UNIT – I

MATRICES

Rank of a matrix :-

Let A be a $m \times n$ matrix, we say that r is the rank of A if (i) Every $(r+1)^{th}$ order minor of A is zero (ii) there exist atleast one r^{th} order minor whis is not zero.

Echelon Form :-

using only Row operations

* The No of non zero rows = Rank of the matrix.

Ex :- 1)
$$\begin{bmatrix} 1 & 0 & -3 & 4 \\ 0 & 2 & -4 & 5 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 10 \end{bmatrix}$$

Ex :- 2)
$$\begin{bmatrix} 3 & 4 & -4 & -6 & 1 \\ 0 & 2 & -5 & 1 & 0 \\ 0 & 0 & 4 & -3 & 4 \\ 0 & 0 & 0 & 4 & 5 \\ 0 & 0 & 0 & 0 & 7 \end{bmatrix}$$

① Reduce the matrix to Echelon form and find its rank

$$\begin{bmatrix} -1 & -3 & 3 & -1 \\ 1 & 1 & -1 & 0 \\ 2 & -5 & 2 & -3 \\ -1 & 1 & 0 & 1 \end{bmatrix}$$

Sol:- Given $\begin{bmatrix} -1 & -3 & 3 & -1 \\ 1 & 1 & -1 & 0 \\ 2 & -5 & 2 & -3 \\ -1 & 1 & 0 & 1 \end{bmatrix}$

$R_1 \leftrightarrow R_2$ $\begin{bmatrix} 1 & 1 & -1 & 0 \\ -1 & -3 & 3 & -1 \\ 2 & -5 & 2 & -3 \\ -1 & 1 & 0 & 1 \end{bmatrix}$

$R_2 \to R_2 + R_1$
$R_3 \to R_3 - 2R_1$
$R_4 \to R_4 + R_1$
$\begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & -2 & 2 & -1 \\ 0 & -7 & 4 & -3 \\ 0 & 2 & -1 & 1 \end{bmatrix}$ $-4(-2)$

$R_2 \to -4R_2 + R_3$ $\begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & -4 & 1 \\ 0 & -7 & 4 & -3 \\ 0 & 2 & -1 & 1 \end{bmatrix}$

$R_3 \to R_3 + 7R_2$
$R_4 \to R_4 - 2R_2$
$\begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & -4 & 1 \\ 0 & 0 & -24 & 4 \\ 0 & 0 & 7 & -1 \end{bmatrix}$

$R_3 \to \frac{R_3}{-4}$ $\begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & -4 & 1 \\ 0 & 0 & 6 & -1 \\ 0 & 0 & 7 & -1 \end{bmatrix}$

$R_3 \to R_4 - R_3$ $\begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & -4 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 7 & -1 \end{bmatrix}$

$R_4 \to R_4 - 7R_3$ $\begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & -4 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$

∴ The no of non zero are 4

∴ The rank of the matrix is 4

pb) Find the rank of the matrix $\begin{bmatrix} 2 & 1 & 3 & 5 \\ 4 & 2 & 1 & 3 \\ 8 & 4 & 7 & 13 \\ 8 & 4 & -3 & -1 \end{bmatrix}$

Sol:- Given $\begin{bmatrix} 2 & 1 & 3 & 5 \\ 4 & 2 & 1 & 3 \\ 8 & 4 & 7 & 13 \\ 8 & 4 & -3 & -1 \end{bmatrix}$

$R_2 \to R_2 - 2R_1$
$R_3 \to R_3 - 4R_1$
$R_4 \to R_4 - 4R_1$
$\begin{bmatrix} 2 & 1 & 3 & 5 \\ 0 & 0 & -5 & -7 \\ 0 & 0 & -5 & -7 \\ 0 & 0 & -15 & -21 \end{bmatrix}$

$R_3 \to R_3 - R_2$ $\begin{bmatrix} 2 & 1 & 3 & 5 \\ 0 & 0 & -5 & -7 \end{bmatrix}$

Normal (canonical) Form :-

using both Row and column operations

Reducing given matrix into Identity matrix

① Find the rank of the matrix $\begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & 3 & 2 & 2 \\ 2 & 4 & 3 & 4 \\ 3 & 7 & 5 & 6 \end{bmatrix}$

by reducing it to normal form.

Sol:-

Given $\begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & 3 & 2 & 2 \\ 2 & 4 & 3 & 4 \\ 3 & 7 & 5 & 6 \end{bmatrix}$

$R_2 \to R_2 - R_1$

$R_3 \to R_3 - 2R_1$

$R_4 \to R_4 - 3R_1$

$$\begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \end{bmatrix}$$

$C_2 \to C_2 - 2C_1$

$C_3 \to C_3 - C_1$

$C_4 \to C_4 - 2C_1$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \end{bmatrix}$$

$$R_4 \to R_4 - R_2 \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$C_3 \to C_3 - C_2 \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$R_4 \to R_4 - R_3 \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ The no of non zero rows $= 3$

$\therefore$ The rank of the matrix $= 3$.

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System of Linear Equations :-

1) If rank of A = rank of $(A|B)$ = The no of unknowns. then the system is Consistent and it has a unique solution.

2) If rank of A = rank of $(A|B)$ < The no of unknowns then the system is Consistent and it has an infinite no of solutions.

3) If rank of $A \neq$ rank of $(A|B)$ Then the system is InConsistent and it has no solution.

① Find whether the ~~stste~~ system of equ are Consistent if so solve them.

$x+y+z=6$, $2x+3y-2z=2$, $5x+y+2z=13$,

Sol:- Given

$$x+y+z=6$$

$$2x+3y-2z=2$$

$$5x+y+2z=13$$

Given system can be written as $AX=B$

$$A=\begin{bmatrix}1 & 1 & 1\\ 2 & 3 & -2\\ 5 & 1 & 2\end{bmatrix};\ X=\begin{bmatrix}x\\ y\\ z\end{bmatrix};\ B=\begin{bmatrix}6\\ 2\\ 13\end{bmatrix}$$

The Augmented matrix $[A \mid B] = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 2 & 3 & -2 & 2 \\ 5 & 1 & 2 & 13 \end{bmatrix}$.

$R_2 \to R_2 - 2R_1$

$R_3 \to R_3 - 5R_1$

$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & -4 & -10 \\ 0 & -4 & -3 & -17 \end{bmatrix}$$

$R_3 \to R_3 + 4R_2$

$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & -4 & -10 \\ 0 & 0 & -19 & -57 \end{bmatrix}$$

$R_3 \to \dfrac{R_3}{-19}$

$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & -4 & -10 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

The rank of $A = 3$

rank of $(A \mid B) = 3$

The no of unknown $= 3$

$\therefore \rho(A) = \rho(A \mid B) =$ The no of unknown

$\therefore$ The Given system is consistent and it has a unique solution.

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad B = \begin{bmatrix} 6 \\ -10 \\ 3 \end{bmatrix}$$

we have $AX = B$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ -10 \\ 3 \end{bmatrix}$$

$x + y + z = 6$

$y - 4z = -10$

$z = 3$

$y - 12 = -10 \Rightarrow y = 2$

$x + 2 + 3 = 6 \Rightarrow x = 1$

$\therefore x = 1,\ y = 2,\ z = 3$

Eigen values and Eigen vectors

The characteristic equ of A is $|A-\lambda I| = 0$

Eigen vector $\Rightarrow$ $(A-\lambda I)X = 0$

Note :- Eigen values = characteristic roots

Eigen vectors = characteristic vectors

1) Find the characteristic roots of the matrix

$$A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$$

Sol:- Given $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$

The characteristic equ of A is $|A-\lambda I| = 0$

$$\begin{vmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)\begin{pmatrix} 3-\lambda & 1 \\ 2 & 2-\lambda \end{pmatrix} - 2\begin{pmatrix} 1 & 1 \\ 1 & 2-\lambda \end{pmatrix} + 1\begin{pmatrix} 1 & 3-\lambda \\ 1 & 2 \end{pmatrix} = 0$$

$$(2-\lambda)\left[(3-\lambda)(2-\lambda) - 2\right] - 2\left[2-\lambda - 1\right] + 1\left[2-(3-\lambda)\right] = 0$$

$$(2-\lambda)\left[6-5\lambda+\lambda^2-2\right] - 2\left[1-\lambda\right] + 1\left[2-3+\lambda\right] = 0$$

$$(2-\lambda)(\lambda^2-5\lambda+4) - 2 + 2\lambda - 1 + \lambda = 0$$

$$2\lambda^2 - 10\lambda + 8 - \lambda^3 + 5\lambda^2 - 4\lambda + 3\lambda - 3 = 0$$

$$-\lambda^3 + 7\lambda^2 - 11\lambda + 5 = 0$$

$$\lambda^3 - 7\lambda^2 + 11\lambda - 5 = 0$$

$\lambda = 5, 1, 1$

$$
\begin{array}{r|rrrr}
\lambda=5 & 1 & -7 & 11 & -5 \\
 & 0 & 5 & -10 & 5 \\
\hline
\lambda=1 & 1 & -2 & 1 & \boxed{0} \\
 & 0 & 1 & -1 & \\
\hline
\lambda=1 & 1 & -1 & \boxed{0} & \\
 & 0 & 1 & & \\
\hline
 & 1 & \boxed{0} & &
\end{array}
$$

$\therefore$ The Eigen value are $\lambda = 1, 1, 5$.

<u>Note</u> :- Sum of the Eigen value is the trace of the matrix.

(P3) Find the Eigen values of and Eigen vector of the matrix
$$\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

<u>Sol</u>:- Given $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

The characteristic equ of A is $|A - \lambda I| = 0$

$$\begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix} = 0$$

$(8-\lambda)\left[(7-\lambda)(3-\lambda) - 16\right] + 6\left[-18 + 6\lambda + 8\right] + 2\left[24 - 14 + 2\lambda\right] = 0$

$(8-\lambda)\left[21 - 10\lambda + \lambda^2 - 16\right] + 6\left[6\lambda - 10\right] + 2\left[2\lambda + 10\right] = 0$

$(8-\lambda)(\lambda^2 - 10\lambda + 5) + 36\lambda - 60 + 4\lambda + 20 = 0$

$8\lambda^2 - 80\lambda + 40 - \lambda^3 + 10\lambda^2 - 5\lambda + 40\lambda - 40 = 0$

$-\lambda^3 + 18\lambda^2 - 45\lambda = 0 \Rightarrow \lambda^3 - 18\lambda^2 + 45\lambda = 0$

$-\lambda(\lambda^2 - 18\lambda + 45) = 0$

$\lambda = 0, \quad \lambda^2 - 3\lambda - 15\lambda + 45 = 0$

$\lambda(\lambda-3) - 15(\lambda-3) = 0$

$\lambda = 0, \lambda = 3, 15 \Rightarrow$

$\therefore \lambda = 0, 3, 15$ are the Eigen values.

The Eigen vector corresponding to $\lambda = 0$:-

we have $(A - \lambda I) X = 0$

$$\begin{bmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Taking $\lambda = 0$ in above

$$\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_1 \to \frac{R_1}{2} \quad \begin{bmatrix} 4 & -3 & 1 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_1 \leftrightarrow R_3 \quad \begin{bmatrix} 2 & -4 & 3 \\ -6 & 7 & -4 \\ 4 & -3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{array}{l} R_2 \to R_2 + 3R_1 \\ R_3 \to R_3 - 2R_1 \end{array} \begin{bmatrix} 2 & -4 & 3 \\ 0 & -5 & 5 \\ 0 & 5 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{array}{l} R_2 \to \frac{R_2}{-5} \\ R_3 \to \frac{R_3}{5} \end{array} \begin{bmatrix} 2 & -4 & 3 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \to R_3 - R_2$ $\begin{bmatrix} 2 & -4 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$2x_1 - 4x_2 + 3x_3 = 0$$
$$x_2 - x_3 = 0$$

put $\underline{x_3 = K}$ $\underline{x_2 = K}$

$2x_1 - 4K + 3K = 0 \Rightarrow 2x_1 = K$

$\Rightarrow x_1 = \frac{K}{2}$

$\therefore X_1 = \begin{bmatrix} \frac{K}{2} \\ K \\ K \end{bmatrix} = \frac{K}{2} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$

$\therefore X_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ is the Eigen correspondig to $\lambda = 0$.

The Eigen vector correspondig to $\lambda = 3$:—

we have $(A - \lambda I)X = 0$

$$\begin{bmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

put $\lambda = 3$ in above eqn

$$\begin{bmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_1 \leftrightarrow \frac{R_3}{2}$

$R_2 \rightarrow \frac{R_2}{2}$

$$\begin{bmatrix} 1 & -2 & 0 \\ -3 & 2 & -2 \\ 5 & -6 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \rightarrow R_2 + 3R_1$

$R_3 \rightarrow R_3 - 5R_1$

$$\begin{bmatrix} 1 & -2 & 0 \\ 0 & -4 & -2 \\ 0 & 4 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \rightarrow R_3 + R_2$

$$\begin{bmatrix} 1 & -2 & 0 \\ 0 & -4 & -2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 - 2x_2 = 0$$

$$-4x_2 - 2x_3 = 0$$

put $x_3 = K$, $-4x_2 - 2K = 0$

$$-4x_2 = 2K \Rightarrow x_2 = \frac{-K}{2}$$

$$x_1 - \cancel{2}\left(\frac{-K}{\cancel{2}}\right) = 0 \Rightarrow x_1 + K = 0 \Rightarrow x_1 = -K$$

$$\therefore X_2 = \begin{bmatrix} -K \\ -\frac{K}{2} \\ K \end{bmatrix} = \frac{K}{2}\begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix}$$

$\therefore X_2 = \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix}$ is the Eigen vector corr. to $\lambda = 3$.

The Eigen vector corresponding to $\lambda = 15$:—

we have $(A - \lambda I) X = 0$

$$\begin{bmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Take $\lambda = 15$

$$\begin{bmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_1 \leftrightarrow \frac{R_3}{2}$

$R_2 \rightarrow \frac{R_2}{-2}$

$$\begin{bmatrix} 1 & -2 & -6 \\ 3 & 4 & 2 \\ -7 & -6 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \rightarrow R_2 - 3R_1$

$R_3 \rightarrow R_3 + 7R_1$

$$\begin{bmatrix} 1 & -2 & -6 \\ 0 & 10 & 20 \\ 0 & -20 & -40 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \rightarrow \frac{R_2}{10}$

$R_3 \rightarrow \frac{R_3}{-20}$

$$\begin{bmatrix} 1 & -2 & -6 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \rightarrow R_3 - R_2$

$$\begin{bmatrix} 1 & -2 & -6 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 - 2x_2 - 6x_3 = 0$$

$$x_2 + 2x_3 = 0$$

Take $x_3 = K$. $\quad x_2 = -2K$

$$x_1 + 4K - 6K = 0$$

$$x_1 - 2K = 0 \Rightarrow x_1 = 2K$$

$$\therefore X_3 = \begin{bmatrix} 2K \\ -2K \\ K \end{bmatrix} = K \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

$\therefore X_3 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$ is the Eigen vector corresponding to $\lambda = 15$.

Cayley hamilton theorem: every square matrix satisfies its own characteristic equation.

(P5) verify Cayley hamilton theorem and find A^{-1} of the matrix

$$\begin{bmatrix} 1 & 0 & 3 \\ 2 & -1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$$

Sol: Given $A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & -1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$

The characteristic eqn of matrix A is

$|A - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 0 & 3 \\ 2 & -1-\lambda & -1 \\ 1 & -1 & 1-\lambda \end{vmatrix} = 0$$

$\Rightarrow (1-\lambda)(-1+\lambda-\lambda+\lambda^2-1) - 0 + 3[\underset{-1}{\underline{-2+1}}+\lambda] = 0$

$\Rightarrow (1-\lambda)(\lambda^2-2) + 3\lambda - 3 = 0$

$\Rightarrow \lambda^2 - 2 - \lambda^3 + 2\lambda + 3\lambda - 3 = 0$

$\Rightarrow -\lambda^3 + \lambda^2 + 5\lambda - 5 = 0$

$\Rightarrow \lambda^3 - \lambda^2 - 5\lambda + 5 = 0$

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put $\lambda = A$

$\Rightarrow A^3 - A^2 - 5A + 5I = 0 \longrightarrow (1)$

$$A \cdot A = A^2 = \begin{bmatrix} 1 & 0 & 3 \\ 2 & -1 & -1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 3 \\ 2 & -1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+3 & -3 & 3+3 \\ 2-2-1 & 1+1 & 6+1-1 \\ 1-2+1 & 1-1 & 3+1+1 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 4 & -3 & 6 \\ -1 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$$

$$A^2 \cdot A = A^3 = \begin{bmatrix} 4 & -3 & 6 \\ -1 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} 1 & 0 & 3 \\ 2 & -1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4-6+6 & 3-6 & 12+3+6 \\ -1+4+6 & -2-6 & -3-2+6 \\ 5 & -5 & 5 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 4 & -3 & 21 \\ 9 & -8 & 1 \\ 5 & -5 & 5 \end{bmatrix}$$

$$5A = 5 \begin{bmatrix} 1 & 0 & 3 \\ 2 & -1 & -1 \\ 1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 15 \\ 10 & -5 & -5 \\ 5 & -5 & 5 \end{bmatrix}$$

$$5I = 5\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} = \begin{bmatrix}5&0&0\\0&5&0\\0&0&5\end{bmatrix}$$

Consider

$$A^3 - A^2 - 5A + 5I = 0$$

$$= \begin{bmatrix}4&-3&21\\9&-8&1\\5&-5&5\end{bmatrix} - \begin{bmatrix}4&-3&6\\-1&2&6\\0&0&5\end{bmatrix} - \begin{bmatrix}5&0&15\\10&-5&-5\\5&-5&5\end{bmatrix} + \begin{bmatrix}5&0&0\\0&5&0\\0&0&5\end{bmatrix}$$

$$= \begin{bmatrix}4-4-5+5 & -3+3-0+0 & 21-6-15\\ 9+1-10+0 & -8-2+5+5 & 1-6+5\\ 5-0-5 & -5-0+5 & 5-5-5+5\end{bmatrix}$$

$$= \begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$$

$\therefore$ cayley hamilton theorem verified.

To find A^{-1}:

Multiplying eqn-① with A^{-1} on l.s

$$A^{-1}(A^3 - A^2 - 5A + 5I) = 0$$

$$A^{-1}A = I$$

$$A^2 - A - 5I + 5A^{-1} = 0$$

$$5A^{-1} = -A^2 + A + 5I$$

$$5A^{-1} = \begin{bmatrix} -4 & 3 & -6 \\ 1 & -2 & -6 \\ 0 & 0 & -5 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 3 \\ 2 & -1 & -1 \\ 1 & -1 & 1 \end{bmatrix} + \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$5A^{-1} = \begin{bmatrix} -4+1+5 & 3+0 & -3 \\ 3 & 2 & -7 \\ 1 & -1 & 1 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{5}\begin{bmatrix} 2 & 3 & -3 \\ 3 & 2 & -7 \\ 1 & -1 & 1 \end{bmatrix}$$

To find A^4:

Multiplying eqn-① with on l.s

$$A(A^3 - A^2 - 5A + 5I) = 0$$

$$A^4 - A^3 - 5A^2 + 5A = 0$$

$$A^4 = A^3 + 5A^2 - 5A$$

$$A^4 = \begin{bmatrix} 4 & -3 & 2 \\ 9 & -8 & 1 \\ 5 & -5 & 5 \end{bmatrix} + \begin{bmatrix} 20 & -15 & 30 \\ -5 & 10 & 30 \\ 0 & 0 & 25 \end{bmatrix} - \begin{bmatrix} 5 & 0 & 15 \\ 10 & -5 & -5 \\ 5 & -5 & 5 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 19 & -18 & 36 \\ -6 & 7 & 36 \\ 0 & 0 & 25 \end{bmatrix}$$

29.1.19

Diagonlization Method

(Pb) (A) If A is symmetric matrix then

$$D = P^T A P$$

$$P = \left[\frac{X_1}{\|X_1\|} \quad \frac{X_2}{\|X_2\|} \quad \frac{X_3}{\|X_3\|} \right]$$

where P is modal matrix

If A is non-symmetric matrix then the diagonal matrix D is equal $= P^{-1}AP$

$$D = P^{-1} A P$$

where $P = [X_1 \; X_2 \; X_3]$

P is modal matrix

(6) Diagonlise the matrix $\begin{bmatrix} 1 & 2 & -2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$

Sol:- Given $A = \begin{bmatrix} 1 & 2 & -2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$

The characteristic eqn of A is $|A - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 2 & -2 \\ 1 & 1-\lambda & 1 \\ 1 & 3 & -1-\lambda \end{vmatrix} = 0$$

$\Rightarrow (1-\lambda)\left[(1-\lambda)(-1-\lambda) - 3\right] - 2\left[-1-\lambda-1\right] - 2\left[3-(1+\lambda)\right] = 0$

$\Rightarrow (1-\lambda)\left[-1-\lambda+\lambda+\lambda^2-3\right] + 4 + 2\lambda - 4 - 2\lambda = 0$

$\Rightarrow \lambda^2 - 4 - \lambda^3 + 4\lambda \cancel{0} = 0$

$\Rightarrow -\lambda^3 + \lambda^2 + 4\lambda - 4 = 0$

$\Rightarrow \lambda^3 - \lambda^2 - 4\lambda + 4 = 0$

$\lambda = -2, 2, 1$

The eigen vector corresponding to $\lambda = 1$

we have $|A - \lambda I| x = 0$

$$\begin{bmatrix} 1-\lambda & 2 & -2 \\ 1 & 1-\lambda & 1 \\ 1 & 3 & -1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Put $\lambda = 1$

$$\begin{bmatrix} 0 & 2 & -2 \\ 1 & 0 & 1 \\ 1 & 3 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \longleftrightarrow R_1$
$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & -2 \\ 1 & 3 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \to R_3 - R_1$

$R_2 \to \frac{R_2}{2}$
$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 3 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \to \frac{R_3}{3}$
$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \to R_3 - R_2$
$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_1 + x_3 = 0$

$x_2 - x_3 = 0$

take $x_3 = k$

$x_2 = k$

$x_1 = -k$

$$X_1 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

The eigen vector corresponding to $\lambda = 2$

we have $|A - \lambda I| x = 0$

$$\begin{bmatrix} -1 & 2 & -2 \\ 1 & -1 & 1 \\ 1 & 3 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 + R_1$

$R_3 \to R_3 + R_1$

$$\begin{bmatrix} -1 & 2 & -2 \\ 0 & 1 & -1 \\ 0 & 5 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \to R_3 - 5R_2$

$R_1 \to \frac{R_1}{-1}$

$$\begin{bmatrix} 1 & -2 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_1 - 2x_2 + 2x_3 = 0$

$x_2 - x_3 = 0$

take $x_3 = k$

$x_2 = k$

$x_1 - 2k + 2k = 0$

$x_1 = 0$

$$X_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

The eigen vector corresponding to $\lambda = -2$ we have $|A - \lambda I| x = 0$

$$\begin{bmatrix} 3 & 2 & -2 \\ 1 & 3 & 1 \\ 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to 3R_2 - R_1$

$R_3 \to 3R_3 - R_1$

$$\begin{bmatrix} 3 & 2 & -2 \\ 0 & 7 & 5 \\ 0 & 7 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\begin{bmatrix} 3 & 2 & -2 \\ 0 & 7 & 5 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$3x_1 + 2x_2 - 2x_3 = 0$

$7x_2 + 5x_3 = 0$

take $x_3 = k$

$7x_2 = -5k$

$x_2 = -\frac{5}{7}k$

$3x_1 - \frac{10}{7}k - 2k = 0$

$3x_1 = \frac{24}{7}k$

$x_1 = \frac{8}{7}k$

$$X_3 = \begin{bmatrix} \frac{8}{7}k \\ -\frac{5}{7}k \\ k \end{bmatrix}$$

$$X_3 = \frac{k}{7}\begin{bmatrix} 8 \\ -5 \\ 7 \end{bmatrix}$$

$$X_3 = \begin{bmatrix} 8 \\ -5 \\ 7 \end{bmatrix}$$

Therefore, the modal matrix $P = [x_1\ x_2\ x_3]$

$$P = \begin{bmatrix} -1 & 0 & 8 \\ 1 & 1 & -5 \\ 1 & 1 & 7 \end{bmatrix}$$

$\therefore D = P^{-1} A P$

$$D = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Quadratic forms

$$Q = X^T A X$$

1) find the symmetric matrix corresponding to the quadratic form $x_1^2 + 6x_1x_2 + 5x_2^2$.

Sol:- Given $x_1^2 + 6x_1x_2 + 5x_2^2$

$$\begin{array}{c} \\ \end{array}\begin{array}{cc} x_1 & x_2 \end{array}$$

$$\begin{bmatrix} 1 & 3 \\ 3 & 5 \end{bmatrix}$$

2) $x^2 + 2y^2 + 3z^2 + 4xy + 5yz + 6zx$.

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & 5/2 \\ 3 & 5/2 & 3 \end{bmatrix}$$

x_1 y_2 z_3

3) $x_1^2 + 2x_2^2 - 7x_3^2 - 4x_1x_2 + 8x_1x_3$

$$\begin{bmatrix} 1 & -2 & 4 \\ -2 & 2 & 0 \\ 4 & 0 & -7 \end{bmatrix}$$

x_1 x_2 x_3

4) $x_1^2 + 2x_2^2 - 7x_3^2 - 4x_1x_2 + 8x_1x_3 + 5x_2x_3$

$$\begin{bmatrix} 1 & -2 & 4 \\ -2 & 2 & \frac{5}{2} \\ 4 & \frac{5}{2} & -7 \end{bmatrix}$$

Canonical form (or) Normal form :-

$$Q = Y^T D Y = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \lambda_3 y_3^2$$

$$D = \text{diag}[\lambda_1, \lambda_2, \lambda_3]$$

Index of the quadratic form :-

The no of positive term in canonical form

Signature of the quadratic form :-

The no of positive terms — The no of negative terms.

Nature of the quadratic form :-

(i) **positive definite :-** If all the eigen values are positive.

(ii) **Negative definite :-** If all the eigen values are Negative.

(iii) **positive semi definite :-** If all the eigen value are positive and atleast one ~~v~~value is zero

(iv) **Negative semi definite :-** If all the eigen values are Negative and atleast one value is zero.

(v) **Indefinite :-** ~~If all~~ other cases.

(16) Identify the nature of the quadratic form & ~~find index & signature~~

$x_1^2 + 4x_2^2 + x_3^2 - 4x_1x_2 + 2x_1x_3 - 4x_2x_3$

Sol: Given $A = \begin{bmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{bmatrix}$

The characteristic eqn of A is $|A - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & -2 & 1 \\ -2 & 4-\lambda & -2 \\ 1 & -2 & 1-\lambda \end{vmatrix} = 0$$

$\Rightarrow (1-\lambda)\left[4 - 4\lambda - \lambda + \lambda^2 - 4\right] + 2\left[-2 + 2\lambda + 2\right] + 1\left[4 - 4 + \lambda\right] = 0$

$\Rightarrow (1-\lambda)\left[\lambda^2 - 5\lambda + 0\right] + 2\left[2\lambda\right] + \lambda = 0$

$\Rightarrow \lambda^2 - 5\lambda - \lambda^3 + 5\lambda^2 + 4\lambda + \lambda = 0$

$\Rightarrow -\lambda^3 + 6\lambda^2 \cancel{- 4\lambda + 4} = 0 \Rightarrow -\lambda^3 + 6\lambda^2 = 0$

$= \lambda^3 - 6\lambda^2 \cancel{+ 4\lambda - 4} = 0 \Rightarrow \lambda^3 - 6\lambda^2 = 0$

$\lambda = 6, 0, 0$