Queuing Theory

- Model 1
- Model 2
- Model 3

Model 1 $(M/M/1) - (\infty/FCFS)$

In this model $\lambda < \mu$

1) probability that a service channel is busy (Traffic intensity (or) Utilization factor)

$$\rho = \frac{\lambda}{\mu}$$

2) probability of an empty (or) idle system

$$P_0 = 1 - \rho = 1 - \frac{\lambda}{\mu}$$

3) probability that there are n customers in the system

$$P_n = (\rho)^n P_0$$

$$= \left(\frac{\lambda}{\mu}\right)^n \left(1 - \frac{\lambda}{\mu}\right)$$

4) Probability that a customer is being served and nobody is waiting

$$P_1 = P_0 \frac{\lambda}{\mu}$$

5) Probability that there are more than two customers in the counter

$$= P_3 + P_4 + P_5 + \cdots$$

$$= 1 - (P_0 + P_1 + P_2)$$

$$= 1 - \left[P_0 \left(1 + \frac{\lambda}{\mu} + \frac{\lambda^2}{\mu^2}\right)\right]$$

6) Probability that there is no customer waiting to be served = probability that there is at most one customer in the counter

$$= P_0 + P_1$$

$$= P_0 + P_0\left(\frac{\lambda}{\mu}\right)$$

$$= \left(1 - \frac{\lambda}{\mu}\right) + \left(1 - \frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)$$

7) Average no. of customers in the system $L_s = \frac{\lambda}{\mu - \lambda}$

8) Average no. of customers in the queue:

$$L_q = \rho L_s = \frac{\lambda}{\mu}\left(\frac{\lambda}{\mu-\lambda}\right)$$

9) Average no. of customers in non-empty queue = Length of queue that is formed time to time

$$L_b = L_n = \frac{\mu}{\mu-\lambda}$$

NOTE: $L_q < L_n$

10) Average waiting time of a customer in the system

$$W_s = \frac{1}{\mu-\lambda} = \frac{L_s}{\lambda}$$

$W_s = W_q +$ service time

11) Average waiting time of a customer in the queue.

$$W_q = \rho W_s = \frac{\lambda}{\mu}\left(\frac{1}{\mu-\lambda}\right)$$

$$W_q = \frac{L_q}{\lambda}$$

12) Average waiting time of a customer in non-empty queue = Average waiting time of customer who has to wait

$$W_n = \frac{1}{\mu - \lambda}$$

13) probability of queue length being greater than or equal to n.

$$P(L_q \geq n) = \left(\frac{\lambda}{\mu}\right)^n$$

$$P(L_q > n) = \left(\frac{\lambda}{\mu}\right)^{n+1}$$

14) probability of waiting for 10 minutes or more

$$\text{probability}[W_q \geq 10] = \int_{10}^{\infty} \frac{\lambda}{\mu}(\mu - \lambda)\, e^{-(\mu-\lambda)t}\, dt$$

15) probability of waiting and service for 10 minutes or more

$$\text{probability}[W_s \geq 10] = \int_{10}^{\infty} (\mu - \lambda)\, e^{-(\mu-\lambda)t}\, dt$$

10.9-4 Assumptions and Limitations of Queuing Model

The various results of section 10.9.2 have been derived under the following simplifying assumptions :

1. The customers arrive for service at a single service facility at random according to Poisson distribution with mean arrival rate λ.
2. The service time has exponential distribution with mean service rate μ.
3. The service discipline followed is first come, first served.
4. Customer behaviour is normal *i.e.*, customers desiring service join the queue, wait for their turn and leave only after getting serviced; they do not resort to balking, reneging or jockeying.
5. Service facility behaviour is normal. It serves the customers continuously, without break, as long as there is queue. Also it serves only one customer at a time.
6. The waiting space available for customers in the queue is infinite.
7. The calling source (population) has infinite size.
8. The elapsed time since the start of the queue is sufficiently long so that the system has attained a steady state or stable state.
9. The mean arrival rate λ is less than the mean service rate μ.

However, in most of the actual business situations the above assumptions are hardly satisfied. The various limitations in a queuing model are :

1. The waiting space for the customers is usually limited.
2. The arrival rate may be state dependent. An arriving customer, on seeing a long queue, may not joint it and go away without getting service.
3. The arrival process may not be stationary. There may be peak period and slack period during which the arrival rate may be more or less than the average arrival rate.
4. The population of customers may not be infinite and the queuing discipline may not be first come, first served.
5. Services may not be rendered continuously. The service facility may breakdown; also the service may be provided in batches rather than individually.
6. The queuing system may not have reached the steady state. It may be, instead, in transient state. It is commonly so when the queue just starts and the elapsed time is not sufficient.

EXAMPLE 10.9-4.1

A self-service store employs one cashier at its counter. Nine customers arrive on an average every 5 minutes while the cashier can serve 10 customers in 5 minutes. Assuming Poisson distribution for arrival rate and exponential distribution for service time, find

1. *Average number of customers in the system.*
2. *Average number of customers in the queue or average queue length.*
3. *Average time a customer spends in the system.*
4. *Average time a customer waits before being served.*

[*P.T.U. B.E., 2001; Karn. U. B.E. (Mech.) 1998, 95*]

Solution

Arrival rate $\lambda = 9/5 = 1.8$ customers/minute,

service rate $\mu = 10/5 = 2$ customers/minute.

1. Average number of customers in the system,

$$L_s = \frac{\lambda}{\mu - \lambda} = \frac{1.8}{2 - 1.8} = 9.$$

2. Average number of customers in the queue,

$$L_q = \frac{\lambda^2}{\mu(\mu - \lambda)} = \frac{\lambda}{\mu} \cdot \frac{\lambda}{(\mu - \lambda)} = \frac{1.8}{2} \times \frac{1.8}{2 - 1.8} = 8.1.$$

3. Average time a customer spends in the system,

$$W_s = \frac{1}{\mu - \lambda} = \frac{1}{2 - 1.8} = 5 \text{ minutes.}$$

4. Average time a costumer waits in the queue,

$$W_q = \frac{\lambda}{\mu}\left(\frac{1}{\mu - \lambda}\right) = \frac{1.8}{2}\left(\frac{1}{2 - 1.8}\right) = 4.5 \text{ minutes}$$

EXAMPLE 10.9-4.2

A person repairing radios finds that the time spent on the radio sets has exponential distribution with mean 20 minutes. If the radios are repaired in the order in which they come in and their arrival is approximately Poisson with an average rate of 15 for 8-hour day, what is the repairman's expected idle time each day ? How many jobs are ahead of the average set just brought in ?

[*P.U.B.E. (T&I.T.) Nov., 2004; B.E. (Mech.) 2002 ; P.T.U. B. (Tech.) 2000; MBA May, 2002 ; IGNOU MBA 2000; G.J.U. B.E. (Mech.) 1996*]

Solution

Arrival rate $\lambda = \frac{15}{8 \times 60} = \frac{1}{32}$ units/minute,

service rate $\mu = \frac{1}{20}$ units/minute.

Number of jobs ahead of the set brought in = Average number of jobs in the system,

$$L_s = \frac{\lambda}{\mu - \lambda} = \frac{1/32}{1/20 - 1/32} = \frac{5}{3}.$$

Number of hours for which the repairman remains busy in an 8-hour day

$$= 8\,\frac{\lambda}{\mu} = 8 \times \frac{1/32}{1/20} = 8 \times \frac{20}{32} = 5 \text{ hours.}$$

$\therefore$ Time for which repairman remains idle in an 8-hour day

$$= 8 - 5 = 3 \text{ hours.}$$

EXAMPLE 10.9-4.3

A branch of Punjab National Bank has only one typist. Since the typing work varies in length (number of pages to be typed), the typing rate is randomly distributed approximating a Poisson distribution with mean service rate of 8 letters per hour. The letters arrive at a rate of 5 per hour during the entire 8-hour work day. If the typewriter is valued at Rs. 1.50 per hour, determine.

1. *Equipment utilization.*
2. *The per cent time that an arriving letter has to wait.*
3. *Average system time.*
4. *Average cost due to waiting on the part of typewriter i.e., it remaining idle.*

[*Nagpur U.B.E. (Mech.) 2003*]

Solution

Arrival rate, $\lambda = 5$ per hour,

service rate, $\mu = 8$ per hour.

1. Equipment utilization, $\rho = \frac{\lambda}{\mu} = \frac{5}{8} = 0.625.$

2. The per cent time an arriving letter has to wait

= per cent time the typewriter remains busy

= 62.5%.

3. Average system time,

$$W_s = \frac{1}{\mu - \lambda} = \frac{1}{8-5} = \frac{1}{3} \text{ hr.} = 20 \text{ minutes.}$$

4. Average cost due to waiting on the part of the typewriter per day

= 8 × (1 – 5/8) × Rs. 1.50 = Rs. 4.50.

EXAMPLE 10.9-4.5

Arrival rate of telephone calls at a telephone booth are according to Poisson distribution, with an average time of 9 minutes between two consecutive arrivals. The length of telephone call is assumed to be exponentially distributed, with mean 3 minutes.

(a) *Determine the probability that a person arriving at the booth will have to wait.*

(b) *Find the average queue length that is formed from time to time.*

(c) *The telephone company will install a second booth when convinced that an arrival would expect to have to wait at least four minutes for the phone. Find the increase in flow rate of arrivals which will justify a second booth.*

(d) *What is the probability that an arrival will have to wait for more than 10 minutes before the phone is free ?*

(e) *What is the probability that he will have to wait for more than 10 minutes before the phone is available and the call is also complete ?*

(f) *Find the fraction of a day that the phone will be in use.*

[*Pbi. U. MCA 1997; Kuru. U.B. Tech. (Mech.) 1990; P.U. B.E (Mech.) May, 1994; P.T.U. MBA June, 2003; Mumbai B.E. 1986*]

Solution

Here, arrival rate, $\lambda = 1/9$ per minute,

service rate, $\mu = 1/3$ per minute.

(*a*) Probability that a person will have to wait

$$= \frac{\lambda}{\mu} = \frac{1/9}{1/3} = \frac{1}{3} = 0.33.$$

(*b*) Average queue length that is formed from time to time

$$= \frac{\mu}{\mu - \lambda} = \frac{1/3}{1/3 - 1/9} = \frac{1/3}{2/9} = \frac{1}{3} \times \frac{9}{2} = 1.5 \text{ persons.}$$

(c) Let λ_1 be the new (increased) arrival rate to justify the installation of the second telephone booth.

$$\text{Average waiting time in the queue} = \frac{\lambda_1}{\mu(\mu - \lambda_1)}.$$

$$\therefore \quad 4 = \frac{\lambda_1}{1/3.(1/3 - \lambda_1)}$$

or $\quad 1/9 - \frac{\lambda_1}{3} = \frac{\lambda_1}{4}$ or $\lambda_1 \times 7/12 = 1/9.$

$$\therefore \lambda_1 = \frac{12}{7 \times 9} = \frac{4}{21} \text{ arrivals/minute.}$$

$$\therefore \text{Increase in flow rate of arrivals} = \frac{4}{21} - \frac{1}{9} = \frac{5}{63} \text{ per minute.}$$

(*d*) Probability [waiting time $\geq$ 10] $= \int_{10}^{\infty} \frac{\lambda}{\mu} . (\mu - \lambda) . e^{-(\mu-\lambda)t} . dt$

$$= \frac{\lambda}{\mu} . (\mu - \lambda) . \left[\frac{e^{-(\mu-\lambda)t}}{-(\mu - \lambda)} \right]_{10}^{\infty}$$

$$= \frac{-\lambda}{\mu} [0 - e^{-(\mu-\lambda).10}]$$

$$= \frac{\lambda}{\mu} e^{-(\mu-\lambda).10}$$

$$= \frac{1/9}{1/3} e^{-\left(\frac{1}{3} - \frac{1}{9}\right).10} = \frac{1}{3} . e^{\frac{-20}{9}} = \frac{1}{30}.$$

(e) Probability [time in system ≥ 10] $= \int_{10}^{\infty} (\mu - \lambda) \cdot e^{-(\mu - \lambda)t} \cdot dt$

$$= (\mu - \lambda) \cdot \left[\frac{e^{-(\mu - \lambda)t}}{-(\mu - \lambda)} \right]_{10}^{\infty}$$

$$= -\left[e^{-(\mu - \lambda)t} \right]_{10}^{\infty}$$

$$= -\left[0 - e^{-(\mu - \lambda).10} \right] = e^{-\left(\frac{1}{3} - \frac{1}{9}\right).10}$$

$$= e^{-20/9} = 0.1.$$

(f) The expected fraction of a day that the phone will be in use

$$= \frac{\lambda}{\mu} = 0.33.$$

EXAMPLE 10.9-4.6

In a large maintenance department, fitters draw parts from the parts-stores which is at present staffed by one storeman. The maintenance foreman is concerned about the time spent by fitters getting parts and wants to know if the employment of a stores labourer to assist the storeman would be worthwhile. On investigation it is found that

(a) a simple queue situation exists,

(b) fitters cost Rs. 2.50 per hour,

(c) the storeman costs Rs. 2 per hour and can deal, on the average, with 10 fitters per hour,

(d) a labourer could be employed at Rs. 1.75 per hour and would increase the service capacity of the stores to 12 per hour,

(e) on the average 8 fitters visit the stores each hour.

[*R.E.C. Hamirpur, 1995 ; Kuru. U. B.E. (Mech.) 1992*]

Solution

The problem can be solved by two methods:

Method 1: Here, we calculate the average *number of customers* in the system before and after the labourer is employed and compare the reduction in the resulting queuing cost with the increase in service cost.

Without labourer :

$\therefore$ Number of customers in the system, $L_s = \dfrac{\lambda}{\mu - \lambda} = \dfrac{8}{10 - 8} = 4.$

$\therefore$ Cost/hr = 4 × Rs. 2.50 = Rs. 10.

With labourer

$$\lambda = 8/\text{hr}, \mu = 12/\text{hr}.$$

$\therefore$ Number of customers in the system, $L_s = \dfrac{\lambda}{\mu - \lambda} = \dfrac{8}{12 - 8} = 2.$

Cost/hr = cost of fitters per hour + cost of labourer per hour

= 2 × Rs. 2.50 + Rs. 1.75 = Rs. 6.75.

Since there is net saving of Rs. 3.25, it is recommended to employ the labourer.

Method 2 : Here, we calculate the average *time* spent by the customers (fitters) in the system before and after the employment of the labourer, and again compare the reduction in the resulting queuing cost with the increase in the service cost.

Without labourer

$$\lambda = 8/\text{hr}, \mu = 10/\text{hr}.$$

$\therefore$ Average time spent by a fitter in the system; $W_s = \dfrac{1}{\mu - \lambda} = \dfrac{1}{10-8} = 0.5$ hr.

$\therefore$ Cost/hr = 8 × 0.5 × Rs. 2.50 = Rs. 10.

With labourer

$$\lambda = 8/\text{hr}, \mu = 12/\text{hr}.$$

$\therefore$ Average time spent by a fitter in the system, $W_s = \dfrac{1}{\mu - \lambda} = \dfrac{1}{12-8} = \dfrac{1}{4}$ hr.

$\therefore$ Cost/hr = 8 × 1/4 × Rs. 2.50 + Rs. 1.75 = Rs. 6.75.

$\therefore$ It is recommended to employ the labourer.

ILLUSTRATION 5

A television repairman finds that the time spent on his jobs has a exponential distribution with mean 30 minutes. If he repairs sets in the order in which they came in, and if the arrival of sets follow a Poisson distribution approximately with an average rate of 10 per 8 hour day, what is the repairman's expected idle time each day? How many jobs are ahead of the average set just brought in? **[JNTU (Mech.) 94/F]**

Solution :

Arrival fashion is Poisson distribution (M); service pattern is exponential distribution (μ), number of servers is 1, no limit is imposed, repair discipline is FIFO. Therefore, according to Kendall-Lee, the queue system $(M/M/1) : (\infty/\text{FIFO})$

and mean arrival rate $(\lambda) = \dfrac{10}{8} = \dfrac{5}{4}$ sets per hour.

and mean service rate $(\mu) = \dfrac{1}{30} \times 60 = 2$ sets per hour

$$\frac{\lambda}{\mu} = \frac{5}{4}/2 = \frac{5}{8}$$

$\dfrac{\lambda}{\mu} < 1$, hence steady state can exist.

(i) Expected idle time of repairman each day

number of hours for which the repairer is busy in 8 hour day is

$$8 \times \frac{\lambda}{\mu} = 8 \times \frac{5}{8} = 5 \text{ hours}$$

$\therefore$ Idle time for repairman in an 8 hour day = 8 – 5 = 3 hours

(ii) Expected number of TV sets in the system

$$L_s = \frac{\lambda}{\mu - \lambda} \text{ or } \frac{\rho}{1-\rho} = \frac{\frac{5}{8}}{1 - \frac{5}{8}} = \frac{5}{3} = 2 \text{ sets approx..}$$

Model -2 (M/M/1) : (N/FCFS)

Single Channel - Restricted Queue Model : This model differs with model-1 in respect of limit of queue. Here the capacity of the system is limited to N. The relevant formulae are given below :

1. $$P_n = \left(\frac{1-\rho}{1-\rho^{n+1}}\right)\rho^n \quad \text{for } 0 \le n \le N, \qquad \rho \ne 1 \text{ i.e, } \lambda \ne \mu$$

$$= \frac{1}{N+1} \text{ for } \rho = 1 \text{ or } \lambda = \mu$$

and $$P_0 = \frac{1-\rho}{1-\rho^{N+1}}; \qquad \rho \ne 1 \text{ and } \rho < 1$$

Where $\rho = \dfrac{\lambda}{\mu}$

N = limit of queue system.

n = number of customers arrived.

2. Expected size of system or expected number of customers in the system

$$L_s = \frac{\rho}{1-\rho} - \frac{(N+1)\,\rho^{N+1}}{1-\rho^{N+1}} \text{ for } \rho \ne 1$$

$$= \frac{N}{2} \text{ for } \rho = 1$$

3. Expected length (size) of queue or expected number of customers waiting in queue.

$$L_q = L_s - \frac{\lambda}{\mu}$$

4. Expected waiting time of customer in the system $\left(W_q + \dfrac{1}{\mu}\right)$

$$W_s = \frac{L_s}{\lambda\,(1-\rho_N)}$$

5. Expected waiting time in queue

$$W_q = \frac{L_q}{\lambda\,(1-\rho_N)}$$

ILLUSTRATION 8

Dr. Raju's out-patient clinic can accommodate six people only in the waiting hall. The patients who arrive when hall is full, balk away. The patients arrive in Poisson fashion at an average rate of 3 per hour and spend an average of 15 minutes in Doctor's chamber which is exponentially distributed. Find

(i) The probability that a patient can get directly into the doctor's chamber upon his arrival.

(ii) Expected number of patients waiting for treatment.

(iii) Effective arrival rate.

(iv) The time a patient can expect to spend in the clinic.

Solution :

From the above problem,

Arrival : Poisson (M); service : Exponential (M);

Number of servers : One; (1) discipline : FCFS;

But the queue is limited

$\therefore$ The model is (M/M/1) : (N/FCFS)

Now capacity of the system

$$N = \text{hall capacity} + \text{doctor's chamber capacity}$$
$$= 6 + 1 = 7$$

Arrival rate $(\lambda) = 3$ per hour.

Service rate $(\mu) = \frac{1}{15}$ per min $= 4$ per hours.

$$\rho = \frac{\lambda}{\mu} = \frac{3}{4} = 0.75 \text{ i.e., } < 1 \text{ (stead state exists)}$$

(i) Patient will directly enter doctor's chamber if the hall is empty on his arrival. The probability at this situation is P_0

$$\therefore \quad P_0 = \frac{1-\rho}{1-\rho^{N+1}} = \frac{1-\left(\frac{3}{4}\right)}{1-\left(\frac{3}{4}\right)^{7+1}} = 0.2778$$

(ii) Expected number of patients waiting for treatment is (L_q)

$$\therefore \quad L_s = \left[\frac{\left(\frac{3}{4}\right)}{1-\left(\frac{3}{4}\right)} - \frac{8\left(\frac{3}{4}\right)^8}{1-\left(\frac{3}{4}\right)^8}\right] = 2.1$$

$$L_q = L_s - \frac{\lambda}{\mu} = 2.1 - \left(\frac{3}{4}\right) = 1.36$$

(iii) Effective arrival rate $(\lambda_{eff}) = \mu\,(1 - P_0)$

$$= 4\,(1 - 0.2778) = 2.89 \cong 3 \text{ per hour.}$$

(iv) The time the patient can expect to spend in the system

$$= \frac{L_q + (1 - P_0)}{\lambda_{eff}} = \frac{1.36 + 1 - 0.2778}{2.89} = 0.72 \text{ hours}$$

$$= 43.2 \text{ minutes.}$$

Model -2 (M/M/1) : (N/FCFS)

$$= P_0 \sum_{n=0}^{N} (\lambda/\mu)^n = P_0 \sum_{n=0}^{N} \rho^n$$

$$= P_0 [1+\rho+\rho^2+\ldots+\rho^N] = P_0 \left[\frac{1-\rho^{N+1}}{1-\rho}\right]$$

and consequently,

$$P_0 = \frac{1-\rho}{1-\rho^{N+1}} \quad ; \ \rho \neq 1 \text{ and } \rho = \frac{\lambda}{\mu} \ (<1)$$

$$P_n = \begin{cases} \left(\dfrac{1-\rho}{1-\rho^{N+1}}\right)\rho^n & ; \ n \le N; \ \dfrac{\lambda}{\mu} \neq 1 \\ \dfrac{1}{N+1} & ; \ \dfrac{\lambda}{\mu} = 1 \end{cases}$$

The steady-state solution in this case exists even for $\rho > 1$. This is due to the limited capacity of the system. If $\lambda < \mu$ and $N \to \infty$, then $P_n = (1 - \lambda/\mu)(\lambda/\mu)^n$, which is the same as in Model I.

1. Expected number of customers in the system:

$$L_s = \sum_{n=1}^{N} nP_n = \sum_{n=1}^{N} n\left(\frac{1-\rho}{1-\rho^{N+1}}\right)\rho^n$$

$$= \frac{1-\rho}{1-\rho^{N+1}} \sum_{n=0}^{N} n\rho^n = \frac{1-\rho}{1-\rho^{N+1}} (\rho+2\rho^2+3\rho^3+\ldots+N\rho^N)$$

i.e.,

$$L_s = \begin{cases} \dfrac{\rho}{1-\rho} - \dfrac{(N+1)\rho^{N+1}}{1-\rho^{N+1}} & ; \ \rho \neq 1 \ (\lambda \neq \mu) \\ \dfrac{N}{2} & ; \ \rho = 1 \ (\lambda = \mu) \end{cases}$$

2. Expected number of customers waiting in the queue:

$$L_q = L_s - \frac{\lambda}{\mu} = L_s - \frac{\lambda(1-P_N)}{\mu}$$

3. Expected waiting time of a customer in the system (waiting + service):

$$W_s = \frac{L_q}{\lambda(1-P_N)} + \frac{1}{\mu} = \frac{L_s}{\lambda(1-P_N)}$$

4. Expected waiting time of a customer in the queue:

$$W_q = W_s - \frac{1}{\mu} \text{ or } \frac{L_q}{\lambda(1-P_N)}$$

5. Potential customers lost (= time for which system is busy):

$$P_N = P_0 \rho^N$$

Effective arrival rate, $\lambda_{eff} = \lambda(1 - P_N)$

Effective traffic intensity, $\rho_{eff} = \lambda_e/\mu$.

Example 16.6 Consider a single server queuing system with Poisson input and exponential service times. Suppose the mean arrival rate is 3 calling units per hour, the expected service time is 0.25 hour and the maximum permissible calling units in the system is two. Derive the steady-state probability distribution of the number of calling units in the system, and then calculate the expected number in the system.

Solution From the data of the problem, we have

$$\lambda = 3 \text{ units per hour}; \ \mu = 4 \text{ units per hour, and } N = 2$$

Then traffic intensity, $\rho = \lambda/\mu = 3/4 = 0.75$

The steady-state probability distribution of the number of n customers (calling units) in the system is:

$$P_n = \frac{(1-\rho)\rho^n}{1-\rho^{N+1}} = \frac{(1-0.75)(0.75)^n}{1-(0.75)^{2+1}} = (0.43)(0.75)^n; \quad \rho \neq 1$$

and

$$P_0 = \frac{(1-\rho)}{1-\rho^{N+1}} = \frac{1-0.75}{1-(0.75)^{2+1}} = \frac{0.25}{1-(0.75)^3} = 0.431$$

The expected number of calling units in the system is given by:

$$L_s = \sum_{n=1}^{N} nP_n = \sum_{n=1}^{2} n(0.43)(0.75)^n$$

$$= 0.43 \sum_{n=1}^{2} n(0.75)^n = 0.43\{(0.75)+2(0.75)^2\} = 0.81.$$

Model -3 (M/M/S) : (INFINITE/FCFS)

Multi Channel - Unrestricted Queue Model : This model can be supposed as the extension to model-1. Here instead of single channel, multi channel (multiple servers) in parallel say 's' lines are assumed. With other notations as usual we use the following formulae for this model.

1. $$P_n = \frac{\rho^n}{\lfloor n} \cdot P_o \quad \text{for} \quad 1 \le n < s$$

$$= \frac{\rho^n}{\lfloor s \cdot s^{n-s}} \cdot P_o \quad \text{for} \quad n \ge s$$

Where $\rho = \frac{\lambda}{\mu}$

$$P_o = \left[\sum_{n=0}^{s-1} \frac{(s\,\rho)^n}{\lfloor n} + \frac{1}{\lfloor s} \frac{(s\,\rho)^2}{1-\rho}\right]^{-1} \quad \text{or} \quad \left[\sum_{n=0}^{s-1} \frac{1}{\lfloor n}\left(\frac{\lambda}{\mu}\right)^n + \frac{1}{\lfloor s}\left(\frac{\lambda}{\mu}\right)^s \cdot \frac{s\,\mu}{s\,\mu - \lambda}\right]^{-1}$$

2. Expected length of queue

$$L_q = \left[\frac{1}{\lfloor (s-1)} \cdot \left(\frac{\lambda}{\mu}\right)^s \cdot \frac{\lambda\,\mu}{(s\,\mu - \lambda)^2}\right] P_o$$

3. Expected number of customers in the system

$$L_s = L_q + \frac{\lambda}{\mu}$$

4. Expected waiting time of the customer in queue

$$W_q = \frac{L_q}{\lambda} = \left[\frac{1}{\lfloor (s-1)} \cdot \left(\frac{\lambda}{\mu}\right)^s \cdot \frac{\mu}{s\,(\mu - \lambda)^2}\right] P_o$$

5. Expected waiting time in the system

$$W_s = W_q + \frac{1}{\mu} \quad \text{or} \quad \frac{L_q}{\lambda} + \frac{1}{\mu}$$

6. The probability that a customer has to wait (busy period)

$$P(n \geq s) = \frac{1}{\lfloor s}\left(\frac{\lambda}{\mu}\right)^s \cdot \frac{s\mu}{s\mu - \lambda} \cdot P_o$$

ILLUSTRATION 9

A super market has two girls ringing up sales at the counters. If the service time for each customer is exponential with mean 4 minutes, and if people arrive in a Poisson fashion at the rate of 10/hour.

(a) What is the probability of having to wait for the service.

(b) What is the expected percentage of idle time for each girl ?

(c) Find the average length and the average number of units in the system ?

[JNTU CSE 97/S JNTU - Mech/Prod/Chem/Mechatronics 2001/S]

Solution :

No. of servers (s) = 2, Arival rate (λ) = $\frac{1}{6}$ and service rate (μ) = $\frac{1}{4}$ per minut

$$\frac{\lambda}{s\mu} = \frac{1}{6} \cdot \frac{1}{2 \cdot \frac{1}{4}} = \frac{1}{3}$$

Therefore,

$$P_0 = \left[\sum_{n=0}^{2-1} \frac{2}{n!}\left(\frac{4}{6}\right)^n + \frac{1}{2!}\left(\frac{4}{6}\right)^2 \frac{2 \cdot \frac{1}{4}}{2 \cdot \frac{1}{4} - \frac{1}{6}}\right]^{-1} = \left[1 + \frac{2}{3} + \frac{1}{3}\right]^{-1} = \frac{1}{2}$$

$$P_1 = \frac{\lambda}{\mu} P_0 = \frac{2}{3} \cdot \frac{1}{2} = \frac{1}{3}$$

(a) $P(n \geq 2) = P(W > 0)$

$$= \sum_{n=2}^{\infty} P_n = \sum_{n=2}^{\infty} \frac{1}{2!\,2^{n-2}}\left(\frac{2}{3}\right)^n P_o = \sum_{n=2}^{\infty}\left(\frac{1}{3}\right)^n = \sum_{n=0}^{\infty}\left(\frac{1}{3}\right)^n - \frac{1}{3} - 1$$

$$= \frac{1}{6} \text{ (or) } 0.167$$

(b) The expected idle time for each girl is $1 - \frac{\lambda}{s\mu}$ i.e., $1 - \frac{1}{3}$ (or) $\frac{2}{3}$ (= 0.67).

Hence, the expected percentage of idle time for each girl is 67%.

(c) Expected length of customers waiting time $= \frac{1}{s\mu - \lambda}$

$$= \frac{1}{0.50 - 0.17} \text{ or 3 minutes.}$$

ILLUSTRATION 10

A bank has two tellers working on savings accounts. The first teller handles withdrawals only. The second teller handles deposits only. It has been found that the service time distribution for deposits and withdrawals both are exponential with mean service time 3 minutees per customer. Depositors are found to arrive in a Poisson fashion through out the day with mean arrival rate 16 per hour. Withdrawals also arrive in a Poisson fashion with mean arrival rate 14 per hour. What would be the effect on the average waiting time for depositors and withdrawers, if each teller could handle both withdrawals and deposits. What would be the effect of this could only be accomplished by increasing the service time to 3.5 minutes?

Solution :

Initially, we can assume two independent queueing systems as follows.

Withdrawers System

Arrivals : Poisson (M)

Service : Eponential (M)

Number of servers : One (1)

Limit of queue : Unlimited (∞)

Discipline : FCFS

Model : (M/M/1) : (∞ / FCFS)

Mean arrival rate : $(\lambda_1) = 14$/hours

Mean service rate : $\mu_1 = \frac{1}{3}$ per min 20/hours

$\left(\rho_1 = \frac{\lambda_1}{\mu_1} = 0.7 < 1, \because \text{ service time } \frac{1}{\mu_1} = 3 \text{ min} \right)$

Average waiting time

$$W_q = \frac{\lambda_1}{\mu_1 (\mu_1 - \lambda_1)} = \frac{14}{20\,(20 - 14)}$$

$$= \frac{7}{60} = 7 \text{ min}$$

Depositors System

Poisson (M)

Exponential (M)

One (1)

Unlimited (∞)

FCFS

(M/M/1) : (∞ / FCFS)

$\lambda_2 = 16$/hours

$\mu_2 = \frac{1}{3}$per min 20/hour

(Mean service time $\frac{1}{\mu_2} = 3$ min)

$$\rho_2 = \frac{\lambda_2}{\mu_2} = \frac{16}{20} = 0.8 < 1$$

Average waiting time

$$W_q = \frac{\lambda_2}{\mu_2 (\mu_2 - \lambda_2)} = \frac{16}{20\,(20 - 16)}$$

$$= \frac{1}{5} \text{ hrs} = 12 \text{ min}$$

Now Consider Combined System :

Arrivals : Poisson (M)

Service : Exponential (M)

No. of servers : two (2)

Limit of queue : Unlimited (∞)

Discipline : FCFS

$\therefore$ Model : (M/M/2) : (∞ / FCFS)

Mean arrival rate : $\lambda = 14 + 16 = 30$/hours

Mean service rate : $\mu = 20$/hours

Number of servers (s) = 2 and $\rho = \frac{\lambda}{s\,\mu} = \frac{30}{2 \times 20} = \frac{3}{4}$

($\therefore \rho < 1$)

Also $\frac{\lambda}{\mu} = \frac{30}{20} = \frac{3}{2}$

Now
$$P_o = \left[\sum_{n=0}^{s-1} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n = \frac{1}{s!}\left(\frac{\lambda}{\mu}\right)^s \left(\frac{s\,\mu}{s\,\mu - \lambda}\right)\right]^{-1}$$

$$= \left[\sum_{n=0}^{1} \frac{1}{n!}\left(\frac{3}{2}\right)^n + \frac{1}{2!}\left(\frac{3}{2}\right)^2 \left(\frac{40}{40-30}\right)\right]^{-1}$$

$$= \left[\frac{1}{0}\left(\frac{3}{2}\right)^o + \frac{1}{1}\left(\frac{3}{2}\right)^1 + \frac{1}{2\times 1} \times \frac{9}{4} \times \frac{40}{10}\right]^{-1}$$

$$= \left[1 + \frac{3}{2} + \frac{9}{2}\right]^{-1} = \frac{1}{7}$$

Average waiting time of arrivals in the queue

$$W_q = \frac{L_q}{\lambda} = \frac{1}{(s-1)!} \cdot \left(\frac{\lambda}{\mu}\right)^s \cdot \frac{\mu}{(s\,\mu - \lambda)^2} \cdot P_o$$

$$= \frac{1}{(2-1)!} \cdot \left(\frac{3}{2}\right)^2 \cdot \frac{20}{(40-30)^2} \cdot \frac{1}{7}$$

$$= \frac{9}{4} \times \frac{20}{100} \cdot \frac{1}{7} \qquad = \frac{9}{140} \text{ hours.}$$

$= 3.86$ min

Combined waiting time with increased service time when

$$\lambda' = 30 \text{ /hour and } \frac{1}{\mu'} = 3.5 \text{ min}$$

or $\mu = \frac{60}{3.5} = \frac{120}{7}$ per hour

$$\frac{\lambda'}{s\,\mu'} = \frac{30}{2\,(120/7)} = \frac{7}{8} \text{ i.e., } < 1 \quad \text{and} \quad \frac{\lambda'}{\mu'} = \frac{30}{120/7} = \frac{7}{4}$$

$$P_o = \left[\sum_{n=0}^{1} \frac{1}{n!} \cdot \left(\frac{7}{4}\right)^n + \frac{1}{2!}\left(\frac{7}{4}\right)^2 \cdot \frac{2 \times (120/7)}{\{2\,(120/7) - 30\}}\right]^{-1}$$

$$= \left[\frac{1}{0!}\left(\frac{7}{9}\right)^o + \frac{1}{1!}\left(\frac{7}{4}\right)^1 + \frac{1}{2\times 1} \times \frac{49}{16} \times \frac{2\,(120/7)}{(30/7)}\right]^{-1}$$

$$= \left[1 + \frac{7}{4} + \frac{49}{4}\right]^{-1} = [15]^{-1} \qquad = \frac{1}{15}$$

Average waiting time of arrivals in the queue

$$W_q = \frac{1}{(s-1)!} \cdot \left(\frac{\lambda'}{\mu'}\right) \cdot \frac{\mu'}{(s\,\mu' - \lambda')^2} \cdot P_0$$

$$= \frac{1}{(2-1)!} \cdot \left(\frac{7}{4}\right)^2 \frac{120/7}{[2\,(120/7) - 30]^2} \times \frac{1}{15}$$

$$= \frac{49}{16} \times \frac{120/7}{(30/7)^2} \times \frac{1}{15} = \frac{49}{16} \times \frac{120}{7} \times \left(\frac{7}{30}\right)^2 \times \frac{1}{15}$$

$$= \frac{343}{30 \times 60} \text{ hours}$$

$$= \frac{343}{30} \text{ min} \quad = 11.433 \text{ min.}$$

Model -3 (M/M/S) : (INFINITE/FCFS)

Thus the probability that the system shall be idle is:

$$P_0 = \left[\sum_{n=0}^{s-1} \frac{(s\rho)^n}{n!} + \frac{1}{s!}\frac{(s\rho)^s}{1-\rho}\right]^{-1} ; \rho = \lambda/s\mu$$

$$= \left[\sum_{n=0}^{s-1} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n + \frac{1}{s!}\left(\frac{\lambda}{\mu}\right)^s \frac{s\mu}{s\mu - \lambda}\right]^{-1}$$

1. The expected number of customers waiting in the queue (length of line):

$$L_q = \sum_{n=s}^{\infty} (n-s)\,P_n = \sum_{n=s}^{\infty} (n-s)\,\frac{\rho^n}{s^{n-s} \cdot s!} P_0$$

$$= \frac{\rho^s P_0}{s!} \sum_{n=s}^{\infty} (n-s)\rho^{n-s} = \frac{\rho^s P_0}{s!} \sum_{m=0}^{\infty} m\rho^m \; ; \quad n-s=m, \quad \rho = \frac{\lambda}{\mu}$$

$$= \frac{\rho^s}{s!} \cdot \rho P_0 \sum_{m=0}^{\infty} m\rho^{m-1} = \frac{\rho^s}{s!} \cdot \rho P_0 \frac{d}{d\rho}\left[\sum_{m=1}^{\infty} \rho^m\right]$$

$$= \frac{\rho^s}{s!} \rho P_0 \frac{1}{(1-\rho)^2} = \left[\frac{1}{s!}\left(\frac{\lambda}{\mu}\right)^s \frac{\lambda \cdot s\mu}{(s\mu-\lambda)^2}\right] P_0$$

$$L_q = \left[\frac{1}{(s-1)!}\left(\frac{\lambda}{\mu}\right)^s \frac{\lambda\mu}{(s\mu-\lambda)^2}\right] P_0$$

2. The expected number of customers in the system:

$$L_s = L_q + \frac{\lambda}{\mu}$$

3. The expected waiting time of a customer in the queue:

$$W_q = \left[\frac{1}{(s-1)!}\left(\frac{\lambda}{\mu}\right)^s \frac{\mu}{(s\mu-\lambda)^2}\right] P_0 = \frac{L_q}{\lambda}$$

4. The expected waiting time that a customer spends in the system:

$$W_s = W_q + \frac{1}{\mu} = \frac{L_q}{\lambda} + \frac{1}{\mu}$$

5. The probability that all servers are simultaneously busy (utilization factor):

$$P(n \geq s) = \sum_{n=s}^{\infty} P_n = \sum_{n=s}^{\infty} \frac{1}{s!\, s^{n-s}} \left(\frac{\lambda}{\mu}\right)^n P_0$$

$$= \frac{1}{s!}\left(\frac{\lambda}{\mu}\right)^s P_0 \sum_{m=0}^{\infty} \left(\frac{\lambda}{\mu}\right) = \frac{1}{s!}\left(\frac{\lambda}{\mu}\right)^s \frac{s\mu}{s\mu-\lambda} P_0$$

Example 16.7 A super market has two sales girls at the sales counters. If the service time for each customer is exponential with a mean of 4 minutes, and if the people arrive in a Poisson fashion at the rate of 10 an hour, then calculate the:

(a) probability that a customer has to wait for being served? [*Banasthali, MSc (Maths), 2000*]

(b) expected percentage of idle time for each sales girl?

(c) if a customer has to wait, what is the expected length of his waiting time?

[*Meerut, MSc (Maths), 2001; Delhi Univ., MBA, 2005*]

Solution From the data of the problem, we have

$\lambda = 1/6$ per minute; $\mu = 1/4$ per minute, $s = 2$ and $\rho = \lambda/s\mu = 1/3$

Therefore, $$P_0 = \left[\sum_{n=0}^{2-1} \frac{1}{n!}(4/6)^n + \frac{1}{2!}(4/6)^2 \frac{2\cdot(1/4)}{\{2\cdot(1/4)-(1/6)\}}\right]^{-1} = \left(1+\frac{2}{3}+\frac{1}{3}\right)^{-1} = \frac{1}{2}$$

and $$P_1 = (\lambda/\mu)P_0 = (4/6)(1/2) = (1/3)$$

(a) The probability of having to wait for service:

$$P(n \geq 2) = \frac{1}{s!}\left(\frac{\lambda}{\mu}\right)^s \cdot \frac{s\mu}{s\mu-\lambda} . P_0 = \frac{1}{2!}\left(\frac{4}{6}\right)^2 \cdot \frac{2(1/4)}{2(1/4)-(1/6)}\left(\frac{1}{2}\right) = \frac{1}{6}$$

(b) The fraction of time the servers are busy, $\rho = \lambda/s\mu = 1/3$. Therefore, the expected idle time for each sales girl is (1 – 1/3) = 2/3 = 67%.

(c) The expected waiting time for a customer in the system:

$$W_s = W_q + \frac{1}{\mu} = \frac{L_q}{\lambda} + \frac{1}{\mu} = \frac{1}{(s-1)!}\left(\frac{\lambda}{\mu}\right)^s \cdot \frac{\mu}{(s\mu-\lambda)^2} \cdot P_0 + \frac{1}{\mu}$$

$$= \left(\frac{4}{6}\right)^2 \frac{1/4}{[(1/2)-(1/6)]^2} \times \frac{1}{2} + 4 = 4.5 \text{ minutes}$$

Example 16.8 A bank has two tellers working on the savings accounts. The first teller only handles withdrawals. The second teller only handles deposits. It has been found that the service time distribution for the deposits and withdrawals, both, are exponential with mean service time 3 minutes per customer. Depositors are found to arrive in a Poisson fashion throughout the day with a mean arrival rate of 16 per hour. Withdrawers also arrive in a Poisson fashion with a mean arrival rate of 14 per hour. What would be the effect on the average waiting time for depositors and withdrawers if each teller could handle both the withdrawals and deposits. What would be the effect if this could only be accomplished by increasing the service time to 3.5 minutes?

Solution Initially there are two independent queuing systems: Withdrawers and Depositors, where arrivals follow Poisson distribution and the service time follows exponential distribution.

For Withdrawers Given that, λ = 14/hour and μ = 3/minute or 20/hour

$$\text{Average waiting time in queue, } W_q = \frac{\lambda}{\mu(\mu-\lambda)} = \frac{14}{20(20-14)} = \frac{7}{60} \text{ hour or 7 minutes}$$

For Depositors Given that, λ = 16/hour; and μ = 3/minute or 20/hour

$$\text{Average waiting time in queue, } W_q = \frac{\lambda}{\mu(\mu-\lambda)} = \frac{16}{20(20-16)} = \frac{1}{5} \text{ hour or 12 minutes}$$

Combined Case In this case there will be a common queue with two servers (tellers). Thus, we have

$$\lambda = 14 + 16 = 30/\text{hour}, \mu = 20/\text{hour}; s = 2; \rho = \lambda/s\mu = 3/4.$$

Now,
$$P_0 = \left[\sum_{n=0}^{s-1} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n + \frac{1}{s!}\left(\frac{\lambda}{\mu}\right)^s \left(\frac{s\mu}{s\mu-\lambda}\right)\right]^{-1} = \left[\sum_{n=0}^{1} \frac{1}{n!}\left(\frac{3}{2}\right)^n + \frac{1}{2!}\left(\frac{3}{2}\right)^s \left(\frac{40}{40-30}\right)\right]^{-1}$$

$$= \left[1 + \frac{3}{2} + \frac{1}{2}\left(\frac{9}{4}\right) \cdot 4\right]^{-1} = \frac{1}{7}$$

Average waiting time of arrivals in the queue:

$$W_q = \frac{L_q}{\lambda} = \left[\frac{1}{(s-1)!}\left(\frac{\lambda}{\mu}\right)^s \frac{\mu}{(s\mu-\lambda)^2}\right] P_0$$

$$= \left(\frac{3}{2}\right)^2 \frac{20}{(40-30)^2} \times \frac{1}{7} = \frac{9}{140} \text{ hour or 3.86 minutes.}$$

Combined waiting time with increased service time, when λ = 30/hour, μ = 60/3.5 or 120/7 per hour, we have:

$$P_0 = \left[\sum_{n=0}^{1} \frac{1}{n!}\left(\frac{21}{12}\right)^n + \frac{1}{2!}\left(\frac{21}{12}\right)^2 \frac{2.(120/7)}{2(120/7)-30}\right]^{-1} = \left[1 + \frac{7}{4} + \frac{49}{4}\right]^{-1} = \frac{1}{15}$$

Average waiting time of arrivals in the queue:

$$W_q = \frac{1}{(s-1)!}\left(\frac{\lambda}{\mu}\right)^s \frac{\mu}{(s\mu-\lambda)^2} \cdot P_0 = \left(\frac{7}{4}\right)^2 \frac{120/7}{(120/7-30)^2} \times \frac{1}{15}$$

$$= \frac{343}{30} \times 60 \text{ hour or 11.43 minutes}$$

Example 16.9 A tax consulting firm has 4 service counters in its office for receiving people who have problems and complaints about their income, wealth and sales taxes. Arrivals average 80 persons in an 8-hour service day. Each tax adviser spends an irregular amount of time servicing the arrivals, which have been found to have an exponential distribution. The average service time is 20 minutes. Calculate the average number of customers in the system, average number of customers waiting to be serviced, average time a customer spends in the system, and average waiting time for a customer. Calculate how many hours each week does a tax adviser spend performing his job. What is the probability that a customer has to wait before he gets service? What is the expected number of idle tax advisers at any specified time?

[*IAS (Maths), 1996*]

Solution Given that, $\lambda = 10$/hour; $\mu = 3$/hour, $s = 4$; and $\rho = \lambda/s\mu = 5/6$

(a) The probability of no customer in the system:

$$P_0 = \left[\sum_{n=0}^{s-1} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n + \frac{1}{s!}\left(\frac{\lambda}{\mu}\right)^s \left(\frac{s\mu}{s\mu - \lambda}\right)\right]^{-1}$$

$$= \left[\sum_{n=0}^{3} \frac{1}{n!}\left(\frac{10}{3}\right)^n + \frac{1}{4!}\left(\frac{10}{3}\right)^4 \frac{12}{12-10}\right]^{-1}$$

$$= \left[1 + \frac{10}{3} + \frac{1}{2}\left(\frac{10}{3}\right)^2 + \frac{1}{6}\left(\frac{10}{3}\right)^3 + \frac{1}{24}\left(\frac{10}{3}\right)^4 \cdot 6\right]^{-1} = 0.021$$

(b) The average number of customers in the system:

$$L_s = L_q + \frac{\lambda}{\mu} = \left[\frac{1}{(s-1)!}\left(\frac{\lambda}{\mu}\right)^s \frac{\lambda\mu}{(s\mu - \lambda)^2}\right] P_0 + \frac{\lambda}{\mu}$$

$$= \left[\frac{1}{3!}\left(\frac{10}{3}\right)^4 \frac{30}{(12-10)^2}\right] \times 0.021 + \frac{10}{3} = 6.57$$

(c) The average number of customers waiting in the queue (queue length):

$$L_q = L_s - \frac{\lambda}{\mu} = 6.57 - (10/3) = 3.24 \text{ customers}$$

(d) The average time a customer spends in the system:

$$W_s = W_q + \frac{1}{\mu} = \frac{L_q}{\lambda} + \frac{1}{\mu} = \frac{3.24}{10} + \frac{1}{3} = 0.657 \text{ hour or } 39.42 \text{ minutes}$$

(e) The average time a customer waits for service in the queue:

$$W_q = \frac{L_q}{\lambda} = \frac{3.24}{10} = 0.324 \text{ hour or } 19.44 \text{ minutes}$$

(f) The time spent by a tax counsellor, i.e. utilization factor:

$$\rho = \frac{\lambda}{s\mu} = \frac{5}{6} = 0.833 \text{ hour or } 50 \text{ minutes}$$

The expected time spent in servicing customers during an 8-hour day is $8 \times 0.833 = 6.66$ hours. Thus, on average, a tax advisor is busy for $6.66 \times (40/8) = 33.30$ hours, based on a 40 hours week.

(g) The probability that a customer has to wait:

$$P_w(n \geq s) = \frac{1}{s!}\left(\frac{\lambda}{\mu}\right)^s \frac{s\mu}{s\mu - \lambda} \cdot P_0 = \frac{1}{4!}\left(\frac{10}{3}\right)^4 \cdot \frac{4\times 3}{4\times 3 - 10}(0.021) = 0.622$$

(h) The expected number of idle advisers at any specified time can be obtained by adding the probability of 3 idle, 2 idle and 1 idle advisers. That is:

$$\begin{aligned}\text{Expected number of idle advisers} &= 4P_0 + 3P_1 + 2P_2 + P_3 \\ &= 4(0.021) + 3(0.070) + 2(0.118) + 0.131 = 0.661\end{aligned}$$

This means, less than one (0.661) adviser is idle on an average at any instance of time.