Fourier Series Analysis of continuous time Signals.

A periodic signal is one which repeats itself periodically over $-\infty < t < \infty$.

For example, a sinusoidal signal $x(t) = A \sin \omega_0 t$ is a periodic signal with period $T = \frac{2\pi}{\omega_0}$

- Sum of two sinusoidal signal is periodic provided that their frequencies are Integer multiples of fundamental frequency ω_0.
- Now let us consider a signal $x(t)$, a sum of sine and cosine functions whose frequencies are integer multiples of ω_0.

$$x(t) = a_0 + a_1 \cos \omega_0 t + a_2 \cos 2\omega_0 t + a_3 \cos 3\omega_0 t + \dots a_K \cos(K\omega_0 t)$$

$$+ b_1 \sin \omega_0 t + b_2 \sin 2\omega_0 t + \dots + b_K \sin(K\omega_0 t)$$

$$x(t) = a_0 + \sum_{n=1}^{K} [a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)] \quad \text{---} (1)$$

$a_0, a_1 \dots a_K$ and $b_0, b_1 \dots b_K$ are constant and ω_0 is fundamental frequency.

If signal $x(t)$ is to be periodic

$$x(t) = x(t+T) \quad \text{---} (2)$$

From (1)

$$x(t+T) = a_0 + \sum_{n=1}^{K} [a_n \cos n\omega_0 (t+T) + b_n \sin n\omega_0 (t+T)]$$

$$= a_0 + \sum_{n=1}^{K} a_n \cos (n\omega_0 t + 2n\pi) + b_n \sin (n\omega_0 t + 2n\pi)]$$

$$(\because T = \frac{2\pi}{\omega_0}).$$

$$= a_0 + \sum_{n=1}^{K} a_n \cos n\omega_0 t + b_n \sin n\omega_0 t$$

$$= x(t).$$

In eqn (1) if $K \to \infty$, we obtain Fourier Series representation of a periodic signals $x(t)$. Thus any periodic signal can be represented as an infinite sum of sine & cosine functions which themselves are periodic signals of angular frequencies $0, \omega_0, 2\omega_0 \dots K\omega_0$.

This series of sine and cosine term is known as trigonometric Fourier Series and can be written as

$$x(t) = a_0 + \sum_{n=1}^{\infty} [a_n \cos n\omega_0 t + b_n \sin n\omega_0 t]. \quad \text{--- (3)}$$

a_n, b_n are constant.

Coefficient a_0 is called dc component

$a_1 \cos\omega_0 t + b_1 \sin\omega_0 t$ the first harmonic

$a_2 \cos 2\omega_0 t + b_2 \sin 2\omega_0 t \rightarrow$ second harmonic.

$a_n \cos n\omega_0 t + b_n \sin n\omega_0 t \rightarrow$ nth harmonic

Evaluation of Fourier Coefficients

$\left.\begin{matrix} a_0, a_1, a_2 \ldots a_n \\ b_0, b_1, b_2 \ldots b_n \end{matrix}\right\}$ Fourier coefficients.

To evaluate a_0 we shall integrate both sides of eqn (3) over one period (t_0, t_0+T) of $x(t)$ an arbitrary time t_0.

$$\int_{t_0}^{t_0+T} x(t)\,dt = a_0 \int_{t_0}^{t_0+T} dt + \int_{t_0}^{t_0+T} \sum_{n=1}^{\infty} [a_n \cos n\omega_0 t + b_n \sin n\omega_0 t]\,dt.$$

$$= a_0 T + \sum_{n=1}^{\infty} a_n \int_{t_0}^{t_0+T} \cos(n\omega_0 t)\,dt + \sum_{n=1}^{\infty} b_n \int_{t_0}^{t_0+T} \sin(n\omega_0 t)\,dt$$

$$\int_{t_0}^{t_0+T} x(t)\,dt = a_0 T$$

$$\boxed{a_0 = \frac{1}{T} \int_{t_0}^{t_0+T} x(t)\,dt.}$$

To evaluate a_n & b_n we can use the following results.

$$\int_{t_0}^{t_0+T} \cos(n\omega_0 t)\cos(m\omega_0 t)\,dt = \begin{cases} 0, & m \neq n \\ T/2, & m = n \neq 0. \end{cases} \quad \text{--- (4)}$$

$$\int_{t_0}^{t_0+T} \sin(n\omega_0 t)\cos(m\omega_0 t)\,dt = 0 \quad \text{for all } m, n. \quad \text{--- (5)}$$

$$\int_{t_0}^{t_0+T} \sin(n\omega_0 t)\sin(m\omega_0 t)\,dt = \begin{cases} 0, & m \neq n \\ T/2, & m = n \neq 0 \end{cases} \quad \text{--- (6).}$$

∴ To find fourier coefficients an eqn (3) by $\cos m\omega_0 t$ and integrate over one period

i.e.

$$\int_{t_0}^{t_0+T} x(t)\cos(m\omega_0 t)\,dt = a_0\int_{t_0}^{t_0+T}\cos(m\omega_0 t)\,dt + \sum_{n=1}^{\infty} a_n\int_{t_0}^{t_0+T}\cos(n\omega_0 t)\cdot\cos(m\omega_0 t)\,dt + \sum_{n=1}^{\infty} b_n\int_{t_0}^{t_0+T}\sin(n\omega_0 t)\cdot\cos(m\omega_0 t)\,dt \quad (7)$$

The first integral on the RHS of (7) is zero because we are integrating over an integer multiples of periods.

sub (4) (5) in (7)

$$\int_{t_0}^{t_0+T} x(t)\cos(m\omega_0 t)\,dt = 0 + a_m\cdot T/2 + 0.$$

$$a_m = \frac{2}{T}\int_{t_0}^{t_0+T} x(t)\cos(m\omega_0 t)\,dt.$$

$$\boxed{a_n = \frac{2}{T}\int_{t_0}^{t_0+T} x(t)\cos(n\omega_0 t)\,dt}$$

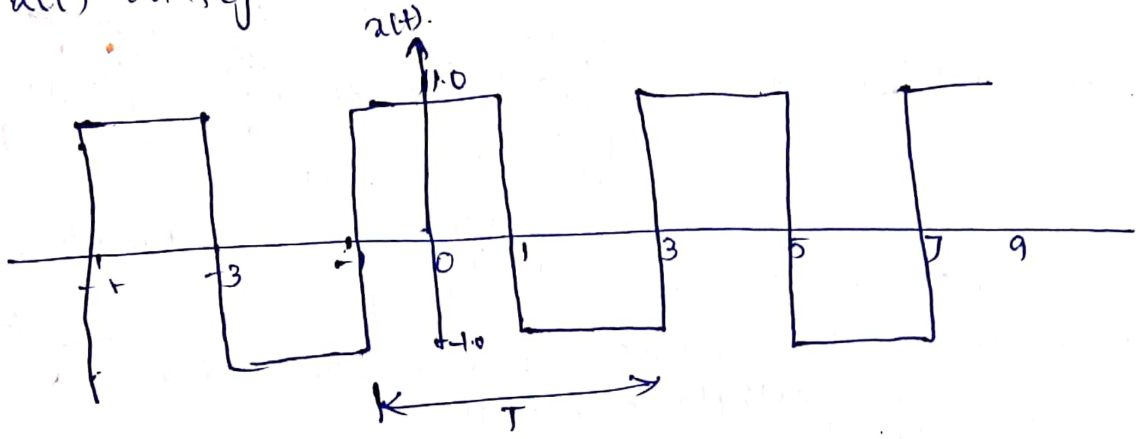
To find b_n multiply eqn (3) by $\sin(m\omega_0 t)$

$$\int_{t_0}^{t_0+T} x(t)\sin(m\omega_0 t)\,dt = \int_{t_0}^{t_0+T} a_0\sin(m\omega_0 t)\,dt + \sum_{n=1}^{\infty} a_n\int_{t_0}^{t_0+T}\cos(n\omega_0 t)\sin(m\omega_0 t)\,dt + \sum_{n=1}^{\infty} b_n\int_{t_0}^{t_0+T}\sin(n\omega_0 t)\sin(m\omega_0 t)\,dt$$

$$= 0 + 0 + b_n\cdot\frac{T}{2}.$$

$$\boxed{b_n = \frac{2}{T}\int_{t_0}^{t_0+T} x(t)\sin(m\omega_0 t)\,dt}$$

① Find the trigonometric Fourier series for periodic signal $x(t)$ in fig.



$T = 4$.

Choose one period of signal from $t=-1$ to $t=3$

$$\text{fundamental frequency} = \omega_0 = \frac{2\pi}{T} = \frac{2\pi}{4} = \pi/2$$

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + b_n \sin n\omega_0 t.$$

$$= a_0 + \sum_{n=1}^{\infty} \left[a_n \cos \frac{n\pi}{2}t + b_n \sin \frac{n\pi}{2}t\right]$$

Given $x(t) = 1$ for $-1 \leq t \leq 1$

-1 for $1 \leq t \leq 3$.

$$a_0 = \frac{1}{T}\int_{t_0}^{t_0+T} x(t)\,dt$$

$$= \frac{1}{4}\int_{-1}^{3} x(t)\,dt = \frac{1}{4}\left[\int_{-1}^{1} dt + \int_{1}^{3} -dt\right]$$

$$\boxed{a_0 = 0}$$

$$a_n = \frac{2}{T}\int_{t_0}^{t_0+T} x(t)\cos(n\omega_0 t)\,dt$$

$$= \frac{2}{4}\int_{-1}^{3} x(t)\cos\frac{n\pi}{2}t\,dt = \frac{1}{2}\left[\int_{-1}^{1}\cos\left(\frac{n\pi}{2}t\right)dt + \int_{1}^{3}(-1)\cos\frac{n\pi}{2}t\,dt\right.$$

$$= \frac{1}{2}\left[\frac{2}{n\pi}\sin\frac{n\pi}{2}t\right]_{-1}^{1} + \left(-\frac{2}{n\pi}\right)\left[\sin\frac{n\pi}{2}t\right]_{1}^{3}$$

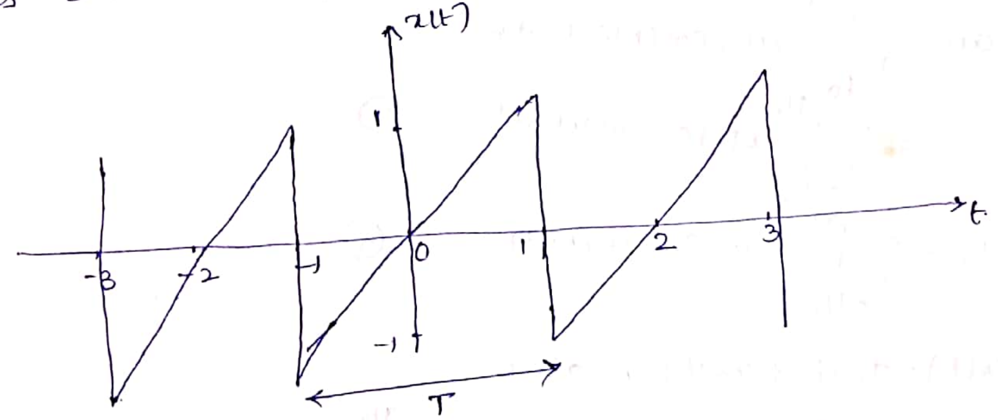
$$a_n = \frac{4}{n\pi}\sin\frac{n\pi}{2} = \begin{cases} 0 & n \text{ even} \\ \frac{4}{n\pi} & n = \cancel{\text{odd}}\ 1,5,9,13\cdots \\ -\frac{4}{n\pi} & n = 3,7,11,15\cdots \end{cases}$$

$$\therefore\ x(t) = 0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{2}t\right)$$

$$= \frac{4}{\pi}\cos\frac{\pi}{2}t + 0 - \frac{4}{3\pi}\cos\frac{3\pi}{2}t + \frac{4}{5\pi}\cos\frac{5\pi}{2}t - \frac{4}{7\pi}\cos\frac{7\pi}{2}t.$$

$$= \frac{4}{\pi}\left[\cos\frac{\pi}{2}t - \frac{1}{3}\cos\frac{3\pi}{2}t + \frac{1}{5}\cos\frac{5\pi}{2}t - \frac{1}{7}\cos\frac{7\pi}{2}t + \cdots\right]$$

② Find the trigonometric Fourier Series for the periodic signals $x(t)$ as shown in fig.



$T = 2$

$\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{2} = \pi$ $\qquad x(t) = t\ ,\ -1 \le t \le t.$

$t = -1$ to $+1$

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t) \quad —①$$

$$= a_0 + \sum_{n=1}^{\infty} a_n \cos(n\pi t) + b_n \sin(n\pi t)$$

$$a_0 = \frac{1}{T}\int_{t_0}^{t_0+T} x(t)\,dt = \frac{1}{2}\int_{-1}^{1} t\,dt = \frac{1}{2}\left[\frac{t^2}{2}\right]_{-1}^{1} = \frac{1}{2}\left[\frac{1}{2} - \frac{1}{2}\right] = 0.$$

$\boxed{a_0 = 0}$

$\int u\,dv = uv - \int v\,du.$

$dv = \cos n\pi t$

$v = \frac{\sin n\pi t}{n\pi}$

$$a_n = \frac{2}{T}\int_{t_0}^{t_0+T} x(t)\cos n\omega_0 t\,dt = \frac{\cancel{2}}{\cancel{2}}\int_{-1}^{1} t.\cos n\pi t\,.dt.$$

$$= t.\frac{\sin n\pi t}{n\pi} - \int \frac{\sin n\pi t}{n\pi}.1\,dt.$$

$$= \left[t\frac{\sin n\pi t}{n\pi}\right]_{-1}^{1} + \left[\frac{\cos n\pi t}{n^2\pi^2}\right]_{-1}^{1}$$

$$= 0 + 0 = 0.$$

$\boxed{a_n = 0}$

$$b_n = \frac{2}{T}\int_{t_0}^{t_0+T} x(t)\sin(n\omega_0 t)dt = \frac{2}{2}\int_{-1}^{1} t.\sin(n\pi t)dt = -t\frac{\cos n\pi t}{n\pi} - \int -\frac{\cos n\pi t}{n\pi}.1\,dt.$$

$$= \left[-t\frac{\cos n\pi t}{n\pi} + \frac{\sin n\pi t}{n^2\pi^2}\right]_{-1}^{1}$$

$$= -\frac{2}{n\pi}\cos n\pi = -\frac{1}{n\pi}\left[\cos n\pi + \cos n\pi + (0-0)\right]$$

$$= \frac{2}{\pi}\left[\frac{-(-1)^n}{n}\right]$$

Sub a_0, a_n, b_n in (1)

$$x(t) = 0 + 0 + \sum_{n=1}^{\infty} \frac{2}{\pi}\left[-\frac{(-1)^n}{n}\right] \sin n\pi t$$

$$= \frac{2}{\pi}\left[\sin \pi t - \frac{1}{2}\sin 2\pi t + \frac{1}{3}\sin 3\pi t - \frac{1}{4}\sin 4\pi t + \cdots\right]$$

Symmetry condition

$$x(t) = x_e(t) + x_o(t). \quad \text{---(1)}$$

$$x_e(t) = \frac{1}{2}[x(t) + x(-t)] \quad \text{--- (2)}$$

$$x_o(t) = \frac{1}{2}[x(t) - x(-t)] \quad \text{--- (3)}$$

For convinient chook interval $-T/2$ to $T/2$

$$a_n = \frac{2}{T}\int_{t_0}^{t_0+T} x(t)\cos(n\omega_0 t)\,dt$$

$$= \frac{2}{T}\int_{-T/2}^{T/2} x(t)\cos(n\omega_0 t)\,dt \quad \text{--- (4)}$$

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\sin(n\omega_0 t)\,dt \quad \text{--- (5)}$$

Sub. $x(t) = x_e(t) + x_o(t)$ in above.

$$a_n = \frac{2}{T}\left[\int_{-T/2}^{T/2} x_e(t)\cos(n\omega_0 t)\,dt + \int_{-T/2}^{T/2} x_o(t)\cos(n\omega_0 t)\,dt\right] \quad \text{--- (6)}$$

Ily $$b_n = \frac{2}{T}\left[\int_{-T/2}^{T/2} x_e(t)\sin(n\omega_0 t)\,dt + \int_{-T/2}^{T/2} x_o(t)\sin(n\omega_0 t)\,dt\right] \quad \text{--- (7)}$$

We know that

odd function	× odd function	= even function
odd function	× even "	= even "
even "	× even "	= odd "
even "	× odd "	

For any even function $x_e(t)$

$$\int_{-t_0}^{t_0} x_e(t)\,dt = 2\int_{0}^{t_0} x_e(t)\,dt \quad \text{--- (8)}$$

For any odd function $x_o(t)$

$$\int_{-t_0}^{t_0} x_o(t)\,dt = 0 \quad \text{--- (9)}$$

If $x(t)$ is an even function, then $x_o(t) = 0$. Sub in (7)

$$b_n = \frac{2}{T}\left[\int_{-T/2}^{T/2} \underbrace{x_e(t)}_{\text{even}} \underbrace{\sin(n\omega_0 t)}_{\text{odd}}\,dt\right]$$

even × odd = odd

So, resulting integral is equal to zero

$\boxed{b_n = 0}$

From (6)

$$a_n = \frac{2}{T}\left[\int_{-T/2}^{T/2} x_e(t)\cos(n\omega_0 t)\,dt\right]$$

$$\boxed{a_n = \frac{4}{T}\int_0^{T/2} x(t)\cos(n\omega_0 t)\,dt}$$

and $\boxed{a_0 = \frac{2}{T}\int_0^{T/2} x(t)\,dt.}$

* The Fourier Series expansion of an even periodic function contains only cosine terms & a constant.

If $x(t)$ is an odd function, then $x_e(t) = 0$. Sub. in (6)

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} \underbrace{x_o(t)}_{\text{odd.}}\ \underbrace{\cos(n\omega_0 t)}_{\text{even.}}\,dt$$

$\boxed{a_n = 0}$ $\left(\because \text{odd (even)} = \text{odd};\ \int \text{odd function} = 0\right).$

$$a_0 = \frac{2}{T}\int_0^{T/2} x_o(t)\,dt$$

$\boxed{a_0 = 0}$

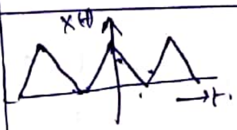
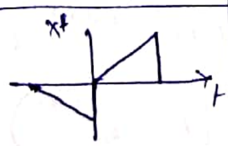
Sub. above conditions in (7)

$$b_n = \frac{2}{T}\left[\int_{-T/2}^{T/2} \underbrace{x_o(t)}_{\text{odd}}\ \underbrace{\sin(n\omega_0 t)}_{\text{odd.}}\,dt\right]$$

$\int_{-T/2}^{T/2}$ even function

$$\boxed{b_n = \frac{4}{T}\int_0^{T/2} x(t)\sin(n\omega_0 t)\,dt.}$$

* Fourier series expansion of an odd periodic functions contains only sine terms.

Type of symmetry	Condition	Example	a_0	a_n	b_n	Property
Even	$x(t) = x(-t)$		$a_0 = \frac{2}{T}\int_0^{T/2} x(t)\,dt$. $\omega_0 = \frac{2\pi}{T}$	$a_n = \frac{4}{T}\int_0^{T/2} x(t)\cos(n\omega_0 t)\,dt$	0	Cosine terms only
odd	$x(t) = -x(-t)$		0	0	$b_n = \frac{4}{T}\int_0^{T/2} x(t)\sin(n\omega_0 t)\,dt$	Sine terms only
Half wave	$x(t) = -x(t \pm \frac{T}{2})$		0	$\frac{4}{T}\int_0^{T/2} x(t)\cos(n\omega_0 t)\,dt$	$\frac{4}{T}\int_0^{T/2} x(t)\sin(n\omega_0 t)\,dt.$	odd n only

Half wave Symmetry

A periodic Signal which satisfy the condition

$$\boxed{x(t) = -x(t \pm T/2)}$$

is said to have a half wave symmetry. The fourier Expansion of such a type of periodic signal contains odd harmonics only.

Cosine representation

The trigonometric Fourier series contains sine & cosine terms of the same frequency. By using trigonometric identity, we can write

$$a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t) = A_n[\cos(n\omega_0 t + \theta_n)]$$

we obtain cosine representation of $x(t)$ which contains sinusoids of frequencies $\omega_0, 2\omega_0, \ldots$

i.e. $$\boxed{x(t) = A_0 + \sum_{n=1}^{\infty} A_n \cos(n\omega_0 t + \theta_n)}$$

where
$$\left.\begin{aligned} A_0 &= a_0 \\ A_n &= \sqrt{a_n^2 + b_n^2} \\ \theta_n &= -\tan^{-1}\left(\frac{b_n}{a_n}\right). \end{aligned}\right\}$$

$A_n \rightarrow$ amplitude coefficients

$\theta_n \rightarrow$ phase coefficients

Exponential Fourier Series

Exponential F.S. is another form of Fourier Series. By using Euler's identity we can write

$$A_n \cos(n\omega_0 t + \theta_n) = A_n\left[\frac{e^{j(n\omega_0 t + \theta_n)} + e^{-j(n\omega_0 t + \theta_n)}}{2}\right] \quad \text{—(1)}$$

Sub (1) in definition of cosine F.S.

$$\therefore x(t) = A_0 + \sum_{n=1}^{\infty} \frac{A_n}{2}\left[e^{j(n\omega_0 t + \theta_n)} + e^{-j(n\omega_0 t + \theta_n)}\right]$$

$$= A_0 + \sum_{n=1}^{\infty} \frac{A_n}{2}\left[e^{jn\omega_0 t} \cdot e^{j\theta_n} + e^{-jn\omega_0 t} \cdot e^{-j\theta_n}\right]$$

$$= A_0 + \sum_{n=1}^{\infty}\left[\frac{A_n}{2} e^{jn\omega_0 t} \cdot e^{j\theta_n} + \frac{A_n}{2} e^{-jn\omega_0 t} \cdot e^{-j\theta_n}\right]$$

$$= A_0 + \sum_{n=1}^{\infty}\left(\frac{A_n}{2} e^{j\theta_n}\right) e^{jn\omega_0 t} + \sum_{n=1}^{\infty}\left(\frac{A_n}{2} e^{j(-\theta_n)}\right) e^{j(-n)\omega_0 t} \quad \text{—(2)}.$$

... for signal shown in fig. (6)

(5).

Let $n = -k$ in second summation of eqn (2) then

$$x(t) = A_0 + \sum_{n=1}^{\infty}\left(\frac{A_n}{2} e^{j\theta_n}\right) e^{jn\omega_0 t} + \sum_{k=-1}^{-\infty}\left(\frac{A_k}{2} e^{j\theta_k}\right) e^{j\omega_0 k t} \quad \text{--- (3)}$$

Comparing (2) & (3)

$$\left.\begin{aligned} A_n &= A_k \\ -\theta_n &= \theta_k \end{aligned}\right\} \begin{aligned} n &> 0 \\ k &< 0 \end{aligned} \quad \text{--- (4)}$$

Let us define

$$\left.\begin{aligned} C_0 &= A_0, \\ C_n &= \frac{A_n}{2} e^{j\theta_n}, \quad n > 0 \end{aligned}\right\} \text{--- (5)}$$

using (4) in (3)

$$x(t) = A_0 + \sum_{n=1}^{\infty}\left(\frac{A_n}{2} e^{j\theta_n}\right) e^{jn\omega_0 t} + \sum_{n=-1}^{-\infty}\left(\frac{A_n}{2} e^{j\theta_n}\right) e^{j\omega_0 n t}.$$

$$\boxed{x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}}$$

This is known as exponential Fourier series

Now we develop expressions for the coefficients of the exponential F.S.

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}, \quad \omega_0 = \frac{2\pi}{T}$$

multiply by $e^{-jk\omega_0 t}$ & integrate over one period

$$\int_{t_0}^{t_0+T} x(t) e^{-jk\omega_0 t} dt = \int_{t_0}^{t_0+T}\left[\sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}\right] e^{-jk\omega_0 t} dt \quad \text{--- (6)}$$

Substituting the relation $\int_{t_0}^{t_0+T} e^{jn\omega_0 t} e^{-jk\omega_0 t} = \begin{cases} 0, & k \neq n \\ T & k = n \end{cases}$

From (6)

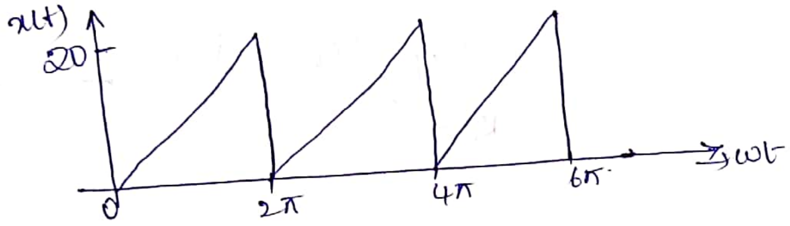
$$\therefore \int_{t_0}^{t_0+T} x(t) e^{-jk\omega_0 t} dt = T C_k.$$

$$C_k = \frac{1}{T}\int_{t_0}^{t_0+T} x(t) e^{-jk\omega_0 t} dt$$

$$\boxed{C_n = \frac{1}{T}\int_{t_0}^{t_0+T} x(t) e^{-jn\omega_0 t} dt.}$$

$C_n \rightarrow$ Fourier Coefficient of Exponential F.S.

① Find the Fourier series for the waveform shown in fig. and plot the spectrum.



$$x(t) = \frac{20}{2\pi} \cdot \omega t \quad \text{—①}$$

$T = 2\pi$.

$\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{2\pi} = 1$.

$$a_0 = \frac{1}{T}\int_0^{2\pi} x(t)\, dt$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{20}{2\pi}\omega t \cdot d\omega t = \frac{20}{(2\pi)^2}\left[\frac{(\omega t)^2}{2}\right]_0^{2\pi} = \frac{20}{(2\pi)^2}\left[\frac{4\pi^2}{2} - 0\right]$$

$$= \frac{20}{4\pi^2} \cdot \frac{4\pi^2}{2} = \underline{10}$$

$$a_n = \frac{2}{T}\int_{t_0}^{t_0+T} x(t)\cos(n\omega_0 t)\, d\omega t.$$

$$= \frac{2}{2\pi}\int_0^{2\pi} \frac{20}{2\pi} \cdot \omega t \cdot \cos(n\omega_0 t)\, d\omega t.$$

$\int u\,dv = uv - \int v\,du$.

$dv = \cos(n\omega_0 t)$

$v = \frac{\sin(n\omega_0 t)}{n\omega_0}$.

$$= \frac{20}{2\pi^2}\left[\frac{\omega t \cdot \sin(n\omega_0 \omega t)}{n\omega_0} + \frac{\cos(n\omega_0 \omega t)}{n^2\omega_0^2}\right]_0^{2\pi}$$

$$= \frac{10}{\pi^2}\left[\frac{(\omega t)\cdot \sin n\omega t}{n} + \frac{\cos n\omega t}{n^2}\right]_0^{2\pi}$$

$$= \frac{10}{\pi^2}\left[0 + \frac{\cos n 2\pi}{n^2} - \frac{1}{n^2}\right]$$

$$= \frac{10}{\pi^2 n^2}\left[\cos n 2\pi - 1\right]$$

$= 0$, for all values of n.

$$b_n = \frac{2}{T}\int_0^{2\pi} x(t)\sin(n\omega_0 \omega t)\, dt$$

$$= \frac{2}{2\pi}\int_0^{2\pi}\frac{20}{2\pi} \cdot \omega t \cdot \sin(n\omega t)\, dt.$$

$dv = \sin(n\omega t)$ -

$v = -\frac{\cos(n\omega t)}{n}$.

$$= \frac{20}{2\pi^2}\left[\frac{-(\omega t)\cdot\cos n\omega t}{n} + \frac{\sin(n\omega t)}{n^2}\right]_0^{2\pi}$$

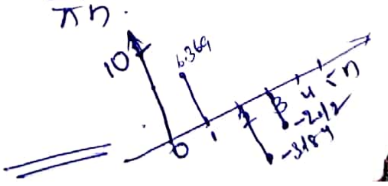
$$= \frac{20}{2\pi^2}\left[-\frac{2\pi\cos n 2\pi}{n} + \underbrace{\frac{\sin n 2\pi}{n^2}}_{=0} - (0+0)\right]$$

$$= \frac{20}{2n\pi^2}\left[-2\pi\cos n 2\pi\right] = \frac{-10}{n\pi^2}\left[+2\pi\right] = \frac{-20}{\pi n}.$$

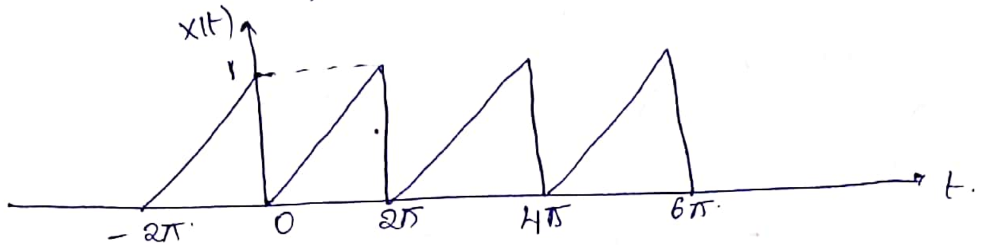
$$\therefore x(t) = 10 + 0 + \sum_{n=1}^{\infty} b_n \sin(n\omega t)$$

$$= 10 - \frac{20}{\pi}\sin\omega t - \frac{20}{2\pi}\sin 2\omega t - \frac{20}{3\pi}\sin 3\omega t \ldots$$

Spectrum



1. Find the cosine representation Fourier series for the signal shown in fig. (6).



Sol

$T = 2\pi$

$\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{2\pi} = 1.$

$x(t) = \frac{1}{2\pi} t$ for $0 \le t \le 2\pi$

$a_0 = \frac{1}{T}\int_{t_0}^{t_0+T} x(t)\,dt = \frac{1}{2\pi}\int_0^{2\pi} \frac{t}{2\pi}\,dt = \frac{1}{4\pi^2}\left[\frac{t^2}{2}\right]_0^{2\pi} = \frac{1}{4\pi^2}\left[\frac{4\pi^2}{2}\right] = \frac{1}{2}.$

$a_n = \frac{2}{T}\int_{t_0}^{t_0+T} x(t)\cos(n\omega_0 t)\,dt$

$= \frac{2}{2\pi}\int_0^{2\pi} \frac{t}{2\pi}\cos(nt)\,dt = \frac{1}{2\pi^2}\int_0^{2\pi} \underset{u}{t}\,\underset{dv}{\cos(nt)}\,dt$ $\qquad dv = \cos nt$, $V = \frac{\sin nt}{n}$.

$= \frac{1}{2\pi^2}\left[\frac{t\sin(nt)}{n} + \frac{\cos nt}{n^2}\right]_0^{2\pi}$

$= \frac{1}{2\pi^2}\left[0 - \frac{1}{n^2} - 0 + \frac{1}{n^2}\right] = 0.$

$b_n = \frac{2}{T}\int_{t_0}^{t_0+T} x(t)\sin(n\omega_0 t)\,dt$

$= \frac{2}{2\pi}\int_0^{2\pi}\frac{t}{2\pi}\sin(n\omega_0 t)\,dt = \frac{1}{2\pi^2}\left[-\frac{t\cos(n\omega_0 t)}{n} + \frac{\sin nt}{n}\right]_0^{2\pi}$

$= \frac{1}{2\pi^2}\left[-\frac{2\pi\cos 2\pi n}{n}\right]$

$= \frac{1}{2\pi^2}\left[-\frac{2\pi}{n}\right] = -\frac{1}{n\pi}$

For cosine representation

$A_0 = a_0 = \frac{1}{2}.$

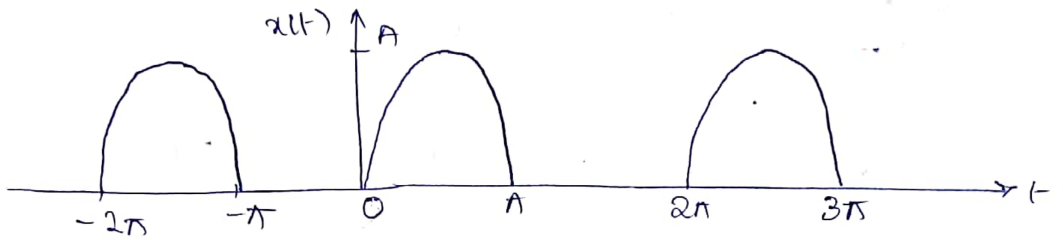
$A_n = \sqrt{a_n^2 + b_n^2} = \frac{1}{n\pi}$

$\theta_n = -\tan^{-1}\left(\frac{b_n}{a_n}\right) = -\tan^{-1}\left(\frac{-1/n\pi}{0}\right) = \frac{\pi}{2}.$

$\therefore x(t) = A_0 + \sum_{n=1}^{\infty} A_n\cos(n\omega_0 t + \theta_n).$

$= \frac{1}{2} + \sum_{n=1}^{\infty}\frac{1}{n\pi}\cos\left(nt + \frac{\pi}{2}\right)$

② Find Cosine Fourier Series of an half wave rectified Sine function



Sol

$x(t) = A \sin \omega_0 t$ for $0 \le t \le \pi$

$= 0$ for $\pi \le t \le 2\pi$

$T = 2\pi$

$\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{2\pi} = 1$

$a_0 = \frac{1}{T}\int_{t_0}^{t_0+T} x(t)\,dt = \frac{1}{2\pi}\int_0^{\pi} A \sin t\, dt = \frac{A}{2\pi}\left[-\cos t\right]_0^{\pi} = -\frac{A}{2\pi}\left[-1-1\right]$

$= A/\pi$

$a_n = \frac{2}{T}\int_{t_0}^{t_0+T} x(t)\cos(nt)\,dt = \frac{2}{2\pi}\int_0^{\pi} A \sin t \cos nt$

$\because \sin A \cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$

$= \frac{A}{\pi}\left[\int_0^{\pi} \frac{1}{2}\left[\sin(t+nt) + \sin(t-nt)\right] dt\right]$

$= \frac{A}{2\pi}\left[-\frac{\cos(1+n)t}{1+n} - \frac{\cos(1-n)t}{1-n}\right]_0^{\pi}$

$= \frac{A}{2\pi}\left[-\frac{\cos(1+n)\pi}{1+n} - \frac{\cos(1-n)\pi}{1-n} + \frac{1}{1+n} + \frac{1}{1-n}\right]$

$= \frac{A}{2\pi}\left[\frac{2}{1+n} + \frac{2}{1-n}\right]$ where n is even.

$= \frac{A}{2\pi}\left[\frac{4^2}{1-n^2}\right] = \frac{2A}{\pi(1-n^2)}$ where n is even

$b_n = \frac{2}{T}\int_{t_0}^{t_0+T} x(t)\sin nt\,dt = \frac{2}{2\pi}\int_0^{\pi} A \sin t \sin nt\,dt$

$= \frac{A}{2\pi}\int_0^{\pi}\{\cos(t-nt) - \cos(t+nt)\}dt$

$= \frac{A}{2\pi}\int_0^{\pi}\left[\cos(1-n)t - \cos(1+n)t\right] dt$

$= \frac{A}{2\pi}\left[\frac{\sin(1-n)t}{1-n} - \frac{\sin(1+n)t}{1+n}\right]_0^{\pi} = 0.$

for $n=1 \Rightarrow a_n = \frac{2}{2\pi}\int_0^{\pi} A \sin t \cos t\, dt = \frac{A}{2\pi}\int_0^{\pi} \sin 2t\, dt = \frac{A}{2\pi}\left[-\frac{\cos 2t}{2}\right]_0^{\pi} = \frac{A}{4\pi}\left[-1+1\right] = 0.$

for $n=1 \rightarrow b_n = \frac{2}{2\pi}\int_0^{\pi} A \sin t \sin t = \frac{2A}{2\pi}\int_0^{\pi} \sin^2 t = \frac{A}{\pi}\int_0^{\pi}\frac{1-\cos 2t}{2} = \frac{A}{2\pi}\left[t - \frac{\sin 2t}{2}\right]_0^{\pi}$

$= \frac{A}{2\pi}[\pi] = A/2$

$A_0 = a_0 = \frac{A}{\pi}$

$A_1 = \sqrt{a_1^2 + b_1^2} = A/2$

$A_n = \sqrt{a_n^2 + b_n^2} = \frac{2A}{\pi(1-n^2)}$

$\theta_1 = -\tan^{-1} b_1/a_1 = -\pi/2$

$\theta_n = \tan^{-1}\frac{b_n}{a_n}$

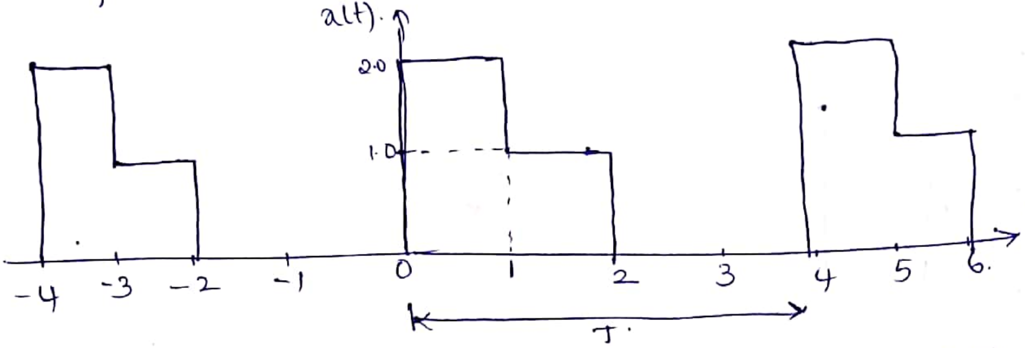
$= -\tan^{-1} 0$

$= 0.$

$\therefore x(t) = A_0 + \sum_{n=1}^{\infty} A_n \cos(nt + \theta_n)$

$= \frac{A}{\pi} + \frac{A}{2}\cos(t - \pi/2) + \sum_{n=2}^{\infty}\frac{2A}{\pi(1-n^2)}\cos nt.$

(3). Compute the exponential Fourier Series of the following signal.



Sol

$T = 4$

$x(t) = 2, \quad 0 \le t \le 1$
$\quad = 1, \quad 1 \le t \le 2$
$\quad = 0, \quad \text{else } 2 \le t \le 4$

$\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{4} = \pi/2$

$C_0 = \frac{1}{T}\int_0^T x(t)\,dt = \frac{1}{4}\int_0^4 x(t)\,dt$

$= \frac{1}{4}\left[\int_0^1 2\,dt + \int_1^2 1\,dt\right] = \frac{1}{4}\left[2\cdot[t]_0^1 + 1[t]_1^2\right]$

$= \frac{1}{4}\left[2+1\right] = 3/4.$

$C_n = \frac{1}{T}\int_0^T x(t)\, e^{-jn\omega_0 t}\,dt.$

$= \frac{1}{4}\int_0^4 x(t)\, e^{-jn\frac{\pi}{2}t}\,dt.$

$= \frac{1}{4}\left[\int_0^1 2\cdot e^{-jn\frac{\pi}{2}t}\,dt + \int_1^2 1\cdot e^{-jn\frac{\pi}{2}\cdot t}\,dt\right]$

$= \frac{1}{4}\left[2\cdot\left[\frac{e^{-jn\frac{\pi}{2}t}}{-jn\frac{\pi}{2}}\right]_0^1 + \left[\frac{e^{-jn\frac{\pi}{2}t}}{-jn\frac{\pi}{2}}\right]_1^2\right]$

$= \frac{1}{4}\left[\frac{2}{-jn\frac{\pi}{2}}\left(e^{-jn\frac{\pi}{2}} - 1\right) + \left(-\frac{1}{jn\frac{\pi}{2}}\right)\left(e^{-jn\pi} - e^{-jn\frac{\pi}{2}}\right)\right]$

$= -\frac{1}{jn\pi}\left[e^{-jn\frac{\pi}{2}} - 1\right] - \frac{1}{2jn\pi}\left[e^{-jn\pi} - e^{-jn\frac{\pi}{2}}\right]$

$= \frac{1}{jn\pi}\left[1 - e^{-jn\frac{\pi}{2}} - \frac{1}{2}e^{-jn\pi} + \frac{1}{2}e^{-jn\frac{\pi}{2}}\right]$

$= \frac{1}{jn\pi}\left[1 - \frac{1}{2}(-1)^n - \frac{1}{2}e^{-jn\pi/2}\right]$

The exponential F.S. of x(t)

$= \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$

$= \sum_{n=-\infty}^{\infty} C_n e^{jn\frac{\pi}{2}t}$

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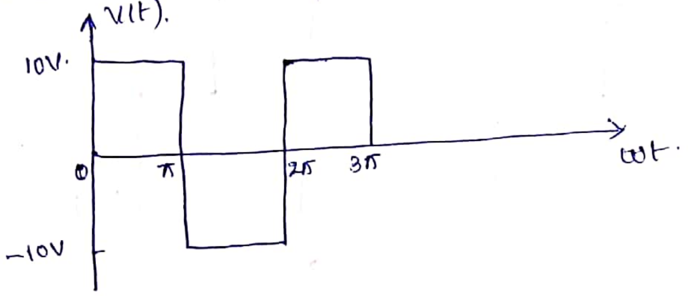
(4) Find the complex Fourier series for the square wave shown in fig and sketch the line spectrum.



$v(t) = 10$ for $0 < \omega t < \pi$

$= -10$ for $-\pi < \omega t < 0$

The avg. value of wave is zero.

$$C_n = \frac{1}{2\pi}\left[\int_{-\pi}^{0} -10\, e^{-jn\omega t}\, d\omega t + \int_{0}^{\pi} (+10)\, e^{-jn\omega t}\, d\omega t\right].$$

$$= \frac{10}{2\pi}\left[-\left[\frac{e^{-jn\omega t}}{-jn}\right]_{-\pi}^{0} + \left[\frac{e^{-jn\omega t}}{-jn}\right]_{0}^{\pi}\right]$$

$$= \frac{10}{-2\pi jn}\left[\cancel{(e^{-j\pi n} - 1) + (e^{-j2\pi n} - e^{-j\pi n})}\right]$$

$$= \frac{10}{-2\pi jn}\left[-(1 - e^{jn\pi}) + (e^{-jn\pi} - 1)\right]$$

$$= \frac{10}{-2\pi jn}\left[-2 + (\cos n\pi + j\sin n\pi) + (\cos n\pi - j\sin n\pi)\right]$$

$$= \frac{j10}{2\pi n}\left[-2(1 - \cos n\pi)\right]$$

$$C_n = -\frac{j10}{\pi n}\left[1 - \cos n\pi\right]$$

n is even $\quad C_n = 0$

n is odd $\quad C_n = -\dfrac{j20}{n\pi}$

The Fourier series is

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega t}.$$

$$= \sum_{n=\infty}^{\infty} C_n e^{jn\omega t}$$

$$= \ldots + j\frac{20}{3\pi} e^{-j3\omega t} - j\frac{20}{\pi} e^{-j\omega t} - j\frac{20}{\pi} e^{j\omega t} - \frac{j20}{3\pi} e^{j3\omega t} - \ldots$$

Complex Fourier Series

The Fourier Series

$$x(t) = a_0 + \sum_{n=1}^{\infty} [a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)] \quad \text{--- (1)}$$

Can be written in complex form

We know that

$$e^{jn\omega t} = \cos n\omega t + j \sin(n\omega t) \quad \text{--- (2)}$$

$$e^{-jn\omega t} = \cos n\omega t - j \sin n\omega t \quad \text{--- (3)}$$

By adding (2) & (3)

$$\cos n\omega t = \frac{1}{2}(e^{jn\omega t} + e^{-jn\omega t})$$

Subtracting (2)-(3)

$$\sin n\omega t = \frac{1}{2j}(e^{jn\omega t} - e^{-jn\omega t}) \quad \text{--- (4)}$$

$$a_n \cos n\omega t + b_n \sin n\omega t = \frac{1}{2} a_n (e^{jn\omega t} + e^{-jn\omega t}) + \frac{1}{2j} b_n (e^{jn\omega t} - e^{-jn\omega t})$$

$$= \frac{1}{2}(a_n - jb_n) e^{jn\omega t} + \frac{1}{2}(a_n + jb_n) e^{-jn\omega t}$$

Consider $a_0 = C_0$

$$\frac{1}{2}(a_n - jb_n) = C_n$$

$$\frac{1}{2}(a_n + jb_n) = C_{-n}$$

Then eqn (1) becomes

$$x(t) = C_0 + \sum_{n=1}^{\infty} C_n e^{jn\omega t} + C_{-n} e^{-jn\omega t} \quad \text{--- (5)}$$

$$C_n = \frac{1}{2}(a_n - jb_n)$$

$$= \frac{1}{2}\left[\frac{2}{T}\int_{-\pi}^{\pi} x(t)\cos n\omega t - j\int_{-\pi}^{\pi} x(t)\sin n\omega t \, dt\right]$$

$$= \frac{\cancel{2}}{\cancel{2}} \cdot \frac{1}{2\pi}\int_{-\pi}^{\pi} x(t)(\cos n\omega t - j \sin n\omega t)\, d\omega t.$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} x(t) e^{-jn\omega t} \cdot d(\omega t)$$

$$C_{-n} = \frac{1}{2}(a_n + jb_n)$$

$$= \frac{\cancel{2}}{\cancel{2}} \frac{1}{2\pi}\int_{-\pi}^{\pi} x(t)(\cos n\omega t + j \sin n\omega t)\, d(\omega t)$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} x(t) e^{jn\omega t} \, d(\omega t).$$

By combining two formulas & writing ~~$C_n = C_{-n}$ we get~~

$$\boxed{x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega t}}$$

where $C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} x(t) e^{-jn\omega t} \, d\omega t$ for $n = 0, \pm 1, \pm 2, \ldots$

This is called Complex form of Fourier Series (or) Complex Fourier Series of $x(t)$

$C_n \to$ complex Fourier coefficient of $x(t)$

(8) Find the Relationship between Fourier coefficients of Trigonometric and exponential form.

Sol The trigonometric F.S. of $x(t)$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \left[\frac{e^{jn\omega_0 t} + e^{-jn\omega_0 t}}{2}\right] + \sum_{n=1}^{\infty} b_n \left[\frac{e^{jn\omega_0 t} - e^{-jn\omega_0 t}}{2j}\right]$$

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

$$\frac{1}{j} = \frac{j}{j^2} = -j$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[\frac{a_n}{2} e^{jn\omega_0 t} + \frac{a_n}{2} e^{-jn\omega_0 t} - j\frac{b_n}{2} e^{jn\omega_0 t} + j\frac{b_n}{2} e^{-jn\omega_0 t}\right]$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[\frac{a_n}{2} - j\frac{b_n}{2}\right] e^{jn\omega_0 t} + \left[\frac{a_n}{2} + j\frac{b_n}{2}\right] e^{-jn\omega_0 t}$$

Let $C_0 = \frac{a_0}{2}$, $C_n = \frac{a_n}{2} - j\frac{b_n}{2}$, $C_n^* = \frac{a_n}{2} + j\frac{b_n}{2}$

$$\therefore x(t) = C_0 + \sum_{n=1}^{\infty} C_n e^{jn\omega_0 t} + C_n^* e^{-jn\omega_0 t}$$

$$= C_0 + \sum_{n=1}^{\infty} C_n e^{jn\omega_0 t} + \sum_{n=1}^{\infty} C_n^* e^{-jn\omega_0 t}$$

$$= C_0 + \sum_{n=1}^{\infty} C_n e^{jn\omega_0 t} + \sum_{-n=1}^{\infty} C_{-n} e^{jn\omega_0 t} \qquad (C_n^* = C_{-n})$$

$$= \sum_{n=-1}^{-\infty} C_n e^{jn\omega_0 t} + C_0 + \sum_{n=1}^{\infty} C_n e^{jn\omega_0 t}$$

$$= \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$$

$$\therefore \boxed{\begin{aligned} C_0 &= \frac{a_0}{2} \\ C_n &= \frac{1}{2}[a_n - jb_n] \\ C_{-n} &= \frac{1}{2}[a_n + jb_n] \end{aligned}}$$

C_n — for $n = 1, 2, 3 \ldots \infty$

C_{-n} — for $-n = -1, -2, -3 \ldots -\infty$.

Signal: Dependant variable or function of one or more independant variables.

$$\underset{\text{Dependant}}{F}\,[\underbrace{x_1, x_2, \ldots x_n}_{\text{independant}}]$$

Ex: height, weight,

ac → signal
Dc → not a s.

$$f = \frac{1}{T} = \frac{1}{\infty} = 0.$$

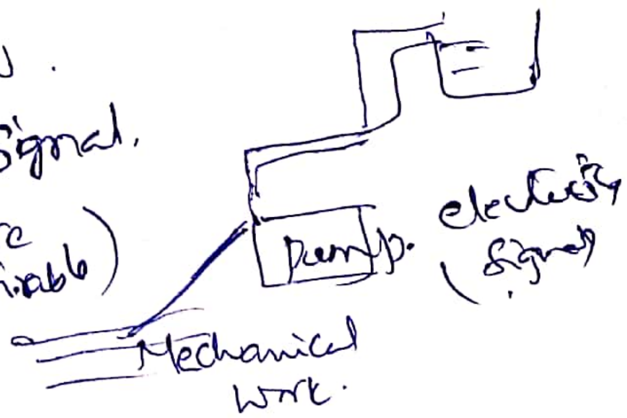
① Single variable signal Ex: $F(x)$, $g(t)$.

② Multi " " Ex: $F(x_1, x_2)$, $g_1(t_1 t_2 t_3)$.

System: The meaningful interconnection

- of physical devices and components is called as system.

i/p signal → [System] → o/p signal (more variable)



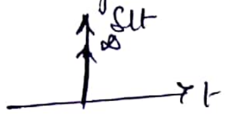
i/p signal → $F(x_1, x_2)$
o/p " → $g(x_1, x_2)$.

- ~~①~~ Analysis problem → i/p; system is there Find o/p ?.

  ② Synthesis "
  i/ps ✓
  system ?
  o/p's ✓

Standard continuous Time Signal

① Impulse sign



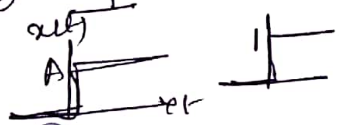
$$\int_{-\infty}^{\infty} \delta(t)\,dt = A$$

$\delta(t) = \infty,\ t=0$
$\quad = 0,\ t \neq 0.$

unit Impulse

$\delta(t) = 1,\ t=0$
$\quad = 0,\ t \neq 0.$

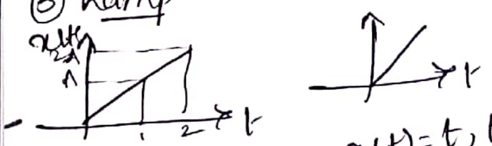
② Step



$x(t) = A,\ t \geq 0$
$u(t)\ \ 0,\ t<0.$

$x(t) = 1,\ t \geq 0$
$\quad = 0,\ t<0.$

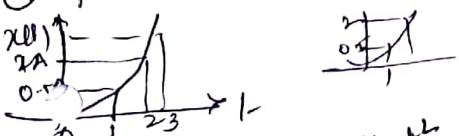
③ Ramp



$x(t) = At,\ t \geq 0$
$r(t)\ \ 0,\ t<0.$

$x(t) = t,\ t \geq 0$
$\quad 0,\ t<0.$

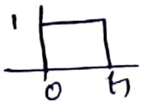
④. parabolic sign



$x(t) = \frac{At^2}{2},\ t \geq 0.$
$\quad = 0,\ t<0.$

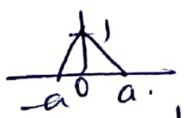
$x(t) = \frac{t^2}{2},\ t \geq 0.$
$\quad = 0,\ t<0.$

⑤ unit pulse signal



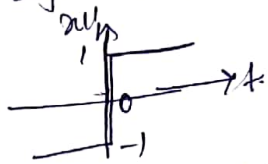
$x(t) = 1,\ 0 \leq t \leq t_1$
$\quad = 0,$ else.

⑥ triangular pulse.



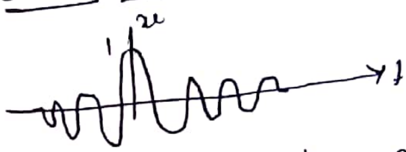
$x(t) = 1 - \frac{|t|}{a},\ |t| \leq a$
$\quad = 0,\ |t| > a.$

⑦ Signum Signal



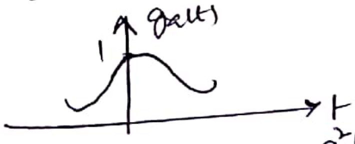
$x(t) = sgn(t) = 1,\ t>0$
$\quad = 0,\ t=0.$
$\quad = -1,\ t<0.$

⑧ Sinc signal



$x(t) = sinc(t) = \frac{\sin t}{t},\ -\infty < t < \infty.$

⑨ Gaussian Signal



$x(t) = g_a(t) = e^{-a^2t^2},\ -\infty < t < \infty.$

→ Mathematical operations on Continuous Time Signals.

i) Scaling < 1) Amp. Scaling
2) Time Scaling.

ii) Folding (Reflection/Transpose)

iii) Time Shifting

iv) Addition

v) Multiplication.

vi) differentiation & Integration

Orthogonality in Signals

Let $f(t)$ represented in terms of $x(t)$ over interval t_1 & t_2

$$f(t) = c\,x(t) + e(t)$$

$$e(t) = f(t) - c\,x(t) \quad \text{—— (1)} \qquad t_1 \le t \le t_2$$

min. value of $e(t)$ will give best approximation of $f(t)$ in $x(t)$

- Rather than min. value of $e(t)$ min energy of $e(t)$ or mean square value of $e(t)$ serves approximate measure.
- Hence for min. energy of $e(t)$, representation of $f(t)$ in $x(t)$ will be better.

Energy of $e(t)$ will be

$$E_e = \int_{t_1}^{t_2} e^2(t)\,dt \quad \text{—— (2)}$$

and mean square value of $e(t)$ will be given as

$$\overline{e^2(t)} = \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} e^2(t)\,dt$$

$$= \frac{E_e}{t_2 - t_1} \quad \text{—— (3)}.$$

Sub (1) in (2)

$$E_e = \int_{t_1}^{t_2} [f(t) - c\,x(t)]^2\,dt$$

Hence the value of 'c' should be selected such that E_e will be min. This can be obtained by differentiating E_e w.r.t c and equating it to zero i.e.

for min E_e

$$\frac{dE_e}{dc} = 0$$

$$\frac{d}{dc}\left[\int_{t_1}^{t_2} [f(t) - c\,x(t)]^2\,dt\right] = 0.$$

$$\frac{d}{dc}\left[\int_{t_1}^{t_2} (f^2(t) + c^2 x^2(t) - 2c\,f(t)x(t))\right]$$

$$\underbrace{\frac{d}{dc}\int_{t_1}^{t_2} f^2(t)\,dt}_{=0} + \frac{d}{dc}\int_{t_1}^{t_2} c^2 x^2(t) - \frac{d}{dc}\int_{t_1}^{t_2} 2c\,f(t)x(t)$$

$$0 + 2c\int_{t_1}^{t_2} x^2(t)\,dt - 2\int_{t_1}^{t_2} f(t)x(t)\,dt = 0$$

$$\boxed{c = \frac{\int_{t_1}^{t_2} f(t)\,x(t)\,dt}{\int_{t_1}^{t_2} x^2(t)\,dt}}$$

→ energy of $x(t)$

It is similar to $c = \frac{f \cdot x}{|x|^2}$

- The same exp can be obtained for min value of $\overline{e^2(t)}$
- Denominator is energy of $x(t)$ it cannot be zero. Hence numerator must be zero to make 'c' zero.
- If c is zero, there will be no component of $f(t)$ along $x(t)$. Then $f(t)$ and $x(t)$ are said to be orthogonal over an interval $[t_1, t_2]$ i.e.

For orthogonality $\boxed{\int_{t_1}^{t_2} f(t)\,x(t)\,dt = 0}$

|||ly if $f(t)$ & $x(t)$ are complex signals, then they are orthogonal

$$\int_{t_1}^{t_2} f(t)\,x^*(t)\,dt = 0 \quad \text{(or)} \quad \int_{t_1}^{t_2} f^*(t)\,x(t)\,dt = 0$$

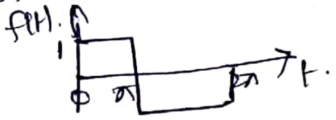
① Show that the following signals orthogonal over an interval [0,1]

$f(t) = 1$

$x(t) = \sqrt{3}(1-2t)$

So $\int_{t_1}^{t_2} f(t)x(t)dt = 0$.

② Fig. shows a square wave. Represent this signal by $\sin t$; plot an error in this representation.



Sol $f(t) \rightarrow$ square wave

$x(t) \rightarrow \sin t$

$f(t) = c\,x(t)$

$= c \sin t$

Value of 'c'

$$c = \frac{\int_{t_1}^{t_2} f(t)x(t)dt}{\int_{t_1}^{t_2} x^2(t)dt}$$

$$\int_{t_1}^{t_2} f(t)x(t)dt = \int_0^{2\pi} f(t)\sin t\, dt.$$

$$= \int_0^{\pi} 1\cdot\sin t\, dt + \int_{\pi}^{2\pi} (-1)\sin t\, dt$$

$$= [-\cos t]_0^{\pi} + [-\cos t]_{\pi}^{2\pi}$$

$$= \underline{4}$$

$$\int_{t_1}^{t_2} x^2(t)dt = \int_{t_1}^{t_2} \sin^2 t\, dt$$

$$= \int_{t_1}^{t_2} \frac{1-\cos 2t}{2} dt$$

$$= \frac{1}{2}\int_0^{2\pi} dt - \frac{1}{2}\int_0^{2\pi} \cos 2t\, dt$$

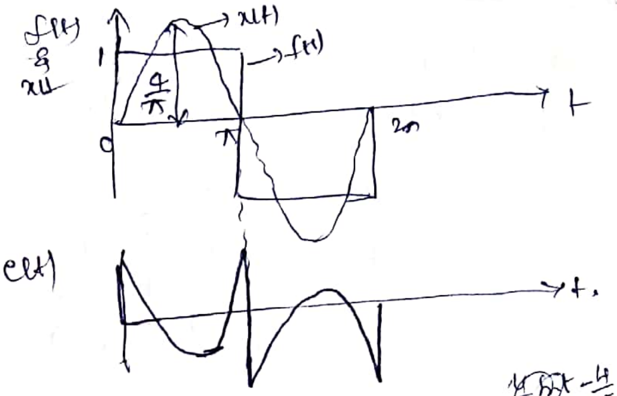
$$= \underline{\pi}$$

$$\therefore C = \frac{4}{\pi}$$

$\therefore$ approximation becomes

$$f(t) = \frac{4}{\pi} x(t)$$

error $e(t) = f(t) - c\,x(t)$.



$\frac{4}{\pi}\sin t - \frac{4}{\pi}$

$1 - \frac{4}{\pi}\sin t$

Signal Representation by a discrete set of orthogonal functions

So eqn for mean square error ϵ

$$\epsilon^2(t) = \frac{1}{t_2-t_1}\left[\int_{t_1}^{t_2} x^2(t)dt - (C_1^2K_1 + C_2^2K_2 + \dots + C_n^2K_n)\right]$$

$$= \frac{1}{t_2-t_1}\left[\int_{t_1}^{t_2} x^2(t)dt - \sum_{r=1}^{n} C_r^2 K_r\right]$$

if we increase 'n' i.e. if we approximate $x(t)$ by a larger number of orthogonal functions, the error will be smaller.

- But mean square value (ε) is positive quantity, hence in the limit as number of terms is made infinity, the sum $\sum_{r=1}^{\infty} C_r^2 K_r$ may converge to integral $\int_{t_1}^{t_2} x^2(t)dt$ and then 'ε' vanishes.

$$\boxed{\int_{t_1}^{t_2} x^2(t)dt = \sum_{r=1}^{\infty} C_r^2 K_r}$$

$x(t) = C_1m_1(t) + C_2m_2(t) + \dots + C_rm_r(t)$

RHS side of above eqn converges to $x(t)$ such that mean square of error is zero. The representation of $x(t)$ is now exact.

Analogy b/w vectors & signals

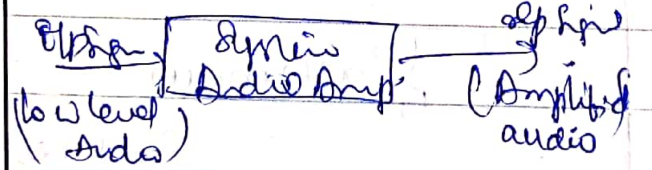
Signal → A function of one (or) more independent variables which contain some information is called signal.

Ex: Electric, voltage (or) current, radio signal, TV signal, telephone signal.

Non-electric → Pressure signal, Sound signal.

System: A system is a set of elements (or) functional block that are connected together and produces an o/p in response to an i/p signal.

Ex: Audio amplifier, attenuator, Tx, Rx. Any m/c (or) engine.



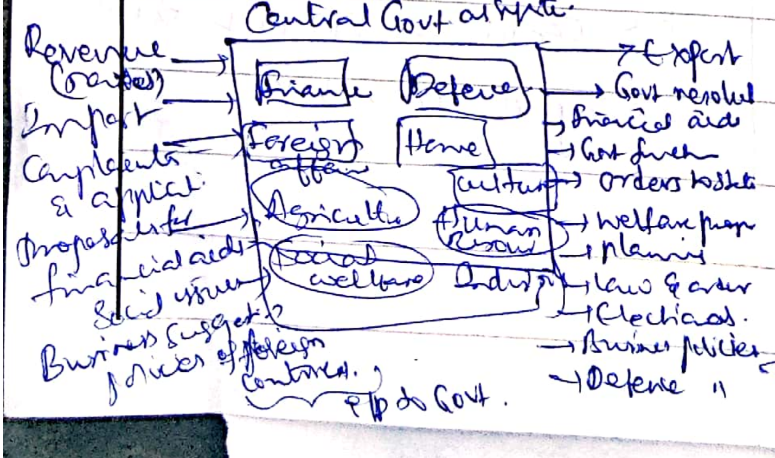
→ every system has one or more i/ps called excitation

" " o/ps called Response.

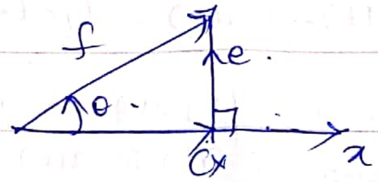
→ i/p & o/p always signals

→ Analogy of signals & system

Central Govt as system.

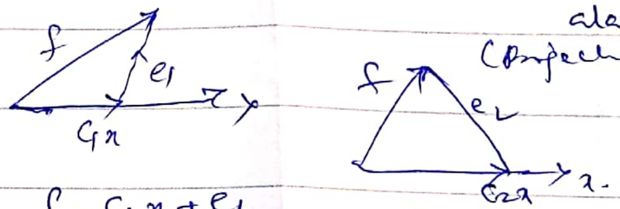


Orthogonality concept in vectors



$f.x = |f||x|\cos\theta$ — ①

$f = cx + e$ — ② $\quad cx \rightarrow$ component of vector f along x. (Projecting f on x)



$f = c_1x + e_1 \qquad f = c_2x + e_2$

$e \rightarrow$ min only when it is vertical

e_1, e_2 greater than e.

e is min only when it intersects

The component of f along x is cx

It is also given as $f\cos\theta$.

i.e. $c|x| = |f|\cos\theta$.

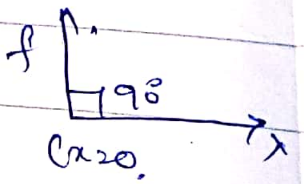
$c|x|^2 = |f||x|\cos\theta$.

$c|x|^2 = f\cdot x$

$c = \frac{f\cdot x}{|x|^2}$

Since $x\cdot x = |x|^2$

$c = \frac{f\cdot x}{x\cdot x}$.



$f.x = |f||x|\cos\theta$.

$= 0$

f & x are said to be orthogonal if their dot product is zero.

→ Vectors are orthogonal if they are mutually perpendicular.

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ECE-A 1, 2, 4, 5, 6, 7, 10, 11, 13, 15, 17, 18, 30, 43, 44, 46, 58, 56

Orthogonality in signals

$f(t) = C x(t) + e(t)$

$e(t) = f(t) - C x(t) \quad t_1 \le t \le t_2$ ①.

Min. value of e(t) will give best approximation of f(t) in x(t)

Completeness (or) Closed set of orthogonal functions

A set of mutually orthogonal functions $m_1(t), m_2(t) \ldots m_r(t)$ over an interval (t_1, t_2) is said to be a complete or a closed set if there exist no function $x(t)$ for which it is true that,

$$\boxed{\int_{t_1}^{t_2} x(t)\, m_k(t)\, dt = 0.}$$

for $k = 1, 2 \ldots$

If a function $x(t)$ could be found such that above integral is zero, then obviously $x(t)$ is orthogonal to each member of the set $[x(t)]$ and as a result, itself a member of the set. So the set cannot be complete without $x(t)$ being its member.

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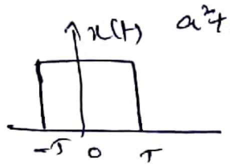
T. Sravan Kumar

1. F.T. of $x(t) = e^{-a|t|}\,\mathrm{sgn}(t)$.

Ans: $\frac{-j2\omega}{a^2+\omega^2}$.

Sampling func $Sa(t) = \frac{\sin t}{t}$.

Sinc func $\mathrm{sinc}(t) = \frac{\sin \pi t}{\pi t}$.

2. 

$= 2T\,\mathrm{sinc}\,\omega T$

3. 

4. $x(t) = te^{-at}u(t)$ Ans: $\frac{1}{(j\omega+a)^2}$

5. Det F.T. of x(t) using time shifting property

$x(t) = e^{-3|t-t_0|} + e^{-3|t+t_0|}$ $\quad F\{e^{-a|t|}\} = \frac{2a}{a^2+(j\omega)^2}$.

Sol: $X(j\omega) = F\{e^{-3|t-t_0|}\} + F\{e^{-3|t+t_0|}\}$

$= e^{-j\omega t_0}F\{e^{-3|t|}\} + e^{j\omega t_0}F\{e^{-3|t|}\}$

$= e^{-j\omega t_0}\cdot\frac{2(3)}{(3)^2+(j\omega)^2} + e^{j\omega t_0}\frac{2(+3)}{(3)^2+(j\omega)^2} = \frac{12\cos\omega t_0}{3^2+(j\omega)^2}$

6. If x(t) & X(jω) are F.T. pairs, det. the F.T. of x(t) sin ω₀t.

Sol: $F\{x(t)\sin\omega_0 t\} = F\{x(t)\left[\frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j}\right]\}$

$= \frac{1}{2j}\left[F\{x(t)e^{j\omega_0 t}\} - F\{x(t)e^{-j\omega_0 t}\}\right]$

$= \frac{1}{2j}X(j(\omega-\omega_0)) - \frac{1}{2j}X(j(\omega+\omega_0))$