UNIT- IV

LAMINAR & TURBULENT FLOW:-

So far, in the previous chapters, primarily the flow of an ideal fluid has been discussed.

In the case of Newtonian fluid, the flow can be classified as (i) Laminar (viscous) (ii) Turbulent, depending upon characteristic Reynold's Number $\frac{\rho V d}{\mu}$

Examples of Laminar flow:-

(i) Flow past tiny bodies (ii) Underground Flow (iii) Movement of blood in the arteries of a human body (iv) Rise of water in plants through their roots ... etc.

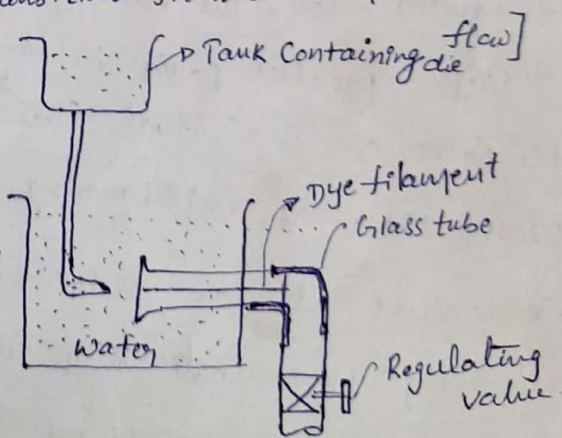
Reynolds Experiment :-

Reynolds with the help of simple experiment demonstrated the existence of following two types of flow.

(i) Laminar flow (Reynolds number, $Re < 2000$)

(ii) Turbulent Flow (Reynolds number, $Re > 4000$)

[Re between 2000 to 4000 indicates transition from Laminar to Turbulent flow]



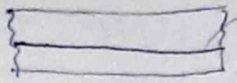
Apparatus:-

1) A const. head tank filled with water
2) A small tank containing dye [Sp.w of die = water]
3) A Glass tube
4) A regulating value.

Procedure:-

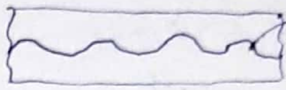
The water was made to flow from the tank through glass tube into the atmosphere and velocity of flow was varied by adjusting value. The liquid die was introduced into the flow at the mouth of glass tube as shown in fig.

Observations made:-

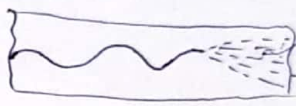


→ when the velocity of flow was Low, the die remained in the form of a straight and stable filament passing through the glass tube so steadily. This was the case of Laminar flow as shown in fig.

→ with increase of velocity a critical state was reached at which the dye filament showed irregularities and began to waver. This was a transition flow as shown in fig.

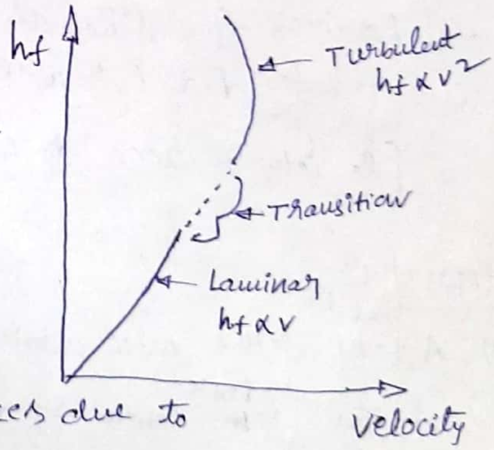


→ With further increase in velocity of flow the fluctuations in the filament of die became more intense and the die diffused over the entire cross section of the tube. This was a Turbulent flow as shown in figure



On the basis of his experiment Reynold's discovered that
(i) In case of Laminar Flow, the loss of pressure head $\propto$ velocity
(ii) In case of turbulent flow, the loss of pressure head $\propto$ velocity^2

Laminar Flow:-



In Laminar flow, the fluid particles move along straight parallel paths in layers such that the paths of individual fluid particles donot cross those of neighbouring particles.

→ It occurs at low velocities so that forces due to viscosity predominate over inertia forces.

→ The viscosity of fluid induces relative motion with in the region fluid as the fluid layers slide over each other, which inturn gives rise of viscous shear stresses.

→ The magnitude of the viscous shear stress so produced, varies from point to point, being max. at the boundary and gradually decreases with increase in distance from the boundary.

→ The shear stresses so produced result in developing a resistance to flow. ②

→ Inorder to overcome the shear resistance, the pressure drops from section to section in the direction of flow, so pressure-gradient exist.

Reynolds number :- [Re]

It is defined as inertia force of a flowing fluid to the viscous force.

$$Re = \frac{F_i}{F_v}$$

Inertia force = Mass x acc^n of flowing fluid

$$= \rho \times \text{volume} \times \frac{\text{velocity}}{\text{sec}}$$

$$= \rho \times \frac{\text{volume}}{\text{sec}} \times \text{velocity}$$

$$= \rho \times \text{Area} \times \text{velocity} \times \text{velocity}$$

$$F_i = \rho A V^2$$

Viscous Force = F_v = Shear stress x Area

$$= \tau \times A$$

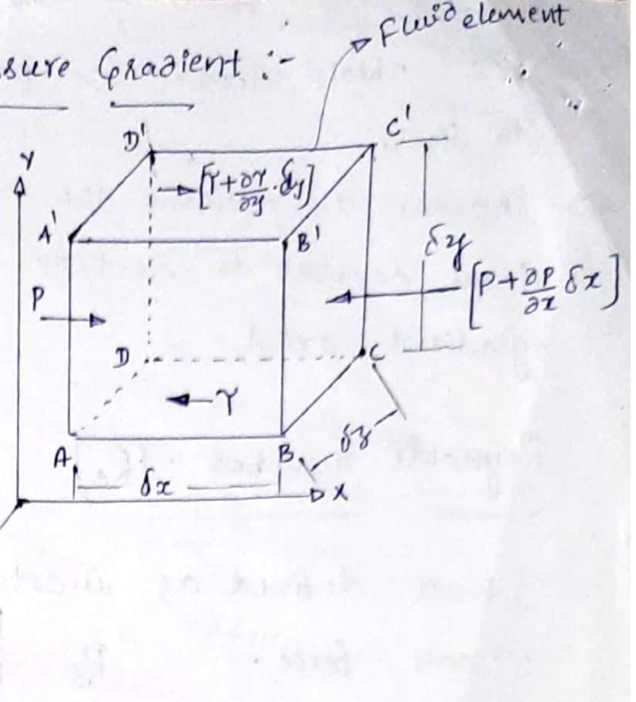
$$= \mu \frac{du}{dy} \times A = \mu \frac{V}{L} \times A$$

$$F_v = \frac{\mu V A}{L}$$

$$Re = \frac{F_i}{F_v} = \frac{\rho A V^2 L}{\mu V A} = \frac{\rho V L}{\mu}$$

Incase of pipe, L is replaced by d, $Re = \frac{\rho V d}{\mu} = \frac{Vd}{\nu}$

Relationship b/w shear stress & pressure Gradient :-



→ let us consider a fluid element having the form of an elementary parallelopiped as shown in fig.

→ The motion of fluid element will be restricted by shear (frictional) forces which must be overcome by maintaining a pressure gradient in the direction of flow.

Assume 2-D flow:

Let pressure acting on left side face = p

pressure force on left side face = $p.\delta y\delta z$

∴ pressure acting on right side face = $\left[p+\frac{\partial p}{\partial x}.dx\right]$

pressure force = $\left[p+\frac{\partial p}{\partial x}.\delta x\right]\delta y\delta z$

Net pressure force = $p\,\delta y\delta z-\left[p+\frac{\delta p}{\partial x}\delta x\right]\delta y\delta z$

$$=-\frac{\delta p}{\delta x}\delta x\delta y\delta z \longrightarrow ①$$

Shear stress acting on the bottom face = τ (on ABCD)

Shear force = $\tau\,\delta x.\delta z$

Shear force on the top face (on $A'B'C'D'$) = $\left[\tau+\frac{\delta\tau}{\partial y}.\delta y\right]$

Shear force = $\left[\tau+\frac{\delta\tau}{\partial y}.\delta y\right]\delta x.\delta z$

Net shear force = $\frac{\delta\tau}{\partial y}.\delta x\delta y\delta z \longrightarrow ②$

For the flow to be steady & uniform, the sum of forces must be zero.

$$\frac{\partial\tau}{\partial y}\delta x\delta y\delta z-\frac{\partial p}{\partial x}\delta x\delta y\delta z=0 \Rightarrow \boxed{\frac{\partial\tau}{\partial y}=\frac{\partial p}{\partial x}}$$

∴ The pressure gradient in the direction of flow is equal to shear gradient in the direction normal to direction of flow. It is valid for all types of flow and all types of boundary geometry.

FLOW OF VISCOUS FLUID IN CIRCULAR PIPES – HAGEN POISEUILLE LAW :- ③

Hagen poiseuille theory is based on following assumptions

1) The fluid follows Newton's Law of viscosity
2) There is no slip of fluid particles adjacent to fluid boundary. [i.e fluid particles adjacent to pipe will have zero velocity]

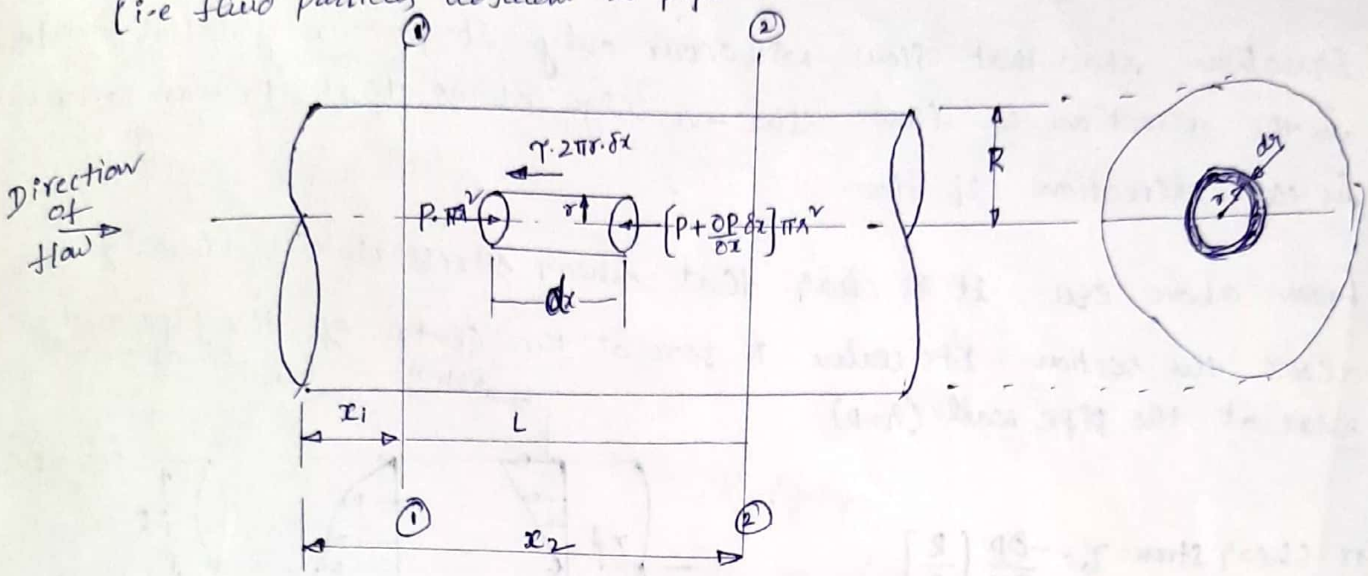


Figure shows a horizontal circular pipe of radius R, having a laminar flow of fluid through it.

Consider a small concentric cylinder (fluid element) of radius 'r' and length dx.

Let p be the intensity of pressure at left end and intensity of pressure at right end $\left[P + \frac{\partial P}{\partial x} \delta x\right]$

τ is the shear stress on the surface of fluid element.

shear force = $\tau \cdot 2\pi r \cdot \delta x$

The forces acting on fluid element are

1) shear force , $\tau \times 2\pi r \times \delta x$
2) pressure force , $p \times \pi r^2$ on left face
3) Pressure force $\left[p + \frac{\partial p}{\partial x} \delta x\right] \pi r^2$ on the right end

For steady flow the net forces on the cylinder must be zero.

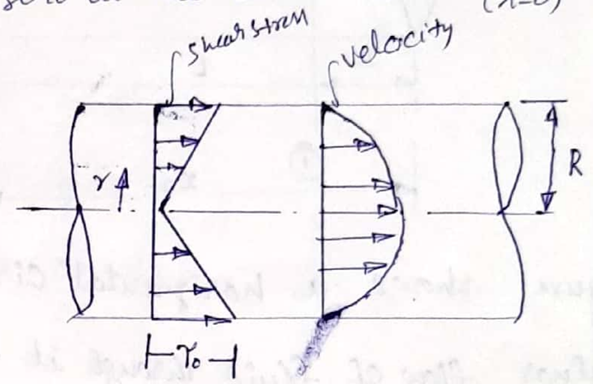
$$\left[P\cdot\pi r^2 - \left[P+\frac{\partial P}{\partial x}\delta x\right]\pi r^2\right] - \tau\cdot 2\pi r\cdot dx = 0$$

$$\Rightarrow -\frac{\partial p}{\partial x}\cdot \delta x\,\pi r^2 - \tau\, 2\pi r\, dx = 0$$

$$\tau = -\frac{\partial p}{\partial x}\cdot\frac{r}{2} \longrightarrow (1)$$

Equation show that flow will occur only if pressure gradient exists in the direction of flow. The -ve sign shows that pressure decreases in the direction of flow.

From above eqn, it is clear that shear stress varies linearly across the section. Its value is zero at the centre of the pipe and $(r=0)$ max at the pipe wall. $(r=R)$



Max. shear stress $= \tau_0 = -\frac{\partial p}{\partial x}\left[\frac{R}{2}\right]$

From Newton's law of viscosity $\tau = \mu\frac{du}{dy}$

In this the distance y is measured from boundary.

The radial distance 'r' is related to distance y by relation

$$y = R - r \Rightarrow dy = -dr$$

$$\therefore \tau = -\mu\frac{du}{dr} \longrightarrow (2)$$

Comparing (1) & (2) $\quad -\mu\frac{du}{dr} = -\frac{\partial p}{\partial x}\frac{r}{2}$

$$du = \frac{1}{2\mu}\left[\frac{\partial p}{\partial x}\right] r\cdot dr$$

Integrating w.r.t 'r'

$$u = \frac{1}{4\mu}\frac{\partial p}{\partial x}\cdot r^2 + C$$

at boundary $r = R,\ u = 0$

$$\therefore C = -\frac{1}{4\mu}\frac{\partial p}{\partial x}\cdot R^2$$

$$\therefore \quad u = \frac{1}{4\mu}\frac{\partial p}{\partial x} r^2 - \frac{1}{4\mu}\frac{\partial p}{\partial x} R^2$$

$$u = \frac{-1}{4\mu}\frac{\partial p}{\partial x}\left[R^2 - r^2\right] \rightarrow (3)$$

equation shows velocity distribution curve is parabola,

the max. velocity occurs at the centre is given by,

$$u_{max} = \frac{-1}{4\mu}\frac{\partial p}{\partial x} \cdot R^2 \rightarrow (4)$$

(3) & (4) $$u = u_{max}\left[1 - \left(\frac{r}{R}\right)^2\right] \rightarrow (5)$$

Equ (5) is most commonly used equ for velocity distribution for laminar flow through pipes.

This equation can be used to calculate discharge as follows

The discharge through an elemental ring of thickness 'dr' at radial distance 'r' is given by

$$dQ = \text{Area} \times \text{vel} = 2\pi r \cdot dr \cdot u$$

$$= u_{max}\left[1 - \left(\frac{r}{R}\right)^2\right] 2\pi r\, dr$$

$$\text{Total discharge} = \int dQ = Q = \int_0^R u_{max}\left[1 - \left(\frac{r}{R}\right)^2\right] 2\pi r\, dr$$

$$= 2\pi u_{max}\int_0^R \left(r - \frac{r^3}{R^2}\right) dr$$

$$Q = 2\pi u_{max}\left[\frac{r^2}{2} - \frac{r^4}{4R^2}\right]_0^R = \frac{\pi}{2} u_{max} R^2$$

$$\text{Avg. velocity of flow} = \bar{u} = \frac{Q}{A} = \frac{\frac{\pi}{2} u_{max} R^2}{\pi R^2} = \frac{u_{max}}{2}$$

$$\therefore \text{Avg. vel} = \bar{u} = \frac{u_{max}}{2} = \frac{-1}{8\mu}\frac{\partial p}{\partial x} \cdot R^2$$

$$-\partial p = \frac{8\mu \bar{u}\, dx}{R^2}$$

The pressure difference b/w two sections ① & ② at distance x_1 & x_2 is given by

$$-\int_{P_1}^{P_2} \partial P = \frac{8\mu\bar{u}}{R^2}\int_{x_1}^{x_2} dx$$

$$\Rightarrow (P_1 - P_2) = \frac{8\mu\bar{u}}{R^2}(x_2 - x_1)$$

$$P_1 - P_2 = \frac{32\mu\bar{u}L}{D^2}$$

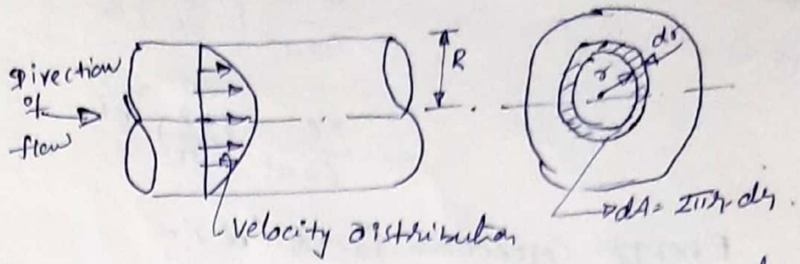
where D = dia of pipe
L = Length of pipe.

The above equation is called Hagen-Poiseuille equation

Momentum correction factor :- (β)

In a circular pipe for laminar flow, the velocity distribution at any radius 'r' is given by

$$u = \frac{-1}{4\mu}\frac{\partial p}{\partial x}\left[R^2 - r^2\right]$$



Consider an elemental area 'dA' in the form of ring at a radius r and of width dr, then

$$dA = 2\pi r \cdot dr$$

Discharge through ring $dQ = u \times 2\pi r\, dr$

Momentum of fluid through the ring per second = mass × velocity

$$= (\rho \cdot \text{Area} \cdot \text{vel})\ \text{vel}.$$

$$= \rho\, 2\pi r\, dr \cdot u^2$$

Total actual momentum of fluid across section per second $= \int_0^R 2\pi \rho u^2 r\, dr$

$$= 2\pi\rho \int_0^R \left[\frac{-1}{4\mu}\frac{\partial p}{\partial x}(R^2 - r^2)\right]^2 r\, dr$$

$$= 2\pi\rho \left[\frac{1}{4\mu}\left(\frac{\partial p}{\partial x}\right)\right]^2 \int_0^R (R^2 - r^2)^2 r \cdot dr$$

$$= \frac{\pi\rho}{8\mu^2}\left(\frac{\partial p}{\partial x}\right)^2 \int_0^R \left[R^4 r + r^5 - 2R^2 r^3\right] dr$$

$$= \frac{\pi\rho}{8\mu^2}\left(\frac{\partial p}{\partial x}\right)^2 \left[\frac{R^4 r^2}{2} + \frac{r^6}{6} - \frac{2R^2 r^4}{4}\right]_0^R$$

$$= \frac{\pi\rho}{8\mu^2}\left(\frac{\partial p}{\partial x}\right)^2 R^6$$

Momentum of fluid based on average velocity = $\frac{\text{mass of fluid}}{\text{sec}}$ × Avg. velocity

$$= \rho A \bar{u} \times \bar{u}$$

$$= \rho \cdot \pi R^2 \cdot \bar{u}^2, \quad \bar{u} = \text{Avg. vel} = \frac{u_{max}}{2}$$

$$= \frac{1}{2}\left[\frac{-1}{4\mu}\frac{\partial p}{\partial x}\right] \times R^2$$

$$= \rho \pi R^2 \cdot \frac{1}{64\mu^2} \cdot \left(\frac{\partial p}{\partial x}\right)^2 \cdot R^4$$

$$= \rho\pi \cdot \frac{1}{64\mu^2}\left(\frac{\partial p}{\partial x}\right)^2 \cdot R^6$$

$$\beta = \frac{\text{momentum/sec based on actual velocity}}{\text{momentum/sec based on average velocity}}$$

$$= \frac{\frac{\pi\rho}{48\mu^2}\left(\frac{\partial p}{\partial x}\right)^2 R^6}{\frac{\pi\rho}{64\mu^2}\left(\frac{\partial p}{\partial x}\right)^2 R^6} = \frac{4}{3} = \beta$$

Energy Correction factor (α):-

K.E of fluid flowing through the elementary ring of radius 'r' and thickness dr per second

$$= \frac{1}{2}\,\text{mass} \times (\text{vel})^2$$

$$= \frac{1}{2}(\rho \cdot A \cdot u) \cdot u^2$$

$$= \frac{1}{2}\rho \cdot 2\pi r\,dr\,u\,u^2$$

$$= \frac{1}{2}\rho\, 2\pi r\, u^3\, dr.$$

Total kinetic energy of flow per second $= \rho\pi \int_0^R r \cdot u^3\, dr$

$$= \rho\pi \int_0^R \left[-\frac{1}{4\mu}\left(\frac{\partial p}{\partial x}\right)(R^2-r^2)\right]^3 \cdot r\, dr$$

$$= -\rho\pi \frac{1}{64\mu^3}\left(\frac{\partial p}{\partial x}\right)^3 \int_0^R \left(R^6 - r^6 - 3R^4r^2 + 3R^2r^4\right) r\, dr$$

$$= -\frac{\rho\pi}{64\mu^3}\left(\frac{\partial p}{\partial x}\right)^3 \left[\frac{R^6r^2}{2} - \frac{r^8}{8} - \frac{3R^4r^4}{4} + \frac{3R^2r^6}{6}\right]_0^R$$

$$= -\frac{\rho\pi}{512\mu^3}\left(\frac{\partial p}{\partial x}\right)^3 \left[R^8\right]$$

K.E of flow based on avg. velocity $= \frac{1}{2}\,\text{mass} \times (\text{Avg. vel})^2$

$$= \frac{1}{2}\rho \cdot A\bar{u}\,\bar{u}^2 = \frac{1}{2}\rho \cdot \pi R^2\,\bar{u}^3$$

$$= \frac{1}{2}\rho\pi R^2 \left[\frac{-1}{8\mu}\left(\frac{\partial p}{\partial x}\right)R^2\right]^3$$

$$= -\frac{1}{2}\rho\pi R^2 \cdot \frac{1}{512\mu^3}\left(\frac{\partial p}{\partial x}\right)^3 \cdot R^6$$

$$= -\frac{\rho\pi}{1024}\left(\frac{\partial p}{\partial x}\right)^3 R^8$$

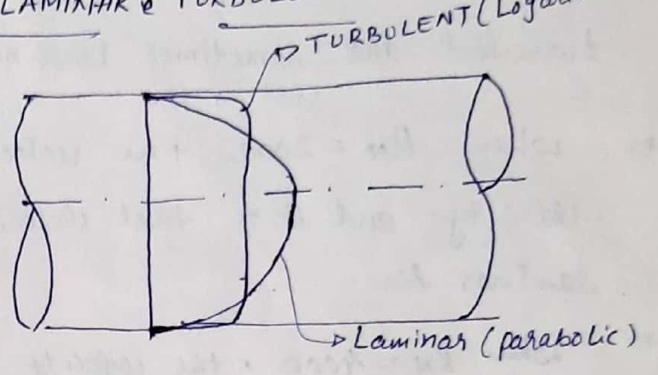
$$\alpha = \frac{\text{K.E/sec based on actual vel.}}{\text{K.E/sec based on avg. vel}} = 2,$$

TURBULENT FLOW IN PIPES:- [Re > 4000]

In a turbulent flow, the fluid motion is irregular and there is complete mixing of fluid due to collision of fluid masses with one another.

The fluid masses are interchanged between adjacent layers. As the fluid masses in adjacent layers have different velocities, interchanging of fluid masses between adjacent layers is accompanied by transfer of momentum which causes additional shear stresses of high magnitude between adjacent layers.

VELOCITY DISTRIBUTION CURVES FOR LAMINAR & TURBULENT FLOW:-



- The velocity distribution in turbulent flow is more uniform than in Laminar flow
- In turbulent flow the velocity gradient near the boundary shall be quite large resulting in more shear
- The velocity distribution which is paraboloid in laminar, Logarithmic law in turbulent flow.
- Random orientation of fluid particles in turbulent flow gives rise to additional stresses, called Reynold's stresses.
- Formation of eddies, mixing and curving of path lines in a turbulent flow, results in much greater frictional losses for the same rate of discharge, viscosity and pipesize.
- TURBULENT MOTION can be classified as follows

1. wall turbulence:- It occurs in the immediate vicinity of solid surfaces and in the boundary layer flows where the fluid has negligible mean acceleration
2. Free turbulence:- It occurs in jets, wakes and mixing layers - - etc
3. Convective turbulence:- It takes place where there is conversion of P.E into K.E by the process of mixing.

CRITICAL VALUES OF REYNOLDS NUMBER:-

→ Reynold found that if $R_N < 2000$, the flow is laminar. Velocity of fluid in laminar remains constant in magnitude & direction.

→ if $R_N > 4000$, the flow is turbulent, velocity of fluid in turbulent have fluctuations, flow becomes unsteady and non-uniform.

→ For values of R_N lying between 2000 & 4000, the flow is unpredi-ctable because, for the same value of R_N it may some times be turbulent and sometimes laminar.

→ when $R_N = 2000$, the velocity of liquid is called lower critical-velocity and it is that velocity below which the liquid will have laminar flow.

→ When $R_N = 4000$, the velocity of fluid is high and is called higher critical velocity. It is the velocity above which the fluid will have turbulent flow.

* LOSSES IN PIPE :-

i) Minor Losses :-

When the fluid is flowing through the pipe, it losses its energy due to change of cross section, obstacle or bend.

The following are the minor losses

a. Sudden enlargement b. Sudden contraction c. at the entrance of pipe d. At the exit from pipe e. obstruction in pipe f. Bend in pipe.

ii) Major losses :-

When fluid flows through the pipe, it gets resistance to the motion due to friction. If the length of the pipe is more, friction is more and major losses are more pre dominant.

* MOODY'S DIAGRAM :-

It is a graph represents the variation of friction factor 'f', Reynolds number 'Re' and Relative roughness '$\frac{\varepsilon_s}{D}$'.

It can be used for non-circular conduits and also for open channels.

* COUETTE FLOW :-

When the narrow gap between parallel plates or surfaces is filled by viscous fluid where the upper plate moves with uniform speed in the direction of flow and lower plate is stationary, such a flow is called 'COUETTE FLOW'.

BOUNDARY LAYER THEORY :- BOUNDARY LAYER FLOW OVER SOLID BODY :-

→ When a fluid flows over a solid body the particles adhere to the boundary and condition of no slip occurs. Which means the velocity of fluid close to the boundary will be same that of boundary.

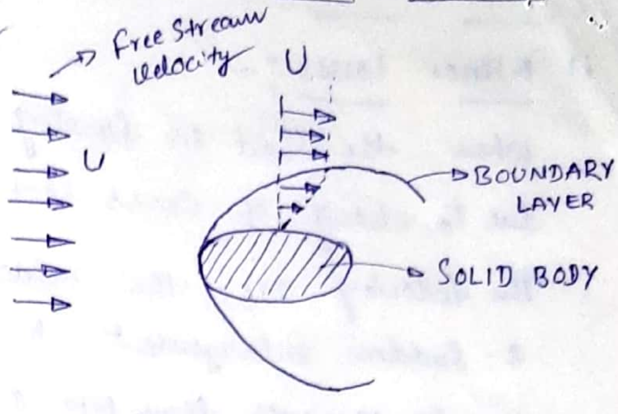


Fig → Flow over Solid Body

→ If the boundary is stationary, the velocity of the fluid at the boundary will be zero.

→ The velocity of fluid increases from zero velocity on the stationary boundary to free stream velocity (U) of the fluid in the direction normal to the boundary takes place in a narrow region in the vicinity of solid boundary. This narrow region of fluid is called boundary layer.

→ According to boundary layer theory, the flow of fluid in the neighbourhood of solid boundary may be divided into 2 regions.

→ (i) A very thin layer of fluid called boundary layer, in the immediate neighbourhood of solid boundary, where the variation of velocity from zero to free stream velocity normal to the boundary takes place.

In this region velocity gradient $\frac{du}{dy}$ exist and hence the fluid exerts a shear stress $\tau = \mu \frac{du}{dy}$

(ii) The remaining fluid which is outside the boundary layer. The velocity outside the boundary layer is constant and is equal to free-stream velocity. Velocity gradient becomes zero. As a result shear stress is zero.

BOUNDARY LAYER ALONG A THIN FLAT PLATE :- (or)

LAMINAR, TURBULENT BOUNDARY LAYER :-

Laminar Boundary Layer :-

→ Consider the flow of fluid having free stream velocity 'U', over a smooth thin plate which is flat and placed parallel to the direction of flow as shown.

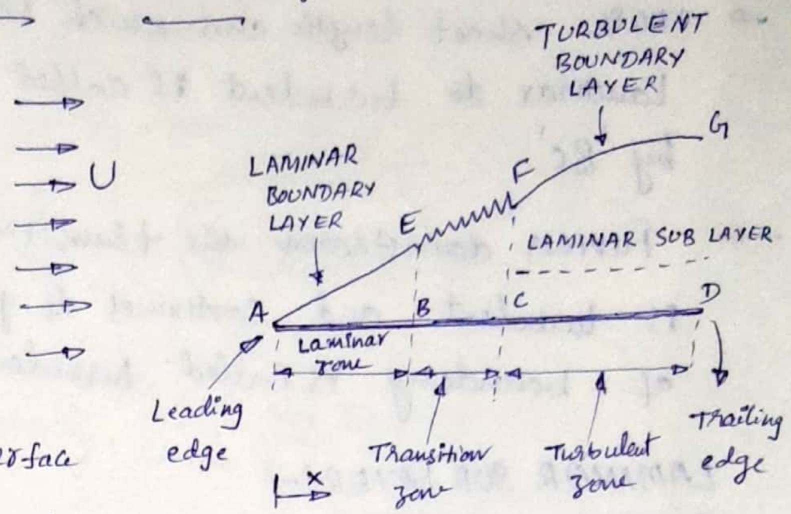


FIG :- Flow over a plate

→ The velocity of fluid on the surface of plate should be equal to the velocity of the plate.

→ But the plate is stationary and hence velocity of fluid on the surface of plate is zero. But at a distance away from the plate, the fluid is having certain velocity.

→ Thus velocity gradient exist in the fluid near the surface of the plate. This velocity gradient develops shear resistance, which retards the flow.

→ Thus the fluid with uniform free stream velocity 'U' is retarded in the vicinity of solid surface of the plate and boundary layer region begins sharp at leading edge.

→ At subsequent points down stream the leading edge, the boundary layer region increases because the retarded fluid further retarded

→ Near the leading edge, where the thickness is small, the flow in the boundary layer is laminar. This is shown by AE.

→ The Reynold's number upto 5×10^5 the boundary layer is laminar.

$$\frac{\rho U x}{\mu} < 5\times10^5$$

Turbulent Boundary Layer :-

If the length of the plate is more than the distance 'x' calculated from eqn $\frac{\rho U x}{\mu} = 5\times10^5$, thickness of boundary layer will go on increasing in the down stream direction.

The Laminar boundary layer becomes unstable, motion of fluid with in it is distributed and irregular which leads to transition from laminar to turbulent boundary layer.

→ This short length over which boundary layer flow changes from laminar to turbulent is called transition zone. It is shown by 'BC'.

→ Further downstream the transition zone, the boundary layer is turbulent and continues to grow in thickness. This layer of boundary is called turbulent boundary layer, shown by 'FG'.

LAMINAR SUB LAYER :-

This is the region in the turbulent boundary zone, adjacent to the solid surface of plate as shown in fig. In this zone velocity variation is only influenced by viscous effects.

Though the velocity distribution would be parabolic in laminar sub-layer zone, but in view of very small thickness we can reasonably assume that velocity variation is linear and $\frac{du}{dy}$ can be considered constant.

BOUNDARY LAYER SEPARATION :-

→ When a solid body immersed in a flowing fluid a thin layer of fluid called boundary layer is formed adjacent to the solid body.

→ In this thin layer of fluid, the velocity varies from zero to free-stream velocity in the direction normal to solid body.

→ Along the length of solid body, the thickness of boundary layer increases.

→ The fluid layer adjacent to the solid surface has to do work against surface friction at the expense of its kinetic energy.

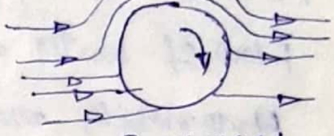
→ This loss of kinetic energy is recovered from the adjacent immediate fluid layer through momentum exchange process.

→ Along the length of the solid body, at a point a stage may come when the boundary layer may not be able to keep sticking on to the solid body, if it cannot provide kinetic energy to overcome resistance offered by solid body.

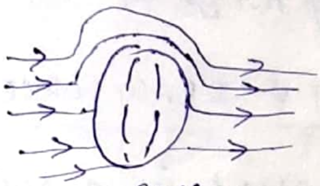
→ In other words the boundary layer will be separated from the surface. This phenomenon is called boundary layer separation.

METHODS TO PREVENT SEPARATION OF BOUNDARY LAYER:-

1. Rotating or rolling the surface of body like sphere or cylinder
2. Providing rough surface or ring on the surface of cylinder

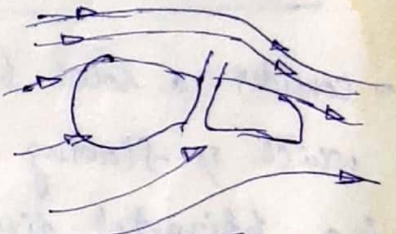


ROTATION



RING

3. Providing suction slot

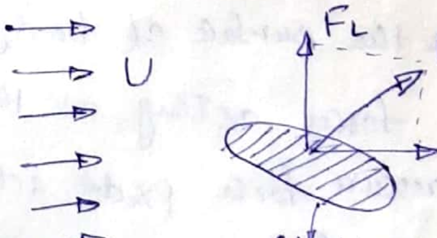


SLOT

4. Providing guid vanes
5. Injecting fluid along the surface.

DRAG & LIFT FORCES:-

Consider a body held stationary in a real fluid, which is flowing at a velocity 'U' as shown in figure.



→ The fluid exerts a force 'F_R' on the body in a direction perpendicular to the surface of body.

→ This force 'F_R' is inclined to the direction of motion. The total force can be resolved into two components, one in the direction of motion and other perpendicular to direction of motion.

Drag:- The component of total force 'F_R' in the direction of motion is called drag. It is denoted by F_D.

LIFT:-

Component of force in the direction normal to the direction of motion is called lift. It is denoted by F_L.

→ Lift force occurs when the axis of body is inclined to the direction of fluid flow.

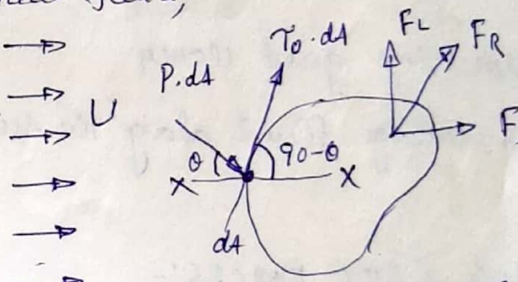
→ If the axis of body is parallel to the direction of motion, lift force zero, only drag force acts.

Examples of Lift & Drag:-

1. Flow of air over buildings
2. Flow of water over bridges
3. Automobiles moving through air.

EXPRESSION FOR LIFT & DRAG FORCES:-

→ Consider a solid body placed in a real fluid, which is flowing with a velocity 'U' in a horizontal direction as shown in figure.



→ Consider a small elemental ~~surface~~ area 'dA' on the surface of body.

→ The forces acting on the dA are

(1) Pressure force $p \times dA$ acting perpendicular to the surface.

(2) Shear force $\tau_0 \times dA$ acting along tangential direction to the surface.

(a) DRAG FORCE:-

The drag force on elemental area = Force due to pressure in the direction of fluid motion + force due to shear stress in the direction of fluid motion

$$dF_D = P.dA \cos\theta + \tau_0 . dA \sin\theta$$

∴ Total drag $= F_D = \int P dA \cos\theta + \int \tau_0 dA \sin\theta$.

$\int p\cos\theta \, dA$ = Pressure drag or form drag; $\int \tau_0 \sin\theta \, dA$ = Skin drag or frictional drag or shear drag

(b) LIFT FORCE (F_L):-

The lift force on elemental area = force due to pressure in the direction perpendicular to dir. of motion + force due to shear in the direction perpendicular to direction of motion

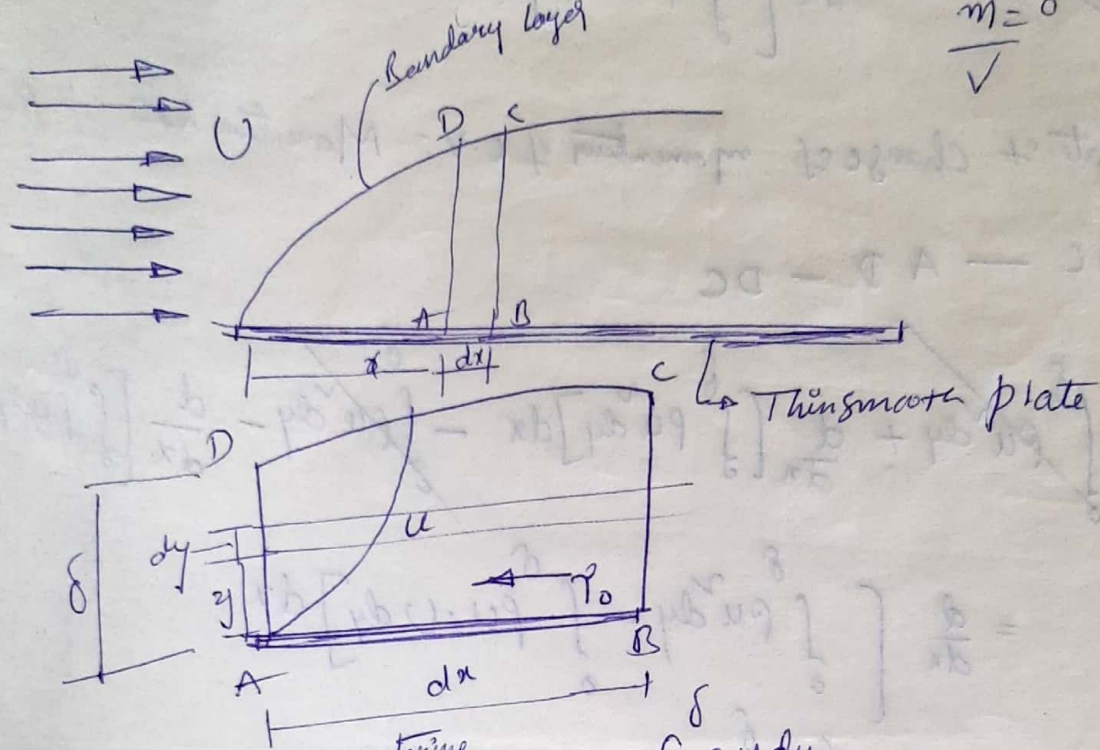
$$dL_D = -p\,dA\sin\theta + \tau_0\,dA\cos\theta$$

$\therefore$ Total lift $= \int \tau_0 \cos\theta\, dA - \int p\sin\theta\, dA$.

Von Karman Integral Momentum Equation :-
(Approx Hydrodynamic Boundary layer Analysis)

Approximate Integral Method → Boundary layer Momentum Equation — Von Karman.

$m = \frac{\delta}{V}$



Mass rate of fluid entering through AD $= \int_0^\delta \rho u\,dy$

Mass rate of fluid leaving through BC $= \int_0^\delta \rho u\,dy + \frac{d}{dx}\left[\int_0^\delta \rho u\,dy\right]dx$

Mass rate of flow entering the control vol through surface DC =

BC − AD

$$= \int_0^\delta \rho u\,dy + \frac{d}{dx}\left[\int_0^\delta \rho u\,dy\right]dx - \int_0^\delta \rho u\,dy$$

$$= \frac{d}{dx}\left[\int_0^\delta \rho u\,dy\right]dx$$

Momentum rate of fluid entering the C.V in X-along AD = $\int_0^\delta \rho u^2\,dy$

$$BC = \int_0^\delta \rho u^2\,dy + \frac{d}{dx}\left[\int_0^\delta \rho u^2\,dy\right]dx$$

Momentum rate DC = ~~BC − AD~~

$$DC = \frac{d}{dx}\left[\int_0^\delta \rho u \, dy\right] dx \times U$$

Rate of change of momentum of C.V = Momentum rate of fluid

$$BC - AD - DC$$

$$= \cancel{\int_0^\delta \rho u^2 dy} + \frac{d}{dx}\left[\int_0^\delta \rho u^2 dy\right] dx - \cancel{\int_0^\delta \rho u^2 dy} - \frac{d}{dx}\left[\int_0^\delta \rho u \, dy\right] dx \, U$$

$$= \frac{d}{dx}\left[\int_0^\delta \rho u^2 dy - \int_0^\delta \rho u \cdot U \, dy\right] dx$$

$$= \frac{d}{dx}\left[\int_0^\delta \rho u^2 dy - \rho u \cdot U dy\right] dx$$

$$= \frac{d}{dx}\left[\int_0^\delta (\rho u^2 - \rho u U) \, dy\right] dx$$

$$= \frac{d}{dx}\left[\rho \int_0^\delta (u^2 - uU) \, dy\right] dx$$

$$= \rho \frac{d}{dx}\left[\int_0^\delta (u^2 - uU) \, dy\right] dx$$

but $mV = F \cdot t$

$$\Delta F_D = \tau_0 \times dx$$
$$= -\tau_0 \times dx$$
$$-\tau_0\, dx = \rho \frac{d}{dx}\left[\int_0^\delta (u^2 - uU)\,dy\right] dx$$
$$\tau_0 = \rho \frac{d}{dx}\left[\int_0^\delta (uU - u^2)\,dy\right]$$
$$\tau_0 = \rho \frac{d}{dx}\left[\int_0^\delta U^2\left[\frac{u}{U} - \frac{u^2}{U^2}\right] dy\right]$$
$$\frac{\tau_0}{\rho U^2} = \frac{d}{dx}\left[\int_0^\delta \frac{u}{U}\left(1 - \frac{u}{U}\right) dy\right]$$

But $\int_0^\delta \frac{u}{U}\left(1 - \frac{u}{U}\right) dy$ = Momentum thickness = θ

$$\therefore \frac{\tau_0}{\rho U^2} = \frac{d\theta}{dx} \rightarrow \text{Von Karman momentum eqn}$$

for boundary layer flow

B.C

1. at the surface of the plate $y = 0,\ u = 0,\ \frac{du}{dy}$ = finite value
2. at the outer edge of B/L $y = \delta,\ u = \alpha$
   $y = \delta,\ \frac{du}{dy} = 0$

ΔF_D = shear ~~stress~~ $\times$ area

$$= \tau_0 \times (B \times dx)$$

Total drag force over a length L

$$F_D = \int_0^L \Delta F_D = \int_0^L \tau_0 B\, dx$$