Compressible Flows

▶ 15.1 INTRODUCTION

Compressible flow is defined as that flow in which the density of the fluid does not remain constant during flow. This means that the density changes from point to point in compressible flow. But in case of incompressible flow, the density of the fluid is assumed to be constant. In the previous chapters, the fluid was assumed incompressible, and the basic equations such as equation of continuity, Bernoulli's equation and impulse momentum equations were derived on the assumption that fluid is incompressible. This assumption is true for flow of liquids, which are incompressible fluids. But in case of flow of [redacted], such as

(*i*) flow of gases through orifices and nozzles,

(*ii*) flow of gases in machines such as compressors, and

(*iii*) projectiles and airplanes flying at high altitude with high velocities, the density of the fluid changes during the flow. The change in density of a fluid is accompanied by the changes in pressure and temperature and hence the thermodynamic behaviour of the fluids will have to be taken into account.

▶ 15.2 THERMODYNAMIC RELATIONS

The thermodynamic relations have been discussed in Chapter 1, which are as follows :

15.2.1 Equation of State. Equation of state is defined as the equation which gives the relationship between the pressure, temperature and specific volume of a gas. For a perfect gas the equation of state is

$$p\forall = RT \qquad \ldots(15.1)$$

where p = Absolute pressure in kgf/m^2 or N/m^2

$\forall$ = Specific volume or volume per unit mass

T = Absolute temperature = 273 + $t°$ (centigrade)

R = Gas constant in kgf-m/kg °K or (J/kg K)

= 29.2 kgf-m/kg °K or 287 J/kg K for air.

In equation (15.1), $\forall$ is the specific volume which is the reciprocal of density or

$$\forall = \frac{1}{\rho}.$$

Substituting this value of $\forall$ in equation (15.1), we get

$$\frac{p}{\rho} = RT \qquad \text{...(15.2)}$$

Note. In the equation of state given by equation (15.2), the dimensions of p, ρ and R should be used with care. The following points must be remembered :

1. If the value of R is given as 29.2 kgf-m/kg °K for air, the corresponding value of p and ρ should be taken in kgf/m^2 and kg/m^3. The mass rate of flow of the gas will be in kg/sec.
2. If the value of R is given as 287 J/kg K, the corresponding value of p and ρ should be in N/m^2 and kg/m^3. The mass rate of flow will be in kg/sec.

Value of $\frac{p}{\rho}$ in Bernoulli's Equation*. (*i*) If the value of p is taken in N/m^2, the corresponding value of ρ is in kg/m^3. And as mentioned above (point number 2), the value of R should be 287 J/kg K.

(*ii*) If the value of p is taken in kgf/m^2 in Bernoulli's equation, the corresponding value of ρ should be in msl/m^3. But as mentioned in point number 1, if the value of R is taken 29.2, the corresponding values of p and ρ are in kgf/m^2 and kg/m^3. Hence the mass density in equation of state is in kg/m^3 while in Bernoulli's equation it is in msl/m^3. The density calculated from equation of state must be converted into msl/m^3.

Note. It is better to use pressure in N/m^2, density in kg/m^3 and value of R = 287 J/kg K. The value of density calculated from equation of state will be in the same dimensions as used in Bernoulli's equation.

15.2.2 Expansion and Compression of Perfect Gas.

When the expansion or compression of a perfect gas takes place, the pressure, temperature and density are changed. The change in pressure, temperature and density of a gas is brought about by the two processes which are known as

1. Isothermal process, and 2. Adiabatic process.

1. Isothermal Process. This is the process in which a gas is compressed or expanded while the temperature is kept constant. The gas obeys Boyle's law, according to which we have

$$p\forall = \text{Constant}, \quad \text{where } \forall = \text{Specific volume}$$

or

$$\frac{p}{\rho} = \text{Constant} \qquad \left(\because \forall = \frac{1}{\rho}\right) \quad \text{...(15.3)}$$

2. Adiabatic Process. If the compression or expansion of a gas takes place in such a way that the gas neither gives heat, nor takes heat from its surrounding, then the process is said to be adiabatic. According to this process,

$$p\forall^k = \text{Constant}$$

where k = Ratio of the specific heat at constant pressure to the specific heat at constant volume

$$= \frac{C_p}{C_v} = 1.4 \text{ for air.}$$

The above relation is also written as $\frac{p}{\rho^k} = \text{Constant.}$ ...(15.4)

If the adiabatic process is reversible (or frictionless), it is known as isentropic process. And if the pressure and density are related in such a way that k is not equal to $\frac{C_p}{C_v}$ but equal to some positive value then the process is known as polytropic. According to which

$$\frac{p}{\rho^n} = \text{Constant} \qquad \text{...(15.5)}$$

where $n \neq k$ but equal to some positive constant.

$pV = mRT$

$p\vartheta = RT$

$\frac{P}{\rho} = \text{Const}$

▶ 15.3 BASIC EQUATIONS OF COMPRESSIBLE FLOW

The basic equations of the compressible flows are

1. Continuity Equation.
2. Bernoulli's Equation or Energy Equation.
3. Momentum Equation.
4. Equation of state.

15.3.1 Continuity Equation. This is based on law of conservation of mass which states that matter cannot be created nor destroyed. Or in other words, the matter or mass is constant. For one-dimensional steady flow, the mass per second = ρAV

where ρ = Mass density, A = Area of cross-section, V = Velocity

As mass or mass per second is constant according to law of conservation of mass. Hence

$$\rho AV = \text{Constant}. \qquad \ldots(15.6)$$

Differentiating equation (15.6), $d(\rho AV) = 0$ or $\rho d(AV) + AVd\rho = 0$

or $\rho[AdV + VdA] + AVd\rho = 0$ or $\rho AdV + \rho VdA + AVd\rho = 0$

Dividing by ρAV, we get

$$\frac{dV}{V} + \frac{dA}{A} + \frac{d\rho}{\rho} = 0. \qquad \ldots(15.7)$$

Equation (15.7) is also known as continuity equation in differential form.

15.3.2 Bernoulli's Equation. Bernoulli's equation has been derived for incompressible fluids in Chapter 6. The same procedure is followed. The flow of a fluid particle along a stream-line in the direction of S is considered. The resultant force on the fluid particle in the direction of S is equated to the mass of the fluid particle and its acceleration. As the flow of compressible fluid is steady, the same Euler's equation as given by equation (6.3) is obtained as

$$\frac{dp}{\rho} + VdV + gdZ = 0 \qquad \ldots(15.8)$$

Integrating the above equation, we get

$$\int \frac{dp}{\rho} + \int VdV + \int gdZ = \text{Constant}$$

or

$$\int \frac{dp}{\rho} + \frac{V^2}{2} + gZ = \text{Constant} \qquad \ldots(15.9)$$

In case of incompressible flow, the density ρ is constant and hence integration of $\frac{dp}{\rho}$ is equal to $\frac{p}{\rho}$.

But in case of compressible flow, the density ρ is not constant. Hence ρ cannot be taken outside the integration sign. With the change of ρ, the pressure p also changes for compressible fluids. This change of ρ and p takes place according to equations (15.3) or (15.4) depending upon the type of process during compressible flow. The value of ρ from these equations in terms of p is obtained and is substituted in $\int \frac{dp}{\rho}$ and then the integration is done. The Bernoulli's equation will be different for isothermal process and for adiabatic process.

(A) Bernoulli's Equation for Isothermal Process. For isothermal process, the relation between pressure (p) and density (ρ) is given by equation (15.3) as

$$\frac{p}{\rho} = \text{Constant} = C_1 \text{ (say)} \qquad \ldots(i)$$

$$\therefore \qquad \rho = \frac{p}{C_1}$$

Hence
$$\int \frac{dp}{\rho} = \int \frac{dp}{p/C_1} = \int \frac{C_1\,dp}{p} = C_1 \int \frac{dp}{p} \qquad (\because C_1 \text{ is constant})$$

$$= C_1 \log_e p = \frac{p}{\rho} \log_e p \qquad \left(\because C_1 = \frac{p}{\rho} \text{ from equation } (i)\right)$$

Substituting the value $\int \frac{dp}{\rho}$ in equation (15.9), we get

$$\frac{p}{\rho} \log_e p + \frac{V^2}{2} + gZ = \text{Constant}$$

Dividing by 'g',
$$\frac{p}{\rho g} \log_e p + \frac{V^2}{2g} + Z = \text{Constant.} \qquad \ldots(15.10)$$

Equation (15.10) is the Bernoulli's equation for compressible flow undergoing isothermal process. For the two points 1 and 2, this equation is written as

$$\frac{p_1}{\rho_1 g} \log_e p_1 + \frac{V_1^2}{2g} + Z_1 = \frac{p_2}{\rho_2 g} \log_e p_2 + \frac{V_2^2}{2g} + Z_2 \qquad \ldots(15.11)$$

(B) Bernoulli's Equation for Adiabatic Process. For the adiabatic process, the relation between pressure (p) and density (ρ) is given by equation (15.4) as

$$\frac{p}{\rho^k} = \text{Constant} = \text{say } C_2 \qquad \ldots(ii)$$

$$\therefore \qquad \rho^k = \frac{p}{C_2} \quad \text{or} \quad \rho = \left(\frac{p}{C_2}\right)^{1/k}$$

Hence
$$\int \frac{dp}{\rho} = \int \frac{dp}{\left(\frac{p}{C_2}\right)^{1/k}} = \int \frac{C_2^{1/k}}{p^{1/k}}\, dp = C_2^{1/k} \int \frac{1}{p^{1/k}}\, dp$$

$$= C_2^{1/k} \int p^{-1/k}\, dp = C_2^{1/k} \frac{p^{\left(-\frac{1}{k}+1\right)}}{\left(-\frac{1}{k}+1\right)}$$

$$= \frac{C_2^{1/k}\, p^{\left(\frac{k-1}{k}\right)}}{\left(\frac{k-1}{k}\right)} = \left(\frac{k}{k-1}\right) C_2^{1/k} p^{\left(\frac{k-1}{k}\right)}$$

$$= \left(\frac{k}{k-1}\right)\left(\frac{p}{\rho^k}\right)^{1/k} p^{\left(1-\frac{1}{k}\right)} \qquad \left\{\because C^{1/k} = \frac{p}{\rho^k} \text{ from } (ii)\right\}$$

$$= \left(\frac{k}{k-1}\right)\frac{p^{1/k}}{\rho^{k \times 1/k}}\, p^{\left(\frac{k-1}{k}\right)} = \left(\frac{k}{k-1}\right)\frac{p^{\frac{1}{k}+\frac{k-1}{k}}}{\rho} = \left(\frac{k}{k-1}\right)\frac{p}{\rho}$$

Substituting the value of $\int \frac{dp}{\rho} = \left(\frac{k}{k-1}\right)\frac{p}{\rho}$ in equation (15.9), we get

$$\left(\frac{k}{k-1}\right)\frac{p}{\rho} + \frac{V^2}{2} + gZ = \text{Constant}$$

Dividing by 'g'

$$\left(\frac{k}{k-1}\right)\frac{p}{\rho g} + \frac{V^2}{2g} + Z = \text{Constant.} \qquad \text{...(15.12)}$$

Equation (15.12) is the Bernoulli's equation for compressible flow undergoing adiabatic process. For the two points 1 and 2, this equation is written as

$$\left(\frac{k}{k-1}\right)\frac{p_1}{\rho_1 g} + \frac{V_1^2}{2g} + Z_1 = \left(\frac{k}{k-1}\right)\frac{p_2}{\rho_2 g} + \frac{V_2^2}{2g} + Z_2 \qquad \text{...(15.13)}$$

Problem 15.1 *A gas is flowing through a horizontal pipe at a temperature of 4°C. The diameter of the pipe is 8 cm and at a section 1-1 in this pipe, the pressure is 30.3 N/cm² (gauge). The diameter of the pipe changes from 8 cm to 4 cm at the section 2-2, where pressure is 20.3 N/cm² (gauge). Find the velocities of the gas at these sections assuming an isothermal process. Take R = 287.14 Nm/kg K, and atmospheric pressure = 10 N/cm².*

Solution. Given :

For the section 1-1,

Temperature, $t_1 = 4°\text{C}$

$\therefore$ Absolute temperature, $T_1 = 4 + 273 = 277°\text{K}$

Diameter pipe, $D_1 = 8 \text{ cm} = 0.08 \text{ m}$

$\therefore$ Area of pipe, $A_1 = \frac{\pi}{4} D_1^2 = \frac{\pi}{4}(.08)^2 = .005026 \text{ m}^2$

Pressure, $p_1 = 30.3 \text{ N/cm}^2$ (gauge)

$= 30.3 + 10 = 40.3 \text{ N/cm}^2 \text{ (absolute)} = 40.3 \times 10^4 \text{ N/m}^2 \text{ (abs.)}$

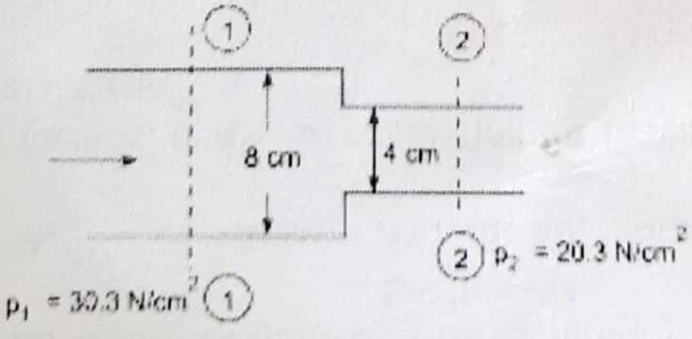


Fig. 15.1

Bulk modulus of Elasticity $K = \frac{\text{Increase in Pressure}}{\text{Change in volume}}$ UNIT-V

$$K = \frac{dp}{-\frac{dv}{v}} \Rightarrow K = -\frac{v\,dp}{dv}$$

for isothermal, ideal gas $pv = \text{const}$

$$p\,dv + v\,dp = 0$$

$$-\frac{v\,dp}{dv} = p$$

$$\Rightarrow \boxed{K = P}$$

Wave motion:-

→ Wave motion in a medium is the movement of a disturbance relative to the medium.

→ The effect of changes at a given point in the medium is communicated to other points through wave motion.

→ Since the molecules in a solid medium are closer to each other compared to liquids & gases a disturbance is communicated (through elastic wave) to other parts comparatively much shorter time

→ The phenomenon of wave propagation in compressible fluids (gases & vapours) is responsible for analysis of compressible flow problems.

→ Various types of waves in closed passages are

(i) Infinitesimal pressure wave (Sound wave)

(ii) Non-steep pr. wave with finite amplitude

(iii) Steep pr. wave (Shock) wave

(iv) Expansion wave.

→ A wave which is at lower pressure than the fluid into which it is moving is called an expansion (rarefraction) wave.

Consequently a wave which is at higher pressure than fluid is referred to as Compression wave.

Expression for velocity of sound wave in a fluid :-

→ The disturbance in gas is transmitted from one point to other.

→ The velocity with which the disturbance is transmitted depends upon distance b/w molecules of medium, the disturbance will be transmitted from one molecule to another molecule.

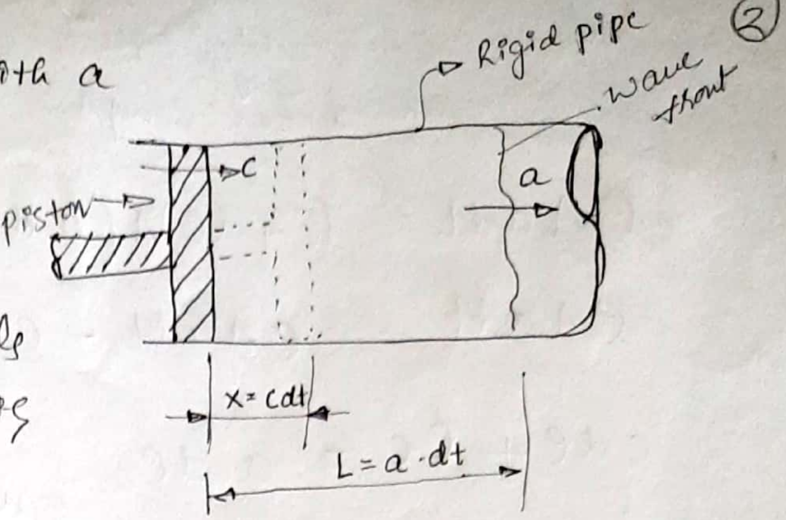
→ The dist b/w molecules is related with density, which inturn depends upon pressure. Hence the velocity of disturbance depends upon the changes of pr. and density of fluid.

→ The disturbance creates the pressure wave in a fluid. These pr. wave travel with a Vel. of Sound in all directions.

→ fig. shows one dimensional model for propagation of the pressure wave.

→ Let the pipe is filled with a compressible fluid, which is at rest initially.

→ The piston is moved towards right and a disturbance is created in the fluid.



→ The disturbance is in the form of pressure wave, which travels in the fluid with a vel. of sound wave.

$C \to a$
$V \to C$

let A = c/s area of pipe
C = vel. of piston
P = Pr of fluid in pipe before movement of piston
ρ = density " " " "
dt = A small interval of time with which piston is moved.
a = vel. of pr or sound wave in the fluid

Distance travelled by piston in time $dt = x = C \cdot dt$

Distance travelled by pressure wave in time $dt = L = a \cdot dt$

Let $P + dp$ = Pr after compression
$\rho + d\rho$ = density "

Mass of fluid before compression $= \rho \cdot A \cdot L = \rho \cdot A \cdot a\,dt$

Mass of fluid after compression $= (\rho + d\rho) A (L - x)$

$= (\rho + d\rho) A (a\,dt - c\,dt)$

from Continuity eqn

mass before comp^n = mass after comp^n

$$\rho A a\,dt = (\rho + d\rho) A (a\,dt - c\,dt)$$

$$\cancel{\rho A a\,dt} = \cancel{\rho A a\,dt} - \rho A c\,dt + A a\,d\rho\,dt - c\,d\rho\,dt\,A$$

$$c\,d\rho + \rho c = a\,d\rho$$

neglecting $c \cdot d\rho$

$$\rho c = a\,d\rho \longrightarrow (1)$$

From Newton's II law

Net force on the fluid = Rate of change of momentum

$$[(P+dP) - P]A = \frac{mv - mu}{t} = \frac{\rho A L}{dt}[c - 0]$$

$$dp\,A = \frac{\rho A L}{dt}[c]$$

$$= \frac{\rho A a\,dt\,c}{dt} \qquad (\because L = a\,dt)$$

$$dp = \rho a c$$

$$a = \frac{dp}{\rho c} \longrightarrow (2)$$

$$\rho c = \cancel{\frac{dp}{\rho c} \cdot d\rho} \quad \frac{dp}{a} = a\,d\rho$$

$$\Rightarrow a^2 = \frac{dp}{d\rho} \Rightarrow a = \sqrt{\frac{dp}{d\rho}}$$

From above eqns vel. of sound wave which is the square root of the ratio of change of pressure to change of density of fluid due to disturbance.

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$$K = \frac{\Delta P}{-\frac{\Delta V}{V}} = \frac{-V dP}{dV}$$

By assuming isothermal, $K = P$ (for ideal gas)

By assuming adiabatic $PV^{\gamma} = \text{Const}$

$$P \cdot \gamma V^{\gamma-1} dV + V^{\gamma} dP = 0$$

$$\frac{dP}{dV} = -\frac{\gamma P}{V}$$

$$K = -V \cdot -\frac{\gamma P}{V} = \gamma P$$

$$a = \sqrt{\frac{P}{e}} = \sqrt{\frac{K}{e}} = \sqrt{\frac{\gamma P}{e}} = \sqrt{\frac{\gamma e R T}{e}} = \sqrt{\gamma R T}$$

$\hookrightarrow K = e a^2$

$C =$ coefficient of compressibility

From above equation,

→ In a given fluid, velocity of sound is higher at higher temp.

→ fluids with higher values of bulk modulus & lower value of coefficient of compressibility have higher velocity of sound.

Mach Number :- (M)

Mach number of a moving object like an aircraft or missile is the ratio of its velocity and velocity of sound in the medium in to which it is moving.

Mach number of a flowing fluid is the ratio of its velocity and velocity of sound at the prevailing temperature.

Definition of mach number can also be obtained from

$$M^2 = \frac{\text{Inertia force}}{\text{Elastic force}}$$

Inertia force $= e A c^2$

Elastic force $= KA$, substituting $K = e a^2$, elastic force $= e a^2 A$

$$M^2 = \frac{e A c^2}{e \cdot a^2 A} \Rightarrow M = \frac{c}{a}$$

→ For a Compressible fluid flow, Mach number is an important non dimensional parameter. On the basis of Mach Number, the flow is defined as

a) Sub-sonic flow b) Sonic flow c) Super sonic flow
d) Hypersonic flow.

Subsonic flow:-

A flow is said to be sub sonic flow if the Mach number is less than 1, which means the velocity of flow is less than the velocity of sound wave $(c<a)$

Sonic flow:-

A flow is said to be sonic flow if the mach number is equal to 1. This means that velocity of flow is equal to the velocity of sound wave $(c=a)$

Super sonic flow:

A flow is said to be super sonic flow if the mach number is greater than 1. This means velocity of flow is more than the velocity of sound wave $(c>a)$

Hypersonic flow:-

A flow is said to be hypersonic flow if the mach number is greater than 5. $(c>a)$.

(4)

(c) With little change in the values of γ for commonly used gases velocity of sound at a given temperature is higher for lower molecular weight gases and vice versa. Hydrogen, with a very low molecular weight has a high of sound of about 1400 m/s, while the velocity of sound in some freons which have higher molecular weights, is only about 150 m/s.

The above figures suggest that Mach number plays an important role in the design and working of machines using higher molecular weight fluids. In contrast to this Mach number has negligible effect on machines using low molecular weight gases like hydrogen; this is because the fluid or gas velocities are very small compared to velocity of sound at the prevailing temperatures.

5.2.1 Mach Number

As stated earlier Mach number (M) of a moving object like an aircraft or a missile is the ratio of its velocity and the velocity of sound in the medium (at a given state) into which it is moving. The Mach number of a flowing fluid is the ratio of its velocity and the velocity of sound at the prevailing temperature.

Definition of Mach number can also be obtained from

$$M^2 = \frac{\text{Inertia force}}{\text{Elastic force}}$$

Inertia force $= \rho A c^2$

Elastic force $= KA$

Substituting for K from Equation (5.13)

Elastic force
$$= \rho A a^2 \tag{5.14}$$

Therefore,
$$M^2 = \frac{\rho A c^2}{\rho A a^2}$$

$$M^2 = \left(\frac{c}{a}\right)^2 \tag{5.15}$$

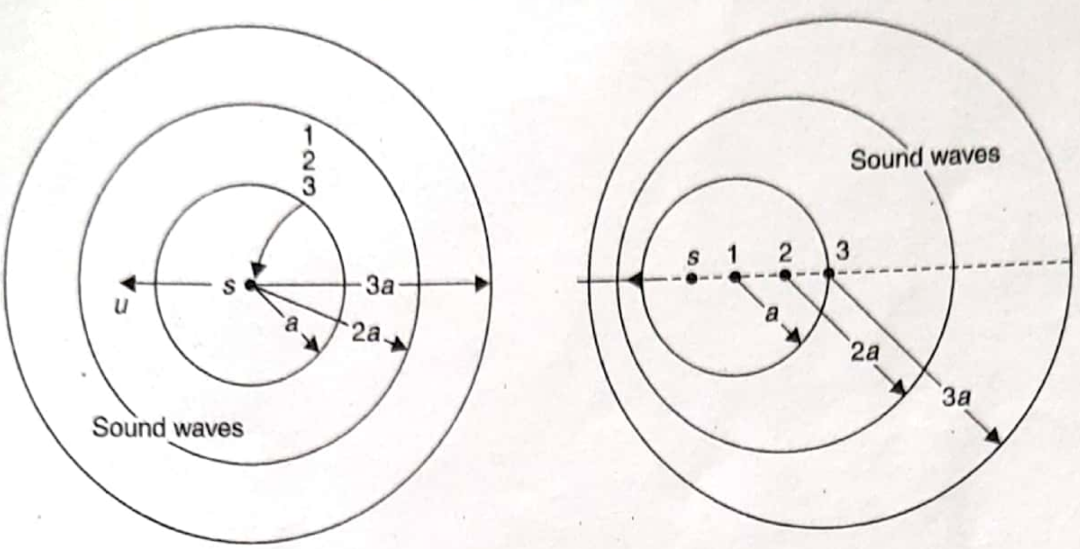
5.2.2 Mach Angle · Propagation of disturbances in a Compressible fluid.

Figures 5.4 ((a), (b), (c) and (d)) show the movement of a source of disturbance (S) at a velocity u in a fluid from right to the left. Point S represents the present position, while 1, 2 and 3 show its positions before 1, 2 and 3 seconds respectively. The disturbance travels distances of a, $2a$ and $3a$ metres in 1, 2 and 3 seconds respectively.

In an incompressible flow model (as shown in Figure 5.4 (a)) velocity u of the source of disturbance is negligibly small compared to the velocity of sound a. Infinitesimal spherical waves (sound waves) are generated which travel at a velocity a in all the directions; here the displacement of the point S during the time considered (3 sec.) is insignificantly small compared to the distance travelled by the pressure waves.

Any disturbance is produced in a compressible fluid, the disturbance is propagated in all directions with a velocity of sound
The nature of propagation of disturbance depends upon the Mach number.

Consider a small particle moving from right to left in a stationary fluid. Due to the movement of particle the disturbance will be created in the fluid. The disturbances will be moving in all directions with a velocity 'a'



(*a*) Incompressible flow
($u/a \approx 0$, $M \approx 0$)

(*b*) Subsonic flow
($a/2u$, $M = 0.5$)

FIGURE 5.4

$\frac{a}{2} =$

Wave propagation in subsonic flow at ($M = u/a = 0.5$) is shown in Figure 5.4 (*b*); here the source of disturbance travels at half the velocity of the wave (disturbance). Spherical sound waves generated at $t = 3$, 2 and 1 secs. before the present position S are shown. It is observed that the wave-fronts move ahead of the point source and the intensity is not symmetrical.

In Figure 5.4 (*c*) the point source travels with the same velocity as that of the wave; the velocity of the point source is sonic ($M = 1$). Under this condition, the wave-fronts always exist at the present position of the point source and cannot move ahead of it. Therefore, the region downstream of the point source, *i.e.*, the zone lying on the left of the wave-front is a zone of silence because the waves do not reach this zone. The zone on the right of the wave-front is traversed by the waves and is therefore a zone of action.

Figure 5.4 (*d*) shows a supersonic flow model. As an example the point source is assumed to be moving at twice the velocity of sound ($M = u/a = 2$). The waves generated at positions 3, 2, 1 and S are shown. The point source is always ahead of the wave-fronts. Tangents drawn from the point S on the spheres define a conical surface referred to as 'Mach cone'.

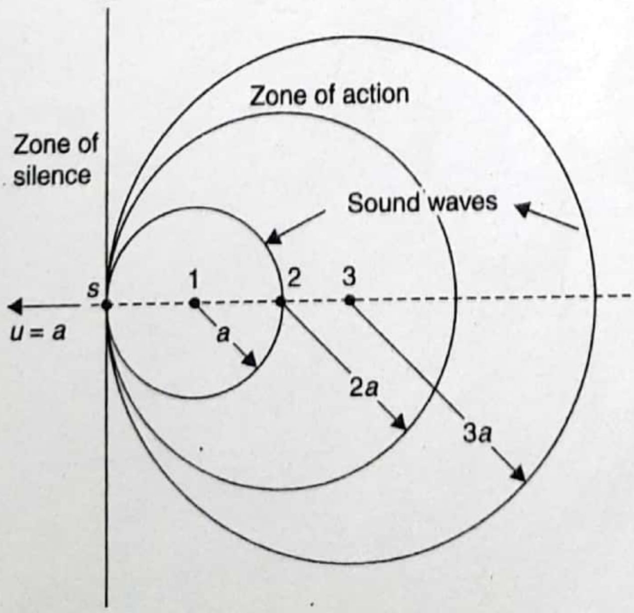


FIGURE 5.4 (*c*) Sonic flow ($a = u$, $M = 1.0$)

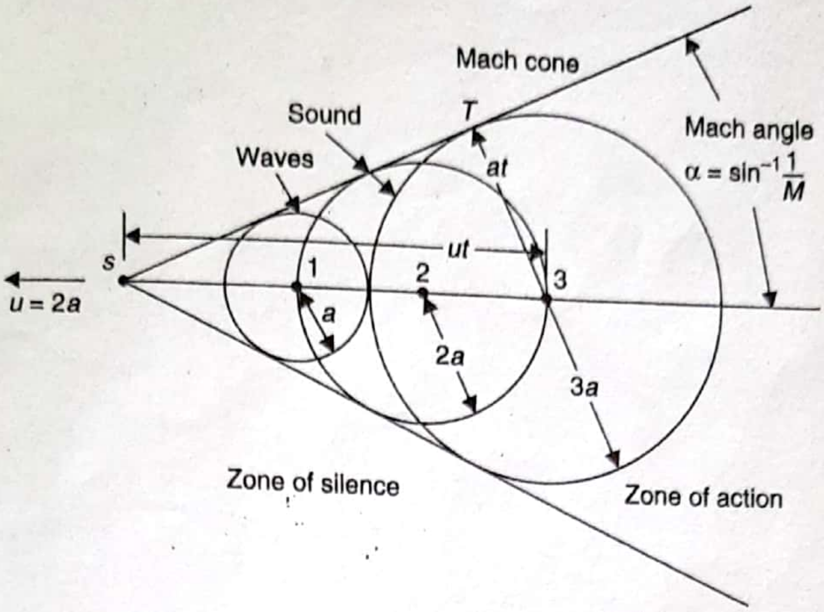


FIGURE 5.4 (*d*) Supersonic flow $\left(a = \frac{1}{2}u, M = 2.0\right)$

All the waves are confined to the region within the Mach cone; therefore, this is referred to as the zone of action. The waves do not reach the region outside the Mach cone; therefore, this zone is known as the zone of silence.

The semi-angle of the cone is known as the Mach angle; this is given by

$$\alpha = \sin^{-1}\frac{3T}{3S} = \sin^{-1}\frac{at}{ut} = \sin^{-1}\frac{1}{u/a}$$

$$\alpha = \sin^{-1}\left(\frac{1}{M}\right). \tag{5.16}$$

5.2.3 Equation of a Sound Wave

In this section equation of motion for a sound wave is derived. Continuity and momentum equations for three-dimensional flow have been derived in Chapter 10. Here the flow is assumed as one-dimensional, inviscid and the body forces are considered as negligibly small.

Figure 5.5 shows an infinitesimal pressure wave (sound wave) moving into the stagnant gas ($c_\infty = 0$) of density ρ_∞. The density and velocity changes along the wave are given by

$$\rho = \rho_\infty + \Delta\rho \tag{5.17}$$

$$c = c_\infty + c' = c' \tag{5.18}$$

The fluid through which the wave has passed moves with a small velocity c'. Momentum equation for this flow is

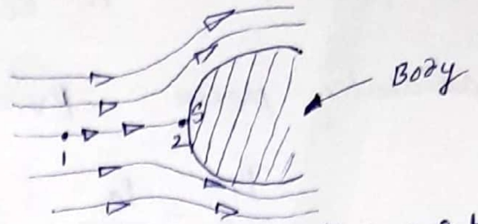
$$\frac{1}{\rho}\frac{\partial p}{\partial x} + c\frac{\partial c}{\partial x} + \frac{\partial c}{\partial t} = 0 \tag{5.19}$$

$$\frac{1}{\rho}\left(\frac{\partial p}{\partial x}\right)_s = \frac{1}{\rho}\left(\frac{\partial p}{\partial \rho}\right)_s\left(\frac{\partial \rho}{\partial x}\right)_s \tag{5.20}$$

Stagnation properties (P_0, T_0, ρ_0) As functions of Mach Number:-

→ Derive an expⁿ of compressible fluid flow for

a) Stagnation pr.
b) Stagnation temp
c) Stagnation density



→ When a fluid is flowing past an immersed body, and at a point on the body if the resultant velocity becomes zero, the values of pressure, temperature and density at that point are called stagnation properties. The point is called stagnation point.

The stagnation properties are denoted by P_0, T_0, ρ_0

(a) Expression for stagnation pressure (P_0):-

Consider a compressible fluid flowing past an immersed body under frictionless adiabatic conditions as shown in fig.

Consider two points 1 and 2 on a stream line as shown in fig

let P_1, T_1, ρ_1 are Pressure, Temp & density at 1
P_2, T_2, ρ_2 " " " at 2

Applying Bernoulli's equ for adiabatic flow at 1 & 2

$$\frac{\gamma}{\gamma-1}\frac{P_1}{\rho_1 g} + \frac{V_1^2}{2g} + Z_1 = \frac{\gamma}{\gamma-1}\frac{P_2}{\rho_2 g} + \frac{V_2^2}{2g} + Z_2 \qquad \text{but } Z_1 = Z_2$$

$$\left(\frac{\gamma}{\gamma-1}\right)\frac{P_1}{\rho_1 g} + \frac{V_1^2}{2g} = \left(\frac{\gamma}{\gamma-1}\right)\frac{P_2}{\rho_2 g} + \frac{V_2^2}{2g}$$

point '2' is a stagnation point. Hence velocity will become zero at stagnation point & pressure, density will be denoted by P_0, ρ_0

$$\therefore V_2 = 0, \quad P_2 = P_0, \quad \rho_2 = \rho_0$$

$$\left(\frac{\gamma}{\gamma-1}\right)\frac{P_1}{\rho_1} + \frac{V_1^2}{2} = \left(\frac{\gamma}{\gamma-1}\right)\frac{P_0}{\rho_0}$$

$$\left(\frac{\gamma}{\gamma-1}\right)\left(\frac{P_1}{\rho_1} - \frac{P_0}{\rho_0}\right) = -\frac{V_1^2}{2}$$

$$\left(\frac{\gamma}{\gamma-1}\right)\frac{P_1}{\rho_1}\left[1-\frac{P_0}{\rho_0}\cdot\frac{\rho_1}{P_1}\right]=-\frac{V_1^2}{2}$$

$$\left(\frac{\gamma}{\gamma-1}\right)\frac{P_1}{\rho_1}\left[1-\frac{P_0}{P_1}\cdot\frac{\rho_1}{\rho_0}\right]=-\frac{V_1^2}{2}$$

for adiabatic process $\quad PV^\gamma = \text{const} \quad \frac{P}{\rho^\gamma} = \text{Const}$

$$\frac{P_1}{\rho_1^\gamma}=\frac{P_0}{\rho_0^\gamma} \Rightarrow \frac{P_1}{P_0}=\left(\frac{\rho_1}{\rho_0}\right)^\gamma \Rightarrow \frac{\rho_1}{\rho_0}=\left(\frac{P_1}{P_0}\right)^{\frac{1}{\gamma}}$$

$$\left(\frac{\gamma}{\gamma-1}\right)\frac{P_1}{\rho_1}\left[1-\frac{P_0}{P_1}\cdot\left(\frac{P_0}{P_1}\right)^{-1/\gamma}\right]=-\frac{V_1^2}{2}$$

$$\left(\frac{\gamma}{\gamma-1}\right)\frac{P_1}{\rho_1}\left[1-\left(\frac{P_0}{P_1}\right)^{\frac{\gamma-1}{\gamma}}\right]=\frac{-V_1^2}{2}$$

$$1+\frac{V_1^2}{2}\left(\frac{\gamma-1}{\gamma}\right)\frac{\rho_1}{P_1}=\left(\frac{P_0}{P_1}\right)^{\frac{\gamma-1}{\gamma}}$$

for adiabatic process $\quad a=\sqrt{\gamma RT}=\sqrt{\frac{\gamma P}{\rho}}$

for 1, $a_1=\sqrt{\frac{\gamma P_1}{\rho_1}} \Rightarrow a_1^2=\frac{\gamma P_1}{\rho_1}$

$$1+\frac{V_1^2}{2}\frac{(\gamma-1)}{a_1^2}=\left(\frac{P_0}{P_1}\right)^{\frac{\gamma-1}{\gamma}}$$

$$1+\frac{V_1^2}{a_1^2}\frac{\gamma-1}{2}=\left(\frac{P_0}{P_1}\right)^{\frac{\gamma-1}{\gamma}}$$

$$1+\frac{M_1^2}{2}(\gamma-1)=\left(\frac{P_0}{P_1}\right)^{\frac{\gamma-1}{\gamma}}$$

$$\frac{P_0}{P_1}=\left[1+\frac{M_1^2}{2}(\gamma-1)\right]^{\frac{\gamma}{\gamma-1}}$$

$$P_0=P_1\left[1+\left(\frac{\gamma-1}{2}\right)M_1^2\right]^{\frac{\gamma}{\gamma-1}}$$

Expression for stagnation density :-

$$\frac{P_0}{P_1} = \left[1 + \left(\frac{\gamma-1}{2}\right) M^2\right]^{\frac{\gamma}{\gamma-1}}$$

for adiabatic

$$\frac{P_1}{\rho_1^{\gamma}} = \frac{P_0}{\rho_0^{\gamma}} \Rightarrow \left(\frac{\rho_0}{\rho_1}\right)^{\gamma} = \frac{P_0}{P_1}$$

$$\therefore \frac{\rho_0}{\rho_1} = \left(\frac{P_0}{P_1}\right)^{\frac{1}{\gamma}} = \left[1 + \left(\frac{\gamma-1}{2}\right) M^2\right]^{\frac{1}{\gamma-1}}$$

Expression for stagnation temp :-

Equation of state = $PV = RT$

$$\frac{P_1 V_1}{T_1} = \frac{P_2 V_2}{T_2} \Rightarrow \frac{P_2}{P_1} \cdot \frac{T_1}{T_2} = \frac{V_1}{V_2}$$

$$P_1 V_1^{\gamma} = P_2 V_2^{\gamma} \Rightarrow \frac{V_1}{V_2} = \left(\frac{P_2}{P_1}\right)^{\frac{1}{\gamma}}$$

$$\frac{T_1}{T_2} = \left(\frac{P_2}{P_1}\right)^{\frac{1}{\gamma}-1}$$

$$\frac{T_1}{T_0} = \left(\frac{P_0}{P_1}\right)^{\frac{1-\gamma}{\gamma}} \Rightarrow \frac{T_0}{T_1} = \left(\frac{P_0}{P_1}\right)^{\frac{\gamma-1}{\gamma}} = \left[1 + \left(\frac{\gamma-1}{2}\right) M^2\right]$$

$$T_0 = T_1 \left[1 + \left(\frac{\gamma-1}{2}\right) M^2\right]$$

(P1) Find the (sound) Sonic velocity for the following fluids (A) (9)

(a) Crude oil of spec gravity 0.8 & bulk modulus $1.5\ GN/m^2$

(b) Mercury having bulk modulus of $27\ GN/m^2$

Sol: Sonic velocity $= a_{oil} = \sqrt{\frac{K}{\rho}} = \sqrt{\frac{1.5\times10^9}{0.8\times1000}} = 1369.3\ m/s$

$$a_{Hg} = \sqrt{\frac{27\times10^9}{13600}} = 1409\ m/s.$$

P2) An aeroplane is flying at a height of 14 Km where temp is $-45^\circ C$. find the speed of plane corresponding to $M=2$, $R=287\ J/kg\,K$ & $\gamma=1.4$

Sol

$T = -45+273 = 228\ K$

$\gamma = 1.4$

$R = 287\ J/kg\,K$

$M = 2$

$M = \frac{C}{a} \Rightarrow C = M\cdot a$

$= M\sqrt{\gamma RT}$

$= 2\times302.67 = 2179.2\ Km/hr$

P3] Find the velocity of a bullet fired in std. air if its Mach angle is 40°.

Sol:- $\alpha = 40^\circ$

for std air, $R = 287\ J/kg\,K$, $\gamma = 1.4$, $T = 15+273 = 288\ K$

velocity of bullet $= C = ?$ $\quad a = \sqrt{\gamma RT} = 340.2\ m/s$

$\sin\alpha = \frac{1}{M} = \frac{a}{c} \Rightarrow c = \frac{a}{\sin\alpha} = \frac{340.2}{\sin 40} = 529.26\ m/s$

P4] A projectile is travelling in air having pr & Temp are $88.3\ KN/m^2$ and $-2^\circ c$. If Mach angle is 40°, find the velocity of ~~particle~~ projectile.

$\gamma = 1.4$, $R = 287\ J/kg\,K$.

Sol:- $P = 88.3\ KN/m^2$, $T = -2+273 = 273\ K$, $\alpha = 40^\circ$

$\sin\alpha = \frac{1}{M}$ $\quad C = \frac{a}{\sin\alpha} = \frac{\sqrt{1.4\times287\times273}}{\sin 40} = \frac{330}{\sin 40} = 513.4\ m/s$

P5] An aeroplane is flying at 1000 km/hr through still air having a pressure of 78.5 KN/m^2 (abs) and temperature $-8°C$. Calculate on the stagnation point on the nose of the plane
(i) Stagnation pr (ii) Stagnation Temp (iii) Stagnation Density

Take for air $\gamma = 1.4$, $R = 287$ J/kg K

Sol: Speed of aeroplane $= \overset{C \text{ or}}{V} = 1000$ km/hr $= 277.77$ m/s

$P = 78.5\ KN/m^2$

$T = -8 + 273 = 265K$

vel. of sound $= a = \sqrt{\gamma RT} = \sqrt{1.4 \times 287 \times 265} = 326.31$ m/s.

Mach Number $= M = \frac{C}{a} = \frac{277.77}{326.31} = 0.851$

(a) $P_0 = P\left[1 + \left(\frac{\gamma-1}{2}\right)M^2\right]^{\frac{\gamma}{\gamma-1}} = 126.1\ KN/m^2$

(ii) $T_0 = T\left[1 + \left(\frac{\gamma-1}{2}\right)M^2\right] = 303.4°K$ or $30.4°C$

(iii) $\rho_0 =$ $\quad P_0 = \rho_0 R T_0 \Rightarrow \rho_0 = \frac{P_0}{RT_0} = 1.448\ kg/m^3$

P6] Air has a vel. of 1000 km/hr at a pressure of 9.81 KN/m^2 vacuum and a temp of $47°C$. Compute its stagnation properties & local mach number. Take atmospheric pressure 98.1 KN/m^2, $R = 287$ J/kg K & $\gamma = 1.4$

Sol: flow vel $= C = 1000$ km/hr $= 277.77$ m/s

$P_{vac} = 9.81\ KN/m^2$

$T = 47 + 273 = 320\ K$

Pressure of air (static) $= P_{atm} - P_{vacuum} = 98.1 - 9.81 = 88.29\ KN/m^2$

$R = 287$ J/kg K $\quad \gamma = 1.4$

Sound vel $= a = \sqrt{\gamma RT} = 358.6$ m/s

Mach Number $= M = \frac{C}{a} = \frac{277.77}{358.6} = 0.7746$

$\therefore$ Stagnation pressure $= P_0 = P\left[1+\left(\frac{\gamma-1}{2}\right)M^2\right]^{\frac{\gamma}{\gamma-1}} = 131.27\ \text{kN/m}^2$

Stagnation temperature $= T_0 = T\left[1+\left(\frac{\gamma-1}{2}\right)M^2\right] = 358.4\ \text{K}$

Stagnation density $= \rho_0 = \frac{P_0}{RT_0} = 1.276\ \text{kg/m}^3$.

Compressibility factor $= 1 + \frac{M^2}{4} + \frac{2-\gamma}{24}M^4 + \cdots$

$$(1+x)^n = 1 + nx + n(n-1)\frac{x^2}{2!} + n(n-1)(n-2)\frac{x^3}{3!} + \cdots$$

$$P_0 = P_1\left[1+\left(\frac{\gamma-1}{2}\right)M_1^2\right]^{\frac{\gamma}{\gamma-1}}$$

$$P_0 = P_1\left[1 + \frac{\gamma}{2}M_1^2 + \frac{\gamma}{8}M_1^4 + \frac{\gamma(2-\gamma)}{48}M_1^6 + \cdots\right]$$

$$P_0 = P_1 + P_1\left[\frac{\gamma}{2}M_1^2 + \frac{\gamma}{8}M_1^4 + \frac{\gamma(2-\gamma)}{48}M_1^6 + \cdots\right]$$

$$\frac{P_0-P_1}{P_1} = \frac{\gamma}{2}M_1^2 + \frac{\gamma}{8}M_1^4 + \frac{\gamma(2-\gamma)}{48}M_1^6 + \cdots$$

$$= \frac{\gamma}{2}M_1^2\left[1 + \frac{M_1^2}{4} + \frac{(2-\gamma)}{24}M_1^4 + \cdots\right]$$

$$= \frac{\gamma}{2}\frac{C_1^2}{\frac{\gamma P_1}{\rho_1}}\left[1 + \frac{1}{4}M_1^2 + \frac{(2-\gamma)}{24}M_1^4 + \cdots\right]$$

$$\frac{P_0-P_1}{P_1} = \frac{1}{2}\frac{\rho_1 C_1^2}{P_1}\left[1 + \frac{1}{4}M_1^2 + \left(\frac{2-\gamma}{24}\right)M_1^4 + \cdots\right]$$

$$P_0 - P_1 = \frac{1}{2}\rho_1 C_1^2\left[1 + \frac{1}{4}M_1^2 + \frac{(2-\gamma)}{24}M_1^4 + \cdots\right]$$

$$P_0 - P_1 = \frac{1}{2}\rho_1 C_1^2\left[\text{compressibility factor}\right]$$

$$P_0 = P_1 + \frac{1}{2}\rho_1 C_1^2\left[1 + \frac{1}{4}M_1^2 + \frac{2-\gamma}{24}M_1^4 + \cdots\right]$$

Area-Velocity Relationship and Effect of Variation of Area for Subsonic, Sonic & Super Sonic flows:- (11) (C)

For compressible flow, the continuity equation is given by

$\rho AV = \text{Const}$, differentiating this

$\rho A dV + \rho V dA + A V d\rho = 0$

Dividing both sides with $m = \rho AV$

$$\frac{dV}{V} + \frac{dA}{A} + \frac{d\rho}{\rho} = 0$$

$$\frac{dA}{A} = -\frac{dV}{V} - \frac{d\rho}{\rho} \longrightarrow (1)$$

Euler's equation for compressible fluid is given by

$$\frac{dp}{\rho} + VdV + gdz = 0$$

Neglecting 'z' terms

$$\frac{dp}{\rho} + VdV = 0$$

$$\frac{dp}{d\rho} \cdot \frac{d\rho}{\rho} + VdV = 0$$

but $a^2 = \frac{dp}{d\rho}$

$$a^2 \frac{d\rho}{\rho} + VdV = 0$$

$$a^2 \frac{d\rho}{\rho} = -VdV \Rightarrow \frac{d\rho}{\rho} = \frac{-VdV}{a^2} \longrightarrow (2)$$

(2) in (1) $\frac{dA}{A} = \frac{-dV}{V} + \frac{VdV}{a^2} \Rightarrow \frac{dV}{V}\left[\frac{V^2}{a^2} - 1\right]$

$$\frac{dA}{A} = \frac{dV}{V}\left[M^2 - 1\right]$$

This equation is due to Hugoniot.

when $M < 1$, $\frac{dA}{A} = -Ve$  , $M > 1$, $\frac{dA}{A} = +Ve$ 

$M = 1$, $\frac{dA}{A} = 0$ 

$$\frac{dA}{A} = \frac{dp}{\rho V^2}(1-M^2) \quad : -$$

$$\frac{dA}{A} + \frac{du}{u} + \frac{d\rho}{\rho} = 0 \quad \text{— Continuity equ}$$

Euler's equ $\frac{dp}{\rho} + V\,dV = 0 \Rightarrow \frac{dV}{V} = \frac{-dp}{\rho V^2}$

$$\frac{dA}{A} = -\frac{dV}{V} - \frac{d\rho}{\rho} = \frac{dp}{\rho V^2} - \frac{d\rho}{\rho}$$

$$= \frac{dp}{\rho V^2}\left[1 - \frac{\cancel{\rho} V^2}{dp}\cdot\frac{d\rho}{\cancel{\rho}}\right]$$

$$\boxed{\frac{dA}{A} = \frac{dp}{\rho u^2}\left[1-M^2\right]} = \frac{dV}{V}\left[M^2-1\right]$$

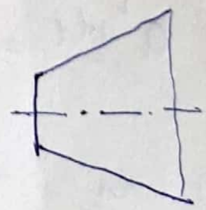
→ In a given conduit flow Mach Number is 1.5. If the velocity undergoes 25% increase, what percent of original area is needed for this

Sol $\frac{dA}{A} = \frac{dV}{V}\left[M^2-1\right] = 0.25\left[(1.5)^2-1\right] = 0.3125$

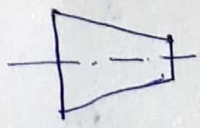
Hence area must be increased by 31.25% than original area.

for subsonic flow ~~pf~~ M<1

$\frac{dA}{A} > 0$, $dp > 0$ $\frac{dV}{V} < 0$ (div/diffuser)



$\frac{dA}{A} < 0$, $dp < 0$, $\frac{dV}{V} > 0$ (con. nozzle)



∴ for super sonic flow:-

D

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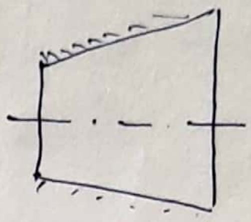
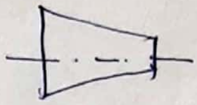
$M>1$ $\qquad \frac{dA}{A} = \frac{dp}{\rho u^2}\left[1-M^2\right] = \frac{dv}{V}\left[M^2-1\right]$

$\frac{dA}{A} > 0,\ dp < 0,\ \frac{dv}{V} > 0$ (div. nozzle)



$\frac{dA}{A} < 0,\ dp > 0,\ \frac{dv}{V} < 0$ (Conv. diffuser)



for sonic flow $M=1$

$\frac{dA}{A} = 0$ (straight flow passage)

$dp = 0,\ \frac{dv}{V} = 0$