0 \\ -1 & 1+2 & -2 \\ 0 & -2 & +2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix}$$

$$\Rightarrow 2\times10^5 \begin{bmatrix} 1 & -1 & 0 \\ -1 & 3 & -2 \\ 0 & -2 & 2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix}$$

∵ Node 3 is fixed $\Rightarrow u_3 = 0$

& $F_2 = 0$ (no force at node 2)

$$2\times10^5 \begin{bmatrix} 1 & -1 & 0 \\ -1 & 3 & -2 \\ 0 & -2 & 2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 1000 \\ 0 \\ F_3 \end{Bmatrix}$$

$$\Rightarrow 2\times10^5 \begin{bmatrix} 1 & -1 \\ -1 & 3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} 1000 \\ 0 \end{Bmatrix}$$

$2\times10^5(u_1 - u_2) = 1000$ — ①

$2\times10^5(-u_1 + 3u_2) = 0$ — ②

Solve ① & ②

$\Rightarrow u_1 = 75\times10^{-4}$ mm
$u_2 = 25\times10^{-4}$ mm } nodal displacement

→ Reaction force :- F_3 at fixed end

$2\times10^5[0\times u_1 + (-2\times u_2) + (2\times 0)] = F_3$

$\Rightarrow F_3 = -1000$ N.

→ Stress & strains : $\varepsilon = \frac{\sigma}{E} \Rightarrow \sigma = E.e$

$e_1 = \frac{u_2 - u_1}{l_1} = \frac{(25-75)10^{-4}}{100} = -50\times10^{-6}$

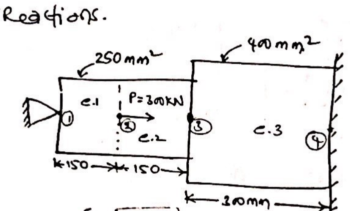
$e_2 = \frac{u_3 - u_2}{l_2} = \frac{(0-25)10^{-4}}{100} = -25\times10^{-6}$

$\therefore \sigma_1 = E\times e_1 = 2\times10^5 \times (-50\times10^{-6}) = -10$ N/mm² (C)

$\sigma_2 = 2\times10^5 \times (-25\times10^{-6}) = -5$ N/mm² (C)

Scanned with CamScanner

④ Find nodal displacements, stresses & strains & Reactions.



Sol: Element ①:

$$K_1 = \frac{A_1 E_1}{l_1} = \frac{25\times 2\times 10^5}{150} = 3.4\times 10^5 \text{ N/mm}$$

element ②:

$$K_2 = 3.4\times 10^5 \text{ N/mm}$$

element ③:

$$K_3 = \frac{A_3 E_3}{l_3} = \frac{400\times 2\times 10^5}{300} = 2.67\times 10^5$$

Stiffness Matrix:

- element ①:

$$3.4\times 10^5 \begin{array}{c|c} & \begin{array}{cc} 1 & 2 \end{array} \\ \begin{array}{c} 1 \\ 2 \end{array} & \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$$

- element ②:

$$3.4\times 10^5 \begin{array}{c|c} & \begin{array}{cc} 2 & 3 \end{array} \\ \begin{array}{c} 2 \\ 3 \end{array} & \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array} \begin{bmatrix} U_2 \\ U_3 \end{bmatrix} = \begin{bmatrix} F_2 \\ F_3 \end{bmatrix}$$

- element ③:

$$2.67\times 10^5 \begin{array}{c|c} & \begin{array}{cc} 3 & 4 \end{array} \\ \begin{array}{c} 3 \\ 4 \end{array} & \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array} \begin{bmatrix} U_3 \\ U_4 \end{bmatrix} = \begin{bmatrix} F_3 \\ F_4 \end{bmatrix}$$

→ combine matrix:

$$10^5 \begin{bmatrix} 3.4 & -3.4 & 0 & 0 \\ -3.4 & 3.4+3.4 & -3.4 & 0 \\ 0 & -3.4 & 3.4+2.67 & -2.67 \\ 0 & 0 & -2.67 & 2.67 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{bmatrix}$$

$$10^5 \begin{bmatrix} 3.4 & -3.4 & 0 & 0 \\ -3.4 & 6.8 & -3.4 & 0 \\ 0 & -3.4 & 6.07 & -2.67 \\ 0 & 0 & -2.67 & 2.67 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{bmatrix}$$

$\because$ $U_1 = 0$ & $U_4 = 0$ (because fixed)

& $F_3 = 0$

$$\begin{bmatrix} 6.8 & -3.4 \\ -3.4 & 6.07 \end{bmatrix} \begin{bmatrix} U_2 \\ U_3 \end{bmatrix} = \begin{bmatrix} F_2 \\ 0 \end{bmatrix} \qquad [\because F_2 = 300\times 10^3]$$

$\Rightarrow$ $6.8U_2 - 3.4U_3 = 300\times 10^3$ — ①

$3.4U_2 + 6.07U_3 = 0$ — ②

solve ① & ②

$U_2 = 0.612$ mm

$U_3 = 0.343$ mm

→ Reactions:- @① & ④

@① :-

$3.4\cancel{U_1}^{0} - 3.4U_2 = F_1$

$-3.4(0.612) = F_1$

@④ :

$-2.67U_3 + 2.67\cancel{U_4}^{0} = F_4$

$-2.67(0.343) = F_4$

→ Stresses & strains:-

$e_1 = \frac{U_2 - U_1}{l_1} = \frac{0.612 - 0}{150} = 4.153\times10^{-3}$

$e_2 = \frac{U_3 - U_2}{l_2} = \frac{0.343 - 0.612}{150} = -1.847\times10^{-3}$

$e_3 = \frac{U_4 - U_3}{l_3} = \frac{0 - 0.346}{300} = -1.153\times10^{-3}$

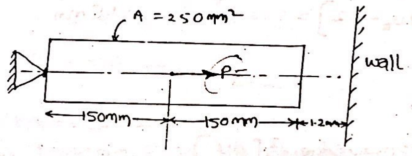
⇒ $\sigma_1 = E\times e_1 = 820\ N/mm^2$

$\sigma_2 = E\times e_2 = -369.4\ N/mm^2$

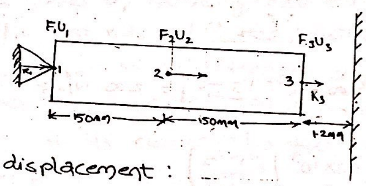
$\sigma_3 = E\times e_3 = -230.6\ N/mm^2$

Q. A rod is subjected to an axial load P = 600 kN. Find the nodal displacements, element stresses & support reactions.

Take E = 200 kN/mm².



Sol:-



nodal displacement :

$K_1 = K_2 = \frac{AE}{l} = \frac{250\times200\times10^3}{150} = \frac{10^6}{3}$ N/mm

$$\frac{10^6}{3}\begin{bmatrix}1 & -1 & 0\\ -1 & 2 & -1\\ 0 & -1 & 1\end{bmatrix}\begin{Bmatrix}U_1\\U_2\\U_3\end{Bmatrix} = \begin{Bmatrix}F_1\\F_2\\F_3\end{Bmatrix}$$

$U_1 = 0$ (∵ fixed)

$U_3 = 1.2$ mm

because when load is applied the beam will touch the wall

$$\frac{10^6}{3}\begin{bmatrix}1&-1&0\\-1&2&-1\\0&-1&1\end{bmatrix}\begin{Bmatrix}0\\U_2\\1.2\end{Bmatrix}=\begin{Bmatrix}F_1\\600\times10^3\\F_3\end{Bmatrix}$$

$\Rightarrow \frac{10^6}{3}\left[2U_2-1.2\right]=600\times10^3 \Rightarrow U_2 = 1.5\,mm$

→ Stresses:

$\sigma_1 = Ee_1 = 200\times10^3\left[\frac{du}{dx}\right]_1$

$= 200\times10^3\left[\frac{U_2-U_1}{l_1}\right]$

$\therefore = 200\times10^3\left[\frac{1.5-0}{150}\right] = 200\ N/mm^2\ (T)$

$\sigma_2 = Ee_2 = 200\times10^3\left[\frac{U_3-U_2}{l_2}\right]$

$= 200\times10^3\left[\frac{1.2-1.5}{150}\right] = -400\ N/mm^2\ (C)$

Support

→ Reaction:- $\boxed{R = K[U]-F}$ $\Rightarrow \boxed{R = KU - F}$ $F = KU$

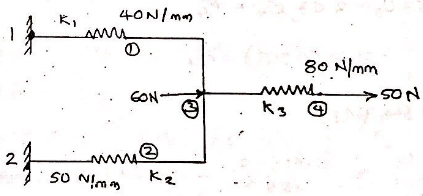
$$\begin{Bmatrix}R_1\\R_2\\R_3\end{Bmatrix}=\frac{10^6}{3}\begin{bmatrix}1&-1&0\\-1&2&-1\\0&-1&1\end{bmatrix}\begin{Bmatrix}0\\1.5\\1.2\end{Bmatrix}-\begin{Bmatrix}0\\600\times10^3\\0\end{Bmatrix}$$

$R_1 = \frac{10^6}{3}\left[(-1\times1.5)-0\right] = -500\,KN$ (left side)

$R_2 = \frac{10^6}{3}\left[2\times1.5+(-1\times1.2)-600\times10^3\right] = 0\,KN$

$R_3 = \frac{10^6}{3}\left[(-1\times1.5)+(1\times1.2)\right] = -100\,KN$ (left side)

Q. Determine the displacement at the nodes of the spring



Sol:- $K_1 = 40\ N/mm$

$K_2 = 50\ N/mm$

$K_3 = 80\ N/mm$

→ for element 1, which is in between node 1 & 3.

F.E. equation is

$$\begin{array}{c|cc} & 1 & 3\\ \hline 1 & K_1 & -K_1\\ 3 & -K_1 & K_1\end{array}\begin{Bmatrix}U_1\\U_3\end{Bmatrix}=\begin{Bmatrix}F_1\\F_3\end{Bmatrix}$$

→ element 2, in between 2 & 3 (nodes)

$$\begin{array}{c c} & \begin{array}{cc} 2 & 3 \end{array} \\ \begin{array}{c} 2 \\ 3 \end{array} & \begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \end{array} \begin{bmatrix} U_2 \\ U_3 \end{bmatrix} = \begin{bmatrix} F_2 \\ F_3 \end{bmatrix}$$

→ element 3, is between nodes 3 & 4

$$\begin{array}{c c} & \begin{array}{cc} 3 & 4 \end{array} \\ \begin{array}{c} 3 \\ 4 \end{array} & \begin{bmatrix} k_3 & -k_3 \\ -k_3 & k_3 \end{bmatrix} \end{array} \begin{bmatrix} U_3 \\ U_4 \end{bmatrix} = \begin{bmatrix} F_3 \\ F_4 \end{bmatrix}$$

∴ Global finite element matrix:

$$\begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} k_1 & 0 & -k_1 & 0 \\ 0 & k_2 & -k_2 & 0 \\ -k_1 & -k_2 & (k_1+k_2+k_3) & -k_3 \\ 0 & 0 & -k_3 & k_3 \end{bmatrix} \end{array} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{bmatrix}$$

$U_1 = U_2 = 0.$

$F_3 = 60\ N$

$F_4 = 50\ N$

$$\begin{bmatrix} 40 & 0 & -40 & 0 \\ 0 & 50 & -50 & 0 \\ -40 & -50 & 170 & -80 \\ 0 & 0 & -80 & 80 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ U_3 \\ U_4 \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \\ 60 \\ 50 \end{bmatrix}$$

(∵ eliminate 1st & 2nd row & column ⇒ $U_1 = U_2 = 0$)

$170\,U_3 - 80\,U_4 = 60$

$-80\,U_3 + 80\,U_4 = 50$

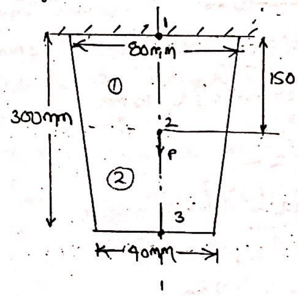
by solving $U_3 = 1.2\,mm$, $U_4 = 1.85\,mm$

Reaction forces F_1 & F_2

$F_1 = -40\,U_3 = -49\,N$

$F_2 = -50\,U_3 = -61\,N.$

*** (6) For a tapered bar of uniform thickness $t = 10\,mm$. Find the displacement at the nodes by forming into two element model. The bar has mass density $\rho = 7800\,kg/m^3$, $E = 2\times10^5\,N/mm^2$. In addition to self weight, the bar is subjected to a point load $P = 1\,KN$ at its centre. Also determine the reaction for at the support?



→ For tapered bar :-

area at node 1 = width × thickness

$= 80 \times 10$

$= 800 \text{ mm}^2$ ——①

area at node 3 $= w \times t$

$= 40 \times 10$

$= 400 \text{ mm}^2$ ——②

area at node 2 $= \frac{① + ②}{2}$

$= \frac{800+400}{2} = 600 \text{ mm}^2$.

→ For the stepped bar :-

C/s area at element 1 :-

$A_1 = \frac{(\text{area at node 1} + \text{area of node 2})}{2}$

$= \frac{800+600}{2} = 700 \text{ mm}^2$.

$= (70 \times 10) \text{ mm}^2$

width = 70 mm, t = 10 mm

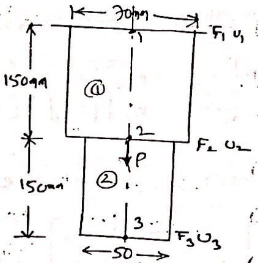
C/s area at element 2 :-

$A_2 = \frac{\text{area at node 2} + \text{area of node 3}}{2}$

$= \frac{600+400}{2} = 500 \text{ mm}^2$

$\therefore w = 50 \text{ mm} ; t = 10 \text{ mm}$

Now, global finite element equation (matrix form)



$$\begin{bmatrix} K_1 & -K_1 & 0 \\ -K_1 & K_1+K_2 & -K_2 \\ 0 & -K_2 & K_2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix}$$

$K_1 = \frac{A_1 E}{l_1} = \frac{700 \times 2 \times 10^5}{150}$

$\therefore k_1 = \frac{14}{15} \times 10^6 \text{ N/mm}$

$K_2 = \frac{A_2 E}{l_2} = \frac{500 \times 2 \times 10^5}{150} = \frac{10}{15} \times 10^6 \text{ N/mm}$

$$\Rightarrow \frac{10^6}{15} \begin{bmatrix} 14 & -14 & 0 \\ -14 & 24 & -10 \\ 0 & -10 & 10 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix}$$

F_1, F_2 & F_3 are unknown

at node 1, $F_1 = \frac{\text{body force of element 1}}{2}$

$= \frac{W_1}{2}$

at node 2 : $F_2 = \frac{\text{body force of ①}}{2} + \frac{\text{body force of ②}}{2} + \text{Point load}$

$= \frac{W_1}{2} + \frac{W_2}{2} + P.$

at node 3 : $F_3 = \frac{\text{body force of } ②}{2} = \frac{W_2}{2}$

$\because \rho = \frac{m}{v} \Rightarrow m = \rho . v$

$\Rightarrow \frac{W}{g} = \rho A l$

$\Rightarrow W = \rho g A l$

$\therefore W_1 = \rho g A_1 l_1 = 7800\times10^{-9}\times9.81\times700\times150 = 8.034\,N$

$W_2 = \rho g A_2 l_2 = 7800\times10^{-9}\times9.81\times500\times150 = 5.739\,N$

$\therefore F_1 = \frac{8.034}{2} = 4.017\,N$

$F_2 = \frac{8.034}{2} + \frac{5.739}{2} + 1000 = 1006.8\,N$

$F_3 = \frac{5.739}{2} = 2.870\,N$

$U_1 = 0$ ($\because$ fixed)

$$\frac{10^6}{15}\begin{bmatrix} 14 & -14 & 0 \\ -14 & 24 & -10 \\ 0 & -10 & 10 \end{bmatrix}\begin{bmatrix} 0 \\ U_2 \\ U_3 \end{bmatrix} = \begin{bmatrix} 4.017 \\ 1006.8 \\ 2.87 \end{bmatrix}$$

$\Rightarrow \frac{10^6}{15}[24U_2 - 10U_3] = 1006.8$

$\Rightarrow \frac{10^6}{15}[-10U_2 + 10U_3] = 2.870$

by solving

$U_2 = 1082\times10^{-6}$ mm

$U_3 = 1086\times10^{-6}$ mm

Reaction force at support :

we know $R_1 = F_1 + F_2 + F_3$

$= 4.107 + 1006.8 + 2.87 = 1013.8\,N$ (upward)

$\therefore$ the reaction will be $-1013.8\,N$ (opp)

check: consider 1st row, [U] & f_1 i.e.,

$[K][U] - [F] = R$

$$\Rightarrow R_1 = \frac{10^6}{15}\begin{bmatrix} 14 & -14 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1082\times10^{-6} \\ 1086\times10^{-6} \end{bmatrix} - 4.107$$

$\Rightarrow R_1 = \frac{10^6}{15}[-14\times1082\times10^{-6}] - 4.107$

$\Rightarrow R_1 = -1013.8\,N$

<u>Galerkin's approach</u> :-

This is one of the FEA technique to obtain the approximate solution to linear and non - linear diff equations. In this method, the trial function $y(x)$ is considered as weighing function $W_i = y(x)$. The domain integral of product of trial function with residue set is equal to zero (0).

$$\int_D W_i R(x, a_i)\,dx = \int_D y(x) R(x, a_i)\,dx = 0$$

$$= \int (y \cdot R)\,dx = 0$$

Problem:

1) Solve the diff equation $\frac{d^2y}{dx^2} + 300x^2 = 0$; $0 \le x \le 1$. Using the trial function $y = ax(1-x^3)$. Use Galerkins method with boundary condition $y(0)=0$; $y(1)=0$. ?

Sol: Given $y = a(x - x^4)$

$$\frac{dy}{dx} = a(1-4x^3)$$

$$\frac{d^2y}{dx^2} = a(-12x^2) = -12ax^2$$

∴ Residue : $R = \frac{d^2y}{dx^2} + 300x^2$

$$= -12ax^2 + 300x^2$$

∴ by Galerkins approach

$$\int_0^1 (y \cdot R)\,dx = 0$$

$$\int_0^1 a(x-x^4)(-12ax^2+300x^2)\,dx = 0$$

$$\Rightarrow \int_0^1 (-12a^2x^3 + 300ax^3 + 12a^2x^6 - 300ax^6)\,dx = 0$$

$$\Rightarrow \left[-12a^2\frac{x^4}{4} + 300a\frac{x^4}{4} + 12a^2\frac{x^7}{7} - 300a\frac{x^7}{7}\right]_0^1 = 0$$

$$-3a^2 + 75a + \frac{12}{7}a^2 - \frac{300}{7}a = 0$$

$$\left[\frac{12}{7} - 3\right]a^2 + \left[75 - \frac{300}{7}\right]a = 0$$

$$-\frac{9}{7}a^2 + \frac{225}{7}a = 0$$

$$\Rightarrow \boxed{a = 25}$$

$$\boxed{\therefore y = 25x(1-x^3)}$$

* Quadratic shape function :-

The noded one dimensional element is known as quadratic shape function.

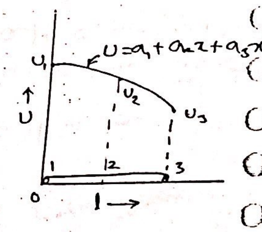
$$U(x) = N_1U_1 + N_2U_2 + N_3U_3$$

3m

$$N_1 = 1 - \frac{3x}{l} + \frac{2x^2}{l^2}$$

$$N_2 = \frac{4x}{l} - \frac{4x^2}{l^2}$$

$$N_3 = -\frac{x}{l} + \frac{2x^2}{l^2}$$



* Natural Co-ordinate system

It is a local co-ordinate system in which the magnitude of nodal values varies from (0 to 1) or (-1 to 1).

Problem on strain – displacement relation.

1) If the displacement field is described by

$U = (-x^2 + 2y^2 + 6xy) \times 10^{-4}$

$V = (3x + 6y - y^2) \times 10^{-4}$

determine e_x, e_y and ϕ_{xy} at the point $x = 1,\ y = 0$?

$$e_x = \frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left(-x^2 + 2y^2 + 6xy\right) \times 10^{-4}$$

$$= (-2x + 6y) \times 10^{-4}$$

$$\Rightarrow e_{x(1,0)} = -2 \times 10^{-4}$$

$$e_y = \frac{\partial v}{\partial y} = \frac{\partial}{\partial y}\left[3x + 6y - y^2\right] \times 10^{-4}$$

$$= (6 - 2y) \times 10^{-4}$$

$$e_{y(1,0)} = 6 \times 10^{-4}$$

$$\phi_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}$$

$$= \frac{\partial}{\partial x}\left[\left[3x + 6y - y^2\right] \times 10^{-4}\right] + \frac{\partial}{\partial y}\left[-x^2 + 2y^2 + 6xy\right] \times 10^{-4}$$

$$= 3 \times 10^{-4} + (4y + 6x) \times 10^{-4}$$

$$\phi_{xy(1,0)} = (3 \times 10^{-4}) + (6 \times 10^{-4})$$

$$= 9 \times 10^{-4}$$

$$\therefore \begin{bmatrix} e_x \\ e_y \\ \phi_{xy} \end{bmatrix} = \begin{bmatrix} -2 \\ 6 \\ 9 \end{bmatrix} 10^{-4}$$

→ Short Answers :-

1) Write the compatibility conditions.

A: Sum of forces in any direction = 0, sum of moments = 0, sum of energies = 0.

2) Compatibility requirement

A: a) Displacement must be known b/w adjacent elements

b) When the elements deforms, their must not be any discontinuity b/w. elements.

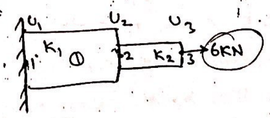
c) There should not be any change in slope of intermediate element

3) Global Cordnts & local co-ord.

4) Treatment of boundary conditions:

Any element can be divided into 'n' no of elements. Each element is specified by 2 nodes. Each node has boundary values from reference pts. The boundary condition may be 0, +ve & -ve

eg:



condition: $U_1 = 0$ (fixed)

U_2, U_3 } May be +ve or -ve

$F_1 = -R = -6\,KN$.

$F_2 = 0$ (no load)

$F_3 = 6\,KN$

UNIT – II

ANALYSIS OF TRUSSES AND FRAMES (BEAMS)

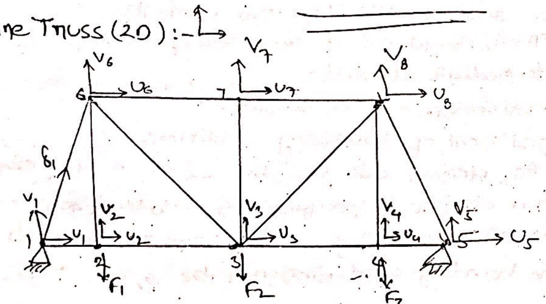
* TRUSSES :–

Truss is a structural member constructed by no of bars & L-angles and are connected each other firmly at their ends by means of bolts or rivets.

Trusses can only transmit forces, & they cannot move individually or relative to each other.

eg. building roof, railway bridges etc

* Plane Truss (2D) :–



The truss element can resists only axial forces (compressive or tensile) & can deform only in the axial direction.

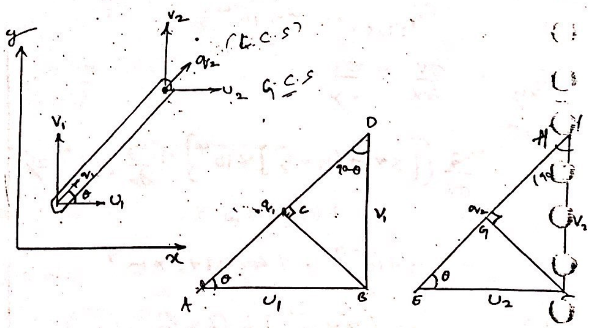
It cannot carry transverse loads or bending moments.

In trusses, it is required that all loads & reactions are applied only at the joints.

The stiffness of truss element depends upon area, youngs modulus, element length & its angular orientation.

* Derivation of Stiffness matrix for Trusses :–

Consider a pin-jointed bar element for Truss analysis. ... to load on the bar element, it will deform in its axial dirn to find nodal displacements, stress induced etc., we have to make use of the relation $[K][\delta]=[F]$



let q_1, q_2 be displacements of the nodes 1&2 in Local Corord system (LCS)

$U_1 V_1$ & $U_2 V_2$ be components of displacement q_1 q_2 in global co-ord system

from fig,

$$\cos\theta = \frac{AC}{AB} \Rightarrow \boxed{AC = AB.\cos\theta}$$

$$\cos(90-\theta) = \frac{CD}{BD} \Rightarrow \boxed{CD = BD.\sin\theta}$$

& $q_1 = AD = AC + CD = AB\cos\theta + BD\sin\theta$

$= U_1\cos\theta + V_1\sin\theta$ — (1)

$q_2 = EH = EG + GH = EF\cos\theta + FH\sin\theta$

$= U_2\cos\theta + V_2\sin\theta$ — (2)

matrix form of (1) & (2)

$$\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \end{bmatrix} \begin{bmatrix} U_1 \\ V_1 \\ U_2 \\ V_2 \end{bmatrix}$$

let $c = \cos\theta$; $\sin = s$

$$\therefore \underbrace{\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} c & s & 0 & 0 \\ 0 & 0 & c & s \end{bmatrix} \begin{bmatrix} U_1 \\ V_1 \\ U_2 \\ V_2 \end{bmatrix}}$$

$\Rightarrow [q] = [L][\delta']$

$[L]$ = Transformation matrix $= \begin{bmatrix} c & s & 0 & 0 \\ 0 & 0 & c & s \end{bmatrix}$

$[q]$ = element displacement vector L.C.S

$[\delta']$ = element displacement vector in G.C.S.

the element stiffness matrix for truss in LCS (1-D)

$$K_1 = \frac{AE}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

* element stiffness matrix as per G.C.S

$K = [L]^T[K_1][L]$

$$= \begin{bmatrix} c & 0 \\ s & 0 \\ 0 & c \\ 0 & s \end{bmatrix} \frac{AE}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} c & s & 0 & 0 \\ 0 & 0 & c & s \end{bmatrix}$$

$$= \frac{AE}{l}\underbrace{\begin{bmatrix} c^2 & cs & -c^2 & -cs \\ cs & s^2 & -cs & -s^2 \\ -c^2 & -cs & c^2 & cs \\ -cs & -s^2 & cs & s^2 \end{bmatrix}}$$

$\therefore [K][\delta'] = [F]$

$$\frac{AE}{l}\begin{bmatrix} c^2 & cs & -c^2 & -cs \\ cs & s^2 & -cs & -s^2 \\ -c^2 & -cs & c^2 & cs \\ -cs & -s^2 & cs & s^2 \end{bmatrix}\begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} f_{1x} \\ f_{1y} \\ f_{2x} \\ f_{2y} \end{bmatrix}$$

$c = \cos\theta$; $s = \sin\theta$; $\theta = \angle^{le}$ of truss element measured from global plus x-axis

If the locations of the primary nodes 1, 2 are assumed as (x_1, y_1) (x_2, y_2) & length of the truss element as l.

then

$$\cos\theta = \frac{x_2 - x_1}{l}$$

$$\sin\theta = \frac{y_2 - y_1}{l}$$

$$l = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Main Note: Consider Nodes from left to right and down to up

Stress calculations:

$$\sigma = E \cdot e$$

$$= E \cdot \left[\frac{q_2 - q_1}{l}\right]$$

$$= E\left[\frac{q_2}{l} - \frac{q_1}{l}\right]$$

$$= E\left[\frac{-q_1}{l} \quad \frac{q_2}{l}\right]$$

$$\boxed{\sigma = \frac{E}{l}\begin{bmatrix} -1 & 1 \end{bmatrix}\begin{bmatrix} q_1' \\ q_2 \end{bmatrix}}$$

$$= \frac{E}{l}\begin{bmatrix} -1 & 1 \end{bmatrix}\begin{bmatrix} c & s & 0 & 0 \\ 0 & 0 & c & s \end{bmatrix}\begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{bmatrix}$$

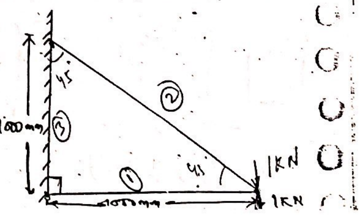
$$\boxed{\therefore \sigma = \frac{E}{l}\begin{bmatrix} -c & -s & c & s \end{bmatrix}\begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{bmatrix}}$$

* Analysing procedure for Truss elements:-

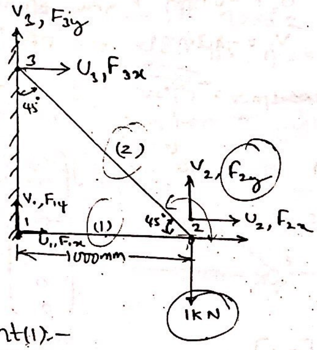
(1) Divide the given truss elements (assembly) into a suitable no of truss elements & name them as (1), (2), (3) etc in counter-clockwise direction from left side usually from left bottom of the assembly, & name the nodes as (1), (2), (3) from left bottom in anti-clockwise direction.

* Problems:-

1) A truss structure is subjected to a load of 1kN, calculate the nodal displacements & forces if the element stiffness of the truss is 10 kN/mm.?



Soln



→ for element(1):-

$K_1 = 10 \times 10^3$ N/mm

$\boxed{\theta_1 = 0°}$ (∵ as it is @ fixed) @ end corner

$C_1 = \cos 0° = 1$; $C_1 S_1 = 0$

$S_1 = \sin 0° = 0$; $C^2 = 1$; $S^2 = 0$

wkt,

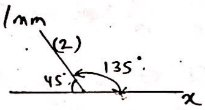
$$[K_1] = \begin{array}{c} \begin{array}{cccc} U_1 & V_1 & U_2 & V_2 \end{array} \\ \begin{bmatrix} c^2 & cs & -c^2 & -cs \\ cs & s^2 & -cs & -s^2 \\ -c^2 & -cs & c^2 & cs \\ -cs & -s^2 & cs & s^2 \end{bmatrix} \end{array} \begin{array}{c} U_1 \\ V_1 \\ U_2 \\ V_2 \end{array}$$

$$\Rightarrow [K_1] = 10 \times 10^3 \begin{array}{c} \begin{array}{cccc} U_1 & V_1 & U_2 & V_2 \end{array} \\ \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{c} U_1 \\ V_1 \\ U_2 \\ V_2 \end{array}$$

→ for element(2):

$K_2 = 10 \times 10^3$ N/mm

$\theta_2 = 135°$



$C_2 = \cos 135$

$= -0.707$

$C_2^2 = 0.49 = 0.5$

$S_2 = \sin 135 = 0.707$, $S_2^2 = 0.49 = 0.5$, $C_2 S_2 = (-0.707) \times (0.707)$

$\therefore C_2 S_2 = -0.5$

$$\therefore [K_2] = 10 \times 10^3 \begin{array}{c} \begin{array}{cccc} U_2 & V_2 & U_3 & V_3 \end{array} \\ \begin{bmatrix} 0.5 & -0.5 & -0.5 & 0.5 \\ -0.5 & 0.5 & 0.5 & -0.5 \\ -0.5 & 0.5 & 0.5 & -0.5 \\ 0.5 & -0.5 & -0.5 & 0.5 \end{bmatrix} \end{array} \begin{array}{c} U_2 \\ V_2 \\ U_3 \\ V_3 \end{array}$$

∴ Global stiffness matrix:

$$[K] = 10^4 \begin{array}{c} \begin{array}{cccccc} U_1 & V_1 & U_2 & V_2 & U_3 & V_3 \end{array} \\ \begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1+0.5 & 0-0.5 & -0.5 & 0.5 \\ 0 & 0 & -0.5 & 0.5 & 0.5 & -0.5 \\ 0 & 0 & -0.5 & 0.5 & 0.5 & 0.5 \\ 0 & 0 & 0.5 & -0.5 & -0.5 & 0.5 \end{bmatrix} \end{array} \begin{array}{c} U_1 \\ V_1 \\ U_2 \\ V_2 \\ U_3 \\ V_3 \end{array}$$

$\therefore [K][\delta'] = [F]$

$$10^4 \begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1.5 & -0.5 & -0.5 & 0.5 \\ 0 & 0 & -0.5 & 0.5 & 0.5 & -0.5 \\ 0 & 0 & -0.5 & 0.5 & 0.5 & -0.5 \\ 0 & 0 & 0.5 & -0.5 & -0.5 & 0.5 \end{bmatrix} \begin{bmatrix} U_1 \\ V_1 \\ U_2 \\ V_2 \\ U_3 \\ V_3 \end{bmatrix} = \begin{bmatrix} F_{1x} \\ F_{1y} \\ F_{2x} \\ F_{2y} \\ F_{3x} \\ F_{3y} \end{bmatrix}$$

Applying boundary conditions

$U_1 = 0;\ V_1 = 0;\ U_3 = 0;\ V_3 = 0;\ F_{2y} = -1000N$

$$10^4 \begin{bmatrix} 1.5 & -0.5 \\ -0.5 & 0.5 \end{bmatrix} \begin{bmatrix} U_2 \\ V_2 \end{bmatrix} = \begin{bmatrix} F_{2x} \\ F_{2y} \end{bmatrix}$$

$$\Rightarrow 10^4 \begin{bmatrix} 1.5 & -0.5 \\ -0.5 & 0.5 \end{bmatrix} \begin{bmatrix} U_2 \\ V_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -1000 \end{bmatrix}$$

$\Rightarrow 10^4 (1.5 U_2 - 0.5 V_2) = 0$

$\Rightarrow 1.5 U_2 - 0.5 V_2 = 0$ —— ①

$-0.5 U_2 + 0.5 V_2 = -1000$ —— ②

by solving $U_2 = -0.1mm$

$V_2 = -0.3mm$

Reaction forces:

$\rightarrow F_{1x} = 10^4 \begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -0.1 \\ -0.3 \\ 0 \\ 0 \end{bmatrix}$

$= 10^4 [0.1]$

$\therefore F_{1x} = 1000\ N$

$\rightarrow F_{1y} = 10^4 \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -0.1 \\ -0.3 \\ 0 \\ 0 \end{bmatrix}$

$= 0N.$

$\rightarrow F_{3x} = 10^4 \begin{bmatrix} 0 & 0 & -0.5 & 0.5 & 0.5 & -0.5 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -0.1 \\ -0.3 \\ 0 \\ 0 \end{bmatrix}$

$= 10^4 (0.5 \times 0.1 - 0.5 \times 0.3)$

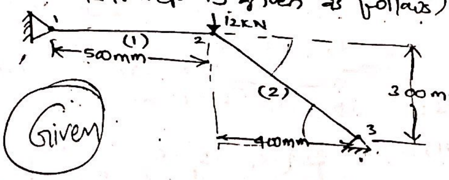
$= -1000N.$

$\rightarrow F_{3y} = 10^4 \begin{bmatrix} 0 & 0 & 0.5 & -0.5 & -0.5 & 0.5 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -0.1 \\ -0.3 \\ 0 \\ 0 \end{bmatrix}$

$= 10^4 [(0.5 \times 0.1) + (0.5 \times 0.3)]$

$= 1000 N.$

Q4 the displacement at node 2 & stresses in both the elements. $A = 200\,mm^2$, $E = 70\times10^3\ N/mm^2$. for both members. (nodes & element rep is given as follows)



Given

Sol: → for element ①:-

$A_1 = 200\,mm^2$, $E = 70\times10^3\ N/mm^2$, $l_1 = 500\,mm$

$\theta_1 = 0^\circ$; $C_1 = \cos 0^\circ = 1$; $S_1 = \sin 0^\circ = 0$; $C_1S_1 = 0$

$C_1^2 = 1$; $S_1^2 = 0$

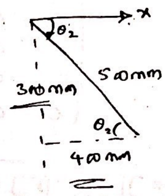
$$[K_1] = \underbrace{\frac{200\times70\times10^3}{500}}_{\frac{AE}{l}} \begin{array}{c} \begin{array}{cccc} u_1 & v_1 & u_2 & v_2 \end{array} \\ \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{c} u_1 \\ v_1 \\ u_2 \\ v_2 \end{array}$$

$$(k_1) = 10^3 \begin{bmatrix} 28 & 0 & -28 & 0 \\ 0 & 0 & 0 & 0 \\ -28 & 0 & 28 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

→ for element ②:-

$A_2 = 200\,mm^2$; $E_2 = 70\times10^3\ N/mm^2$

$\theta_2 = \sin^{-1}(0.6) = 36.87$.



[∵ from fig,
$\sin\theta_2 = \frac{300}{500} = 0.6$]

$l_2 = \sqrt{(400)^2 + (300)^2} = 500\,mm$

$C_2 = \cos 36.87 = 0.79 \Rightarrow C_2^2 = 0.64$

$S_2 = \sin(36.87) = 0.60 \Rightarrow S_2^2 = 0.36$; $C_2S_2 = 0.474$

$$K_2 = \frac{200\times70\times10^3}{500} \begin{array}{c} \begin{array}{cccc} u_2 & v_2 & u_3 & v_3 \end{array} \\ \begin{bmatrix} 0.64 & 0.47 & -0.64 & -0.47 \\ 0.47 & 0.36 & -0.47 & -0.36 \\ -0.64 & -0.47 & 0.64 & 0.47 \\ -0.47 & -0.36 & 0.47 & 0.64 \end{bmatrix} \end{array} \begin{array}{c} u_2 \\ v_2 \\ u_3 \\ v_3 \end{array}$$

$$= 10^3 \begin{bmatrix} 17.92 & 13.16 & -17.92 & -13.16 \\ 13.16 & 10.08 & -13.16 & -10.08 \\ -17.92 & -13.16 & 17.92 & 13.16 \\ -13.16 & -10.08 & 13.16 & 10.08 \end{bmatrix}$$

→ Global stiffness matrix is $K = K_1 + K_2$

$$K = 10^3 \begin{pmatrix} 28 & 0 & -28 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -28 & 0 & 45.92 & 13.16 & -17.9 & 13.16 \\ 0 & 0 & 13.16 & 10.08 & -13.16 & -10.08 \\ 0 & 0 & -17.92 & -13.16 & 17.92 & 13.16 \\ 0 & 0 & -13.16 & -10.08 & 13.16 & 10.08 \end{pmatrix}$$

$$\therefore \boxed{[K][\delta] = [F]}$$

$$10^3 \begin{bmatrix} 28 & 0 & -28 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -28 & 0 & 45.92 & 13.16 & -17.92 & -13.16 \\ 0 & 0 & 13.16 & 10.08 & -13.16 & -10.08 \\ 0 & 0 & -17.92 & -13.16 & 17.92 & 13.16 \\ 0 & 0 & -13.16 & -10.08 & 13.16 & 10.08 \end{bmatrix} \begin{pmatrix} U_1 \\ V_1 \\ U_2 \\ V_2 \\ U_3 \\ V_3 \end{pmatrix} = \begin{pmatrix} F_{1x} \\ F_{1y} \\ F_{2x} \\ F_{2y} \\ F_{3x} \\ F_{3y} \end{pmatrix}$$

boundary conditions:

$U_1 = 0$; $V_1 = 0$; $U_3 = 0$; $V_3 = 0$

$F_{2y} = -12000\,N$, $F_{2x} = 0$

$$10^3 \begin{pmatrix} 45.92 & 13.16 \\ 13.16 & 10.08 \end{pmatrix} \begin{pmatrix} U_2 \\ V_2 \end{pmatrix} = \begin{pmatrix} 0 \\ -12 \times 10^3 \end{pmatrix}$$

$10^3 (45.92 U_2 + 13.16 V_2) = 0$

$10^3 (13.6 U_2 + 10.08 V_2) = -12 \times 10^3$

by solving $U_2 = 0.545$ mm

$V_2 = -1.9$ mm

→ stresses :-

element (1) :- $\sigma_1 = \frac{E_1}{l_1} \left[-C_1 \quad -S_1 \quad C_1 \quad S_1 \right] \begin{pmatrix} U_1 \\ V_1 \\ U_2 \\ V_2 \end{pmatrix}$

$= \frac{70 \times 10^3}{500} \left[-1 \quad 0 \quad 1 \quad 0 \right] \begin{pmatrix} 0 \\ 0 \\ 0.545 \\ -1.90 \end{pmatrix}$

$= 0.14 \times 10^3 (0.545)$

$= 76.3\ N/mm^2$ (T)

element (2) :- $\sigma_2 = \frac{E_2}{l_2} \left[-C_2 \quad -S_2 \quad C_2 \quad S_2 \right] \begin{pmatrix} U_2 \\ V_2 \\ U_3 \\ V_3 \end{pmatrix}$

$= \frac{70 \times 10^3}{500} \left(-0.79 \quad 0.6 \quad 0.79 \quad 0.6 \right) \begin{pmatrix} 0.545 \\ -1.90 \\ 0 \\ 0 \end{pmatrix}$

$\therefore \sigma_2 = 0.14 \times 10^3 (-0.79 \times 0.545 + 0.6 \times 1.9)$

$= 99.3\ N/mm^2$ (T)

(3) The loading & other parameters for a two-bar truss element as shown in fig. Det

(a) The element stiffness matrix for each element

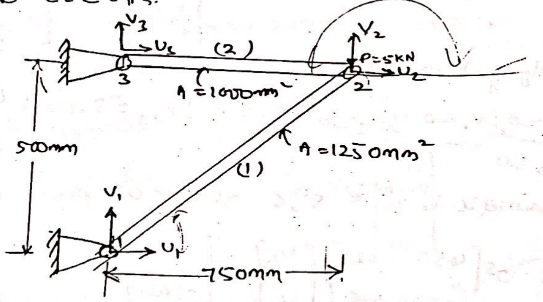
(b) Global stiffness matrix

(c) nodal displacements

(d) Reaction forces &

(e) Stress induced in the elements

Assume E = 200 GPa.



Sol:- for element (1):-

$A_1 = 1250\,mm^2$; $l_1 = \sqrt{500^2 + 750^2} = 901.4\,mm$

$E_1 = 200 \times \frac{10^9}{10^6}\,N/mm^2 = 2 \times 10^5\,N/mm^2$

$\theta_1 = \tan^{-1}\left(\frac{500}{750}\right) = 33.7°$

$c_1 = \cos 33.7 = 0.832 \Rightarrow c_1^2 = 0.692$

$s_1 = \sin 33.7 = 0.555 \Rightarrow s_1^2 = 0.308$

$c_1 s_1 = 0.462$

$$[K_1^0] = \frac{A_1 E_1}{l_1}\begin{bmatrix} c_1^2 & c_1 s_1 & -c_1^2 & -c_1 s_1 \\ c_1 s_1 & s_1^2 & -c_1 s_1 & -s_1^2 \\ -c_1^2 & -c_1 s_1 & c_1^2 & c_1 s_1 \\ c_1 s_1 & -s_1^2 & c_1 s_1 & s_1^2 \end{bmatrix}$$

$$= \frac{1250 \times 20 \times 10^3}{90.4}\begin{bmatrix} 0.692 & 0.462 & -0.692 & -0.462 \\ 0.462 & 0.308 & -0.462 & -0.308 \\ -0.692 & -0.462 & 0.612 & 0.462 \\ -0.462 & -0.308 & 0.462 & 0.308 \end{bmatrix}$$

$$= 10^5 \begin{array}{c} \begin{array}{cccc} u_1 & v_1 & u_2 & v_2 \end{array} \\ \begin{bmatrix} 1.92 & 1.28 & -1.92 & -1.28 \\ 1.28 & 0.85 & -1.28 & -0.85 \\ -1.92 & -1.28 & 1.92 & 1.28 \\ -1.28 & -0.85 & 1.28 & 0.85 \end{bmatrix} \end{array} \begin{array}{c} \\ u_1 \\ v_1 \\ u_2 \\ v_2 \end{array}$$

→ for element (2):-

$A_2 = 1000\,mm^2$; $l_2 = 750\,mm$; $E_2 = 2 \times 10^5\,N/mm^2$

$\theta_2 = 180°$, $c_2 = \cos 180 = -1 \Rightarrow c_2^2 = 1$; $c_2 s_2 = 0$

$s_2 = \sin 180 = 0 \Rightarrow s_2^2 = 0$

$$\therefore [K_2] = \frac{1000 \times 2 \times 10^5}{750} \begin{array}{c} \begin{array}{cccc} u_2 & v_2 & u_3 & v_3 \end{array} \\ \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{c} u_2 \\ v_2 \\ u_3 \\ v_3 \end{array}$$

$$= 10^5 \begin{array}{c} \begin{array}{cccc} u_2 & v_2 & u_3 & v_3 \end{array} \\ \begin{bmatrix} 2.67 & 0 & -2.67 & 0 \\ 0 & 0 & 0 & 0 \\ -2.67 & 0 & 2.67 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{c} u_2 \\ v_2 \\ u_3 \\ v_3 \end{array}$$

→ (b) Global stiffness matrix :- $K = K_1 + K_2$

$$K = 10^5 \begin{bmatrix} 1.92 & 1.28 & 1.92 & 1.28 & 0 & 0 \\ 1.28 & 0.85 & -1.28 & -0.85 & 0 & 0 \\ -1.92 & -1.28 & 4.59 & 1.28 & -2.67 & 0 \\ -1.28 & -0.85 & 1.28 & 0.85 & 0 & 0 \\ 0 & 0 & -2.67 & 0 & 2.67 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(c) nodal displacement :-

$$[K][\delta'] = [F]$$

$$10^5 \begin{bmatrix} 1.92 & 1.28 & -1.92 & -1.28 & 0 & 0 \\ 1.28 & 0.85 & -1.28 & -0.85 & 0 & 0 \\ -1.92 & -1.28 & 4.59 & 1.28 & -2.67 & 0 \\ -1.28 & -0.85 & 1.28 & 0.85 & 0 & 0 \\ 0 & 0 & -2.67 & 0 & 2.67 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{bmatrix} = \begin{bmatrix} F_{1x} \\ F_{1y} \\ F_{2y} \\ F_{2x} \\ F_{3x} \\ F_{3y} \end{bmatrix}$$

Applying boundary conditions

$u_1 = 0$; $v_1 = 0$; $u_3 = 0$; $v_3 = 0$; $F_{2y} = 5000N$

$F_{2x} = 0$.

eliminate 1st, 2nd & 5th, 6th row & column

$$\therefore 10^5 \begin{bmatrix} 4.59 & 1.27 \\ 1.28 & 0.85 \end{bmatrix} \begin{bmatrix} u_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -5000 \end{bmatrix}$$

$$\Rightarrow 10^5 [4.59 u_2 + 1.28 v_2] = 0$$

$$10^5 [1.28 u_2 + 0.85 v_2] = -5000N$$

by solving $u_2 = 0.028$ mm

$v_2 = -0.101$ mm

(d) Reaction forces:-

$$\therefore F_{1x} = 10^5 \begin{bmatrix} 1.92 & 1.28 & -1.92 & -1.28 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0.028 \\ 0.101 \\ 0 \\ 0 \end{bmatrix}$$

$$= 10^5 \left[-1.92 \times 0.028 + (-1.28 \times -0.1) \right]$$

$$= 7528 \text{ N.}$$

$$F_{1y} = 10^5 \left[-1.28 \times 0.028 + 0.85 \times 0.101 \right]$$

$$= 5000 \text{ N}$$

$$F_{3x} = 10^5 \left[-2.67 \times 0.028 \right] = -7476 \text{ N}$$

$$F_{3y} = 10^5 [0] = 0 \text{ N.}$$

(e) for element (1):-

$$\sigma_1 = \frac{E_1}{l_1} \begin{bmatrix} -c_1 & -s_1 & c_1 & s_1 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix}$$

$$= \frac{2 \times 10^5}{901.4} \begin{bmatrix} -0.832 & -0.555 & 0.832 & 0.555 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0.028 \\ -0.101 \end{Bmatrix}$$

$$= -7.32 \text{ N/mm}^2 \text{ (C)}$$

for element (2):

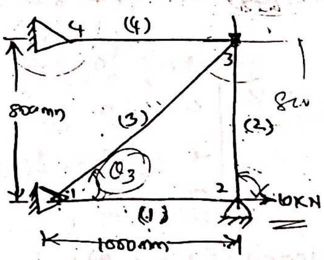
$$\sigma_2 = \frac{E}{l_2} \begin{bmatrix} -c_2 & -s_2 & +c_2 & s_2 \end{bmatrix} \begin{Bmatrix} u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}$$

$$= \frac{2 \times 10^5}{750} \begin{bmatrix} 1 & 0 & -1 & 0 \end{bmatrix} \begin{Bmatrix} 0.028 \\ -0.101 \\ 0 \\ 0 \end{Bmatrix}$$

$$= 7.47 \text{ N/mm}^2 \text{ (T)}$$

Note: consider points (links) from left to right and down to up.

✱ all the elements Det
a) nodal displacement
b) support reactions
c) element stresses



Sol. Element (1):

$A_1 = 500 \text{ mm}^2$

$E_1 = 2 \times 10^5 \text{ N/mm}^2$

$l_1 = 1000 \text{ mm}$, $\theta_1 = 0°$.

$c_1 = \cos 0 = 1 \Rightarrow c_1^2 = 1$

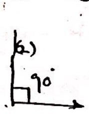
$s_1 = \sin 0 = 0 \Rightarrow s_1^2 = 0$

$$[K_1] = \frac{A_1 E_1}{l_1} \begin{bmatrix} c_1^2 & c_1 s_1 & -c_1^2 & -c_1 s_1 \\ c_1 s_1 & s_1^2 & -c_1 s_1 & -s_1^2 \\ -c_1^2 & -c_1 s_1 & c_1^2 & c_1 s_1 \\ -c_1 s_1 & -s_1^2 & c_1 s_1 & s_1^2 \end{bmatrix}$$

$$
= \frac{500\times10^5\times2}{1000}
\begin{array}{c}
\begin{array}{cccc} u_1 & v_1 & u_2 & v_2 \end{array} \\
\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}
\end{array}
\begin{array}{c} u_1 \\ v_1 \\ u_2 \\ v_2 \end{array}
$$

$$
= 10^5
\begin{array}{c}
\begin{array}{cccc} u_1 & v_1 & u_2 & v_2 \end{array} \\
\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}
\end{array}
\begin{array}{c} u_1 \\ v_1 \\ u_2 \\ v_2 \end{array}
$$

→element(2) :-



$A_2 = 500\,mm^2$

$E_2 = 2\times10^5\,N/mm^2$; $l_2 = 800\,mm$

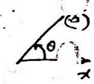
$\theta_2 = 90^\circ$; $C_2 = \cos 90 = 0$; $C_2^2 = 0$; $C_2S_2 = 0$

$S_2 = \sin 90 = 1$; $S_2^2 = 1$

$$
[K_2] = \frac{500\times2\times10^5}{800}
\begin{array}{c}
\begin{array}{cccc} u_2 & v_2 & u_3 & v_3 \end{array} \\
\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}
\end{array}
\begin{array}{c} u_2 \\ v_2 \\ u_3 \\ v_3 \end{array}
$$

$$
[K_2] = 10^5 \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1.25 & 0 & -1.25 \\ 0 & 0 & 0 & 0 \\ 0 & -1.25 & 0 & 1.25 \end{bmatrix}
$$

→Element(3):- $A_3 = 500\,mm^2$; $E_3 = 2\times10^5\,N/mm^2$



$l_3 = \sqrt{800^2+1000^2} = 1281\,mm$; $\theta_3 = \tan^{-1}\left(\frac{800}{1000}\right) = 38.7$

$C_3 = \cos(38.7) = 0.78 \Rightarrow C_3^2 = 0.608$

$S_3 = \sin(38.7) = 0.63 \Rightarrow S_3^2 = 0.397$; $C_3S_3 = 0.491$

$$
[K_3] = \frac{500\times2\times10^5}{1281}
\begin{bmatrix} 0.608 & 0.491 & -0.608 & -0.491 \\ 0.491 & 0.397 & -0.491 & -0.397 \\ -0.608 & -0.491 & 0.608 & 0.491 \\ -0.491 & -0.397 & 0.491 & 0.397 \end{bmatrix}
$$

$$
[K_3] = 10^5
\begin{array}{c}
\begin{array}{cccc} u_1 & v_1 & u_3 & v_3 \end{array} \\
\begin{bmatrix} 0.47 & 0.38 & -0.47 & -0.38 \\ 0.38 & 0.31 & -0.38 & -0.31 \\ -0.47 & -0.38 & 0.47 & 0.38 \\ -0.38 & -0.31 & 0.38 & 0.31 \end{bmatrix}
\end{array}
\begin{array}{c} u_1 \\ v_1 \\ u_3 \\ v_3 \end{array}
$$

→ element(4):- $A_4 = 500\,mm^2$; $E_4 = 2\times10^5\ N/mm^2$

$l_4 = 1000\,mm$; $\theta_4 = 180°$; $C_4 = \cos 180 = -1 \Rightarrow C_4^2 = 1$

$S_4 = \sin 180 = 0$; $S_4^2 = 0$

$C_4 S_4 = 0$

$$[K_4] = \frac{500\times2\times10^5}{1000}\begin{array}{c} \begin{array}{cccc} u_4 & v_4 & u_3 & v_3 \end{array} \\ \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}\begin{array}{c} u_4 \\ v_4 \\ u_3 \\ v_3 \end{array}$$

$$\therefore K_4 = 10^5 \begin{array}{c} \begin{array}{cccc} u_4 & v_4 & u_3 & v_3 \end{array} \\ \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}\begin{array}{c} u_4 \\ v_4 \\ u_3 \\ v_3 \end{array}$$

→ Global stiffness matrix: $K = K_1 + K_2 + K_3 + K_4$ & $[K][\delta] = F$

$$\begin{bmatrix} 1.47 & 0.38 & -1 & 0 & -0.47 & -0.38 & 0 & 0 \\ 0.38 & 0.31 & 0 & 0 & -0.38 & -0.31 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.25 & 0 & -1.25 & 0 & 0 \\ -0.47 & -0.38 & 0 & 0 & 1.47 & 0.38 & -1 & 0 \\ -0.38 & -0.31 & 0 & -1.25 & 0.38 & 1.56 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{bmatrix} = \begin{bmatrix} F_{1x} \\ F_{1y} \\ F_{2x} \\ F_{2y} \\ F_{3x} \\ F_{3y} \\ F_{4x} \\ F_{4y} \end{bmatrix}$$

applying boundary conditions:

$u_1 = 0$, $v_1 = 0$, $u_2 = 0$, $v_2 = 0$, $u_4 = 0$, $v_4 = 0$

$F_{3y} = -15\times10^3\,N$, $F_{3x} = 0$; $F_{2x} = 10\times10^3\,N$, $F_{2y} = 0$

$$10^5 \begin{bmatrix} 1.47 & 0.38 \\ 0.38 & 1.56 \end{bmatrix}\begin{bmatrix} u_3 \\ v_3 \end{bmatrix} = \begin{bmatrix} F_{3x} \\ F_{3y} \end{bmatrix} = \begin{bmatrix} 0 \\ -1.5\times10^3 \end{bmatrix}$$

$\Rightarrow 10^5(1.47u_3 + 0.38v_3) = 0$

$10^5(0.38u_3 + 1.56v_3) = -15\times10^3$

$$\left.\begin{array}{l} 1.47u_3 + 0.38v_3 = 0 \\ 0.38u_3 + 1.56v_3 = -15\times10^{-2} \end{array}\right\} \begin{array}{l} \text{by solving} \\ u_3 = 0.026\,mm \\ v_3 = -0.103\,mm \end{array}$$

→ (b) support Reactions:

$F_{1x} = 10^5(-0.47\times0.026 + 0.38\times0.103)$

$= 10^5(-0.01222 + 0.03914)$

$= 2692\,N$

$F_{1y} = 10^5(-0.38\times0.026 + 0.31\times0.103)$

$= 10^5(-0.00988 + 0.03193) = 2205\,N.$

$F_{2y} = 10^5(1.25\times0.103) = 12875\,N$

$F_{4x} = 10^5(-1\times0.026) = -2600\,N$

$F_{4y} = 10^5(0) = 0.$

→ Element stresses:-

element(1):-

$$\sigma_1 = \frac{E_1}{l_1}\begin{bmatrix} C_1 & -S_1 & C_1 & S_1 \end{bmatrix}\begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{bmatrix}$$

$$= \frac{2\times10^5}{1000}\begin{bmatrix} -1 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = 0$$

element(2):-

$$\sigma_2 = \frac{2\times10^5}{800}\begin{bmatrix} 0 & -1 & 0 & 1 \end{bmatrix}\begin{array}{c}\begin{bmatrix} 0 \\ 0 \\ 0.026 \\ -0.103 \end{bmatrix}\end{array}\begin{array}{l} u_2 \\ v_2 \\ u_3 \\ v_3 \end{array} = -25.75\ N/mm^2\ (C)$$

element(3):-

$$\sigma_3 = \frac{2\times10^5}{1281}\begin{bmatrix} 0.78 & -0.63 & 0.78 & 0.63 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 0.026 \\ -0.103 \end{bmatrix}\begin{array}{l} u_1 \\ v_1 \\ u_3 \\ v_3 \end{array}$$

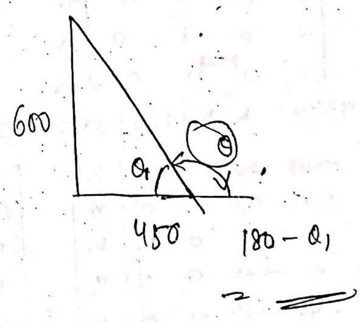
$$= 156128\,(0.0203 - 0.0649)$$

$$= -6.96\ N/mm^2\ (C)$$

element (4)

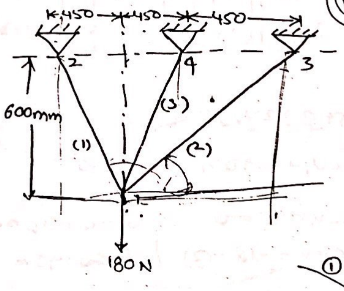
$$\sigma_4 = \frac{2\times10^5}{1050}\begin{bmatrix} -1 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 0.026 \\ -0.103 \end{bmatrix}\begin{array}{l} u_4 \\ v_4 \\ u_3 \\ v_3 \end{array}$$

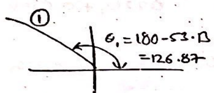
$$= 5.2\ N/mm^2$$



Q) Determine the horizontal & vertical displacement of the nodes & stresses in each element. all elements have $E = 200\,GPa$, $A = 250\,mm^2$?

data given





Sol:- element(1):

$A_1 = 250\,mm^2$; $E_1 = 2\times10^5\,N/mm^2$

$l_1^2 = \sqrt{600^2 + 450^2} = 750\,mm$; $\theta_e = \tan^{-1}\left(\frac{600}{450}\right) = 53.13^\circ$

(or)

$\Rightarrow \boxed{\theta_1 = 126.87}$

$C_1 = \cos 126.87 = -0.6$; $C_1^2 = 0.36$

$S_1 = \sin 126.87 = 0.8$; $S_1^2 = 0.62$ $\Big\}$ $C_1S_1 = -0.48$

$$\therefore K_1 = \frac{A_1 E_1}{l_1}\begin{bmatrix} c^2 & cs & -c^2 & -cs \\ cs & s^2 & -cs & -s^2 \\ c^2 & -cs & c^2 & cs \\ cs & -s^2 & cs & s^2 \end{bmatrix}$$

$$= \frac{250\times2\times10^5}{750}\begin{bmatrix} 0.36 & -0.47 & -0.86 & 0.47 \\ -0.47 & 0.62 & 0.47 & -0.62 \\ -0.36 & 0.47 & 0.36 & -0.47 \\ 0.47 & -0.62 & -0.47 & 0.62 \end{bmatrix}$$

$$= 10^5 \begin{array}{c} \begin{array}{cccc} U_1 & V_1 & U_2 & V_2 \end{array} \\ \begin{bmatrix} 0.24 & -0.31 & -0.24 & 0.31 \\ -0.31 & 0.42 & 0.31 & -0.42 \\ -0.24 & 0.31 & 0.24 & -0.31 \\ 0.31 & -0.42 & -0.31 & 0.42 \end{bmatrix} \end{array} \begin{array}{c} U_1 \\ V_1 \\ U_2 \\ V_2 \end{array}$$

→ element (2) :-

$A_2 = 250\,mm^2$; $E = 2\times10^5\,N/mm^2$

$l_2 = \sqrt{600^2 + 900^2} = 1082\,mm$

$\theta_2 = \tan^{-1}\left(\frac{600}{900}\right) = 33.7^\circ$

$C_2 = \cos 33.7 = 0.83$; $C_2^2 = 0.69$; $C_2S_2 = 0.46$

$S_2 = \sin 33.7 = 0.55$; $S_2^2 = 0.31$

$$K_2 = \frac{250\times2\times10^5}{1082}\begin{bmatrix} 0.69 & 0.46 & -0.69 & -0.46 \\ 0.46 & 0.31 & -0.46 & -0.31 \\ -0.69 & -0.46 & 0.69 & 0.46 \\ -0.46 & -0.31 & 0.46 & 0.31 \end{bmatrix}$$

$$
= 10^5 \begin{array}{c} \begin{array}{cccc} u & v_1 & u_3 & v_3 \end{array} \\ \begin{bmatrix} 0.32 & 0.22 & -0.32 & -0.22 \\ 0.22 & 0.14 & -0.22 & -0.14 \\ -0.32 & -0.22 & 0.32 & 0.22 \\ -0.22 & -0.14 & 0.22 & 0.14 \end{bmatrix} \end{array} \begin{array}{c} u \\ v_1 \\ u_3 \\ v_3 \end{array}
$$

$\rightarrow$ Element (3): $A_3 = 250\,mm^2$; $E_3 = 2\times10^5\ N/mm^2$

$l_3 = 750\,mm$; $\theta_3 = \tan^{-1}\frac{600}{450} = 53.13$

$$
K_3 = 10^5 \begin{array}{c} \begin{array}{cccc} u_1 & v_1 & u_4 & v_4 \end{array} \\ \begin{bmatrix} 0.24 & 0.31 & -0.24 & -0.31 \\ 0.31 & 0.42 & -0.31 & -0.42 \\ -0.24 & -0.31 & 0.24 & 0.31 \\ -0.31 & -0.42 & 0.31 & 0.42 \end{bmatrix} \end{array} \begin{array}{c} u \\ v_1 \\ u_4 \\ v_4 \end{array}
$$

$\therefore$ Global stiffness matrix $[K] = [K_1] + [K_2] + [K_3]$

$$
K = 10^5 \begin{bmatrix} 0.8 & 0.22 & -0.24 & 0.31 & -0.32 & -0.22 & -0.24 & -0.31 \\ 0.22 & 0.98 & 0.31 & -0.42 & -0.22 & -0.14 & -0.31 & -0.42 \\ -0.24 & 0.31 & 0.24 & -0.31 & 0 & 0 & 0 & 0 \\ 0.31 & -0.42 & -0.31 & 0.42 & 0 & 0 & 0 & 0 \\ -0.32 & -0.22 & 0 & 0 & 0.32 & 0.22 & 0 & 0 \\ -0.22 & -0.14 & 0 & 0 & 0.22 & 0.14 & 0 & 0 \\ -0.24 & -0.31 & 0 & 0 & 0 & 0 & 0.24 & 0.31 \\ -0.31 & -0.42 & 0 & 0 & 0 & 0 & 0.31 & 0.42 \end{bmatrix}
$$

Applying boundary conditions:

$u_2 = 0, v_2 = 0$; $v_4 = 0, u_4 = 0$

$u_3 = 0, v_3 = 0$; $F_{1y} = -18\times10^3 N$; $F_{1x} = 0$

$$
\therefore 10^5 \begin{bmatrix} 0.8 & 0.22 \\ 0.22 & 0.98 \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \end{bmatrix} = \begin{bmatrix} F_{1x} \\ F_{1y} \end{bmatrix} = \begin{bmatrix} 0 \\ -18\times10^3 \end{bmatrix}
$$

$\Rightarrow 10^5 [0.8u_1 + 0.22v_1] = 0$

$\Rightarrow 10^5 [0.22u_1 + 0.98v_1] = -18\times10^3$

$$
\left.\begin{array}{l} 0.8u_1 + 0.22v_1 = 0 \quad \text{—(1)} \\ 0.22u_1 + 0.98v_1 = -18 \quad \text{—(2)} \end{array}\right\} \begin{array}{l} u_1 = 0.054\,mm \\ v_1 = 0.196\,mm \end{array}
$$

$\rightarrow$ stresses in each element

$$
\sigma_1 = \frac{E_1}{l_1}\begin{bmatrix} -c_1 & -s_1 & c_1 & s_1 \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{bmatrix}
$$

$$
= \frac{2\times10^5}{750}\begin{bmatrix} 0.6 & -0.8 & -0.6 & 0.8 \end{bmatrix} \begin{bmatrix} 0.054 \\ -0.196 \\ 0 \\ 0 \end{bmatrix}
$$

$= 266.67(0.6\times0.054 + 0.8\times0.196)$

$= 50.5\ N/mm^2\ (T)$

$$\sigma_2 = \frac{2\times10^5}{1082}\begin{bmatrix}-0.83 & -0.55 & 0.83 & +0.55\end{bmatrix}\begin{array}{c|c} \begin{bmatrix}0.054\\-0.196\\0\\0\end{bmatrix} & \begin{array}{c}u_1\\v_1\\u_3\\v_3\end{array}\end{array}$$

$$= 184.84(-0.0448 + 0.1078)$$

$$= 11.6\ N/mm^2\ (T)$$

$$\sigma_3 = \frac{2\times10^5}{750}\begin{bmatrix}-0.6 & -0.79 & 0.6 & 0.79\end{bmatrix}\begin{array}{c|c} \begin{bmatrix}0.054\\-0.196\\0\\0\end{bmatrix} & \begin{array}{c}u_1\\v_1\\u_4\\v_4\end{array}\end{array}$$

$$= 266.67(-0.0324 + 0.1548)$$

$$= 32.7\ N/mm^2\ (T)$$

*SAQ's :

1) What is a truss....

A:- Structural member constructed by no of bars & L-angles.

2) Type of truss

A:- a. Simple truss (b) Compound truss (c) Complex Truss

3) Shape functions for a 2D-truss

$$u = (N_1 q_1 + N_2 q_2)\cos\theta \quad ; q = \text{displacmt} = u$$

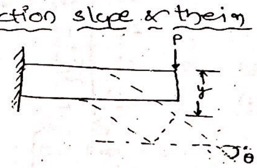
$$v = (N_1 q_1 + N_2 q_2)\sin\theta$$

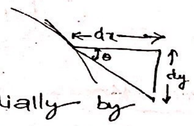
Here $N_1 = 1 - \frac{x}{l}$; $N_2 = \frac{x}{l}$.

*Analysis of Beams

Beam is a structural member having very long length compared to other two dimensials width & thickness. It is another type of one-dimensional structure member in which loads are applied vertically on the horizontal member. These horizontal member (beams) are used in buildings, bridges, shafts, railway bridges etc.

→ Deflection slope & their relations:-





When a beam is loaded radially by a vertical load acting downwards the beam will bend downwards & also the beam will try to rotate. The displacement of the particles by applied load is called deflection (y), & the angle through which the cross section at any location of the beam rotates w.r.t. the original position is called as angular rotation or slope (θ).

let y = deflection ; θ = slope

$\tan\theta = \frac{dy}{dx}$; If 'θ' is small

$\Rightarrow \theta = \frac{dy}{dx}$

Note : The displacements are translation & rotational. eg : v_1 θ_1 v_2 θ_2 etc
(v = vertical displacement)

* Stiffness matrix for beams :-

$$[K] = \frac{EI}{l^3} \begin{array}{c} \begin{array}{cccc} v_1 & \theta_1 & v_2 & \theta_2 \end{array} \\ \begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix} \end{array} \begin{array}{c} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{array}$$

* Formulation of F.E.A. equations to beams :-

$[K][\delta] = [F]$

here

$$[K] = \frac{EI}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix} ; \quad [\delta] = \begin{bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{bmatrix}$$

$$F = \begin{bmatrix} F_1 \\ M_1 \\ F_2 \\ M_2 \end{bmatrix}$$

F_1, M_1, F_2, M_2 are the shear force & bending moments developed at the two ends of the element.

If uniformly distributed loads are present then consider the reactions at the elements due to distributed loading. The reactions are $\frac{Wl}{2}$ for force & $\frac{Wl^2}{2}$ due to bending moment.

If distributed load acting downwards then force vector (global)

$$F = \begin{bmatrix} F_1 + \frac{Wl}{2} \\ M_1 + \frac{Wl^2}{12} \\ F_2 + \frac{Wl}{2} \\ M_2 + \frac{Wl^2}{12} \end{bmatrix}$$

If distributed load acts upwards then change the signs like $\frac{Wl}{2}, \frac{Wl^2}{12}, \frac{Wl}{2}, \frac{Wl}{12}$

→ Moment depends on hogging (-ve) & sagging (+)

→ Force always -ve (F_d).

Scanned with CamScanner

→ Beam Properties:-

1) for S.S.B; only reactions, no moment (F, M = 0)

2) for fixed beam; consider both reaction & moment (F, M)

3) load at a point; consider reaction (force); no moment (F, M = 0)

*→ deflections:-

1). at fixed point; $v = 0$; $\theta = 0$

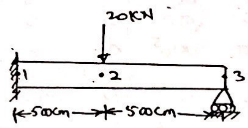
2) at point load; v, θ are considered

3) at roller support; $v = 0$; θ is considered

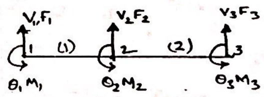
→ Problems:-

1) A beam fixed at one end & supported by a roller at the other end, has a 20 KN concentrated load applied at the centre of span. Calculate the deflection under the load? $E = 20 \times 10^6\, N/cm^2$

$I = 2500\, cm^4$



sol:



nodal deflections:-

→ element 1:-

$$[K]_1 = \frac{E_1 I_1}{l_1^3} \begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}$$

$$\frac{E_1 I_1}{l_1^3} = \frac{20 \times 10^6 \times 2500}{500^3} = 400\, N/cm$$

$$[K_1] = 400 \begin{array}{c} \begin{array}{cccc} v_1 & \theta_F & V_2 & \theta_2 \end{array} \\ \begin{bmatrix} 12 & 3000 & -12 & 3000 \\ 3000 & 10^6 & -3000 & 5 \times 10^5 \\ -12 & -3000 & 12 & -3000 \\ 3000 & 5 \times 10^5 & -3000 & 10^6 \end{bmatrix} \end{array} \begin{array}{c} \\ v_1 \\ \theta_1 \\ V_2 \\ \theta_2 \end{array}$$

→ element 2:-

$$[K]_2 = \frac{E_2 I_2}{l_2^3} \begin{bmatrix} 12 & 3000 & -12 & 3000 \\ 3000 & 10^6 & -3000 & 5 \times 10^5 \\ -12 & 3000 & 12 & -3000 \\ 3000 & 5 \times 10^5 & -3000 & 10^6 \end{bmatrix}$$

$$\Rightarrow \frac{E_2 I_2}{l_2^3} = 400\, N/cm$$

Global stiffness matrix: $K = K_1 + K_2$

$$
[K] = 400 \begin{array}{c} \\ \\ \\ \\ \\ \\ \\ \end{array}
\begin{array}{cccccc|c}
v_1 & \theta_1 & v_2 & \theta_2 & v_3 & \theta_3 & \\
\hline
12 & 3000 & -12 & 3000 & 0 & 0 & v_1 \\
3000 & 10^6 & -3000 & 5\times10^5 & 0 & 0 & \theta_1 \\
-12 & -3000 & 24 & 0 & -12 & 3000 & v_2 \\
3000 & 5\times10^5 & 0 & 2\times10^6 & -3000 & 5\times10^5 & \theta_2 \\
0 & 0 & -12 & -3000 & 12 & -3000 & v_3 \\
0 & 0 & 3000 & 5\times10^5 & -3000 & 10^6 & \theta_3
\end{array}
$$

$\therefore$ finite element equation $[K][\delta] = [F]$

$$
400 \begin{bmatrix} 12 & 3000 & -12 & 3000 & 0 & 0 \\ 3000 & 10^6 & -3000 & 5\times10^5 & 0 & 0 \\ -12 & -3000 & 24 & 0 & -12 & 3000 \\ 3000 & 5\times10^5 & 0 & 2\times10^6 & -3000 & 5\times10^5 \\ 0 & 0 & -12 & -3000 & 12 & -3000 \\ 0 & 0 & 3000 & 5\times10^5 & -3000 & 10^6 \end{bmatrix} \begin{bmatrix} v_1 \\ \theta_1 \\ V_2 \\ \theta_2 \\ V_3 \\ \theta_3 \end{bmatrix} = \begin{bmatrix} F_1 \\ M_1 \\ F_2 \\ M_2 \\ F_3 \\ M_3 \end{bmatrix}
$$

Nodal conditions :- $v_1 = 0$; $\theta_1 = 0$; $V_3 = 0$

$F_2 = -20000\,N$; $M_2 = 0$; $M_3 = 0$

$$
400 \begin{bmatrix} 24 & 0 & 3000 \\ 0 & 2\times10^6 & 5\times10^5 \\ 3000 & 5\times10^5 & 10^6 \end{bmatrix} \begin{bmatrix} V_2 \\ \theta_2 \\ \theta_3 \end{bmatrix} = \begin{bmatrix} -20{,}000 \\ 0 \\ 0 \end{bmatrix}
$$

$\Rightarrow 400(24V_2 + 3000\theta_3) = -20{,}000$

$400(2\times10^6\theta_2 + 5\times10^5\theta_3) = 0$

$400(3000V_2 + 5\times10^5\theta_2 + 10^6\theta_3) = 0$

$24V_2 + 3000\theta_3 = -50$ — (1)

$2\times10^6\theta_2 + 0.5\times10^6\theta_3 = 0$ — (2)

$3000V_2 + 0.5\times10^6\theta_2 + 10^6\theta_3 = 0$ — (3)

eqⁿ (2) $\Rightarrow \cancel{\theta_3 = 0}$ $\Rightarrow \theta_3 = -4\theta_2$ sub in (3)

$\Rightarrow 3000V_2 - 3.5\times10^6\theta_2 = 0$ — (4)

eq (1) + eq (2)

$24V_2 + 3000\theta_3 + 2\times10^6\theta_2 + 0.5\times10^6\theta_3 = -50$

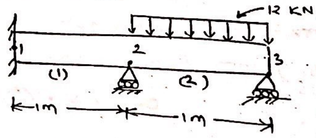
$\Rightarrow 24V_2 - 12000\theta_2 = -50$ — (5)

solving (4) & (5)

$\Rightarrow \theta_2 = -0.003125$ rad

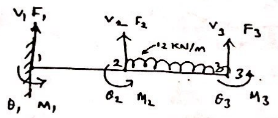
$V_2 = -3.646$ cm

(2) For the beam loaded determine the slopes at nodes 2 & 3 & the vertical deflection at the end mid-point of the distributed load.

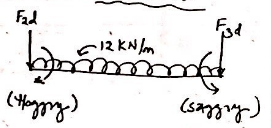


Sol:- let F_{2d}, F_{3d} be nodal forces due to distributed load at 2 & 3.

M_{2d}, M_{3d} be nodal bending moments at 2 & 3.



element (2)



→ determination of nodal displacements:-

- element (1):-

$$[K]_1 = \frac{E_1 I_1}{l_1^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}$$

$$\Rightarrow \frac{E_1 I_1}{l_1^3} = \frac{200\times10^9\times4\times10^{-6}}{(1)^3} = 8\times10^5\ N/m$$

$$[K]_1 = 8\times10^5 \begin{array}{c} \begin{array}{cccc} v_1 & \theta_1 & v_2 & \theta_2 \end{array} \\ \begin{bmatrix} 12 & 6 & -12 & 6 \\ 6 & 4 & -6 & 2 \\ -12 & -6 & 12 & -6 \\ 6 & 2 & -6 & 4 \end{bmatrix} \end{array} \begin{array}{c} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{array}$$

- element (2):

$$[K_2] = 8\times10^5 \begin{array}{c} \begin{array}{cccc} v_2 & \theta_2 & v_3 & \theta_3 \end{array} \\ \begin{bmatrix} 12 & 6 & -12 & 6 \\ 6 & 4 & -6 & 2 \\ -12 & -6 & 12 & -6 \\ 6 & 2 & -6 & 4 \end{bmatrix} \end{array} \begin{array}{c} v_2 \\ \theta_2 \\ v_3 \\ \theta_3 \end{array}$$

- Global stiffness matrix:-

$$[K] = 8\times10^5 \begin{array}{c} \begin{array}{cccccc} v_1 & \theta_1 & v_2 & \theta_2 & v_3 & \theta_3 \end{array} \\ \begin{bmatrix} 12 & 6 & -12 & 6 & 0 & 0 \\ 6 & 4 & -6 & 2 & 0 & 0 \\ -12 & -6 & 24 & 0 & -12 & 6 \\ 6 & 2 & 0 & 8 & -6 & 2 \\ 0 & 0 & -12 & -6 & 12 & -6 \\ 0 & 0 & 6 & 2 & -6 & 4 \end{bmatrix} \end{array} \begin{array}{c} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \\ v_3 \\ \theta_3 \end{array}$$

$$F_{2d} = F_{3d} = \frac{Wl}{2} = \frac{12000\times1}{2} = 6000\,N$$

$$M_{2d} = M_{3d} = \frac{Wl^2}{12} = \frac{12000\times1^2}{12} = 1000\ N\text{-}m$$

∴ finite element equations in matrix form

$$[K][\delta] = [F]$$

$$8\times10^5 \begin{bmatrix} 12 & 6 & -12 & 6 & 0 & 0 \\ 6 & 4 & -6 & 2 & 0 & 0 \\ -12 & -6 & 24 & 0 & -12 & 6 \\ 6 & 2 & 0 & 8 & -6 & 2 \\ 0 & 0 & -12 & -6 & 12 & -6 \\ 0 & 0 & 6 & 2 & -6 & 4 \end{bmatrix} \begin{bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \\ v_3 \\ \theta_3 \end{bmatrix} = \begin{bmatrix} F_1+F_{1d} \\ M_1+M_{1d} \\ F_2+F_{2d} \\ M_2+M_{2d} \\ F_3+F_{3d} \\ M_3+M_{3d} \end{bmatrix}$$

→ nodal displacements :- $v_1=0$, $\theta_1=0$, $v_2=0$, $v_3=0$

$F_{1d}=0$, $M_{1d}=0$, $F_{2d}=-6000\,N$

$M_{2d} = -1000$ N-m.

$F_{3d} = -6000$ N, $M_{3d} = 1000$ N-m

$M_2 = 0$, $M_3 = 0$

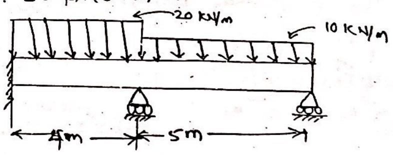
Applying the conditions

$$8\times10^5 \begin{bmatrix} 12 & 6 & -12 & 6 & 0 & 0 \\ 6 & 4 & -6 & 2 & 0 & 0 \\ -12 & -6 & 24 & 0 & -12 & 6 \\ 6 & 2 & 0 & 8 & -6 & 2 \\ 0 & 0 & -12 & -6 & 12 & -6 \\ 0 & 0 & 6 & 2 & -6 & 4 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ \theta_2 \\ 0 \\ \theta_3 \end{bmatrix} = \begin{bmatrix} F_1 \\ M_1 \\ F_2-6000 \\ -1000 \\ F_3-6000 \\ 1000 \end{bmatrix}$$

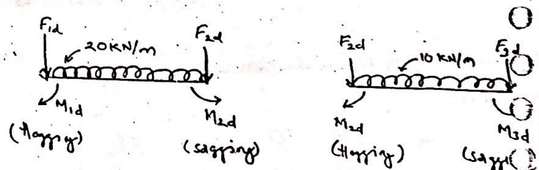
$$8\times10^5 \begin{bmatrix} 8 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} \theta_2 \\ \theta_3 \end{bmatrix} = \begin{bmatrix} -1000 \\ 1000 \end{bmatrix}$$

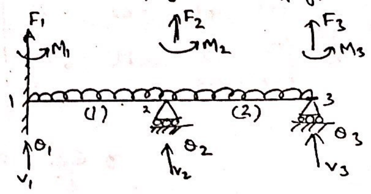
$$\Rightarrow \left.\begin{aligned} 8\times10^5(8\theta_2+2\theta_3) &= -1000 \\ 8\times10^5(2\theta_2+4\theta_3) &= 1000 \end{aligned}\right\} \text{by solving} \quad \begin{aligned} \theta_2 &= -2.67\times10^{-4}\ \text{rad} \\ \theta_3 &= 4.46\times10^{-4}\ \text{rad} \end{aligned}$$

③ For the beam shown in fig. compute slope at the hinged support points. $E = 200\,GPa$. $I = 4\times10^{-6}\,m^2$. use two beam elements



Sol:-





→ determination of nodal displacements:-

- element(1) :- $l_1 = 4m$

$$\frac{E_1 I_1}{l_1^3} = \frac{200\times10^9 \times 4\times10^{-6}}{(4)^3} = 12.5\times10^3 \text{ N/m}$$

$$[K_1] = \begin{bmatrix} 12 & 24 & -12 & 24 \\ 24 & 64 & -24 & 32 \\ -12 & -24 & 12 & -24 \\ 24 & 32 & -24 & 64 \end{bmatrix} 12.5\times10^3$$

$$= 10^3 \begin{array}{c} \begin{array}{cccc} v_1 & \theta_1 & v_2 & \theta_2 \end{array} \\ \begin{bmatrix} 150 & 300 & -150 & 300 \\ 300 & 800 & -300 & 400 \\ -150 & 300 & 150 & -300 \\ 300 & 400 & -300 & 800 \end{bmatrix} \end{array} \begin{array}{c} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{array}$$

- element(2):-

$$\frac{E_2 I_2}{l_2^3} = \frac{200\times10^9 \times 4\times10^{-6}}{5^3}$$

26.4×10^3 N/m

$$[K_2] = 6.4\times10^3 \begin{bmatrix} 12 & 30 & -12 & 30 \\ 30 & 100 & -30 & 50 \\ -12 & -30 & 12 & -30 \\ 30 & 50 & -30 & 100 \end{bmatrix}$$

$$[K_2] = 10^3 \begin{array}{c} \begin{array}{cccc} v_2 & \theta_2 & v_3 & \theta_3 \end{array} \\ \begin{bmatrix} 76.8 & 192 & -76.8 & 192 \\ 192 & 640 & -192 & 320 \\ -76.8 & -192 & 76.8 & -192 \\ 192 & 320 & -192 & 640 \end{bmatrix} \end{array} \begin{array}{c} v_2 \\ \theta_2 \\ v_3 \\ \theta_3 \end{array}$$

- Global stiffness matrix:

$$[K] = [K_1] + [K_2]$$

$$K = 10^3 \begin{bmatrix} 150 & 300 & -150 & 300 & 0 & 0 \\ 300 & 800 & -300 & 400 & 0 & 0 \\ -150 & -300 & 226.8 & -108 & -76.8 & 192 \\ 300 & 400 & -108 & 1440 & -192 & 320 \\ 0 & 0 & -76.8 & -192 & 76.8 & -192 \\ 0 & 0 & 192 & 320 & -192 & 640 \end{bmatrix}$$

nodal forces:- Element(1):

- the forces at node 1 & 2 due to distributed load

$$F_{1d} = F_{2d} = \frac{W_1 l_1}{2} = \frac{20000\times4}{2} = 40,000 \text{ N}$$

- moments at 1 & 2 :

$$M_{1d} = M_{2d} = \frac{W_1 l_1^2}{12} = \frac{20,000\times16}{12} = 26667 \text{ N-m}.$$

→ element(2):

$$F_{2d} = F_{3d} = \frac{W_2 l_2}{2} = \frac{10{,}000 \times 5}{2} = 25000\,N$$

$$M_{2d} = M_{3d} = \frac{W_2 l_2^2}{12} = \frac{10{,}000 \times 25}{12} = 20833\ N\text{-}m.$$

→ Resultant nodal forces & moments :-

$$F = \begin{bmatrix} F_1 + F_{1d} \\ M_1 + M_{1d} \\ F_2 + (F_{2d})_1 + (F_{2d})_2 \\ M_2 + (M_{2d})_1 + (M_{2d})_2 \\ F_3 + F_{3d} \\ M_3 + M_{3d} \end{bmatrix}$$

$F_{1d} = (F_{2d})_{(1)} = -4000N$

$M_{1d} = -26667\ N\text{-}m$

$(M_{2d})_{(1)} = +26667\ N\text{-}m$

$(F_{2d})_{(2)} = F_{3d} = -25000N$

$(M_{2d})_2 = -20833\ N\text{-}m$

$M_{3d} = +20833\,N\text{-}m$

$$= \begin{bmatrix} F_1 - 40000 \\ M_1 - 26667 \\ F_2 - 40000 - 25000 \\ M_2 + 26667 - 20833 \\ F_3 - 25000 \\ M_3 + 20833 \end{bmatrix} = \begin{bmatrix} F_1 - 40000 \\ M_1 - 26667 \\ F_2 - 65000 \\ M_2 + 5834 \\ F_3 - 2500 \\ M_3 + 20833 \end{bmatrix}$$

nodal conditions :- $V_1 = 0, \theta_1 = 0, V_2 = 0, V_3 = 0$

$F_2 = 0,\ F_3 = 0,\ M_2 = 0,\ M_3 = 0$

$$10^3 \begin{bmatrix} 1440 & 320 \\ 320 & 640 \end{bmatrix} \begin{bmatrix} \theta_2 \\ \theta_3 \end{bmatrix} = \begin{bmatrix} 5834 \\ 20833 \end{bmatrix}$$

$$\Rightarrow \left.\begin{aligned} 10^3(1440\theta_2 + 320\theta_3) &= 5834 \\ 10^3(320\theta_2 + 640\theta_3) &= 20833 \end{aligned}\right\}$$

by solving

$\theta_2 = -3.58 \times 10^{-3}$ rad

$\theta_3 = 34.3 \times 10^{-3}$ rad.

16/08/2017.

Unit-3

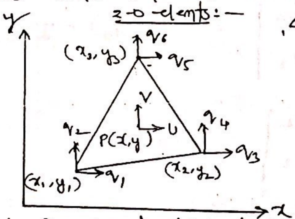
FINITE ELEMENT MODELING

→ **F.E.M. of 2-D stress analysis:-**

For the stress analysis of any machine component or structure using FEM, the component must be divided theoretically into many no. of very small elements of almost regular shapes like triangle, rectangle, parallelogram etc. the 2D surface may be flat or curved, & the problems related to these surfaces are known as 2-D problems.

2-D elements:-

C.S.T :-



→ **Constant strain triangles:-**

The element (triangular) has 3-nodes & 6-unknown nodal displacement. The shape functions for the element in natural co-ordinate system is given by $N_1 = \xi$, $N_2 = \eta$; $N_3 = 1-\xi-\eta$

Note: $N_1 + N_2 + N_3 = 1$ (sum of all shape fun^n)

→ **Derivation of shape functions for 2-D linear elements (C.S.T.) :-**

Consider a two dimensional straight sided triangular element with three corner nodes. let ϕ be the field variable. The nodes are named as 1,2,3 usually in counter clockwise direction & their global co-ordinates are (x_1, y_1) (x_2, y_2) & (x_3, y_3)

∴ The polynomial series to find the value of ϕ at any point is

$$\phi(x,y) = a_1 + a_2 x + a_3 y \quad \text{——} (1)$$

a_1, a_2, a_3 are unknown polynomial coeff.

at $\phi = \phi_1$: $x = x_1$ & $y = y_1$

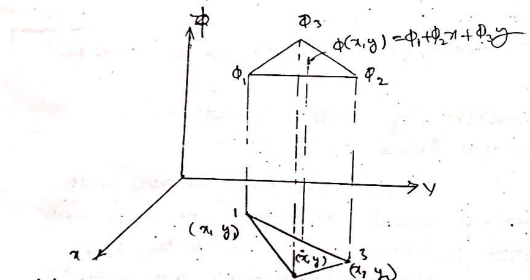
$\phi = \phi_2$: $x = x_2$ & $y = y_2$

$\phi = \phi_3$: $x = x_3$ & $y = y_3$

$$\Rightarrow \phi_1 = a_1 + a_2 x_1 + a_3 y_1$$

$$\phi_2 = a_1 + a_2 x_2 + a_3 y_2$$

$$\phi_3 = a_1 + a_2 x_3 + a_3 y_3$$



matrix form:

$$\begin{bmatrix}\phi_1\\ \phi_2\\ \phi_3\end{bmatrix} = \begin{bmatrix}1 & x_1 & y_1\\ 1 & x_2 & y_2\\ 1 & x_3 & y_3\end{bmatrix}\begin{bmatrix}a_1\\ a_2\\ a_3\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}a_1\\ a_2\\ a_3\end{bmatrix} = \begin{bmatrix}1 & x_1 & y_1\\ 1 & x_2 & y_2\\ 1 & x_3 & y_3\end{bmatrix}^{-1}\begin{bmatrix}\phi_1\\ \phi_2\\ \phi_3\end{bmatrix}$$

$$\Rightarrow [a] = [D]^{-1}[\phi] \quad \text{—— (A)}$$

where,

$$D = \begin{bmatrix}1 & x_1 & y_1\\ 1 & x_2 & y_2\\ 1 & x_3 & y_3\end{bmatrix}$$

co-ordinate matrix

but $D^{-1} = \dfrac{[C]^T}{|D|}$

Here $[C]$ is the cofactor matrix of $[D]$

here,

$$C = \begin{bmatrix}C_{11} & C_{12} & C_{13}\\ C_{21} & C_{22} & C_{23}\\ C_{31} & C_{32} & C_{33}\end{bmatrix} \qquad C_{ij} = (-1)^{i+j}\,|D_m|,$$

$i \rightarrow$ row
$j \rightarrow$ coloumn

here,

$$C_{11} = (-1)^{1+1}\begin{vmatrix}x_2 & y_2\\ x_3 & y_3\end{vmatrix} = x_2y_3 - x_3y_2$$

$$C_{12} = (-1)^{1+2}\begin{vmatrix}1 & y_2\\ 1 & y_3\end{vmatrix} = -(y_3 - y_2) = y_2 - y_3$$

$$C_{13} = (-1)^{1+3}\begin{vmatrix}1 & x_2\\ 1 & x_3\end{vmatrix} = x_3 - x_2$$

$$C_{21} = (-1)^{2+1}\begin{vmatrix}x_1 & y_1\\ x_3 & y_3\end{vmatrix} = -(x_1y_3 - x_3y_1) = x_3y_1 - x_1y_3$$

$$C_{22} = (-1)^{2+2}\begin{vmatrix}1 & y_1\\ 1 & y_3\end{vmatrix} = y_3 - y_1$$

$$C_{23} = (-1)^{2+3}\begin{vmatrix} 1 & x_1 \\ 1 & x_3 \end{vmatrix} = -(x_3 - x_1) = x_1 - x_3$$

$$C_{31} = (-1)^{3+1}\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} = x_1 y_2 - x_2 y_1$$

$$C_{32} = (-1)^{3+2}\begin{vmatrix} 1 & y_1 \\ 1 & y_2 \end{vmatrix} = -(y_2 - y_1) = y_1 - y_2$$

$$C_{33} = (-1)^{3+3}\begin{vmatrix} 1 & x_1 \\ 1 & x_2 \end{vmatrix} = x_2 - x_1$$

$$\therefore C = \begin{bmatrix} (x_2y_3 - x_3y_2) & (y_2 \cdot y_3) & (x_3 - x_2) \\ (x_3y_1 - x_1y_3) & (y_3 - y_1) & (x_1 - x_3) \\ (x_1y_2 - x_2y_1) & (y_1 - y_2) & (x_2 - x_1) \end{bmatrix}$$

$$\Rightarrow C^T = \begin{bmatrix} (x_2y_3 - x_3y_2) & (x_3y_1 - x_1y_3) & (x_1y_2 - x_2y_1) \\ (y_2 - y_3) & (y_3 - y_1) & (y_1 - y_2) \\ (x_3 - x_2) & (x_1 - x_3) & (x_2 - x_1) \end{bmatrix}$$

& determinant of matrix [D] can be given as

$$[D] = \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix} = 1(x_2y_3 - x_3y_2) - x_1(y_3 - y_2) + y_1(x_3 - x_2)$$

$$= 2A \qquad \text{ie, } (D = 2A)$$

∴ area of triangle 'A' can be expressed as

~~$A = \frac{1}{2}$~~

$$D = 2A$$

$$A = \frac{1}{2}[D]$$

$$= \frac{1}{2}\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}$$

eq (A) $\Rightarrow$

$$[a] = [D^{-1}][\phi]$$

$$[a] = \frac{[C^T]}{[D]}[\phi]$$

$$\Rightarrow \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \frac{1}{2A}\begin{Bmatrix} (x_2y_3 - x_3y_2) & (x_3y_1 - x_1y_3) & (x_1y_2 - x_2y_1) \\ (y_2 - y_3) & (y_3 - y_1) & (y_1 - y_2) \\ (x_3 - x_2) & (x_1 - x_3) & (x_2 - x_1) \end{Bmatrix}\begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{Bmatrix}$$

$$\Rightarrow \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \frac{1}{2A} \begin{Bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{Bmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{Bmatrix} \quad \text{— (15)}$$

here,

$\alpha_1 = x_2y_3 - x_3y_2$; $\beta_1 = y_2 - y_3$; $\gamma_1 = x_3 - x_1$

$\alpha_2 = x_3y_1 - x_1y_3$; $\beta_2 = y_3 - y_1$; $\gamma_2 = x_1 - x_3$

$\alpha_3 = x_1y_2 - x_2y_1$; $\beta_3 = y_1 - y_2$; $\gamma_3 = x_2 - x_1$

eq (1)

$\phi(x,y) = a_1 + a_2x + a_3y$ can be written as

$$\phi(x,y) = \begin{bmatrix} 1 & x & y \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

$$\therefore \phi(x,y) = \begin{bmatrix} 1 & x & y \end{bmatrix} \frac{1}{2A} \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix}$$

$$= \frac{1}{2A} \begin{bmatrix} \alpha_1 + \beta_1 x + \gamma_1 y & \alpha_2 + \beta_2 x + \gamma_2 y & \alpha_3 + \beta_3 x + \gamma_3 y \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix}$$

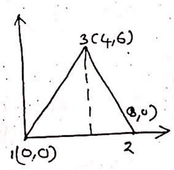
$$= \begin{bmatrix} \frac{\alpha_1 + \beta_1 x + \gamma_1 y}{2A} & \frac{\alpha_2 + \beta_2 x + \gamma_2 y}{2A} & \frac{\alpha_3 + \beta_3 x + \gamma_3 y}{2A} \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix}$$

$$= \begin{bmatrix} N_1 & N_2 & N_3 \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix}$$

$\Rightarrow \phi(x,y) = [N][\phi]$

$\Rightarrow \phi(x,y) = N_1\phi_1 + N_2\phi_2 + N_3\phi_3$

Note: Prove that $|D| = \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix} = 2A$



$$|D| = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 8 & 0 \\ 1 & 4 & 6 \end{vmatrix}$$

$= 1(8\times6 - 4\times0) - 0 + 0$

$= 48$

$\because A = \frac{1}{2}(6)(8) = 24$ sq.

$\Rightarrow |D| = 48$

$= 2\times A$

$= 2\times 24$

Summary :-

$$\phi(x,y) = N_1\phi_1 + N_2\phi_2 + N_3\phi_3$$

$$N_1 = \frac{1}{2A}(\alpha_1 + \beta_1 x + \gamma_1 y)$$

$$N_2 = \frac{1}{2A}(\alpha_2 + \beta_2 x + \gamma_2 y)$$

$$N_3 = \frac{1}{2A}(\alpha_3 + \beta_3 x + \gamma_3 y)$$

Here,

$\alpha_1 = (x_2y_3 - x_3y_2)$; $\alpha_2 = (x_3y_1 - x_1y_3)$; $\alpha_3 = x_1y_2 - x_2y_1$

$\beta_1 = y_2 - y_3$; $\beta_2 = y_3 - y_1$; $\beta_3 = y_1 - y_2$

$\gamma_1 = x_3 - x_2$; $\gamma_2 = x_1 - x_3$; $\gamma_3 = x_2 - x_1$

* Summary :-

$$\phi(x,y) = N_1\phi_1 + N_2\phi_2 + N_3\phi_3$$

$$N_1 = \frac{1}{2A}(\alpha_1 + \beta_1 x + \gamma_1 y)$$

$$N_2 = \frac{1}{2A}(\alpha_2 + \beta_2 x + \gamma_2 y)$$

$$N_3 = \frac{1}{2A}(\alpha_3 + \beta_3 x + \gamma_3 y)$$

Here, $\alpha_1 = (x_2y_3 - x_3y_2)$; $\alpha_2 = (x_3y_1 - x_1y_3)$; $\alpha_3 = (x_1y_2 - x_2y_1)$

$\beta_1 = y_2 - y_3$; $\beta_2 = y_3 - y_1$; $\beta_3 = y_1 - y_2$

$\gamma_1 = x_3 - x_2$; $\gamma_2 = x_1 - x_3$; $\gamma_3 = x_2 - x_1$

* Problems on Triangular element :-

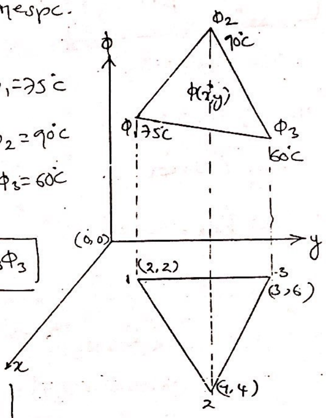
(1) Determine the temperature at a point P(4, 3) given that the temperatures at nodes 1, 2, & 3 are 75°, 90° & 60°C respc.

Sol :- at

node 1 : $x_1 = 2, y_1 = 2$; $\phi_1 = 75°C$

node 2 : $x_2 = 7, y_2 = 4$; $\phi_2 = 90°C$

node 3 : $x_3 = 3, y_3 = 6$; $\phi_3 = 60°C$



$$\boxed{\phi(x,y) = N_1\phi_1 + N_2\phi_2 + N_3\phi_3}$$

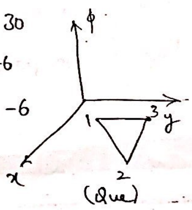
$$2A = \begin{vmatrix} 1 & 2 & 2 \\ 1 & 7 & 4 \\ 1 & 3 & 6 \end{vmatrix}$$

$$= 1(42-12) - 2(6-4) + 2(3-7) = 18 \Rightarrow 2A = 18$$

$\alpha_1 = (x_2y_3 - x_3y_2) = (7\times 6) - (3\times 4) = 30$

$\alpha_2 = (x_3y_1 - x_1y_3) = (3\times 2) - (2\times 6) = -6$

$\alpha_3 = (x_1y_2 - x_2y_1) = (2\times 4) - (7\times 2) = -6$



$\beta_1 = y_2 - y_3 = 4 - 6 = -2$; $\gamma_1 = x_3 - x_2 = -4$

$\beta_2 = y_3 - y_1 = 6 - 2 = 4$; $\gamma_2 = x_1 - x_3 = -1$

$\beta_3 = y_1 - y_2 = 2 - 4 = -2$; $\gamma_3 = 7 - 2 = 5$

$$\rightarrow \boxed{\therefore N_1 = \frac{1}{2A}(\alpha_1 + \beta_1 x + \gamma_1 y)}$$

$$= \frac{1}{18}(30 - 2x - 4y)$$

$$\rightarrow N_2 = \frac{1}{2A}(\alpha_2 + \beta_2 x + \gamma_2 y)$$

$$= \frac{1}{2A}(-6 + 4x - y)$$

$$\rightarrow N_3 = \frac{1}{2A}(\alpha_3 + \beta_3 x + \gamma_3 y)$$

$$= \frac{1}{18}(-6 - 2x + 5y)$$

$$\therefore \phi(4,3) = N_1\phi_1 + N_2\phi_2 + N_3\phi_3$$

$$= \frac{1}{18}\left[(30 - 2x - 4y)(75)\right] + \frac{1}{18}\left[-6 + 4x - y\right](90)$$

$$+ \frac{1}{18}\left[-6 - 2x + 5y\right](60)$$

$$= \frac{1}{18}\left[(30 - 2\times4 - 4\times3)(75)\right] + \frac{1}{18}\left[(-6 + 4(4) - 3)(90)\right]$$

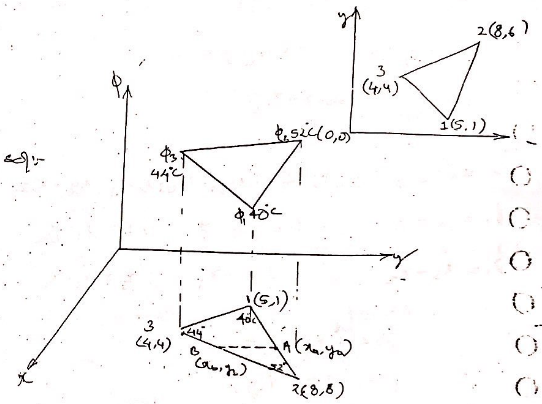
$$+ \frac{1}{18}\left[-(6 - 2(4) + 5(3)\right] 60$$

$\therefore$ Temp at the point $P(4,3) = 80°C$

Given

(2) For a 3 noded triangular element. Determine isotherm corresponding to 46°C. The temperature at nodes 1, 2 & 3 are 40°C, 52°C & 44°C respc.

Sol:-



Node 1 : $\phi_1 = 40°C$

Node 2 : $\phi_2 = 52°C$

Node 3 : $\phi_3 = 44°C$

We need the points (x_a, y_a) & (x_b, y_b) at temp 46°C

A is in b/w 1 & 2 ; B is in b/w 3 & 2.

40 - 46 - 52   44 - 46 - 52

Consider side 1&2 (1-2) ; Consider side 2-3

Slope eqn:

$$\frac{x_a - x_1}{x_2 - x_1} = \frac{\phi_a - \phi_1}{\phi_2 - \phi_1} \qquad \frac{x_b - x_3}{x_2 - x_3} = \frac{\phi_b - \phi_3}{\phi_2 - \phi_3}$$

$$\Rightarrow \frac{x_a - 5}{8-5} = \frac{46-40}{52-40} \qquad \Rightarrow \frac{x_b - 4}{8-4} = \frac{46-44}{52-44}$$

$$\Rightarrow x_a = 6.5 \qquad \Rightarrow x_b = 5$$

My &

$$\frac{y_a - y_1}{y_2 - y_1} = \frac{\phi_a - \phi_1}{\phi_2 - \phi_1} \qquad \frac{y_b - y_3}{y_2 - y_3} = \frac{\phi_b - \phi_3}{\phi_2 - \phi_3}$$

$$\frac{y_a - 1}{6-1} = \frac{46-40}{52-40} \qquad \Rightarrow \frac{y_b - 4}{6-4} = \frac{46-44}{52-44}$$

$$\Rightarrow y_a = 3.5 \qquad \Rightarrow y_b = 4.5$$

$\therefore (x_a, y_a) = (6.5, 3.5)$ ...... & $\therefore (x_b, y_b) = (5, 4.5)$

for 2 points it is called isotherm line

Note:-

sometimes the fn ϕ_1, ϕ_2, ϕ_3 can be in terms of stresses like $\phi_1 = 40 MPa$; $\phi_2 = 34 MPa$; $\phi_3 = 46 MPa$.
& he will ask stress at a point (2, 15)
[same procedure ; in terms of Temp sub stress]
(no change)

* Isoparametric representation:-

$\because$ we know $\phi = N_1\phi_1 + N_2\phi_2 + N_3\phi_3$

If $\delta_1, \delta_2, \delta_3 \rightarrow$ are displacements at nodes 1, 2, 3

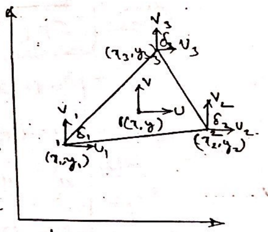
$u_1, u_2, u_3 \rightarrow$ components of $\delta_1, \delta_2, \delta_3$ along x-axis

$v_1, v_2, v_3 \rightarrow$ components of $\delta_1, \delta_2, \delta_3$ along y-axis

shape function

Values of N_1, N_2 & N_3 are same as previous

$$\left.\begin{aligned} \therefore u &= N_1u_1 + N_2u_2 + N_3u_3 \\ v &= N_1v_1 + N_2v_2 + N_3v_3 \end{aligned}\right\} \text{①}$$



for the triangular element (i.e, CST element) the coordinates x, y of the point P can also be represented in terms of nodal co-ordinates using same shape functions. This is referred to isoparametric representation

$$\therefore x = N_1x_1 + N_2x_2 + N_3x_3$$

$$y = N_1y_1 + N_2y_2 + N_3y_3$$

also $N_1 + N_2 + N_3 = 1 \Rightarrow N_3 = 1 - N_1 - N_2$

$$\therefore x = N_1x_1 + N_2x_2 + (1 - N_1 - N_2)x_3$$
$$= N_1(x_1 - x_3) + N_2(x_2 - x_3) + x_3$$

My $y = N_1(y_1 - y_3) + N_2(y_2 - y_3) + y_3$

displacement $\delta(x,y) = \begin{bmatrix} u \\ v \end{bmatrix}$

$$= \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{bmatrix}$$

*stress-strain displacement relation matrix :- (constitute matrix). [D] (or) (elasticity matrix) for plane strain conditions :

stress $\boxed{[\sigma] = [D][e]}$

here,

$$[D] = \frac{E}{(1+\mu)(1-2\mu)} \begin{bmatrix} (1-\mu) & \mu & \mu & 0 & 0 & 0 \\ \mu & (1-\mu) & \mu & 0 & 0 & 0 \\ \mu & \mu & (1-\mu) & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2\mu}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2\mu}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\mu}{2} \end{bmatrix}$$

6x6 Matrix

$$\sigma = \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{bmatrix} ; \quad e = \begin{bmatrix} e_x \\ e_y \\ e_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{bmatrix}$$

$\boxed{\frac{1-2\mu}{2} = (0.5-\mu)}$

(3x3) matrix :-

$$\sigma = [D][e]$$

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \underbrace{\frac{E}{(1-\mu^2)} \begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & \frac{1-\mu}{2} \end{bmatrix}}_{[D] \text{ matrix}} \begin{bmatrix} e_x \\ e_y \\ \gamma_{xy} \end{bmatrix}$$

constitute matrix for plane stress conditions.

* Plane stress :-

A state of plane stress is said to exist when elastic body is very thin & there are no loads applied in co-ordinate direction parallel to thickness.

(or)

For some two-dimensional objects the stresses can be produced only in 2-dim^c & not possible in the third direction.

*Plane strain :-

3rd direction strain is negligible or almost zero.

constitotative matrix for plane strain conditions (3x3) matrix

$$[D] = \frac{E}{(1+\mu)(1-2\mu)} \begin{bmatrix} (1-\mu) & \mu & 0 \\ \mu & (1-\mu) & 0 \\ 0 & 0 & \left(\frac{1-2\mu}{2}\right) \end{bmatrix}$$

$$\sigma = [D][e]$$

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \frac{E}{(1+\mu)(1-2\mu)} \begin{bmatrix} (1-\mu) & \mu & 0 \\ \mu & (1-\mu) & 0 \\ 0 & 0 & \left(\frac{1-2\mu}{2}\right) \end{bmatrix} \begin{bmatrix} e_x \\ e_y \\ \gamma_{xy} \end{bmatrix}$$

here

$$\begin{bmatrix} e_x \\ e_y \\ e_z \end{bmatrix} = \underbrace{\frac{1}{2A} \begin{bmatrix} \beta_1 & 0 & \beta_2 & 0 & \beta_3 & 0 \\ 0 & \gamma_1 & 0 & \gamma_2 & 0 & \gamma_3 \\ \gamma_1 & \beta_1 & \gamma_2 & \beta_2 & \gamma_3 & \beta_3 \end{bmatrix}}_{\text{known as } [B] \text{ gradient matrix or strain disp}'\text{mt matrix}} \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{bmatrix}$$

$\therefore$ $\boxed{\sigma = [D][B][\delta]}$

* Stiffness matrix for 2-D (CST) element :-

$$[K] = \int_v [B]^T[D][B]\, dv$$

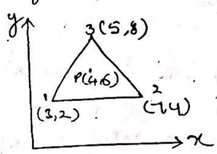
$$[K] = [B]^T[D][B]\, At$$

A = Area (triangular)
t = thickness of element

Problems :-

① Evaluate the shape functions N_1, N_2 & N_3 at the interior point P for the Δ^{lar} element?

(a) Using shape function expression
(b) Using Isoparametric representation



$x_1 = 3$; $y_1 = 2$
$x_2 = 7$; $y_2 = 4$
$x_3 = 5$; $y_3 = 8$

sol :-
(a) using shape function exp :

$\alpha_1 = 36$; $\beta_1 = -4$; $\gamma_1 = -2$
$\alpha_2 = -14$; $\beta_2 = 6$; $\gamma_2 = -2$
$\alpha_3 = -2$; $\beta_3 = -2$; $\gamma_3 = 4$

$2A = 20$; $N_1 = 0.4$
$N_2 = -0.1$
$N_3 = 0.7$

b) Isoparametric representation:-

we know

$x = (x_1 - x_3)N_1 + (x_2 - x_3)N_2 + x_3$ —— (1)

$y = (y_1 - y_3)N_1 + (y_2 - y_3)N_2 + y_3$ —— (2)

eq (1) ⇒ $4 = (3-5)N_1 + (7-5)N_2 + 5$

⇒ $4 = -2N_1 + 2N_2 + 5$ ⇒ $2N_1 - 2N_2 = 1$ —— (A)

eq (2) ⇒ $6 = (2-8)N_1 + (4-8)N_2 + 8$

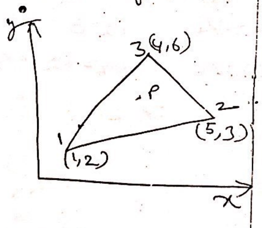
⇒ $6N_1 + 4N_2 = 2$ —— (B)

solving $N_1 = 0.4$ & $N_2 = -0.1$

∴ $N_1 + N_2 + N_3 = 1$

$0.4 - 0.1 + N_3 = 1$ ⇒ $N_3 = 0.7$

(2) The nodal co-ordinates of a Δ^{ler} element are shown in fig. at the point 'p' inside the element the x-cordinate is 3.3 & shape func $N_1 = 0.3$. Determine the shape functions N_2, N_3 & y-cordinate of the point 'p'.



Sol. $x_1 = 1$, $y_1 = 2$; $N_1 = 0.3$

$x_2 = 5$, $y_2 = 3$; $x = 3.3$

$x_3 = 4$, $y_3 = 6$

we know,

$x = (x_1 - x_3)N_1 + (x_2 - x_3)N_2 + x_3$

⇒ $3.3 = (1-4)0.3 + (5-4)N_2 + 4$ ⇒ $N_2 = 0.2$

also $N_1 + N_2 + N_3 = 1$ ⇒ $N_3 = 0.5$

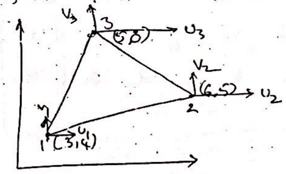
we know

$y = N_1 y_1 + N_2 y_2 + N_3 y_3$

$= 0.3 \times 2 + 0.2 \times 3 + 0.5 \times 6$

$= 4.2$

(3) compute the strain-displacement matrix for the element also det the element strains.



Sol: the nodal displacements are

$u_1 = 0.002$, $u_2 = 0.001$, $u_3 = -0.003$

$v_1 = 0.001$, $v_2 = -0.004$, $v_3 = 0.007$

all dimensions are in mm?

Sol: We know the Gradient matrix.
i.e, the element strains related with nodal displacements

$$[e]=[B][\delta]$$

i.e,

$$\begin{bmatrix} e_x \\ e_y \\ e_z \end{bmatrix} = \frac{1}{2A} \underbrace{\begin{bmatrix} \beta_1 & 0 & \beta_2 & 0 & \beta_3 & 0 \\ 0 & q_1 & 0 & q_2 & 0 & q_3 \\ q_1 & \beta_1 & q_2 & \beta_2 & q_3 & \beta_3 \end{bmatrix}}_{\text{strain-displacement matrix}} \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{bmatrix}$$

$$2A = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} = 1(48-25)-3(8-5)+4(5-6)$$

$$= 23-9-4$$

$$= 10 \text{ cm}^2$$

$\beta_1 = y_2 - y_3 = -3$; $q_1 = x_3 - x_2 = -1$

$\beta_2 = y_3 - y_1 = 4$; $q_2 = x_1 - x_3 = -2$

$\beta_3 = y_1 - y_2 = -1$; $q_3 = x_2 - x_1 = 3$

∴ strain displ^c matrix is given by

$$[B] = \frac{1}{10}\begin{bmatrix} -3 & 0 & 4 & 0 & -1 & 0 \\ 0 & -1 & 0 & -2 & 0 & 3 \\ -1 & -3 & -2 & 4 & 3 & -1 \end{bmatrix}$$

element strains :-

we know

$$\begin{bmatrix} e_x \\ e_y \\ \gamma_{xy} \end{bmatrix} = \frac{1}{10}\begin{bmatrix} -3 & 0 & 4 & 0 & -1 & 0 \\ 0 & -1 & 0 & -2 & 0 & 3 \\ -1 & -3 & -2 & 4 & 3 & -1 \end{bmatrix}\begin{bmatrix} 0.002 \\ 0.001 \\ 0.001 \\ -0.004 \\ -0.003 \\ 0.007 \end{bmatrix}$$

$$\Rightarrow e_x = \frac{1}{10}\left[-3(0.002)+4(0.001)-1(-0.003)\right]$$

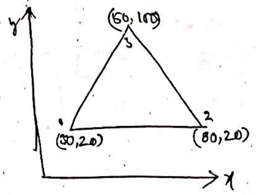
$$= 0.0001$$

$$\Rightarrow e_y = \frac{1}{10}\left[-1(0.001)-2(-0.001)+3(0.007)\right] = 0.0028$$

$$\Rightarrow \gamma_{xy} = \frac{1}{10}\left[-1(0.002)-3(0.001)-2(0.001)+4(-0.004) + 3(-0.003)-1(0.007)\right]$$

$$= -0.0039$$

③ for the plane stress element, evaluate the stiffness matrix assume $E = 210 \times 10^3$ N/mm², $\mu = 0.3$ & element thickness $t = 10$ mm. ?



Sol) Given $x_1 = 30\,mm$; $y_1 = 20\,mm$

$x_2 = 80\,mm$; $y_2 = 20\,mm$

$x_3 = 50\,mm$; $y_3 = 100\,mm$.

$E = 210 \times 10^3\ N/mm^2$; $\mu = 0.25$; $t = 10\,mm$

we know, stiffness matrix $[K] = [B]^T[D][B]At$

we know

$$A = \frac{1}{2}\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix} = 2000\ mm^2$$

for strain-displ^c matrix:

$\beta_1 = y_2 - y_3 = 20 - 100 = -80$; $\gamma_1 = x_3 - x_2 = -30$

$\beta_2 = y_3 - y_1 = 100 - 20 = 80$; $\gamma_2 = 30 - 50 = -20$

$\beta_3 = y_1 - y_2 = 20 - 20 = 0$; $\gamma_3 = 80 - 30 = 50$

$$[B] = \frac{1}{2 \times 2000}\begin{bmatrix} -80 & 0 & 80 & 0 & 0 & 0 \\ 0 & -30 & 0 & -20 & 0 & 50 \\ -30 & -80 & -20 & 80 & 50 & 0 \end{bmatrix}$$

$$[B] = \frac{1}{400}\begin{bmatrix} -8 & 0 & 8 & 0 & 0 & 0 \\ 0 & -3 & 0 & -2 & 0 & 5 \\ -3 & -8 & -2 & 8 & 5 & 0 \end{bmatrix}$$

$$[B]^T = \begin{bmatrix} -8 & 0 & -3 \\ 0 & -3 & -8 \\ 8 & 0 & -2 \\ 0 & -2 & 8 \\ 0 & 0 & 5 \\ 0 & 5 & 0 \end{bmatrix}$$

stress-strain relationship matrix for plane stress element:

$$[D] = \frac{E}{1-\mu^2}\begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 1 & \frac{1-\mu}{2} \end{bmatrix}$$

$$= \frac{210 \times 10^3}{1-(0.25)^2}\begin{bmatrix} 1 & 0.25 & 0 \\ 0.25 & 1 & 0 \\ 0 & 0 & \left(\frac{1-0.25}{2}\right) \end{bmatrix}$$

$$= 56 \times 10^3\begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 1.5 \end{bmatrix}$$

Stiffness matrix $[K] = [B]^T[D][B]At$

$$[K] = \frac{1}{400}\begin{bmatrix} -8 & 0 & -3 \\ 0 & -3 & -8 \\ 8 & 0 & -2 \\ 0 & -2 & 8 \\ 0 & 0 & 5 \\ 0 & 5 & 0 \end{bmatrix} 56 \times 10^3 \begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 1.5 \end{bmatrix} \frac{1}{400}\begin{bmatrix} -8 & 0 & 8 & 0 & 0 & 0 \\ 0 & -3 & 0 & -2 & 0 & 5 \\ -3 & -8 & -2 & 8 & 5 & 0 \end{bmatrix}$$

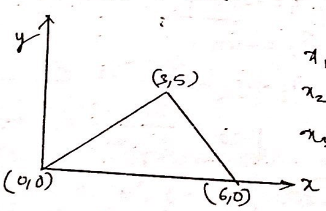
$\times 2000 \times 10$

$$\therefore [K] = 7\times10^{3}\begin{bmatrix} 269.5 & 60 & -247 & -20 & -22.5 & -40 \\ 60 & 132 & 0 & -72 & -60 & -60 \\ -247 & 0 & 262 & -40 & -15 & 40 \\ -20 & -72 & -40 & 112 & 60 & -40 \\ -22.5 & -60 & -15 & 60 & 37.5 & 0 \\ -40 & -60 & 40 & -40 & 0 & 100 \end{bmatrix}$$

symmetric ⇒ correct N/mm

① Evaluate the element stiffness matrix for the 1st element under plane strain condition

$E = 200\,GPa$, $\mu = 0.25$, $t = 1\,mm$



$x_1 = 0$, $y_1 = 0$

$x_2 = 6$, $y_2 = 0$

$x_3 = 3$; $y_3 = 5$

Sol: $E = 2\times10^{5}\ N/mm^2$; $\mu = 0.25$, $t = 1\,mm$

we know $A = \frac{1}{2}\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix} = 15\,mm^2$

$\beta_1 = y_2 - y_3 = -5$; $\gamma_1 = x_3 - x_2 = -3$

$\beta_2 = y_3 - y_1 = 5$; $\gamma_2 = x_1 - x_3 = -3$

$\beta_3 = y_1 - y_2 = 0$; $\gamma_3 = x_2 - x_1 = 6$

$$[B] = \frac{1}{30}\begin{bmatrix} -5 & 0 & 5 & 0 & 0 & 0 \\ 0 & -3 & 0 & -3 & 0 & 6 \\ -3 & -5 & -3 & 5 & 6 & 0 \end{bmatrix}$$

$$[B]^T = \frac{1}{30}\begin{bmatrix} -5 & 0 & -3 \\ 0 & -3 & -5 \\ 5 & 0 & -3 \\ 0 & -3 & 5 \\ 0 & 0 & 6 \\ 0 & 6 & 0 \end{bmatrix}$$

Plane strain condition:

$$[D] = \frac{E}{(1+\mu)(1-2\mu)}\begin{bmatrix} (1-\mu) & \mu & 0 \\ \mu & (1-\mu) & 0 \\ 0 & 0 & \left(\frac{1-2\mu}{2}\right) \end{bmatrix}$$

$$= \frac{2\times10^{5}}{(1+0.25)(1-0.5)}\begin{bmatrix} 1-0.25 & 0.25 & 0 \\ 0.25 & 1-0.25 & 0 \\ 0 & 0 & \frac{1-0.5}{2} \end{bmatrix}$$

$$= 8\times10^{4}\begin{bmatrix} 3 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

∴ Stiffness matrix is obtained by

$$\boxed{[K] = [B]^T[D][B]At}$$

$$[K] = \frac{1}{30}\begin{bmatrix} -3 & 0 & -3 \\ 0 & -3 & -5 \\ 5 & 0 & -3 \\ 0 & -3 & 5 \\ 0 & 0 & 6 \\ 0 & 6 & 0 \end{bmatrix} 8\times10^{4}\begin{bmatrix} 3 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}\frac{1}{30}\begin{bmatrix} -5 & 0 & 5 & 0 & 0 & 0 \\ 0 & -3 & 0 & -3 & 0 & 6 \\ -3 & -5 & -3 & 5 & 6 & 0 \end{bmatrix}$$

$\times 15 \times 1$

$$= 1.3\times10^{3}\begin{bmatrix} 84 & 30 & -66 & 0 & -18 & -30 \\ 30 & 52 & 0 & 2 & -30 & -54 \\ -66 & 0 & 84 & -30 & -18 & 30 \\ 0 & 2 & -30 & 52 & 30 & -54 \\ -18 & -30 & -18 & 30 & 36 & 0 \\ -30 & -54 & 30 & -54 & 0 & 108 \end{bmatrix} \text{N/mm}$$

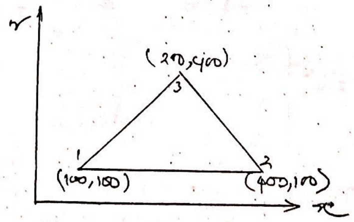
?

(5) for the plane stress element the nodal displacement are

$u_1 = 2\,mm$, $v_1 = 1\,mm$, $E = 200\,GN/m^2$

$u_2 = 1\,mm$ $v_2 = 1.5\,mm$, $\mu = 0.3$, $t = 10\,mm$

$u_3 = 2.5\,mm$ $v_3 = 0.5\,mm$



Sol:- Given $x_1 = 100$, $y_1 = 100$, $E = 200\times10^{3}\ N/mm^2$

$x_2 = 400$ $y_2 = 100$ $\mu = 0.3$

$x_3 = 200$ $y_3 = 400$ $t = 10\,mm$

element stress matrix :

$[\sigma] = [D][e]$

$\Rightarrow [\sigma] = [D][B][\delta]$

$$[D] = \frac{2\times10^{6}}{91}\begin{bmatrix} 10 & 3 & 0 \\ 3 & 10 & 0 \\ 0 & 0 & 3.5 \end{bmatrix}$$

by plane stress formula

here

$A = 45000\ mm^2$

$\beta_1 = -300$; $\gamma_1 = -200$

$\beta_2 = 300$ $\gamma_2 = -100$

$\beta_3 = 0$ $\gamma_3 = 300$

$$[B] = \frac{1}{2\times45000}\begin{bmatrix} -300 & 0 & 300 & 0 & 0 & 0 \\ 0 & -200 & 0 & -100 & 0 & 300 \\ -200 & -300 & -100 & 300 & 300 & 0 \end{bmatrix}$$

$$\Rightarrow [B] = \frac{1}{900}\begin{bmatrix} -3 & 0 & 3 & 0 & 0 & 0 \\ 0 & -2 & 0 & -1 & 0 & 3 \\ -2 & -3 & -1 & 3 & 3 & 0 \end{bmatrix}$$

$$[\delta] = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 1.5 \\ 2.5 \\ 0.5 \end{bmatrix}$$

$$\therefore \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \frac{2\times10^6}{91}\begin{bmatrix} 10 & 3 & 0 \\ 3 & 10 & 0 \\ 0 & 0 & 3.5 \end{bmatrix} \frac{1}{900}\begin{bmatrix} -3 & 0 & 3 & 0 & 0 & 0 \\ 0 & -2 & 0 & -1 & 0 & 3 \\ -2 & -3 & -1 & 3 & 3 & 0 \end{bmatrix} \times \begin{bmatrix} 2 \\ 1 \\ 1 \\ 1.5 \\ 2.5 \\ 0.5 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \begin{bmatrix} -839.12 \\ -706.18 \\ 34188 \end{bmatrix} N/mm^2$$

* **Axis symmetric problems for triangular elements:-**

same as isoparametric & shape fun^c replace 'x' with 'r' & 'y' with 'z'

i.e,

$$[D] = \begin{vmatrix} 1 & r_1 & z_1 \\ 1 & r_2 & z_2 \\ 1 & r_3 & z_3 \end{vmatrix} = 2A$$

$\alpha_1 = r_2 z_3 - r_3 z_2$; $\beta_1 = (z_2 - z_3)$; $\gamma_1 = r_3 - r_2$

$\alpha_2 = r_3 z_1 - r_1 z_3$ $\beta_2 = z_3 - z_1$ $\gamma_2 = r_1 - r_3$

$\alpha_3 = r_1 z_2 - r_2 z_1$ $\beta_3 = z_1 - z_2$ $\gamma_3 = r_2 - r_1$

but

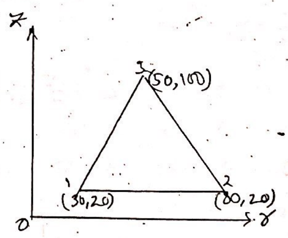
$$[B] = \frac{1}{2A}\begin{bmatrix} \beta_1 & 0 & \beta_2 & 0 & \beta_3 & 0 \\ \frac{\alpha_1 + \beta_1 r + \gamma_1 z}{r} & 0 & \frac{\alpha_2 + \beta_2 r + \gamma_2 z}{r} & 0 & \frac{\alpha_3 + \beta_3 r + \gamma_3 z}{r} & 0 \\ 0 & \gamma_1 & 0 & \gamma_2 & 0 & \gamma_3 \\ \gamma_1 & \beta_1 & \gamma_2 & \beta_3 & \gamma_3 & \beta_3 \end{bmatrix}$$

& stiffness matrix $[K] = 2\pi r A [B]^T [D] [B]$

here, $r = \frac{r_1 + r_1 + r_3}{3}$, $z = \frac{z_1 + z_2 + z_3}{3}$

Problems:-

① for an axi-symmetric Δ^{lar} element. $E = 200 GPa$.
evaluate the stiffness matrix.
$\mu = 0.25$



Q1) Given $r_1 = 30\,mm$, $z_1 = 20\,mm$

$r_2 = 80\,mm$ $\quad z_2 = 20\,mm$

$r_3 = 50\,mm$ $\quad z_3 = 100\,mm$

$E = 200 \times 10^3\ N/mm^2$

$\mu = 0.25$

Here

$$r = \frac{r_1 + r_2 + r_3}{3} = \frac{160}{3} \; ; \; z = \frac{z_1 + z_2 + z_3}{3} = \frac{140}{3}$$

$A = 2000\,mm^2$

$\alpha_1 = r_2 z_3 - r_3 z_2 = 7000$; $\beta_1 = z_2 - z_3 = -80$

$\alpha_2 = r_3 z_1 - r_1 z_3 = -2000$; $\beta_2 = z_3 - z_1 = 80$

$\alpha_3 = r_1 z_2 - r_2 z_1 = -1000$; $\beta_3 = z_1 - z_2 = 0$

$\gamma_1 = r_3 - r_2 = -30$

$\gamma_2 = r_1 - r_3 = -20$

$\gamma_3 = r_2 - r_1 = 50$

$$\frac{\alpha_1 + \beta_1 r + \gamma_1 z}{r} = 25 , \quad \frac{\alpha_2 + \beta_2 r + \gamma_2 z}{r} = 25$$

$$\frac{\alpha_3 + \beta_3 r + \gamma_3 z}{r} = 25$$

$$\therefore [B] = \frac{1}{4000}\begin{bmatrix} -80 & 0 & 80 & 0 & 0 & 0 \\ 25 & 0 & 25 & 0 & 25 & 0 \\ 0 & -30 & 0 & -20 & 0 & 50 \\ -30 & -80 & -20 & 80 & 50 & 0 \end{bmatrix}$$

$$[B]^T = \frac{1}{800}\begin{bmatrix} -16 & 5 & 0 & -6 \\ 0 & 0 & -6 & -6 \\ 16 & 5 & 0 & -4 \\ 0 & 0 & -4 & 16 \\ 0 & 5 & 0 & 10 \\ 0 & 0 & 10 & 0 \end{bmatrix}$$

$$[D] = \frac{E}{(1+\mu)(1-2\mu)}\begin{bmatrix} 1-\mu & \mu & \mu & 0 \\ \mu & 1-\mu & \mu & 0 \\ \mu & \mu & 1-\mu & 0 \\ 0 & 0 & 0 & \frac{1-2\mu}{2} \end{bmatrix}$$

$$= 84 \times 10^3 \begin{bmatrix} 3 & 1 & 1 & 0 \\ 1 & 3 & 1 & 0 \\ 1 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\therefore$ stiffness matrix $[K] = 2\pi r A [B]^T [D] [B]$

$$[K] = 2\pi \times \frac{160}{3} \times 2000 \times \frac{1}{800}\begin{bmatrix} -16 & 5 & 0 & -6 \\ 0 & 0 & -6 & -16 \\ 16 & 5 & 0 & -4 \\ 0 & 0 & -4 & 16 \\ 0 & 5 & 0 & 10 \\ 0 & 0 & 10 & 0 \end{bmatrix} 84 \times 10^3 \times$$

$$\times \begin{bmatrix} 3 & 1 & 1 & 0 \\ 1 & 3 & 1 & 0 \\ 1 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \frac{1}{800}\begin{bmatrix} -16 & 0 & 16 & 0 & [illegible] & \\ 5 & 0 & 5 & 0 & 5 & 0 \\ 0 & -6 & 0 & -4 & 0 & 1[illegible] \\ -6 & -16 & -4 & 16 & 1[illegible] & [illegible] \end{bmatrix}$$

$$\therefore [K] = 87465 \begin{bmatrix} 719 & 162 & -669 & -52 & -65 & -110 \\ 162 & 364 & -62 & -184 & -190 & -180 \\ -669 & -62 & 1019 & -148 & 115 & 210 \\ -52 & -184 & -148 & 304 & 140 & -120 \\ -65 & 190 & 115 & 140 & 170 & 50 \\ -110 & -180 & 210 & -120 & 50 & 300 \end{bmatrix} N/mm$$

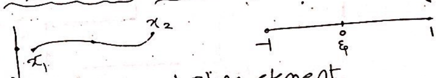
Unit-4

→ 2-Dimensional four noded Iso-parametric elements:-

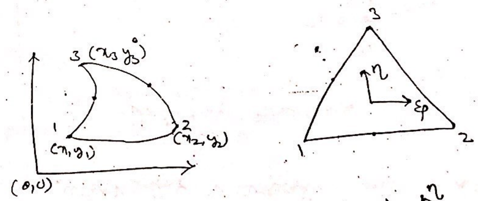
let x, y be co-ordinate values of cartesian co-ordinate system

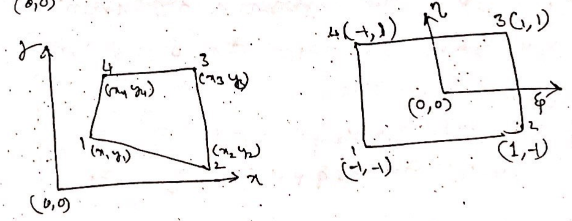
ξ, η be co-ordinate values of single natural co-ordinate system

→ Transformation of actual elements into master (Isoparametric) elements:



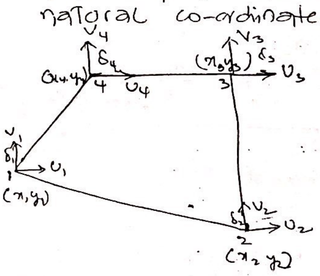
1-D line element

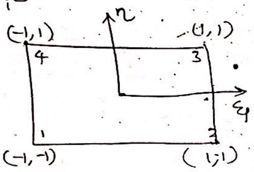




let x_1, x_2 be global co-ordnts of nodes 1&2
u_1, u_2 be displacemts of nodes 1&2 due to axial force (load) F_1 & F_2

* Derivation of shape functions for a 4-noded quadrilateral element using natural co-ordinates :-





let δ_1 δ_2 δ_3 & δ_4 are displacemts. at nodes 1,2,3&4

u_1, u_2 u_3 & u_4 are components of δ in x-dirn
v_1 v_2 v_3 & v_4 are components of δ in y-dirn

Polynomial expressions:

$u(\xi,\eta) = a_1 + a_2\xi + a_3\eta + a_4\xi\eta$ — (A)

$v(\xi,\eta) = a_5 + a_6\xi + a_7\eta + a_8\eta$ — (B)

eqn (A) $\boxed{u(\xi,\eta) = a_1 + a_2\xi + a_3\eta + a_4\xi\eta}$

nodal conditions:

node 1 : $\xi=-1, \eta=-1, u=u_1 \Rightarrow u_1 = a_1 - a_2 - a_3 + a_4$ — (1)

node 2 : $\xi=1, \eta=-1, u=u_2 \Rightarrow u_2 = a_1 + a_2 - a_3 - a_4$ — (2)

node 3 : $\xi=1, \eta=1, u=u_3 \Rightarrow u_3 = a_1 + a_2 + a_3 + a_4$ — (3)

node 4 : $\xi=-1, \eta=1, u=u_4 \Rightarrow u_4 = a_1 - a_2 + a_3 - a_4$ — (4)

add (1) & (2) $\Rightarrow u_1 + u_2 = 2a_1 - 2a_3$ — (5)

add (3) & (4) $\Rightarrow u_3 + u_4 = 2a_1 + 2a_3$ — (6)

add eqn (5) & (6) $\Rightarrow u_1 + u_2 + u_3 + u_4 = 4a_1$

$\Rightarrow a_1 = \frac{1}{4}(u_1 + u_2 + u_3 + u_4)$ — (7)

subc eqn (5) & (6) $\Rightarrow -u_1 - u_2 + u_3 + u_4 = 4a_3$

$\Rightarrow a_3 = \frac{1}{4}(-u_1 - u_2 + u_3 + u_4)$ — (8)

similarly,

sub eqn (1) & (2) $\Rightarrow u_1 - u_2 = -2a_2 + 2a_4$ — (9)

sub eqn (3) & (4) $\Rightarrow u_3 - u_4 = 2a_2 + 2a_4$ — (10)

add eqn (9) & (10) $\Rightarrow 4a_4 = u_1 - u_2 + u_3 - u_4$

$a_4 = \frac{1}{4}(u_1 - u_2 + u_3 - u_4)$ — (11)

sub (9) & (10) $\Rightarrow a_2 = \frac{1}{4}(-u_1 + u_2 + u_3 - u_4)$ — (12)

now sub the values of a_1, a_2, a_3 & a_4 in (A)

$$u = a_1 + a_2\xi + a_3\eta + a_4\xi\eta$$

$$\Rightarrow u = \frac{1}{4}(u_1+u_2+u_3+u_4) + \frac{1}{4}(-u_1+u_2+u_3-u_4)\xi + \frac{1}{4}(-u_1-u_2+u_3+u_4)\eta + \frac{1}{4}(u_1-u_2+u_3-u_4)\xi\eta$$

$$\Rightarrow u = \frac{1}{4}(1-\xi-\eta+\xi\eta)u_1 + \frac{1}{4}(1+\xi-\eta-\xi\eta)u_2 + \frac{1}{4}(1+\xi+\eta+\xi\eta)u_3 + \frac{1}{4}(1-\xi+\eta-\xi\eta)u_4$$

$$\Rightarrow u = \frac{1}{4}(1-\xi)(1-\eta)u_1 + \frac{1}{4}(1+\xi)(1-\eta)u_2 + \frac{1}{4}(1+\xi)(1+\eta)u_3 + \frac{1}{4}(1-\xi)(1+\eta)u_4$$

$$\Rightarrow u = N_1u_1 + N_2u_2 + N_3u_3 + N_4u_4$$

$\therefore$ Shape func^c

$$N_1 = \frac{1}{4}(1-\xi)(1-\eta)$$

$$N_2 = \frac{1}{4}(1+\xi)(1-\eta)$$

$$N_3 = \frac{1}{4}(1+\xi)(1+\eta)$$

$$N_4 = \frac{1}{4}(1-\xi)(1+\eta)$$

Mly for $v = N_1v_1 + N_2v_2 + N_3v_3 + N_4v_4$

& $N_1 + N_2 + N_3 + N_4 = 1$

Jacobian matrix :

$$J = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}$$

here,

$$J_{11} = \frac{1}{4}\left[(1-\eta)(x_2-x_1) + (1+\eta)(x_3-x_4)\right]$$

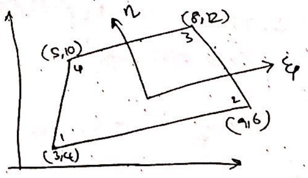
$$J_{12} = \frac{1}{4}\left[(1-\eta)(y_2-y_1) + (1+\eta)(y_3-y_4)\right]$$

$$J_{21} = \frac{1}{4}\left[(1-\xi)(x_4-x_1) + (1+\xi)(x_3-x_2)\right]$$

$$J_{22} = \frac{1}{4}\left[(1-\xi)(y_4-y_1) + (1+\xi)(y_3-y_2)\right]$$

* Problems :-

① determine the cartesian co-ordinates of the point 'P' which has local co-ordinates $\xi = 0.8$ & $\eta = 0.6$



Given $x_1 = 3$, $y_1 = 4$

$x_2 = 9$, $y_2 = 6$

$x_3 = 8$, $y_3 = 12$

$x_4 = 5$, $y_4 = 10$

we know

$x = N_1 x_1 + N_2 x_2 + N_3 x_3 + N_4 x_4$

$y = N_1 y_1 + N_2 y_2 + N_3 y_3 + N_4 y_4$

$N_1 = \frac{1}{4}(1-\xi)(1-\eta) = \frac{1}{4}(1-0.8)(1-0.6) = 0.02$

$N_2 = \frac{1}{4}(1+\xi)(1-\eta) = \frac{1}{4}(1+0.8)(1-0.6) = 0.18$

$N_3 = \frac{1}{4}(1+\xi)(1+\eta) = \frac{1}{4}(1+0.8)(1+0.6) = 0.72$

$N_4 = \frac{1}{4}(1-\xi)(1+\eta) = \frac{1}{4}(1-0.8)(1+0.6) = 0.08$

$\therefore x = N_1 x_1 + N_2 x_2 + N_3 x_3 + N_4 x_4$

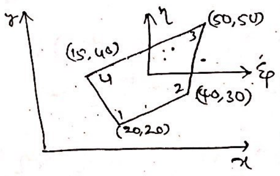
$= 7.84$ units

$y = N_1 y_1 + N_2 y_2 + N_3 y_3 + N_4 y_4$

$= 10.6$ units

<u>Note</u>:- sometimes (ξ, η) can be replaced as (r, s) ⇒ same procedure.

② For the 4 noded quadrilateral element. Determine the jacobian & evaluate its value at the point $(1/2, 1/2)$.



Given $\xi = 1/2$, $\eta = 1/2$.

$x_1 = 20$, $y_1 = 20$

$x_2 = 40$ $y_2 = 30$

$x_3 = 50$ $y_3 = 50$

$x_4 = 15$ $y_4 = 40$

$J_{11} = \frac{1}{4}\left[(1-\eta)(x_2-x_1) + (1+\eta)(x_3-x_4)\right]$

$= \frac{1}{4}\left[(1-0.5)(40-20) + (1+0.5)(50-15)\right]$

$= 15.625$

$J_{12} = \frac{1}{4}\left[(1-\eta)(y_2-y_1) + (1+\eta)(y_3-y_4)\right]$

$= \frac{1}{4}\left[(1-0.5)(30-20) + (1+0.5)(50-40)\right] = 5$

$J_{21} = \frac{1}{4}\left[(1-\xi)(x_4-x_1) + (1+\xi)(x_3-x_2)\right]$

$= \frac{1}{4}\left[(1-0.5)(15-20) + (1+0.5)(50-40)\right] = 3.125$

$J_{22} = \frac{1}{4}\left[(1-\xi)(y_4-y_1) + (1+\xi)(y_3-y_2)\right]$

$= \frac{1}{4}\left[(1-0.5)(40-20) + (1+0.5)(50-30)\right]$

$= 10.$

$\therefore$ Jacobian matrix, $J = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}$

$$= \begin{vmatrix} 15.625 & 5 \\ 3.125 & 10 \end{vmatrix} = 15.625(10) - 3.125(5) = 140.625$$

* STEADY STATE HEAT TRANSFER ANALYSIS.

→ one dimensional analysis:

The steady state heat transfer problems like temperature distribution through fin, heat conduction along the wall thickness, axially loaded bars, the fluid flow through hydraulic network, elastic spring systems etc. are treated as 1-D problems

the Governing one-dim heat conduction is given by $q_x = -k\frac{dT}{dx}$, $q = \frac{Q}{A}$

or

$Q_x = -KA\frac{dT}{dx}$; $\frac{dT}{dx} \rightarrow$ Temp gradient

$K \rightarrow$ thermal conductivity

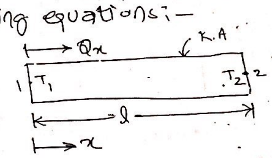
* shape functions:-

$$T(x) = N_1T_1 + N_2T_2 \Rightarrow T(x) = [N_1 \; N_2]\begin{bmatrix} T_1 \\ T_2 \end{bmatrix}$$

here

$$N_1 = 1 - \frac{x}{L} \; ; \; N_2 = \frac{x}{L} \Rightarrow [T_x] = [N][T]$$

* Derivation of stiffness matrix using Governing equations:-



Consider a 1-D bar element of length 'l', c/s area 'A', having two nodes 1 & 2. assume the heat flow from node 1 to 2

we know $Q_x = -KA\frac{dT}{dx}$

→ heat flow at node 1: $Q_1 = \frac{KA}{l}(T_1 - T_2)$ —①

→ heat flow at node 2: $Q_2 = \frac{KA}{l}(T_2 - T_1)$

$\Rightarrow Q_2 = \frac{-KA}{l}(T_1 - T_2)$ —②

... form :

$$\underbrace{\begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}}_{\text{Heat flow vector}} = \frac{KA}{l}\underbrace{\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}}_{\text{thermal stiffness matrix}}\underbrace{\begin{bmatrix} T_1 \\ T_2 \end{bmatrix}}_{\text{Temp Vector}}$$

$$\Rightarrow [Q] = [K][T]$$

here $[K] = \frac{KA}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

* F.E. equation for 1-D pure heat conduction problems:-

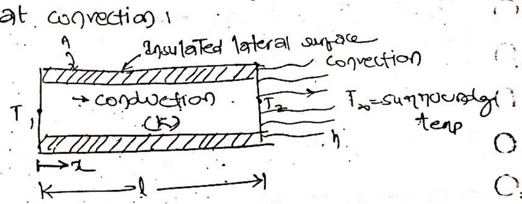
If Q_1 & Q_2 (nodal heat flux) are considered as node thermal forces for analysis then

$$[K][T] = [F]$$

$$\Rightarrow \frac{KA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} T_1 \\ T_2 \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$$

* Derivation of thermal stiffness matrix for 1-D element subjected to conduction & convection:-

(i) Heat conduction along with free end heat convection :



the conduction part of thermal stiffness matrix can be derived from relation

$$[K]_K = \int_V [B]^T[D][B]\,dv$$

$$= \frac{KA}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad \text{(already derived)}$$

→ Convection can be derived from

$$\boxed{[K]_h = \int_A h\,[N]^T[N]\,dA}$$

$h \rightarrow$ convection heat transfer

$[N] \rightarrow$ shape function matrix

we know

$$[N] = [N_1 \quad N_2] = \left[1-\frac{x}{l} \quad \frac{x}{l}\right]$$

$\because$ convection takes place at free end $x=l$.

$$\Rightarrow N_1 = 1-\frac{l}{l} = 1-1 = 0$$

$$N_2 = \frac{l}{l} = 1$$

$$\therefore [N] = [N_1, N_2] = [0 \quad 1]$$

$$\Rightarrow [N]^T = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\therefore [K]_h = \int_A h [N]^T [N] \, dA$$

$$= hA \begin{bmatrix} 0 \\ 1 \end{bmatrix} [0 \quad 1]$$

$$= hA \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$\therefore$ Combined thermal stiffness matrix is

$$[K] = [K]_k + [K]_h$$

$$\therefore [K] = \frac{KA}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + hA \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \longrightarrow (1)$$

Now convection force from free end:

$$[F_h]_{end} = \int_A h T_\infty [N]^T dA$$

$$[F_h]_{end} = hAT_\infty \begin{bmatrix} 0 \\ 1 \end{bmatrix} \longrightarrow (2)$$

$\therefore$ finite element equation

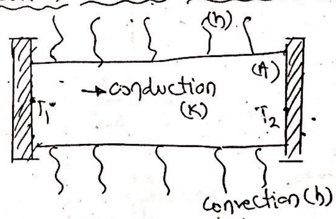
$$[K][T] = [F]$$

$$\boxed{\left(\frac{KA}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + hA \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} T_1 \\ T_2 \end{bmatrix} = hAT_\infty \begin{bmatrix} 0 \\ 1 \end{bmatrix}}$$

conduction + convection = convection force @ free end

5M

ii) thermal stiffness matrix for 1-D heat conduction with lateral surface convection & with internal heat generation:



we know

$$[K]_k = \frac{KA}{L}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} \quad \text{—①}$$

convection for lateral surface area

$$[K]_h = \int_s h[N]^T[N]\,ds$$

{ ∵ $ds = P\,dx$
⇒ surface = Perimeter × thickness }

$$\Rightarrow [K]_h = \int_0^l h[N]^T[N]P\,dx$$

$$= hP\int_0^l \begin{bmatrix}(1-\frac{x}{l})\\ (\frac{x}{l})\end{bmatrix}\begin{bmatrix}(1-\frac{x}{l}) & (\frac{x}{l})\end{bmatrix}dx$$

$$= hP\int_0^l \begin{bmatrix}(1-\frac{x}{l})^2 & (1-\frac{x}{l})\frac{x}{l}\\ (1-\frac{x}{l})(\frac{x}{l}) & (\frac{x}{l})^2\end{bmatrix}dx$$

$$= hP\begin{bmatrix} x+\frac{x^3}{3l^2}-\frac{2x^2}{2l} & \frac{x^2}{2l}-\frac{x^3}{3l^2}\\ \frac{x^2}{2l}-\frac{x^3}{3l^2} & \frac{x^3}{3l^2}\end{bmatrix}$$

$$= hP\begin{bmatrix} l+\frac{l}{3}-l & \frac{l}{2}-\frac{l}{3}\\ \frac{l}{2}-\frac{l}{3} & \frac{l}{3}\end{bmatrix}$$

$$\Rightarrow [K]_h = hP\begin{bmatrix}\frac{l}{3} & \frac{l}{6}\\ \frac{l}{6} & \frac{l}{3}\end{bmatrix} = \frac{hPl}{6}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}$$

* Combined thermal stiffness matrix:

$$[K] = [K]_k + [K]_h$$

$$\Rightarrow [K] = \frac{KA}{l}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} + \frac{hP}{l}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}$$

* thermal forces due to lateral surface heat convection

$$[F]_h = \frac{hPl\,T_\infty}{2}\begin{bmatrix}1\\ 1\end{bmatrix}$$

* Thermal force due to ~~lateral surface~~ internal heat generation:

$$[F]_Q = \frac{QAL}{2}\begin{bmatrix}1\\ 1\end{bmatrix}$$

∴ Total thermal force ⇒ $F = [F]_h + [F]_Q$

$$= \frac{hPLT_\infty}{2}\begin{bmatrix}1\\ 1\end{bmatrix} + \frac{QAL}{2}\begin{bmatrix}1\\ 1\end{bmatrix}$$

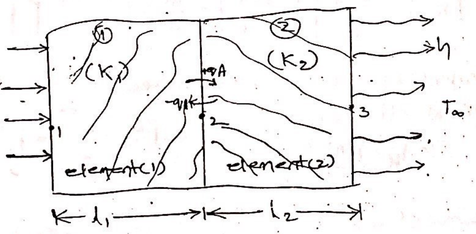
$$\therefore [F] = \frac{hPLT_\infty + QAL}{2}\begin{bmatrix}1\\ 1\end{bmatrix}$$

∴ finite element equation:

$$[K][T] = [F]$$

$$\boxed{\left\{\frac{KA}{L}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} + \frac{hPL}{6}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}\right\}\begin{bmatrix}T_1\\ T_2\end{bmatrix} = \frac{hPLT_\infty + QAL}{2}\begin{bmatrix}1\\ 1\end{bmatrix}}$$

* Heat Transfer analysis for Composite Wall :-



let T_1, T_2 & T_3 → Temp's @ nodes 1, 2 & 3

q = heat inflow at inner side of elemt ①

k_1, k_2 = thermal conductivities of elemt (1) & (2)

h = heat convection coeff

T_∞ = atmospheric (ambient) temp

A ⊥ c/s area ⊥ to dirn of heat flow

l_1, l_2 = wall thickness of elemt (1) & (2)

→ consider element (1)

∴ the heat is transferred through conduction only

$$\therefore [K]_1[T]_1 = [F]_1$$

$$\Rightarrow \frac{K_1 A}{L_1}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{bmatrix}T_1\\ T_2\end{bmatrix} = \begin{bmatrix}F_1\\ F_2\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}\frac{K_1 A}{L_1} & -\frac{K_1 A}{L_1}\\ -\frac{K_1 A}{L_1} & \frac{K_1 A}{L_1}\end{bmatrix}\begin{bmatrix}T_1\\ T_2\end{bmatrix} = \begin{bmatrix}F_1\\ F_2\end{bmatrix} \quad \text{— ①}$$

also $F_1 = qA$ & $F_2 = -qA$

→ element (2):

heat transfer through conduction along with end surface convection.

$$[K]_2[T]_2 = [F]_2$$

$$\Rightarrow \left\{\frac{K_2 A}{l_2}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} + hA\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}\right\}\begin{bmatrix}T_2\\ T_3\end{bmatrix} = \begin{bmatrix}F_2\\ F_3\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}\frac{K_2 A}{l_2} & -\frac{K_2 A}{l_2}\\ -\frac{K_2 A}{l_2} & \frac{K_2 A}{l_2} + hA\end{bmatrix}\begin{bmatrix}T_2\\ T_3\end{bmatrix} = \begin{bmatrix}F_2\\ F_3\end{bmatrix}$$

here $F_2 = qA$ & $F_3 = hAT_\infty$

If heat gen' is given consider F_1 F_2

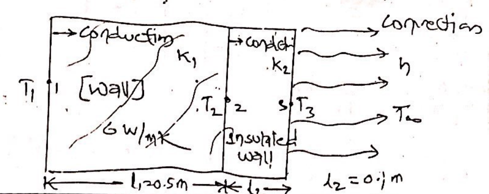
... composite wall the combined stiffness matrix $[K][T] = [F]$

$$\begin{bmatrix} \frac{K_1 A}{l_1} & -\frac{K_1 A}{l_1} & 0 \\ -\frac{K_1 A}{l_1} & \frac{K_1 A}{l_1} + \frac{K_2 A}{l_2} & -\frac{K_2 A}{l_2} \\ 0 & -\frac{K_2 A}{l_2} & \frac{K_2 A}{l_2} + hA \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} = \begin{bmatrix} q_A \\ 0 \\ hAT_\infty \end{bmatrix}$$

$q_A - q_A = 0$

* Prob's one one-dim composite walls:-

1. A wall of 0.5m thickness having thermal conductivity of 6 W/mK the wall is to be insulated with a material of thickness 0.1m having an average thermal conductivity of 0.3 W/mK. The inner surface temp is 1200°C & the outside of insulation is exposed to atm air @ 30°C, with 40 W/m²K. Calculate nodal temp's.



element (1):
$K_1 = 6$ W/mK, thickness $l_1 = 0.5$m

element (2):
$K_2 = 0.3$ W/mK, thickness $l_2 = 0.1$m, $h = 40$ W/m²K

Given $T_1 = 1200°C = 1200 + 273 = 1473K$
$T_\infty = 30°C = 30 + 273 = 303K$

→ consider element (1):
→ conduction: let $A = 1m^2$

$$[K_1] = \frac{K_1 A_1}{l_1}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{6(1)}{0.5}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 12 & -12 \\ -12 & 12 \end{bmatrix}$$

→ element (2): conduction + convection
→ conduction:

$$[K]_K = \frac{K_2 A_2}{l_2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{0.3}{0.1}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix}$$

→ convection:

$$[K]_h = hA\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = 40\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 40 \end{bmatrix}$$

∴ ~~Global stiffness matrix is [K]~~

element (2) ⇒ $K_2 = [K]_K + [K]_h$

$$= \begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 40 \end{bmatrix}$$

$$\therefore [K]_2 = \begin{array}{cc} \begin{array}{cc} 2 & 3 \end{array} & \\ \begin{bmatrix} 3 & -3 \\ -3 & 43 \end{bmatrix} & \begin{array}{c} 2 \\ 3 \end{array} \end{array}$$

$\therefore$ Global stiffness matrix is $[K] = [K]_1 + [K]_2$

$$K = \begin{array}{cc} \begin{array}{ccc} 1 & 2 & 3 \end{array} & \\ \begin{bmatrix} 12 & -12 & 0 \\ -12 & 15 & -3 \\ 0 & -3 & 43 \end{bmatrix} & \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \end{array}$$

We know $[K][T] = [F]$

$$\Rightarrow \begin{bmatrix} 12 & -12 & 0 \\ -12 & 15 & -3 \\ 0 & -3 & 43 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix}$$

Note: due to cond^c there wont be any thermal force.

→ boundary conditions:

$T_1 = 1473\,K$

$F_1 = F_2 = 0$ ($\because$ no heat gen^n)

$F = qA = 0$

$F_3 = hAT_\infty$

$= 40 \times 1 \times 303 = 12120\,W$

$$\Rightarrow \begin{bmatrix} 12 & -12 & 0 \\ -12 & 15 & -3 \\ 0 & -3 & 43 \end{bmatrix} \begin{bmatrix} 1473 \\ T_2 \\ T_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 12120 \end{bmatrix}$$

$\Rightarrow (-12 \times 1473) + 15T_2 - 3T_3 = 0$

$\Rightarrow 15T_2 - 3T_3 = 17676$ — (1)

$-3T_2 + 43T_3 = 12120$ — (2)

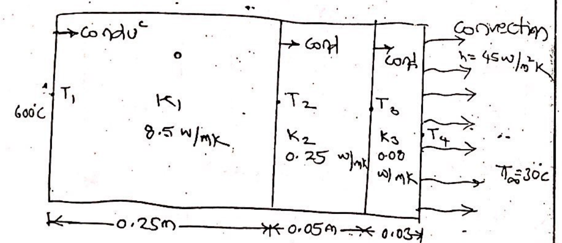
solving (1) & (2)

$T_2 = 18738.69/15$; $T_3 = 369.23\,K$

$= 1252.25\,K$

$\Rightarrow T_2 = 979.25^\circ C$ & $T_3 = 96.23^\circ C$

(2) A furnace wall is made up of 3 layers, inside layer with thermal conductivity of 8.5 W/mK the middle layer with conductivity 0.25 & the outer layer 0.08 W/mK. The respective thickness of the inner, middle & outer layer are 25 cm, 5 cm & 3 cm resp. The inside temp of wall is 600°C & outside of the wall is exposed to atmosphere air of 30°C with heat transfer coeff 45 W/m²K. Determine the nodal temp's. ? ($A = 1\,m^2$)



→ element (1) :– (conduction)

$$[K]_1 = \frac{K_1 A_1}{l_1}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} = \frac{8.5}{0.25}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$$

$$= 34\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} = \begin{bmatrix}34 & -34\\ -34 & 34\end{bmatrix}$$

→ element (2) :– (condu^c)

$$[K]_2 = \frac{k_2 A_2}{l_2}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} = \frac{0.25}{0.05}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} = \begin{bmatrix}5 & -5\\ -5 & 5\end{bmatrix}$$

→ element (3) : (cond^c + conve^c)

$$[K]_3 = [K]_K + [K]_h$$

$$= \frac{K_3 A_3}{l_3}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} + hA\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}$$

$$= \frac{0.08}{0.03}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} + 45\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}$$

$$= 2.67\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} + \begin{bmatrix}0 & 0\\ 0 & 45\end{bmatrix}$$

$$[K]_3 = \begin{bmatrix}2.67 & -2.67\\ -2.67 & 47.67\end{bmatrix}$$

∴ combined stiffness matrix : $[K] = [K]_1 + [K]_2 + [K]_3$

$T_1 = 873K$, $F_1 = F_2 = F_3 = 0$, $F_4 = hAT_\infty$

$= 45 \times 1 \times 303$

$= 13635\ W.$

$$\begin{bmatrix}34 & -34 & 0 & 0\\ -34 & 39 & -5 & 0\\ 0 & -5 & 7.67 & -2.67\\ 0 & 0 & -2.67 & 47.67\end{bmatrix}\begin{bmatrix}873\\ T_2\\ T_3\\ T_4\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 0\\ 13635\end{bmatrix}$$

$\Rightarrow -34(873) + 39(T_2) - 5T_3 = 0$

$-5T_2 + 7.67T_3 - 2.67T_4 = 0$

$-2.67T_3 + 47.67T_4 = 13635$

by solving

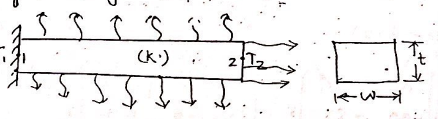
$T_2 = 846.2\,K = 573.2^\circ C$

$T_3 = 664.1\,K = 391.1^\circ C$

$T_4 = 323.2\,K = 50.2^\circ C$

* Heat Transfer Analysis for fins :-
(extended surfaces) (heat sink)

→ Straight fin analysis :-



$T_1, T_2 \rightarrow$ Temp @ node 1 & 2

$l, w, t \rightarrow$ length, width & thickness of fin

$K \rightarrow$ thermal conductivity

→ Conduction :

$$[K]_k = \frac{KA}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

→ Convection through perimeter surface :

$$[K]_{hp} = \frac{hPl}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

→ Convection through end surface :

$$[K]_{he} = hA\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

∴ overall thermal stiffness matrix.

$$[K] = [K]_k + [K]_{hp} + [K]_{he}$$

$$[K] = \frac{KA}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{hPl}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} + hA\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

∴ Thermal force vector

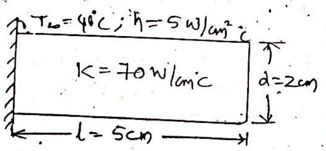
$$[F] = \left[\frac{hPlT_\infty + QAL}{2}\right]\begin{bmatrix} 1 \\ 1 \end{bmatrix} + hAT_\infty\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

∴ $[K][T] = [F]$

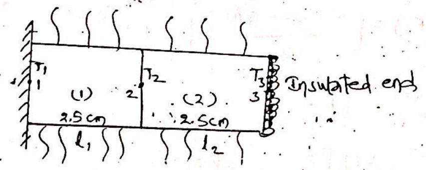
$$\frac{KA}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{hPl}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} + hA\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \frac{hPlT_\infty + QAL}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + hAT_\infty\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \frac{KA}{l} + \frac{hPl}{3} & \frac{-KA}{l} + \frac{hPl}{6} \\ \frac{-KA}{l} + \frac{hPl}{6} & \frac{KA}{l} + \frac{hPl}{3} + hA \end{bmatrix}\begin{bmatrix} T_1 \\ T_2 \end{bmatrix} = \begin{bmatrix} \frac{hPlT_\infty + QAL}{2} \\ \frac{hPlT_\infty + QAL}{2} + hAT_\infty \end{bmatrix}$$

① Find the temp distribution in a straight fin with the physical properties $K = 70\ W/cm^{\circ}C$. $h = 5\ W/cm^2\,^{\circ}C$. Temp @ root of the fin: $T_0 = 140^{\circ}C$, surrounding temp $T_\infty = 40^{\circ}C$. Assume that free end of fin is insulated. (Consider two element)



Sol:-



Given $K_1 = K_2 = 70\ W/cm^{\circ}C$

$l_1 = l_2 = 2.5cm$

$h_1 = h_2 = 5\ W/cm^2\,^{\circ}C$

$A_1 = A_2 = \frac{\pi}{4}(2)^2 = \pi\ cm^2$

$\boxed{T_0 = T_1 = 140^{\circ}C}$

$P = \pi d = \pi \times 2 = 2\pi\ cm$

→ element(1) :- (conduc + Perimeter surface convection)

$$\left\{\frac{K_1A_1}{l_1}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} + \frac{h_1P_1l_1}{6}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}\right\}\begin{bmatrix}T_1\\ T_2\end{bmatrix} = \frac{h_1P_1l_1T_\infty}{2}\begin{bmatrix}1\\ 1\end{bmatrix}$$

$$\left[\frac{70\times\pi}{2.5}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} + \frac{5\times2\pi\times2.5}{6}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}\right]\begin{bmatrix}T_1\\ T_2\end{bmatrix} = \frac{5\times2\pi\times25\times40}{2}\begin{bmatrix}1\\ 1\end{bmatrix}$$

$$\left[\begin{array}{c}\begin{array}{cc}1 & 2\end{array}\\ \begin{bmatrix}88 & -88\\ -88 & 88\end{bmatrix}\end{array}\begin{array}{c}1\\2\end{array} + \begin{array}{c}\begin{array}{cc}1 & 2\end{array}\\ \begin{bmatrix}26 & 13\\ 13 & 26\end{bmatrix}\end{array}\right]\begin{bmatrix}140\\ T_2\end{bmatrix} = \begin{bmatrix}1571\\ 1571\end{bmatrix}\begin{array}{c}1\\2\end{array} \quad — ①$$

→ element (2) :-

$$\left\{\begin{array}{c}\begin{array}{cc}2 & 3\end{array}\\ \begin{bmatrix}88 & -88\\ -88 & 88\end{bmatrix}\end{array}\begin{array}{c}2\\3\end{array} + \begin{array}{c}\begin{array}{cc}2 & 3\end{array}\\ \begin{bmatrix}26 & 13\\ 13 & 26\end{bmatrix}\end{array}\right\}\begin{bmatrix}T_2\\ T_3\end{bmatrix}\begin{array}{c}2\\3\end{array} = \begin{bmatrix}1571\\ 1571\end{bmatrix} \quad — ②$$

→ combine matrix :-

$$\left[\begin{bmatrix}88 & -88 & 0\\ -88 & 176 & -88\\ 0 & -88 & 88\end{bmatrix} + \begin{bmatrix}26 & 13 & 0\\ 13 & 52 & 13\\ 0 & 13 & 26\end{bmatrix}\right]\begin{bmatrix}140\\ T_2\\ T_3\end{bmatrix} = \begin{bmatrix}1571\\ 3142\\ 1571\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}114 & -75 & 0\\ -75 & 228 & -75\\ 0 & -75 & 114\end{bmatrix}\begin{bmatrix}140\\ T_2\\ T_3\end{bmatrix} = \begin{bmatrix}1571\\ 3142\\ 1571\end{bmatrix}$$

$$(-75\times 140) + 228T_2 - 75T_3 = 3742 \quad \text{---} (1)$$

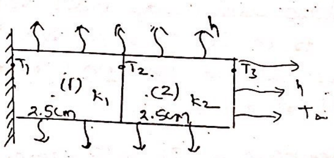
$$-75T_2 + 114T_3 = 1571 \quad \text{---} (2)$$

solving (1) & (2)

$$\Rightarrow T_2 = 87.2^\circ C$$

$$T_3 = 67.7^\circ C.$$

(2) Consider the heat convection through end surface. also a part from perimeter surface heat convection & body conduction.



Sol: Given $T_0 = T_1 = 140^\circ C$

$T_\infty = 40^\circ C$

$h = 5\ W/cm^2{}^\circ C = h_1 = h_2$

$k_1 = k_2 = 70\ W/cm^\circ C$

$l_1 = l_2 = 2.5\ cm$

$A_1 = A_2 = \frac{\pi}{4}(2)^2 = \pi cm^2$

$hA = 16\ W/^\circ C$

→ element (1) :- (Conduction + Peri. Convection):

$$\left\{\frac{k_1A_1}{l_1}\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix} + \frac{h_1P_1l_1}{6}\begin{bmatrix}2 & 1\\1 & 2\end{bmatrix}\right\}\begin{bmatrix}T_1\\T_2\end{bmatrix} = \frac{h_1P_1l_1T_\infty}{2}\begin{bmatrix}1\\1\end{bmatrix}$$

$$\Rightarrow \left\{\frac{70\times\pi}{2.5}\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix} + \frac{5\times 2\pi\times 2.5}{6}\begin{bmatrix}2 & 1\\1 & 2\end{bmatrix}\right\}\begin{bmatrix}T_1\\T_2\end{bmatrix} = \frac{5(2\pi)(2.5)40}{2}\begin{Bmatrix}1\\1\end{Bmatrix}$$

$$\Rightarrow \left\{\begin{bmatrix}88 & -88\\-88 & 88\end{bmatrix} + \begin{bmatrix}26 & 13\\13 & 26\end{bmatrix}\right\}\begin{bmatrix}T_1\\T_2\end{bmatrix} = \begin{bmatrix}1571\\1571\end{bmatrix} \quad \text{---} (1)$$

→ element (2) :- (Cond^c + Perimeter convection + end conv^c)

$$\left\{\frac{k_2A_2}{l_2}\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix} + \frac{h_2P_2l_2}{6}\begin{bmatrix}2 & 1\\1 & 2\end{bmatrix} + h_2A_2\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\right\}\begin{bmatrix}T_2\\T_3\end{bmatrix} =$$

$$\frac{h_2P_2l_2T_\infty}{2}\begin{bmatrix}1\\1\end{bmatrix} + h_2A_2T_\infty\begin{bmatrix}0\\1\end{bmatrix}$$

$$\Rightarrow \left\{\begin{array}{c}\begin{array}{cc}2 & 3\end{array}\\\begin{bmatrix}88 & -88\\-88 & 88\end{bmatrix}\end{array} + \begin{array}{c}\begin{array}{cc}2 & 3\end{array}\\\begin{bmatrix}26 & 13\\13 & 26\end{bmatrix}\end{array} + \begin{array}{c}\begin{array}{cc}2 & 3\end{array}\\\begin{bmatrix}0 & 0\\0 & 16\end{bmatrix}\end{array}\right\}\begin{bmatrix}T_2\\T_3\end{bmatrix} = \begin{bmatrix}1571\\1571\end{bmatrix} + \begin{bmatrix}0\\628\end{bmatrix}$$

→ combine stiffness matrix:

$$\left\{\begin{bmatrix}88 & -88 & 0\\-88 & 176 & -88\\0 & -88 & 88\end{bmatrix} + \begin{bmatrix}26 & 13 & 0\\13 & 52 & 13\\0 & 13 & 26\end{bmatrix} + \begin{bmatrix}0 & 0 & 0\\0 & 0 & 0\\0 & 0 & 16\end{bmatrix}\right\}\begin{bmatrix}140\\T_2\\T_3\end{bmatrix} =$$

$$= \begin{bmatrix} 1571 \\ 3142 \\ 2199 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 114 & -75 & 0 \\ -75 & 228 & -75 \\ 0 & -75 & 130 \end{bmatrix} \begin{bmatrix} 140 \\ T_2 \\ T_3 \end{bmatrix} = \begin{bmatrix} 1571 \\ 3142 \\ 2199 \end{bmatrix}$$

$$\Rightarrow \left.\begin{aligned} -75(140) + 228T_2 - 75T_3 &= 3142 \\ -75T_2 + 130T_3 &= 2199 \end{aligned}\right\}$$

$$\Rightarrow T_2 = 80.7^\circ C$$
$$T_3 = 63.5^\circ C$$

Unit-5

DYNAMIC ANALYSIS

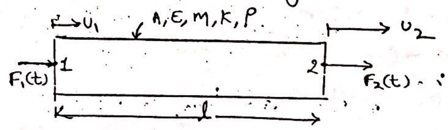
The analysis of fast moving mass & the resultant displacement is known as Dynamic analysis.

In dynamic analysis problems, the displacements, velocities, strains, stresses & loads are time dependent i.e., their magnitude vary w.r.t time.

Any structure consists of mass, stiffness subjected to loads causing acceleration or vibrations, so here the mass & acce^n effect will be seen.

→ Natural frequency:- It is the frequency of free vibration of the system.

* Derivation of dynamic equation of motion for 1-D bar element using Newton's 2nd law:-



Consider a bar element subjected to a time dependent load (dynamic load).

let F_1, F_2 are applied external forces @node 1 & 2 (nodal forces)

$u_1\ u_2$ are displacements produced @node 1 & 2.

Accd to Newton's 2nd law

$\boxed{\text{Applied force} - \text{Resisting force} = \text{Inertia force}}$

@node 1,

$$F_1(t) - K(u_1 - u_2) = m_1 \ddot{u}_1 \qquad \ddot{u} = \frac{d^2u}{dt^2} \rightarrow acc'$$

$$\Rightarrow m_1 \ddot{u}_1 + K(u_1 - u_2) = F_1(t)$$

@node 2,

$$\Rightarrow m_2 \ddot{u}_2 + K(-u_1 + u_2) = F_2(t)$$

∴ matrix form

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{bmatrix} + \begin{bmatrix} K & -K \\ -K & K \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} F_1(t) \\ F_2(t) \end{bmatrix}$$

here:

m_1 & m_2 are lumped masses

$$m_1 = m_2 = \frac{m}{2} = \frac{\rho AL}{2} \ \& \ k = \frac{AE}{l} \qquad \left\{ \rho = \frac{m}{V},\ m = \rho \cdot V = \rho AL \right.$$

$$\Rightarrow \underbrace{\frac{\rho AL}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{\text{lumped mass matrix}} \overbrace{\begin{bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{bmatrix}}^{\text{Acc vector}} + \underbrace{\frac{AE}{l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}}_{\text{stiffness matrix}} \underbrace{\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}}_{\text{displacement vector}} = \underbrace{\begin{bmatrix} F_1(t) \\ F_2(t) \end{bmatrix}}_{\text{force vector}}$$

To get accurate solution, another type of mass-matrix called consistent mass matrix must be derived

consistent mass matrix can be derived from

$$[M] = \rho \int_V [N]^T [N] \, dV$$

$$= \rho \int_0^l [N]^T [N] \cdot A \, dx$$

$$= \rho \int_0^l \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} [N_1 \ \ N_2] \, A \, dx$$

$$= \rho A \int_0^l \begin{bmatrix} N_1^2 & N_1 N_2 \\ N_1 N_2 & N_2^2 \end{bmatrix} dx$$

$$= \rho A \begin{bmatrix} l/3 & l/6 \\ l/6 & l/3 \end{bmatrix}$$

$$[M] = \frac{\rho AL}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

∴ by using consistent mass matrix eqⁿ of motion is given by

** $$\boxed{\underbrace{\frac{\rho A L}{6}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}}_{\text{consistent mass matrix}}\begin{bmatrix}\ddot{U}_1\\ \ddot{U}_2\end{bmatrix} + \underbrace{\frac{AE}{l}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}}_{\text{stiffness matrix}}\underbrace{\begin{bmatrix}U_1\\ U_2\end{bmatrix}}_{\text{displacement vector}} = \underbrace{\begin{bmatrix}F_1(t)\\ F_2(t)\end{bmatrix}}_{\text{force vector}}}$$

∴ Global Stiffness matrix ; Global Mass Matrix

$$\therefore [K] = \frac{AE}{l}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} \qquad [M] = \frac{\rho A l}{6}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}$$

→ characteristic eqⁿ:

$$\boxed{\left|[K] - \lambda [M]\right| = 0}$$

here $\lambda = \omega^2$

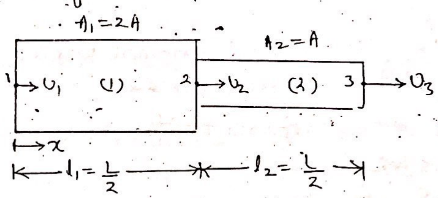
$\omega = \sqrt{\frac{K}{m}}$

dynamic eqⁿ of motion is given by

$$[K - \lambda M][U] = 0$$

Problems:-

① Find the natural frequencies of longitudinal vibration of the unconstrained stepped bar.



Sol: we know

dynamic eqⁿ of motion $[K - \lambda M][U] = 0$

→ element (1):

→ Global stiffness matrix

$$[K]_1 = \frac{A_1 E}{l_1}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} = \frac{2AE}{L/2}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$$

$$= \frac{4AE}{L}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$$

→ element mass matrix

$$[M]_1 = \frac{\rho_1 A_1 l_1}{6}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}$$

$$= \frac{\rho_1 (2A) L/2}{6}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix} = \frac{\rho A L}{6}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}$$

→ element(3) :

$$[K_2] = \frac{A_2 E_2}{l_2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{AE}{l/2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= \frac{2AE}{L}\begin{array}{c} \begin{array}{cc} 2 & 3 \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array}\begin{array}{c} 2 \\ 3 \end{array}$$

$$[M]_2 = \frac{\rho A(L/2)}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \frac{\rho AL}{12}\begin{array}{c} \begin{array}{cc} 2 & 3 \end{array} \\ \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \end{array}\begin{array}{c} 2 \\ 3 \end{array}$$

→ Combine stiffness matrix & mass matrix:

$$[K] = \frac{2AE}{l}\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 2 & -2 & 0 \\ -2 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix} \end{array}\begin{array}{c} 1 \\ 2 \\ 3 \end{array}$$

$$[M] = \frac{\rho AL}{12}\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 4 & 2 & 0 \\ 2 & 6 & 1 \\ 0 & 1 & 2 \end{bmatrix} \end{array}\begin{array}{c} 1 \\ 2 \\ 3 \end{array}$$

∴ Subⁿ in eqⁿ ①

$$\left[\frac{2AE}{l}\begin{bmatrix} 2 & -2 & 0 \\ -2 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix} - \frac{\rho AL}{12}\begin{bmatrix} 4 & 2 & 0 \\ 2 & 6 & 1 \\ 0 & 1 & 2 \end{bmatrix}\omega^2\right]\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = 0$$

⇒ characteristic eqⁿ

$$\left|\begin{bmatrix} 2 & -2 & 0 \\ -2 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix} - \frac{\rho L^2 \omega^2}{24E}\begin{bmatrix} 4 & 2 & 0 \\ 2 & 6 & 1 \\ 0 & 1 & 2 \end{bmatrix}\right| = 0$$

Let $\lambda = \dfrac{\rho L^2 \omega^2}{24E}$

$$\Rightarrow \begin{bmatrix} 2 & -2 & 0 \\ -2 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix} - \lambda\begin{bmatrix} 4 & 2 & 0 \\ 2 & 6 & 1 \\ 0 & 1 & 2 \end{bmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} (2-4\lambda) & (-2-2\lambda) & 0 \\ (-2-2\lambda) & (3-6\lambda) & (-1-\lambda) \\ 0 & (-1-\lambda) & (1-2\lambda) \end{vmatrix} = 0$$

$$\Rightarrow (2-4\lambda)\left[(3-6\lambda)(1-2\lambda) - (-1-\lambda)(-1-\lambda)\right] - (-2-2\lambda)\left[(-2-2\lambda)(1-2\lambda) - (-1-\lambda)(0)\right] + 0[0] = 0$$

$$\Rightarrow \lambda(1-2\lambda)(\lambda-2) = 0$$

If $\lambda = 0 \Rightarrow \dfrac{\rho L^2 \omega^2}{24E} = 0 \Rightarrow \omega_1^2 = 0 \Rightarrow \omega_1 = 0$

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when $(1-2\lambda)=0$; $\lambda = 0.5$

$$\Rightarrow 1-2\left[\frac{\rho L^2 \omega_2^2}{24E}\right] = 0$$

$$\Rightarrow \omega_2^2 = \frac{12E}{\rho L^2} \Rightarrow \omega_2 = 3.46\left[\frac{E}{\rho L^2}\right]^{1/2} \text{ rad/sec}$$

when $(\lambda-2)=0$; $\lambda = 2$

$$\Rightarrow \frac{\rho L^2 \omega_3^2}{24E} - 2 = 0$$

$$\Rightarrow \omega_3^2 = \frac{48E}{\rho L^2} \Rightarrow \omega_3 = 6.93\left[\frac{E}{\rho L^2}\right]^{1/2} \text{ rad/sec}$$

are natural frequencies

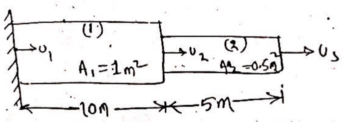
To get values of λ

$$\lambda(1-2\lambda)(\lambda-2)=0$$

$$\Rightarrow \left.\begin{matrix} \lambda_1 = 0 \\ \lambda_2 = 0.5 \\ \lambda_3 = 2 \end{matrix}\right\} \text{Eigen values.}$$

(2) Det eigen values & frequencies for the stepped bar as shown. $E = 30\times10^{10}\ N/m^2$

$\rho = 8500\ kg/m^3$



Sol:- we know

$$[K-\omega^2 M][U] = 0 \longrightarrow (A)$$

→ element (1):

$$[K]_1 = \frac{A_1E_1}{l_1}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} = \frac{1\times 3\cancel{0}\times10^{10}}{1\cancel{0}}\begin{array}{c} \begin{matrix}1 & 2\end{matrix} \\ \begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{matrix}1\\2\end{matrix}\end{array}$$

~~[K]~~

$$[M]_1 = \frac{\rho_1 A_1 L_1}{6}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix} = \frac{8500\times1\times10}{6}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}$$

$$= \frac{85\times10^3}{6}\begin{array}{c} \begin{matrix}1 & 2\end{matrix} \\ \begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}\begin{matrix}1\\2\end{matrix}\end{array}$$

→ element (2):

$$[K]_2 = \frac{A_2E_2}{l_2}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} = \frac{0.5\times30\times10^{10}}{5}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$$

$$= 3\times10^{10}\begin{array}{c} \begin{matrix}2 & 3\end{matrix} \\ \begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{matrix}2\\3\end{matrix}\end{array}$$

$$[M]_2 = \frac{\rho_2 A_2 l_2}{6}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix} = \frac{8500\times0.5\times5}{6}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}$$

$$\Rightarrow [M]_2 = \frac{85\times10^3}{24}\begin{array}{c c} \begin{array}{cc} 2 & 3 \end{array} & \\ \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} & \begin{array}{c} 2 \\ 3 \end{array} \end{array}$$

∴ Global stiffness & mass matrix :-

$$[K] = 3\times10^{10}\begin{array}{c c} \begin{array}{ccc} 1 & 2 & 3 \end{array} & \\ \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} & \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \end{array} \ N/m$$

&

$$[M] = \frac{85\times10^3}{24}\begin{array}{c c} \begin{array}{ccc} 1 & 2 & 3 \end{array} & \\ \begin{bmatrix} 8 & 4 & 0 \\ 4 & 10 & 1 \\ 0 & 1 & 2 \end{bmatrix} & \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \end{array} \ kg$$

Sub in eq(1)

$$\left[3\times10^{10}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} - \frac{85\times10^3\times\omega^2}{24}\begin{bmatrix} 8 & 4 & 0 \\ 4 & 10 & 1 \\ 0 & 1 & 2 \end{bmatrix}\right]\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} - 1.187\times10^{-7}\omega^2\begin{bmatrix} 8 & 4 & 0 \\ 4 & 10 & 1 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = 0.$$

let $\lambda = 1.187\times10^{-7}\omega^2$

$$\Rightarrow \begin{bmatrix} 1-8\lambda & (-1-4\lambda) & 0 \\ -1-4\lambda & (2-10\lambda) & (-1-\lambda) \\ 0 & (-1-\lambda) & (1-2\lambda) \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = 0.$$

∴ node 1 is fixed $\Rightarrow u_1 = 0$

$$\begin{bmatrix} (2-10\lambda) & (-1-\lambda) \\ (-1-\lambda) & (1-2\lambda) \end{bmatrix}\begin{bmatrix} u_2 \\ u_3 \end{bmatrix} = 0$$

determinent

$$|K - \lambda M| = 0$$

$$\Rightarrow (2-10\lambda)(1-2\lambda) - (-1-\lambda)(-1-\lambda) = 0$$

$$\Rightarrow 19\lambda^2 - 16\lambda + 1 = 0$$

$$\Rightarrow \lambda = \frac{16 \pm \sqrt{16^2 - 4(19)(1)}}{2(19)} \quad \text{Eigen Values}$$

$$\lambda = 0.7742 \text{ (or) } 0.0679$$

$$\therefore \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} 0.0679 \\ 0.7742 \end{bmatrix}$$

when $\lambda = 0.0679$,

$\Rightarrow 1.181\times10^{-7}\omega_1^2 = 0.0679$

$\Rightarrow \omega_1 = 758.3$ rad/sec

when $\lambda = 0.7742$

$\Rightarrow 1.181\times10^{-7}\omega_2^2 = 0.7742$

$\Rightarrow \omega_2 = 2560.3$ rad/sec

} Natural freq^n

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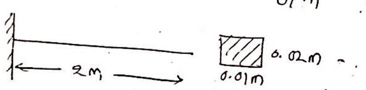
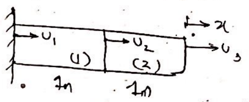
(3) For the cantilever beam show. Determine the natural frequencies & given $E = 200\,GPa$ & $\rho = 7800\ kg/m^3$.



Sol:-



$$A = 0.02 \times 0.01 = 0.0002\ m^2$$

$$\Rightarrow A = 2\times10^{-4}\ m^2 = A_1 = A_2$$

$$\rho_1 = \rho_2 = 7800\ kg/m^3$$

$$E_1 = E_2 = 2\times10^{11}\ N/m^2$$

→ element (1) :

$$[K]_1 = \frac{A_1 E_1}{l_1}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} = \frac{2\times10^{-4}\times2\times10^{11}}{1}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$$

$$= 4\times10^{7}\begin{array}{c} \begin{array}{cc}1 & 2\end{array} \\ \begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\end{array}\begin{array}{c}1\\2\end{array}$$

$$[M]_1 = \frac{\rho_1 A_1 L_1}{6}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix} = \frac{7800\times2\times10^{-4}\times1}{6}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}$$

$$= 0.26\begin{array}{c} \begin{array}{cc}1 & 2\end{array} \\ \begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}\end{array}\begin{array}{c}1\\2\end{array}$$

→ element (2) :-

$$[K]_2 = 4\times10^{7}\begin{array}{c} \begin{array}{cc}2 & 3\end{array} \\ \begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\end{array}\begin{array}{c}2\\3\end{array}$$

$$[M]_2 = 0.26\begin{array}{c} \begin{array}{cc}2 & 3\end{array} \\ \begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}\end{array}\begin{array}{c}2\\3\end{array}$$

→ combine stiffness matrix & mass matrix :-

$$[K] = \begin{array}{c} \begin{array}{ccc}1 & 2 & 3\end{array} \\ \begin{bmatrix}1 & -1 & 0\\ -1 & 2 & -1\\ 0 & -1 & 1\end{bmatrix}\end{array}\begin{array}{c}1\\2\\3\end{array} \times 4\times10^{7}$$

& $$[M] = 0.26\begin{array}{c} \begin{array}{ccc}1 & 2 & 3\end{array} \\ \begin{bmatrix}2 & 1 & 0\\ 1 & 4 & 1\\ 0 & 1 & 2\end{bmatrix}\end{array}\begin{array}{c}1\\2\\3\end{array}$$

∴ Dynamic eqn $[K - \lambda M][U] = 0$

$$\Rightarrow \left[4\times10^{7}\begin{bmatrix}1 & -1 & 0\\ -1 & 2 & -1\\ 0 & -1 & 1\end{bmatrix} - \omega^2\begin{bmatrix}2 & 1 & 0\\ 1 & 4 & 1\\ 0 & 1 & 2\end{bmatrix}0.26\right]\begin{bmatrix}u_1\\ u_2\\ u_3\end{bmatrix} = 0$$

$$\Rightarrow \left(\begin{bmatrix}1 & -1 & 0\\ -1 & 2 & -1\\ 0 & -1 & 1\end{bmatrix} - 6.5\times10^{-9}\omega^2\begin{bmatrix}2 & 1 & 0\\ 1 & 4 & 1\\ 0 & 1 & 2\end{bmatrix}\right)\begin{bmatrix}u_1\\ u_2\\ u_3\end{bmatrix} = 0$$

let $6.5\times10^{-9}\omega^2=\lambda$

$$\begin{bmatrix}1-2\lambda & -1-\lambda & 0\\ -1-\lambda & 2-\lambda & -1-\lambda\\ 0 & -1-\lambda & 1-2\lambda\end{bmatrix}=0 \qquad [\because u_1=0]$$

$\Rightarrow (2-\lambda)(1-2\lambda)-(-1-\lambda)(-1-\lambda)=0$

$\Rightarrow 2-4\lambda-\lambda+2\lambda^2-(1+\lambda+\lambda+\lambda^2)=0$

$\Rightarrow 2-4\lambda-\lambda+2\lambda^2-1-2\lambda-\lambda^2=0$

$\Rightarrow \lambda^2-7\lambda+1=0$

$\Rightarrow \lambda=6.875$ & $\lambda=0.15$

$$\begin{bmatrix}\lambda_1\\ \lambda_2\end{bmatrix}=\begin{bmatrix}0.15\\ 6.85\end{bmatrix}$$

When,

$\lambda_1=0.15 \Rightarrow 6.5\times10^{-9}\omega_1^2=0.15$

$\omega_1^2=2307692 \Rightarrow \omega_1=1519$ rad/sec

$\lambda_2=6.85 \Rightarrow 6.5\times10^{-9}\omega_2^2=6.85$

$\omega_2=10265$ rad/sec

→ also find mode shape (eigen vectors)

@ $\lambda=0.15$

$$\begin{bmatrix}2-\lambda & -1-\lambda\\ -1-\lambda & 1-2\lambda\end{bmatrix}\begin{bmatrix}u_2\\ u_3\end{bmatrix}=0$$

$$\begin{bmatrix}2-0.15 & -1-0.15\\ -1-0.15 & 1-2(0.15)\end{bmatrix}\begin{bmatrix}u_2\\ u_3\end{bmatrix}=0$$

$\Rightarrow \left.\begin{array}{l}1.75u_2-1.15u_3=0\\ -1.15u_2+0.7u_3=0\end{array}\right\}$ solving

$$\begin{bmatrix}u_2\\ u_3\end{bmatrix}=\begin{bmatrix} & \end{bmatrix}$$

@ $\lambda=6.85$

$$\begin{bmatrix}2-\lambda & -1-\lambda\\ -1-\lambda & 1-2\lambda\end{bmatrix}\begin{bmatrix}u_2\\ u_3\end{bmatrix}=0$$

$$\Rightarrow \begin{bmatrix}2-6.85 & -1-6.85\\ -1-6.85 & 1-2(6.85)\end{bmatrix}\begin{bmatrix}u_2\\ u_3\end{bmatrix}=0$$

$\Rightarrow \left.\begin{array}{l}-4.85u_2-7.85u_3=0\\ -7.85u_2-12.7u_3=0\end{array}\right\}$ solving

$$\begin{bmatrix}u_2\\ u_3\end{bmatrix}=\begin{bmatrix} & \end{bmatrix}$$