FOURIER TRANSFORM

Development of F.T from Fourier Series

The exponential form of Fourier series representation of periodic signals is given by

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t} \quad \text{——— (1)}$$

$$\text{where } C_n = \frac{1}{T}\int_{-T/2}^{T/2} x(t) e^{-jn\omega_0 t} dt \quad \text{——— (2)}$$

In Fourier representation using eqn ①, the C_n for various values of n are the spectral components of the signal x(t), located at intervals of fundamental frequency ω_0. Therefore the frequency spectrum is discrete in nature.

- The Fourier representation of a signal using eqn ① is applicable for periodic signals.
- For Fourier representation of non periodic signals, Let us consider that the fundamental period tends to infinity, when fundamental period (T) tends to infinity, the fundamental frequency (ω_0) tends to zero (or) becomes very small.
- Since ω_0 is very small, the spectral components will lie very close to each other and so frequency spectrum becomes continuous.
- In order to obtain the Fourier representation of a non-periodic signals let us consider that the fundamental frequency ω_0 is very small.

Let $\omega_0 \rightarrow \Delta\omega$

Replace ω_0 by $\Delta\omega$ in eqn ①

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\Delta\omega t}$$

$$= \sum_{n=-\infty}^{\infty} \left[\frac{1}{T}\int_{-T/2}^{T/2} x(t) e^{-jn\Delta\omega t} dt\right] e^{jn\Delta\omega t} \quad \text{——— (3)}$$

$$\omega_0 = 2\pi f_0 = \frac{2\pi}{T}$$

$$\frac{1}{T} = \frac{\omega_0}{2\pi}$$

Since $\omega_0 \to \Delta\omega$

$$\frac{1}{T} = \frac{\Delta\omega}{2\pi}$$

Sub. these in eqn (3)

$$x(t) = \sum_{n=-\infty}^{\infty} \left[\frac{\Delta\omega}{2\pi} \int_{-T/2}^{T/2} x(t)\, e^{-jn\Delta\omega t}\, dt \right] e^{jn\Delta\omega t}$$

$$= \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \left[\int_{-T/2}^{T/2} x(t)\, e^{-jn\Delta\omega t}\, dt \right] e^{jn\Delta\omega t} \cdot \Delta\omega$$

For Non periodic signals $T \to \infty$

$$x(t) = \lim_{T\to\infty} \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \left[\int_{-T/2}^{T/2} x(t)\, e^{-jn\Delta\omega t}\, dt \right] e^{jn\Delta\omega t} \cdot \Delta\omega$$

When $T \to \infty$; $\Sigma \to \int$; $\Delta\omega \to \omega$

$$\therefore x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \underbrace{\left[\int_{-\infty}^{\infty} x(t)\, e^{-jn\omega t}\, dt \right]}_{X(j\omega)} e^{jn\omega t} \cdot d\omega$$

$$\boxed{x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega)\, e^{jn\omega t} \cdot d\omega} \quad \text{— (4)} \rightarrow \text{IFT.}$$

where. $X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-jn\omega t}\, dt.$

$$\boxed{X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-jn\omega t} \cdot dt} \quad \text{— (5)}$$

F.T.

eqn (5) is Fourier Transform of $x(t)$ → ~~Synthesis~~ Analysis of signal $x(t)$.

eqn (4) is Inverse Fourier transform of $x(t)$. → Synthesis ~~analysis~~ of the Signal $x(t)$

Definition of F.T.

Let $x(t)$ = continuous time signal

$X(j\omega)$ = Fourier Transform of $x(t)$

The F.T. of Continuous time signal $x(t)$ is defined as

$$\boxed{FT\{x(t)\} = X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt.}$$

$\omega \to$ radian frequency.

$$X(jF) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi F t}\, dt$$

$F \to$ cyclic frequency.

Condition for Existence of F.T

The F.T. of x(t) exists if it satisfies the following conditi.

1) The x(t) is absolutely integrable.

i.e. $\int_{-\infty}^{\infty} x(t)\, dt < \infty$

2) The x(t) should have a finite number of maxima and minima within any finite interval.

3) The x(t) can have a finite number of discontinuities within any interval.

Definition of Inverse F.T.

The IFT of X(jω) is defined as

$$\boxed{F^{-1}\{X(j\omega)\} = x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega) \cdot e^{j\omega t}\, d\omega}$$

x(t) and X(jω) are called Fourier Transform pairs

$$\boxed{x(t) \underset{F^{-1}}{\overset{F}{\rightleftharpoons}} X(j\omega)}$$

Frequency Spectrum using F.T.

The X(jω) is a complex function of ω. Hence it can be expressed as a sum of real part and Imaginary part.

$\therefore X(j\omega) = X_r(j\omega) + jX_i(j\omega)$

The magnitude of X(jω) is called magnitude spectrum

$$\therefore \boxed{\text{Magnitude spectrum} = |X(j\omega)| = \sqrt{X_r^2(j\omega) + X_i^2(j\omega)}}$$

or.

$$|X(j\omega)| = \sqrt{X(j\omega)\, X^*(j\omega)}$$

where $X^*(j\omega)$ = conjugate of X(jω)

The phase of X(jω) is called phase spectrum

$$\boxed{\therefore \text{phase spectrum} = \angle X(j\omega) = \tan^{-1}\frac{X_i(j\omega)}{X_r(j\omega)}}$$

* The magnitude spectrum will always have even symmetry and phase spectrum will have odd symmetry. The magnitude & phase spectrum together called frequency spectrum.

properties of Fourier Transform

1) Linearity

Let $F\{x_1(t)\} = X_1(j\omega)$

$F\{x_2(t)\} = X_2(j\omega)$

The linearity property of F.T. say that

$$\boxed{F\{a_1 x_1(t) + a_2 x_2(t)\} = a_1 X_1(j\omega) + a_2 X_2(j\omega)}$$

Proof: From definition of F.T.

$$X_1(j\omega) = \int_{-\infty}^{\infty} x_1(t)\, e^{-j\omega t}\, dt$$

and $X_2(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$

LHS:

$$F\{a_1 x_1(t) + a_2 x_2(t)\} = \int_{-\infty}^{\infty} \{a_1 x_1(t) + a_2 x_2(t)\}\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} a_1 x_1(t)\, e^{-j\omega t}\, dt + \int_{-\infty}^{\infty} a_2 x_2(t)\, e^{-j\omega t}\, dt$$

$$= a_1 X_1(j\omega) + a_2 X_2(j\omega).$$

$= \underline{\underline{RHS}}$

2) Time Shifting

If $F(x(t)) = X(j\omega)$

$$F\{x(t-t_0)\} = e^{-j\omega_0 t_0} X(j\omega).$$

Proof: LHS.

$$F\{x(t)\} = X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt.$$

$$F\{x(t-t_0)\} = \int_{-\infty}^{\infty} x(t-t_0)\, e^{-j\omega t}\, dt.$$

let $t - t_0 = \nu$

$t = \nu + t_0$

$$= \int_{-\infty}^{\infty} x(\nu)\, e^{-j(\nu+t_0)\omega}\, d\nu$$

$$= e^{-j\omega t_0} \int_{-\infty}^{\infty} x(\nu)\, e^{-j\omega\nu}\, d\nu$$

$$= e^{-j\omega t_0} X(j\omega). \quad = RHS$$

3) Time Scaling

If $F\{x(t)\} = X(j\omega)$

Then $F\{x(at)\} = \frac{1}{|a|} X\left(\frac{j\omega}{a}\right)$

proof:

$$F\{x(t)\} = X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$F\{x(at)\} = \int_{-\infty}^{\infty} x(at) e^{-j\omega t} dt$$

put $at = \tau$

$t = \tau/a$

$dt = \frac{1}{a} d\tau$

$$= \int_{-\infty}^{\infty} x(\tau) e^{-j\omega\left(\frac{\tau}{a}\right)} \cdot \frac{1}{a} d\tau$$

$$= \frac{1}{a} \int_{-\infty}^{\infty} x(\tau) e^{-j\left(\frac{\omega}{a}\right)\tau} \cdot d\tau$$

$$= \frac{1}{a} X\left(\frac{j\omega}{a}\right)$$

If a is +ve then $F\{x(at)\} = \frac{1}{a} X\left(\frac{j\omega}{a}\right)$

If a is -ve then $F\{x(at)\} = \frac{-1}{a} X\left(\frac{j\omega}{a}\right)$

In general $F\{x(at)\} = \frac{1}{|a|} X\left(\frac{j\omega}{a}\right)$.

4) Time Reversal

If $F\{x(t)\} = X(j\omega)$

Then $F\{x(-t)\} = X(-j\omega)$

Proof: From time scaling property

$$F\{x(at)\} = \frac{1}{|a|} X\left(\frac{j\omega}{a}\right)$$

Let $a = -1$

$\therefore F\{x(-t)\} = X(-j\omega)$

5) Conjugation

If $F\{x(t)\} = X(j\omega)$

Then $F\{x^*(t)\} = X^*(-j\omega)$

proof: $F\{x(t)\} = X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$

LHS $= F\{x^*(t)\} = \int_{-\infty}^{\infty} x^*(t) e^{-j\omega t} dt$

$$= \left[\int_{-\infty}^{\infty} x(t) e^{j\omega t} dt\right]^* = \left[\int_{-\infty}^{\infty} x(t) e^{-j(-j\omega)t} dt\right]^*$$

$$= [X(-j\omega)]^* = X^*(-j\omega)$$

= RHS

6) Frequency shifting

If $F\{x(t)\} = X(j\omega)$

Then $F\{e^{j\omega_0 t} x(t)\} = X(j(\omega-\omega_0))$

Proof: $F\{x(t)\} = X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$

LHS = $F\{e^{j\omega_0 t} x(t)\} = \int_{-\infty}^{\infty} e^{j\omega_0 t} \cdot x(t)\, e^{-j\omega t}\, dt$

$= \int_{-\infty}^{\infty} x(t)\, e^{-j(\omega-\omega_0)t}\, dt$

$= X(j(\omega-\omega_0))$

= RHS

7) Time Differentiation

If $F\{x(t)\} = X(j\omega)$

Then $F\left\{\frac{d}{dt} x(t)\right\} = j\omega X(j\omega)$

Proof:

$F\{x(t)\} = X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$

$\int u\, dv = uv - \int v\, du.$

LHS. ∴ $F\left\{\frac{d}{dt} x(t)\right\} = \int_{-\infty}^{\infty} \frac{d}{dt}(x(t))\, e^{-j\omega t}\, dt$

$dv = \frac{d}{dt} x(t)$
$v = x(t).$

$= \int_{-\infty}^{\infty} e^{-j\omega t}\left(\frac{d}{dt} x(t)\right) dt.$

$= \left[e^{-j\omega t} \cdot x(t)\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t} (-j\omega)\, dt.$

$= 0 - e^{\infty} x(-\infty) + j\omega \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt.$

$= 0 + j\omega \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt.$ $(\because x(-\infty) = 0.$

$= j\omega X(j\omega)$

= RHS.

8) Frequency Differentiation

If $F\{x(t)\} = X(j\omega)$

Then $F\{t\,x(t)\} = j\frac{d}{d\omega}X(j\omega)$.

Proof:

$$F\{x(t)\} = X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

Differentiate above eqn w.r.t. ω

$$\frac{d}{d\omega}X(j\omega) = \frac{d}{d\omega}\left[\int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt\right]$$

$$= \int_{-\infty}^{\infty} x(t)\left[\frac{d}{d\omega}e^{-j\omega t}\right]dt.$$

$$= \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\,.\,(-jt)\, dt.$$

$$= \frac{1}{j}\int_{-\infty}^{\infty} t\,x(t)\, e^{-j\omega t}.dt$$

$$= \frac{1}{j}F\{t\,x(t)\}$$

$$\therefore F\{t\,x(t)\} = j\frac{d}{d\omega}X(j\omega).$$

9) Time Integration

If $F\{x(t)\} = X(j\omega)$ and $X(0)=0$

Then $F\left\{\int_{-\infty}^{t} x(\tau)\,d\tau\right\} = \frac{1}{j\omega}X(j\omega)$

Proof:

$$\frac{d}{dt}\left[\int_{-\infty}^{t} x(\tau)\,d\tau\right] = x(t).$$

on taking F.T. of above eqn

$$F\left\{\frac{d}{dt}\left[\int_{-\infty}^{t} x(\tau)\,d\tau\right]\right\} = F\{x(t)\}$$

$$j\omega\, F\left\{\int_{-\infty}^{t} x(\tau)\,d\tau\right\} = F\{x(t)\}$$

$$\therefore F\left\{\int_{-\infty}^{t} x(\tau)\,d\tau\right\} = \frac{1}{j\omega}X(j\omega)$$

10) Convolution theorem

It says that F.T. of Convolution of two Signals is given by the product of the F.T of the individual signals.

if $F\{x_1(t)\} = X_1(j\omega)$ and $F\{x_2(t)\} = X_2(j\omega)$

then $F\{x_1(t) * x_2(t)\} = X_1(j\omega)\, X_2(j\omega)$.

Proof:

$$x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau$$

Let $x_1(t)$ and $x_2(t)$ be two time domain Signals.

$$F\{x_1(t)\} = X_1(j\omega) = \int_{-\infty}^{\infty} x_1(t)\, e^{-j\omega t} dt$$

$$F\{x_2(t)\} = X_2(j\omega) = \int_{-\infty}^{\infty} x_2(t)\, e^{-j\omega t} dt$$

$$LHS = F\{x_1(t) * x_2(t)\} = \int_{-\infty}^{\infty} [x_1(t) * x_2(t)]\, e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau\right] e^{-j\omega t} dt.$$

put $t - \tau = M$

$dt = dM$.

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1(\tau)\, x_2(M)\, e^{-j\omega(M+\tau)}\, d\tau . dM$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1(\tau)\, x_2(M)\, e^{-j\omega M}. e^{-j\omega \tau}. d\tau . dM$$

$$= \int_{-\infty}^{\infty} x_1(\tau)\, e^{-j\omega\tau} d\tau . \int_{-\infty}^{\infty} x_2(M)\, e^{-j\omega M} . dM$$

$$= X_1(j\omega) . X_2(j\omega).$$

$= RHS.$

iii) Frequency Convolution

Let $F\{x_1(t)\} = X_1(j\omega)$ and $F\{x_2(t)\} = X_2(j\omega)$

then frequency convolution property of F.T. says that

$$F\{x_1(t)\,x_2(t)\} = \frac{1}{2\pi}\int_{\lambda=-\infty}^{\infty} X_1(j\lambda)\,X_2(j(\omega-\lambda))\,d\lambda$$

Proof:

$$F\{x(t)\} = X(j\omega) = \int_{-\infty}^{\infty} x(t)\,e^{-j\omega t}\,dt$$

$$\therefore F\{x_1(t)\,x_2(t)\} = \int_{t=-\infty}^{t=\infty} x_1(t)\,x_2(t)\,e^{-j\omega t}\,dt \quad \text{--- (1)}$$

From the definition of IFT

$$x_1(t) = F^{-1}\{X_1(j\omega)\} = \frac{1}{2\pi}\int_{\omega=-\infty}^{\infty} X_1(j\omega)\,e^{j\omega t}\,d\omega$$

$$= \frac{1}{2\pi}\int_{\lambda=-\infty}^{\infty} X_1(j\lambda)\,e^{j\lambda t}\,d\lambda \quad (\because \text{replace } \omega\to\lambda) \quad \text{--- (2)}$$

Sub (2) in (1)

$$F\{x_1(t)\,x_2(t)\} = \int_{t=-\infty}^{\infty}\left[\frac{1}{2\pi}\int_{\lambda=-\infty}^{\infty} x_1(j\lambda)\,e^{j\lambda t}\,d\lambda\right] x_2(t)\,e^{-j\omega t}\,dt$$

$$= \frac{1}{2\pi}\int_{\lambda=-\infty}^{\infty} x_1(j\lambda)\left[\int_{t=-\infty}^{\infty} x_2(t)\,e^{-j\omega t}\,e^{j\lambda t}\,dt\right] d\lambda$$

$$= \frac{1}{2\pi}\int_{\lambda=-\infty}^{\infty} X_1(j\lambda)\left[\int_{t=-\infty}^{\infty} x_2(t)\,e^{-j(\omega-\lambda)t}\,dt\right] d\lambda$$

$$= \frac{1}{2\pi}\int_{\lambda=-\infty}^{\infty} X_1(j\lambda)\,X_2(j(\omega-\lambda))\,d\lambda$$

$= \underline{RHS}$.

12) Parseval's Relation

The Parseval's relation says that

If $F\{x(t)\} = X(j\omega)$

Then $\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega$

Proof: Let $x(t)$ be a continuous time signal and $x^*(t)$ be conjugate of $x(t)$

Now $|x(t)|^2 = x(t)\, x^*(t)$

on Integrating the above eqn w.r.t. t

$$\int_{t=-\infty}^{\infty} |x(t)|^2 dt = \int_{t=-\infty}^{\infty} x(t)\, x^*(t)\, dt \quad \text{——— (1)}$$

By definition of I.F.T.

$$x(t) = F^{-1}\{X(j\omega)\} = \frac{1}{2\pi}\int_{\omega=-\infty}^{\infty} X(j\omega)\, e^{j\omega t}\, d\omega$$

on taking conjugate of the above eqn

$$x^*(t) = \frac{1}{2\pi}\int_{\omega=-\infty}^{\infty} X^*(j\omega)\, e^{-j\omega t}\, d\omega \quad \text{——— (2)}$$

using (2) in (1)

$$\int_{t=-\infty}^{\infty} |x(t)|^2 dt = \int_{t=-\infty}^{\infty} x(t)\left[\frac{1}{2\pi}\int_{\omega=-\infty}^{\infty} X^*(j\omega)\, e^{-j\omega t}\, d\omega\right] dt$$

$$= \frac{1}{2\pi}\int_{\omega=-\infty}^{\infty} X^*(j\omega)\underbrace{\left[\int_{t=-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt\right]}_{X(j\omega).} d\omega$$

$$= \frac{1}{2\pi}\int_{\omega=-\infty}^{\infty} X(j\omega)\, X^*(j\omega)\, d\omega.$$

$$= \frac{1}{2\pi}\int_{\omega=-\infty}^{\infty} |X(j\omega)|^2 d\omega.$$

13). Duality

If $F\{x_1(t)\} = X_1(j\omega)$ and $F\{x_2(t)\} = X_2(j\omega)$.

and if $x_2(t) = X_1(j\omega)$ i.e. $x_2(t)$ and $X_1(j\omega)$ are similar functions

Then $X_2(j\omega) = 2\pi x_1(-j\omega)$ i.e. $x_2(j\omega)$ and $2\pi x_2(-j\omega)$ are similar functions.

Alternatively duality property is expressed as

If $x_2(t) \Leftrightarrow X_1(j\omega)$

Then $X_2(j\omega) \Leftrightarrow 2\pi x_1(-j\omega)$

Proof:

$F\{x_1(t)\} = X_1(j\omega)$

$F\{x_2(t)\} = X_2(j\omega)$

let $x_2(t)$ and $X_1(j\omega)$ are similar ~~functions~~ in form.

$\therefore x_2(t) = X_1(j\omega)|_{j\omega = t}$

By definition of I.F.T

$$x_1(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X_1(j\omega)\, e^{j\omega t}\, d\omega$$

$$\therefore x_1(-t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X_1(j\omega)\, e^{-j\omega t}\, d\omega.$$

$$x_1(-t)|_{t=j\omega} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \{X_1(j\omega)|_{j\omega=t}\}\, e^{-j\omega t}\, d\omega$$

$$\therefore x_1(-j\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} x_2(t)\, e^{-j\omega t}\, d\omega$$

$$\int_{-\infty}^{\infty} x_2(t)\, e^{-j\omega t}\, d\omega = 2\pi X_1(-j\omega).$$

$$X_2(j\omega) = 2\pi x_1(-j\omega).$$

For even function $x_1(-j\omega) = X_1(j\omega)$

$\therefore X_2(j\omega) = 2\pi x_1(j\omega)$.

14). Area under a time domain signal.

Area under $x(t) = \int_{-\infty}^{\infty} x(t)\,dt$

If $x(t)$ and $X(j\omega)$ are F.T. pairs

Then $\int_{-\infty}^{\infty} x(t)\,dt = X(0)$

where $X(0) = \lim_{j\omega \to 0} X(j\omega)$.

Proof.

$X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\,dt$

$\therefore X(0) = \lim_{j\omega \to 0} X(j\omega) = \lim_{j\omega \to 0} \int_{-\infty}^{\infty} x(t) e^{-j\omega t}\,dt$

$= \int_{-\infty}^{\infty} x(t)\, e^{0}\, dt = \int_{-\infty}^{\infty} x(t)\,dt.$

$\therefore \int_{-\infty}^{\infty} x(t)\,dt = X(0)$

15) Area under a Frequency Domain Signal

Area under $X(j\omega) = \int_{-\infty}^{\infty} X(j\omega)\,d\omega$

If $x(t)$ and $X(j\omega)$ are F.T. pair

Then $\int_{-\infty}^{\infty} X(j\omega)\,d\omega = 2\pi x(0)$

where $x(0) = \lim_{t \to 0} x(t)$

Proof:

From IFT

$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t}\,d\omega$

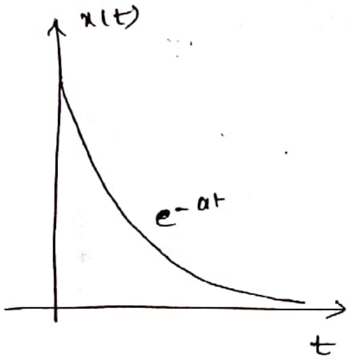
$\therefore x(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega)\, e^{0}\,d\omega$

$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega)\,d\omega.$

$\therefore \int_{-\infty}^{\infty} X(j\omega)\,d\omega = 2\pi x(0)$

Pb1: Find the F.T of $x(t) = e^{-at}u(t)$ $a>0$. Plot mag & phase spect

Sol:



By Def

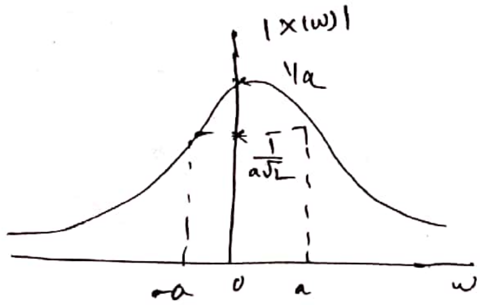
$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} e^{-at} u(t) \cdot e^{-j\omega t}\, dt$$

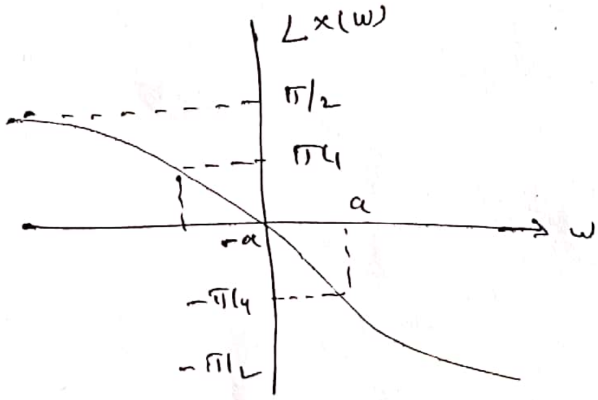
$$= \int_{0}^{\infty} e^{-(a+j\omega)t}\, dt.$$

$$X(\omega) = \frac{-1}{a+j\omega}\, e^{-(a+j\omega)}\Big|_0^{\infty} = \frac{-1}{a+j\omega}(0-1) = \frac{1}{a+j\omega}$$

$$|X(\omega)| = \frac{1}{\sqrt{a^2+\omega^2}} \qquad \angle X(\omega) = -\tan^{-1}\frac{\omega}{a}$$

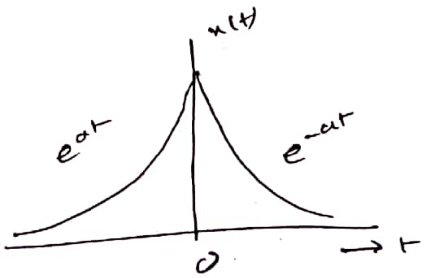


magnitude spec



phase spec.

2) Find F.T of $x(t) = e^{-a|t|}$ $a>0$.



$$x(t) = e^{-a|t|} = \begin{cases} e^{at} & t<0 \\ e^{-at} & t\geq 0. \end{cases}$$

$$= e^{at}u(-t) + e^{-at}u(t).$$

Time domain signal. $\therefore$ $X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$

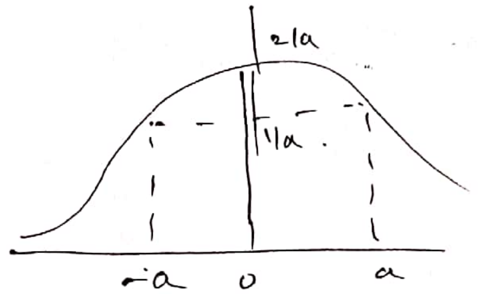
$$= \int_{-\infty}^{\infty} \left[e^{at}u(-t) + e^{-at}u(t)\right] e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{0} e^{(a-j\omega)t}\, dt + \int_{0}^{\infty} e^{-(a+j\omega)t}\, dt.$$

$$= \frac{1}{a-j\omega} e^{(a-j\omega)t} \Big|_{-\alpha}^{0} + \frac{1}{a+j\omega} e^{-(a+j\omega)} \Big|_{0}^{\alpha}$$

$$= \frac{1}{a-j\omega}(1-0) - \frac{1}{(a+j\omega)}(0-1) = \frac{1}{a-j\omega} + \frac{1}{a+j\omega}$$

$$X(\omega) = \frac{2a}{a^2+\omega^2}$$



spectrum.

③ $x(t) = \delta(t)$.

$$X(\omega) = \int_{-\alpha}^{\alpha} x(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\alpha}^{\alpha} \delta(t)\, e^{-j\omega t}\, dt$$

$$= e^{-j\omega t}\Big|_{t=0} = 1$$

$$\boxed{\delta(t) \longleftrightarrow 1}$$

④ I.F.T of $X(\omega) = \delta(\omega)$.

$$x(t) = \frac{1}{2\pi}\int_{-\alpha}^{\alpha} X(\omega)\, e^{j\omega t}\, d\omega$$

$$F^{-1}[\delta(\omega)] = \frac{1}{2\pi}\int_{-\alpha}^{\alpha} \delta(\omega)\, e^{j\omega t}\, d\omega$$

$$= \frac{1}{2\pi} e^{j\omega t}\Big|_{\omega=0} = \frac{1}{2\pi}$$

$$\boxed{\frac{1}{2\pi} \longleftrightarrow \delta(\omega)}$$

⑤ $X(\omega) = \delta(\omega-\omega_0)$

$$x(t) = \frac{1}{2\pi}\int_{-\alpha}^{\alpha} X(\omega)\, e^{j\omega t}\, d\omega$$

$$F^{-1}[\delta(\omega-\omega_0)] = \frac{1}{2\pi}\int_{-\alpha}^{\alpha} \delta(\omega-\omega_0)\, e^{j\omega t}\, d\omega$$

$$= \frac{1}{2\pi} e^{j\omega t}\Big|_{\omega=\omega_0} = \frac{1}{2\pi} e^{j\omega_0 t}$$

$$\therefore \frac{1}{2\pi} e^{j\omega_0 t} \longleftrightarrow \delta(\omega-\omega_0)$$

$$e^{j\omega_0 t} \longleftrightarrow 2\pi\,\delta(\omega-\omega_0)$$

$$e^{-j\omega_0 t} \longleftrightarrow 2\pi\,\delta(\omega+\omega_0)$$

⑥ $x(t) = \cos\omega_0 t$

$$\Rightarrow x(t) = \text{''} = \frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}$$

$$X(\omega) = F(x(t)) = F\left[\frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}\right]$$

$$= \frac{1}{2}\left[F(e^{j\omega_0 t}) + F[e^{-j\omega_0 t}]\right]$$

$$X(\omega) = \frac{1}{2}\left[2\pi\delta(\omega-\omega_0) + 2\pi\delta(\omega+\omega_0)\right]$$

$$= \pi\left[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)\right]$$

$$\cos\omega_0 t \longleftrightarrow \pi\left[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)\right]$$

⑦ $x(t) = \sin w_0 t = \frac{e^{jw_0t} - e^{-jw_0t}}{2j}$

$$X(w) = F\left[\frac{e^{jw_0t} - e^{-jw_0t}}{2j}\right]$$

$$= \frac{1}{2j}\left[F(e^{jw_0t}) - F(e^{-jw_0t})\right]$$

$$X(w) = \frac{1}{2j}\left[2\pi\delta(w-w_0) - 2\pi\delta(w+w_0)\right]$$

$$= \frac{\pi}{j}\left[\delta(w-w_0) - \delta(w+w_0)\right]$$

$$\boxed{\sin(w_0 t) \longleftrightarrow \frac{\pi}{j}\left[\delta(w-w_0) - \delta(w+w_0)\right]}$$

⑧ F.T of $u(t)$.

let $2u(t) = 1 + \mathrm{sgn}(t)$

$$u(t) = \frac{1}{2} + \frac{1}{2}\mathrm{sgn}(t)$$

$$F(u(t)) = U(w) = F\left[\frac{1}{2} + \frac{1}{2}\mathrm{sgn}(t)\right]$$

$$U(w) = \frac{1}{2}F(1) + \frac{1}{2}F[\mathrm{sgn}(t)]$$

$$= \frac{1}{2}2\pi\delta(w) + \frac{1}{2}\cdot\frac{2}{jw}$$

$$= \pi\delta(w) + \frac{1}{jw}$$

$$\boxed{u(t) \longleftrightarrow \pi\delta(w) + \frac{1}{jw}}$$

(10) $x(t) = \mathrm{sgn}(t) = \begin{cases} 1 & t>0 \\ 0 & t=0 \\ -1 & t<0 \end{cases}$

$$= u(t) - u(-t).$$

This is not absolutely integrable.

Thus

$$\mathrm{sgn}(t) = \lim_{a\to 0}\left[e^{-at}u(t) - e^{at}u(-t)\right]$$

$$X(w) = F[\mathrm{sgn}(t)] = \lim_{a\to 0}\left[F(e^{-at}u(t) - e^{at}u(-t))\right]$$

$$= \lim_{a\to 0}\left[F(e^{-at}u(t)) - F[e^{at}u(-t)]\right]$$

$$= \lim_{a\to 0}\left[\frac{1}{a+jw} - \frac{1}{a-jw}\right]$$

$$= \lim_{a\to 0}\left(\frac{-2jw}{a^2+w^2}\right) = \frac{-2jw}{w^2}$$

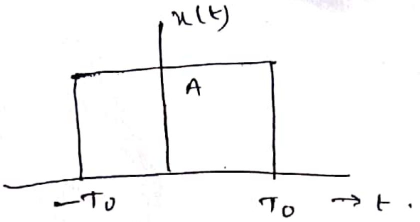
$$X(w) = F[\mathrm{sgn}(t)] = \frac{2}{jw}$$

$$\boxed{\mathrm{sgn}(t) \longleftrightarrow \frac{2}{jw}}$$

Scanned by CamScanner

27.22.20,

(11) $x(t) = A\, rec\left(\frac{t}{2T_0}\right) = \begin{cases} A & |t| < T_0 \\ 0 & |t| > T_0 \end{cases}$



$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt = \int_{-T_0}^{T_0} A e^{-j\omega t} dt$$

$$= A \frac{1}{-j\omega} e^{-j\omega t} \Big|_{-T_0}^{T_0} = \frac{A}{-j\omega}\left[e^{-j\omega T_0} - e^{j\omega T_0}\right] = \frac{2A}{\omega}\left[\frac{e^{j\omega T_0} - e^{-j\omega T_0}}{2j}\right]$$

$$= \frac{2A}{\omega} \sin(\omega T_0)$$

$$X(\omega) = 2AT_0 \frac{\sin\left(\pi \frac{\omega T_0}{\pi}\right)}{\frac{\pi \omega T_0}{\pi}} = 2AT_0 \operatorname{sinc}\left(\frac{\omega T_0}{\pi}\right)$$

(12) $g(t) = e^{-t} \sin(\omega_c t)\, u(t) = x(t) \cdot \sin\omega_c t$ where $x(t) = e^{-t} u(t)$

F.T of $x(t) = X(\omega) = \frac{1}{1+j\omega}$

Applying multiplication property

$$F[g(t)] = \frac{1}{2\pi}\left[F(x(t)) * F(\sin \omega_c t)\right]$$

$$G(\omega) = \frac{1}{2\pi}\left(X(\omega) * \frac{\pi}{j}\left[\delta(\omega-\omega_c) + \delta(\omega+\omega_c)\right]\right)$$

$$= \frac{1}{2j}\left[X(\omega-\omega_c) - X(\omega+\omega_c)\right]$$

Substituting $X(\omega) = \frac{1}{1+j\omega}$

$$G(\omega) = \frac{1}{2j}\left[\frac{1}{1+j(\omega-\omega_c)} - \frac{1}{1+j(\omega+\omega_c)}\right]$$

$$G(\omega) = \frac{\omega_c}{(1+j\omega)^2 + \omega_c^2}$$

25.27.22.20,

(13) F.T of $g(t) = e^{-at}\cos(\omega_c t)\, u(t)$.

$= x(t) \cdot \cos(\omega_c t)$

$x(t) = e^{-at} u(t)$ F.T of $x(t)$ $X(\omega) = \frac{1}{a+j\omega}$.

Applying multiplication property

$$F[g(t)] = \frac{1}{2\pi}\left[F(x(t)) * F(\cos\omega_c t)\right]$$

$$G(\omega) = \frac{1}{2\pi}\left[X(\omega) * \pi\left(\delta(\omega-\omega_c) + \delta(\omega-\omega_c)\right)\right]$$

$$= \frac{1}{2}\left[X(\omega-\omega_c) + X(\omega+\omega_c)\right]$$

Subituing $X(\omega) = \frac{1}{a+j\omega}$

$$G(\omega) = \frac{1}{2}\left[\frac{1}{a+j(\omega-\omega_c)} + \frac{1}{a+j(\omega+\omega_c)}\right]$$

$$= \frac{a+j\omega}{(a+j\omega)^2+\omega_c^2}$$

(14) $g(t) = t \cdot e^{-at} u(t)$.

Let $g(t) = t \cdot x(t)$

from differentiation in F.D we have

$$t \cdot x(t) \longleftrightarrow j\frac{dX(\omega)}{d\omega}$$

$$\therefore F[g(t)] = F[t \cdot x(t)] = j\frac{dX(\omega)}{d\omega}$$

Here $x(t) = e^{-at} u(t)$. F.T $X(\omega) = \frac{1}{a+j\omega}$

$$G(\omega) = F[t \cdot e^{-at} u(t)] = j\frac{d}{d\omega}\left(\frac{1}{a+j\omega}\right)$$

$$= \frac{1}{(a+j\omega)^2}$$

$$\boxed{t\, e^{-at} u(t) \longleftrightarrow \frac{1}{(a+j\omega)^2}}$$

2.5 27,22,20,

Find and sketch F.T of $x(t) = \delta_{T_0}(t)$.

Sol: $x(t)$ is periodic with $T = T_0$ fundamental freq $\omega_0 = \frac{2\pi}{T_0}$

$$x(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_0).$$

Impulse train for one period

$$x(t) = \delta(t) \quad -T_0/2 < t < T_0/2$$

The exponential F.S

$$X_n = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, e^{-jn\omega_0 t}\, dt$$

$$= \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)\, e^{-jn\omega_0 t}\, dt = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} \delta(t)\, e^{-jn\omega_0 t}\, dt$$

$$X_n = \frac{1}{T_0}.$$

F.T of periodic signal $x(t)$ is

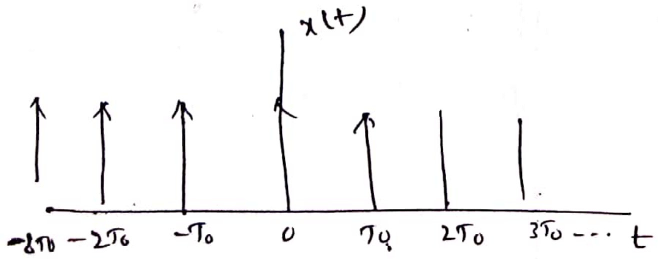
$$X(\omega) = 2\pi \sum_{n=-\infty}^{\infty} X_n\, \delta(\omega - n\omega_0)$$

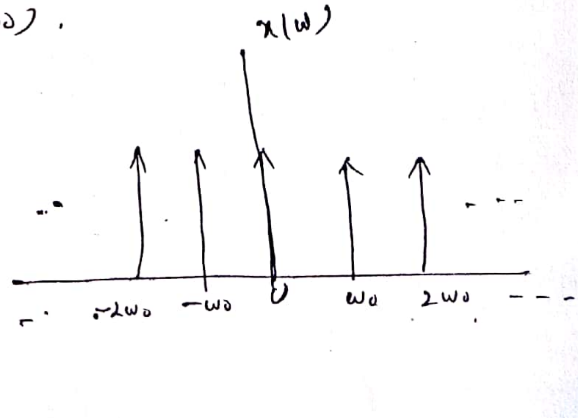
$$X_n = \frac{1}{T_0}$$

$$X(\omega) = 2\pi \sum_{n=-\infty}^{\infty} \frac{1}{T_0}\, \delta(\omega - n\omega_0)$$

$$= \frac{2\pi}{T_0} \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_0).$$

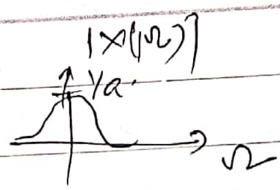
$$= \omega_0 \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_0).$$



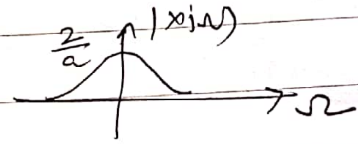


$\delta(t) \longleftrightarrow 1$

$e^{-at}u(t) \longleftrightarrow \dfrac{1}{j\Omega + a}$

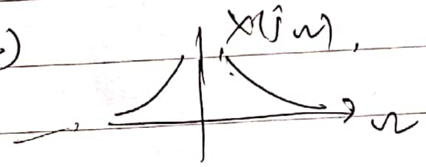


$e^{-a|t|} \longleftrightarrow \dfrac{2a}{\Omega^2 + a^2}$



$A \longleftrightarrow 2\pi A\,\delta(\Omega)$

$\mathrm{Sgn}(t) \longleftrightarrow \dfrac{2}{j\Omega}$



$\mathrm{Sgn}(t) = 2u(t) - 1$ ✓

or $u(t) - u(-t)$

$e^{-at}u(t) - e^{at}u(-t)$

$u(t) \longleftrightarrow \dfrac{1}{j\Omega}$ (or) $\pi\delta(\omega) + \dfrac{1}{j\omega}$.

$u(t) = \frac{1}{2}\left[1 + \mathrm{Sgn}(t)\right]$.

$t\,u(t) \longleftrightarrow \dfrac{1}{(j\Omega)^2}$

$t^n u(t) \longrightarrow \dfrac{n!}{(j\Omega)^{n+1}}$

$\dfrac{e^{j\theta} + e^{-j\theta}}{2} = \cos\theta$

$\dfrac{e^{j\theta} - e^{-j\theta}}{2j} = \sin\theta$

$e^{-at}\cos\Omega_0 t \longrightarrow \dfrac{j\omega + a}{(j\omega + a)^2 + \Omega_0^2}$.

$e^{-at}\sin\Omega_0 t \longrightarrow \dfrac{\Omega_0}{(j\Omega + a)^2 + \Omega_0^2}$.

① Det IFT using Partial fraction Method

$X(j\Omega) = \dfrac{3(j\Omega) + 14}{(j\Omega)^2 + 7(j\Omega) + 12}$

Ans: $x(t) = 5e^{-3t}u(t) - 2e^{-4t}u(t)$

② Det. Convolution of $x_1(t) = e^{-2t}u(t)$ & $x_2(t) = e^{-3t}u(t)$, using FT.

Ans: $y(t) = e^{-2t}u(t) - e^{-3t}u(t)$.

③ The impulse response of an LTI system is $h(t) = e^{-3t}u(t)$. Find the response of system for the I/P $x(t) = e^{-2t}u(t)$ using FT.

Ans: $e^{-2t}u(t) - e^{-3t}u(t)$

(2)

LAPLACE TRANSFORM

- The Laplace Transform is used to transform a time signal to complex frequency domain.
- Frequency domain is also known as Laplace domain or S-domain.

Complex frequency:

Complex frequency is defined as

Complex frequency, $s = \sigma + j\omega$

where σ = Neper frequency, in neper per second.

ω = Radian (or Real) frequency in radian per second.

The complex frequency is involved in the time domain signal of the form Ke^{st}.

Let $x(t) = Ae^{st} = Ae^{(\sigma + j\omega)t}$ —— (1)

Let us analyse the signal of equation for various choice of σ and ω.

Case(i) when $\sigma = 0$, $\omega = \omega_0$

$$\therefore x(t) = Ae^{(0 + j\omega_0)t}$$
$$= A[\cos\omega_0 t + j\sin\omega_0 t]$$
$$= A\cos\omega_0 t + jA\sin\omega_0 t$$

$Re\{x(t)\} = A\cos\omega_0 t$ → cosinusoidal signal

$Im\{x(t)\} = A\sin\omega_0 t$ → sinusoidal signal

Case ii) when $\omega = 0$

$$x(t) = Ae^{\sigma t}$$

if σ is +ve then the signal is exponentially increasing signal

σ is −ve " " " decreasing signal

i.e. $x(t) = Ae^{\sigma t}$ → Exponentially increasing signal.

$= Ae^{-\sigma t}$ → " decreasing "

Case iii) when $\sigma = 0$ and $\omega = 0$

$x(t) = Ae^{0} = A$ This represents a step signal.

Case iv) when $\sigma \neq 0$, $\omega \neq 0$ and $\omega = \omega_0$

$$x(t) = Ae^{(\sigma + j\omega_0)t}$$
$$= Ae^{\sigma t} \cdot e^{j\omega_0 t}$$
$$= Ae^{\sigma t}\left[\cos\omega_0 t + j\sin\omega_0 t\right]$$
$$= Ae^{\sigma t}\cos\omega_0 t + jAe^{\sigma t}\sin\omega_0 t$$

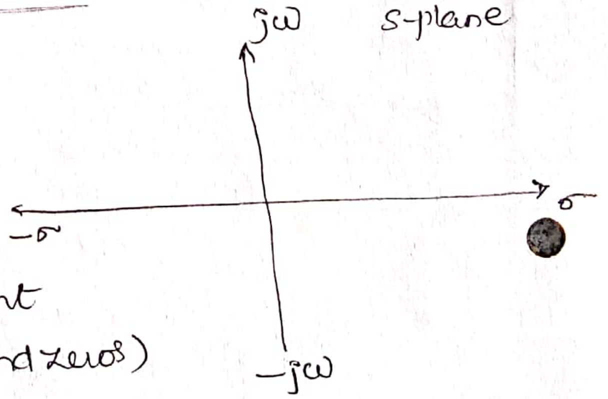
$Re\{x(t)\} = Ae^{\sigma t}\cos\omega_0 t$

- $Ae^{\sigma t}\cos\omega_0 t \rightarrow$ Exponentially increasing cosinusoidal signal
- $Ae^{-\sigma t}\cos\omega_0 t \rightarrow$ " decreasing "

$Im\{x(t)\} = Ae^{\sigma t}\sin\omega_0 t$

- $Ae^{\sigma t}\sin\omega_0 t \rightarrow$ Exponentially ~~De~~ increasing sinusoidal signal
- $Ae^{-\sigma t}\sin\omega_0 t \rightarrow$ " Decreasing "

Complex frequency plane (or) S-plane



Complex frequency is defined as

$$s = \sigma + j\omega$$

σ and ω values from $-\infty$ to $+\infty$.

- The s-plane is used to ~~pr~~ represent various critical frequencies (poles and zeros) of signals which are functions of s and to study the path taken by these critical frequencies when some parameters of the signals are varied.
- This study will be useful to design systems for a desired response.

Definition of Laplace Transform

$$\boxed{L\{x(t)\} = X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt} \rightarrow$$ Two sided L.T. (or) Bilateral L.T.

If $x(t)$ is defined for $t \geq 0$ (i.e. $x(t)$ is causal) then

$$\boxed{L\{x(t)\} = X(s) = \int_{0}^{\infty} x(t)\, e^{-st}\, dt} \rightarrow$$ one sided L.T. (or) Unilateral L.T.

Definition of Inverse Laplace Transform

$$\boxed{L^{-1}\{X(s)\} = x(t) = \frac{1}{2\pi j}\int_{s=\sigma-j\omega}^{s=\sigma+j\omega} X(s)\, e^{st}\, ds}$$

$$x(t) \underset{L^{-1}}{\overset{L}{\rightleftarrows}} X(s)$$

Existence of Laplace Transform

1) L.T. of a signal exists if the integral $\int_{-\infty}^{\infty} x(t)\, e^{-st} dt$ converges (i.e. finite)

2) The integral will converge if the signal $x(t)$ is sectionally continuous in every finite interval of t and if it is of exponential order as t approaches infinity.

3) A causal signal $x(t)$ is said to be exponential order if a real, positive constant σ (where σ is real part of s) exists such that the function, $e^{-\sigma t}|x(t)|$ approaches zero as t approaches to infinity.

i.e. if $\lim_{t\to\infty} |x(t)| = 0$, when $x(t)$ is of exponential order.

4) For a causal signal, if $\lim_{t\to\infty} e^{-\sigma t}|x(t)| = 0$ for $\sigma > \sigma_c$ and if $\lim_{t\to\infty} e^{-\sigma t}|x(t)| = \infty$ for $\sigma < \sigma_c$, then σ_c is called abscissa of convergence. (where σ_c is a point on real axis in s-plane)

5) $\int_{-\infty}^{\infty} x(t)\, e^{-st} dt$ converges only if the real part of s is greater than the abscissa of convergence σ_c.

The values of s for which the integral $\int_{-\infty}^{\infty} x(t)\, e^{-st} dt$ converges is called Region of convergence (ROC). Therefore for a causal signal the ROC includes all points on the s-plane to the right of abscissa of convergence

Region of Convergence

The values of s for which the integral $\int_{-\infty}^{\infty} x(t)\, e^{-st} dt$ converges is called Region of Convergence (ROC).

The ROC for the following three types of signals are discussed here.

Case(i) Right sided (causal) signal

case ii) Left sided (anticausal) signal

case iii) Two sided signal.

Case(i) Right sided (causal) signal.

Let $x(t) = e^{-at} u(t)$, where $a > 0$

$= e^{-at}$ for $t \geq 0$.

$$L\{x(t)\} = X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st} dt = \int_{-\infty}^{\infty} e^{-at} u(t)\, e^{-st} dt$$

$$= \int_0^{\infty} e^{-at} e^{-st} dt = \int_0^{\infty} e^{-(s+a)t} dt$$

$$= \left[\frac{e^{-(s+a)t}}{-(s+a)}\right]_0^{\infty}$$

$$= \left[\frac{e^{-(\sigma+j\omega+a)\infty}}{-(s+a)} - \frac{e^{0}}{-(s+a)}\right]$$

$$= -\frac{e^{-(\sigma+a)\infty} \cdot e^{-j\omega\times\infty}}{s+a} + \frac{1}{s+a}$$

$$L\{x(t)\} = -\frac{e^{-k\infty} \cdot e^{-j\omega\times\infty}}{s+a} + \frac{1}{s+a}$$

where $k = \sigma + a = \sigma - (-a)$

when $\sigma > -a$, $k = \sigma - (-a) = $ positive, $\therefore e^{-k\infty} = e^{-\infty} = 0$.

when $\sigma < -a$, $k = \sigma - (-a) = $ Negative $\therefore e^{-k\infty} = e^{\infty} = \infty$

Hence we can say that $X(s)$ converges, when $\sigma > -a$ and does not converge for $\sigma < -a$.

$\therefore$ Abscissa of convergence, $\sigma_c = -a$

when $\sigma > -a$, the $X(s)$ is given by

$$L\{x(t)\} X(s) = -\frac{e^{-k\infty} e^{-j\omega\infty}}{s+a} + \frac{1}{s+a} = \frac{1}{s+a}.$$

Therefore for a causal signal ROC includes all points on s-plane to the right of abscissa of convergence $\sigma_c = -a$

Case ii) Left sided (anticausal) signal

Let $x(t) = e^{-bt}u(-t) = e^{-bt}$ for $t \le 0$, where $b > 0$.

$$L\{x(t)\} = X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt = \int_{-\infty}^{\infty} e^{-bt}u(-t)\, e^{-st}\, dt$$

$$= \int_{-\infty}^{0} e^{-bt} e^{-st}\, dt$$

$$= \int_{-\infty}^{0} e^{-(s+b)t}\, dt = \left[\frac{e^{-(s+b)t}}{-(s+b)}\right]_{-\infty}^{0}$$

$$= \frac{e^{0}}{-(s+b)} - \frac{e^{+(\sigma+j\omega+b)\infty}}{-(s+b)}$$

$$= -\frac{1}{s+b} + \frac{e^{(\sigma+b)\times\infty} \cdot e^{j\omega\times\infty}}{s+b}$$

$$= -\frac{1}{s+b} + \frac{e^{K\infty} \cdot e^{j\omega\times\infty}}{s+b}$$

where $K = \sigma + b = \sigma - (-b)$

When $\sigma > -b$, $K = \sigma - (-b) =$ Positive, $\therefore e^{K\infty} = e^{\infty} = \infty$

When $\sigma < -b$, $K = \sigma - (-b) =$ Negative, $\therefore e^{-K\infty} = e^{-\infty} = 0$

Hence we can say that, $X(s)$ converges when $\sigma < -b$, and does not converge for $\sigma > -b$

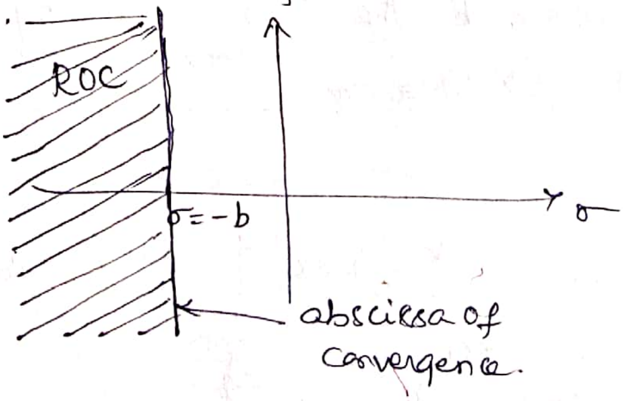
$\therefore$ Abscissa of convergence, $\sigma_c = -b$.

When $\sigma < -b$, the $X(s)$ is given by

$$L\{x(t)\} = X(s) = -\frac{1}{s+b} + \frac{e^{K\infty} \cdot e^{j\omega\infty}}{s+b}$$

$$= -\frac{1}{s+b} + \frac{0 \cdot e^{j\omega\infty}}{s+b} = -\frac{1}{s+b}$$

Therefore for an anticausal signal ROC includes all points on the s-plane to the left of abscissa of convergence $\sigma_c = -b$.



case iii) Two sided signal.

Let $x(t) = e^{-at}u(t) + e^{-bt}u(-t)$ where $a>0$, $b>0$ and $a>b$
i.e. $(-a<-b)$

Now Laplace transform of $x(t)$

$$L\{x(t)\} = X(S) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

$$= \int_{-\infty}^{\infty} [e^{-at}u(t) + e^{-bt}u(-t)]e^{-st}dt$$

$$= \int_{0}^{\infty} e^{-at}e^{-st}dt + \int_{-\infty}^{0} e^{-bt}e^{-st}dt$$

$$= \left[\frac{e^{-(s+a)t}}{-(s+a)}\right]_0^{\infty} + \left[\frac{e^{-(b+s)t}}{-(s+b)}\right]_{-\infty}^{0}$$

$$= \frac{e^{-(\sigma+j\omega+a)\infty}}{-(s+a)} - \frac{e^0}{-(s+a)} + \frac{e^0}{-(s+b)} + \frac{e^{+(\sigma+j\omega+b)\infty}}{-(s+b)}$$

$$= \frac{e^{-(p+j\omega)\infty}}{-(s+a)} + \frac{e^{(q+j\omega)\infty}}{-(s+b)} + \frac{1}{s+a} - \frac{1}{s+b} \quad \left\{\begin{array}{l}\text{where} \\ p=\sigma+a \\ q=\sigma+b\end{array}\right\}$$

$$= \frac{e^{-p\times\infty}\,e^{-j\omega\times\infty}}{-(s+a)} + \frac{1}{s+a} - \frac{1}{s+b} + \frac{e^{q\times\infty}\cdot e^{j\omega\times\infty}}{+(s+b)}$$

where $p = \sigma + a = \sigma - (-a)$

and $q = \sigma + b = \sigma - (-b)$

when $\sigma > -a$, $p = \sigma - (-a) =$ positive, $\therefore e^{-p\infty} = e^{-\infty} = 0$

when $\sigma < -a$, $p = \sigma - (-a) =$ Negative, $\therefore e^{-p\infty} = e^{+\infty} = \infty$

when $\sigma > -b$, $q = \sigma - (-b) =$ positive, $\therefore e^{q\infty} = e^{\infty} = \infty$

when $\sigma < -b$, $q = \sigma - (-b) =$ Negative, $\therefore e^{q\infty} = e^{-\infty} = 0$.

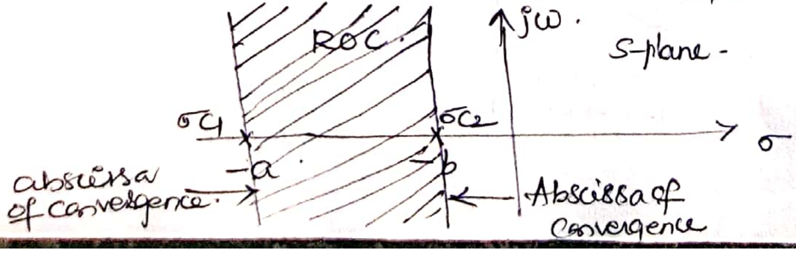
Hence we can say that $X(s)$ converges, when σ lies b/w $-a$ and $-b$ (i.e. $-a<\sigma<-b$) and does not converge for $\sigma<-a$ and $\sigma>-b$

$\therefore$ Abscissa of convergences $\sigma_{c_1} = -a$, and $\sigma_{c_2} = -b$

when $-a<\sigma<-b$, the $X(s)$ is given by

$$L\{x(t)\} = X(s) = -\frac{e^{-p\times\infty}\,e^{-j\omega\times\infty}}{s+a} + \frac{1}{s+a} + \frac{1}{s+b} + \frac{e^{q\times\infty}\cdot e^{j\omega\times\infty}}{s+b}$$

$$= -0 + \frac{1}{s+a} - \frac{1}{s+b} + 0. \quad = \frac{1}{s+a} - \frac{1}{s+b}.$$



Scanned by CamScanner

Properties and Theorems of Laplace Transforms.

1) Amplitude Scaling

If $L\{x(t)\} = X(S)$

Then $L\{A\,x(t)\} = A\,X(S)$

Proof: From Definition of L.T.

$$X(S) = L\{x(t)\} = \int_{-\infty}^{\infty} x(t)\, e^{-St}\, dt$$

$$L\{A\,x(t)\} = \int_{-\infty}^{\infty} A\, x(t)\, e^{-St}\, dt$$

$$= A\,X(S)$$

2) Linearity

If $L\{x_1(t)\} = X_1(S)$ and $L\{x_2(t)\} = X_2(S)$

Then $L\{a_1 x_1(t) + a_2 x_2(t)\} = a_1 X_1(S) + a_2 X_2(S)$

Proof:

$$X_1(S) = L\{x_1(t)\} = \int_{-\infty}^{\infty} x_1(t)\, e^{-St}\, dt \quad \text{— ①}$$

$$X_2(S) = L\{x_2(t)\} = \int_{-\infty}^{\infty} x_2(t)\, e^{-St}\, dt \quad \text{— ②}$$

$$L\{a_1 x_1(t) + a_2 x_2(t)\} = \int_{-\infty}^{\infty} [a_1 x_1(t) + a_2 x_2(t)]\, e^{-St}\, dt$$

$$= a_1 \int_{-\infty}^{\infty} x_1(t)\, e^{-St}\, dt + a_2 \int_{-\infty}^{\infty} x_2(t)\, e^{-St}\, dt$$

$$= a_1 X_1(S) + a_2 X_2(S)$$

3) Time Differentiation

If $L\{x(t)\} = X(S)$

Then $L\{\frac{d}{dt} x(t)\} = S\,X(S) - x(0)$; where $x(0)$ is value of $x(t)$ at $t=0$.

Proof:

$$L\{x(t)\} = X(S) = \int_0^{\infty} x(t)\, e^{-St}\, dt$$

$$L\{\frac{d}{dt} x(t)\} = \int_0^{\infty} \{\underbrace{\frac{d}{dt} x(t)}_{dv}\}\underbrace{e^{-St}}_{u}\, dt$$

$$= [e^{-St} x(t)]_0^{\infty} - [\int_0^{\infty} x(t)\, e^{-St} (-s)\, dt] = e^{-\infty} x(\infty) - e^{0} x(0) + S\int_0^{\infty} x(t) e^{-St} dt$$

$$= S\,X(S) - x(0)$$

4) Time Integration

The time integration property sates that, if a causal signal $x(t)$ is continuous and Laplace transform of $x(t)$ is $X(S)$, then

i.e. If $L\{x(t)\} = X(S)$

then $L\{\int x(t)\,dt\} = \frac{X(S)}{S} + \frac{[\int x(t)\,dt]_{t=0}}{S}$

Proof: $X(S) = L\{x(t)\} = \int_0^\infty x(t)\,e^{-st}\,dt$

$$L\{\int x(t)\,dt\} = \int_0^\infty \underbrace{[\int x(t)\,dt]}_{u}\ \underbrace{e^{-st}\,dt}_{dv}$$

$$= \left[[\int x(t)\,dt]\,\frac{e^{-st}}{-s}\right]_0^\infty - \left[\int_0^\infty \frac{e^{-st}}{-s}\cdot x(t)\,dt\right]$$

$$= [\int x(t)\,dt]_{t=\infty}\frac{e^{-\infty}}{-S} - [\int x(t)\,dt]_{t=0}\frac{e^{0}}{-S} + \frac{1}{S}\int_0^\infty x(t)\,e^{-st}\,dt$$

$$= 0 + \frac{1}{S}[\int x(t)\,dt]_{t=0} + \frac{1}{S}\int_0^\infty x(t)\,e^{-st}\,dt$$

$$= \frac{X(S)}{S} + \frac{[\int x(t)\,dt]_{t=0}}{S}$$

5) Frequency Shifting

If $L\{x(t)\} = X(S)$

Then $L\{e^{\pm at}x(t)\} = X(S\mp a)$ [i.e. $L\{e^{at}x(t)\} = X(S-a)$ and $L\{e^{-at}x(t)\} = X(S+a)$]

Proof:

$$X(S) = L\{x(t)\} = \int_{-\infty}^{\infty} x(t)\,e^{-st}\,dt$$

$$L\{e^{\pm at}x(t)\} = \int_{-\infty}^{\infty} e^{\pm at}\,x(t)\,e^{-st}\,dt$$

$$= \int_{-\infty}^{\infty} x(t)\,e^{-(S\mp a)t}\,dt$$

$$= X(S\mp a)$$

6) Time Shifting

If $L\{x(t)\} = X(s)$

Then $L\{x(t \pm a)\} = e^{\pm as} X(s)$ [i.e. $L\{x(t+a)\} = e^{as} X(s)$ & $L\{x(t-a)\} = e^{-as} X(s)$]

Proof:

$$X(s) = L\{x(t)\} = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

$$\therefore L\{x(t \pm a)\} = \int_{-\infty}^{\infty} x(t \pm a) e^{-st} dt$$

Let $t \pm a = \tau$

$t = \tau \mp a$

$dt = d\tau$

$$= \int_{-\infty}^{\infty} x(\tau) e^{-s(\tau \mp a)} dt$$

$$= \int_{-\infty}^{\infty} x(\tau) e^{-s\tau} . e^{\pm as} dt$$

$$= e^{\pm as} \int_{-\infty}^{\infty} x(\tau) e^{-s\tau} d\tau$$

$$= e^{\pm as} X(s)$$

7) Frequency Differentiation

i.e. If $L\{x(t)\} = X(s)$

Then $L\{t\,x(t)\} = -\frac{d}{ds} X(s)$

Proof:

$$X(s) = L\{x(t)\} = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

Differentiate above

$$\frac{dX(s)}{ds} = \frac{d}{ds}\left[\int_{-\infty}^{\infty} x(t) e^{-st} dt\right]$$

$$= \int_{-\infty}^{\infty} x(t) \left(\frac{d}{ds} e^{-st}\right) dt$$

$$= \int_{-\infty}^{\infty} x(t)\, e^{-st} (-t)\, dt$$

$$= \int_{-\infty}^{\infty} \{-t\, x(t)\} e^{-st} dt$$

$$= -L\{t\,x(t)\}$$

$$\therefore L\{t\,x(t)\} = -\frac{d}{ds} X(s)$$

8) <u>Frequency Integration</u>

If $L\{x(t)\} = X(s)$

Then $L\{\frac{1}{t}x(t)\} = \int_s^{\infty} X(s)\,ds$

<u>Proof</u>:

$$X(s) = L\{x(t)\} = \int_{-\infty}^{\infty} x(t)e^{-st}\,dt$$

$$\int_s^{\infty} X(s)\,ds = \int_s^{\infty}\left[\int_{-\infty}^{\infty} x(t)e^{-st}\,dt\right]ds$$

$$= \int_{-\infty}^{\infty} x(t)\left[\int_s^{\infty} e^{-st}\,ds\right]dt$$

$$= \int_{-\infty}^{\infty} x(t)\left[\frac{e^{-st}}{-t}\right]_s^{\infty} dt$$

$$= \int_{-\infty}^{\infty} x(t)\left[\frac{e^{-\infty}}{-t} - \frac{e^{-st}}{-t}\right]dt = \int_{-\infty}^{\infty} x(t)\left[0 + \frac{e^{-st}}{t}\right]dt$$

$$= L\left\{\frac{1}{t}x(t)\right\}$$

9) <u>Time Scaling</u>

If $L[x(t)] = X(s)$

Then $L\{x(at)\} = \frac{1}{|a|}X\left(\frac{s}{a}\right)$

<u>Proof</u>

$$X(s) = L\{x(t)\} = \int_{-\infty}^{\infty} x(t)e^{-st}\,dt$$

$$\therefore L\{x(at)\} = \int_{-\infty}^{\infty} x(at)e^{-st}\,dt$$

Let $at = \tau$

$dt = \frac{d\tau}{a}$

$$= \int_{-\infty}^{\infty} x(\tau)e^{-\left(\frac{s}{a}\right)\tau}\frac{d\tau}{a}$$

$$= \frac{1}{a}X\left(\frac{s}{a}\right)$$

The above transform is applicable for positive values of 'a'

If 'a' be -ve then $L\{x(at)\} = \frac{1}{-a}X\left(\frac{s}{a}\right)$

Hence in general

$$L\{x(at)\} = \frac{1}{|a|}X\left(\frac{s}{a}\right)$$

10) Periodicity

If $x(t) = x(t+nT)$ and $x_1(t)$ be one period of $x(t)$

and $L\{x_1(t)\} = \int_0^T x_1(t)\, e^{-st}\, dt$

Then $L\{x(t+nT)\} = \frac{1}{1-e^{-sT}} \int_0^T x_1(t)\, e^{-st}\, dt$

Proof:

$$L\{x(t+nT)\} = \int_0^\infty x(t+nT)\, e^{-st}\, dt$$

$$= \int_0^T x_1(t)\, e^{-st} dt + \int_T^{2T} x_1(t-T)\, e^{-s(t+T)} dt + \int_{2T}^{3T} x_1(t-2T)\, e^{-s(t+2T)} dt + \dots + \int_{PT}^{(P+1)T} x_1(t-PT)\, e^{-s(t+PT)} dt + \dots$$

$$= \sum_{P=0}^{\infty} \int_{PT}^{(P+1)T} x_1(t-PT)\, e^{-s(t+PT)}\, dt$$

$$= \sum_{P=0}^{\infty} \int_0^T x_1(t)\, e^{-st}\, e^{-PST}\, dt$$

$$= \int_0^T x_1(t)\, e^{-st} \left(\sum_{P=0}^{\infty} e^{-PST} \right) dt$$

$$= \int_0^T x_1(t)\, e^{-st} \left[\sum_{P=0}^{\infty} e^{-ST} \right]^P dt$$

$$= \int_0^T x_1(t)\, e^{-st} \left(\frac{1}{1-e^{-ST}} \right) dt$$

$$= \frac{1}{1-e^{-ST}} \int_0^T x_1(t)\, e^{-st}\, dt$$

11) Initial Value Theorem

It states that, if $x(t)$ and its derivative are Laplace Transformable then

$$\boxed{\lim_{t\to 0} x(t) = \lim_{s\to\infty} SX(S)}$$

i.e. Initial value of signal, $x(0) = \lim_{t\to 0} x(t)$
$= \lim_{S\to\infty} SX(S)$

Proof: We know that $L\left\{\frac{dx(t)}{dt}\right\} = SX(S) - x(0)$

on taking limit $S\to\infty$ on both sides

$$\lim_{S\to\infty} L\left\{\frac{dx(t)}{dt}\right\} = \lim_{S\to\infty} [SX(S) - x(0)]$$

$$\lim_{S\to\infty} \int_0^\infty \frac{dx(t)}{dt}\, e^{-st}\, dt = \lim_{S\to\infty} [SX(S) - x(0)]$$

$$\int_0^\infty \frac{dx(t)}{dt} \Big(\underbrace{\lim_{S\to\infty} e^{-St}}_{=0}\Big)\, dt = \lim_{S\to\infty} SX(S) - x(0).$$

$$0 = \lim_{S\to\infty} SX(S) - x(0)$$

$$\therefore x(0) = \lim_{S\to\infty} SX(S)$$

$$\therefore \lim_{t\to 0} x(t) = \lim_{S\to\infty} SX(S)$$

12) Final Value Theorem

It states that if $x(t)$ and derivative are Laplace Transformable

Then $\boxed{\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s)}$

i.e. Final value of Signal, $x(\infty) = \lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s)$

Proof:

$$L\left\{\frac{dx(t)}{dt}\right\} = sX(s) - x(0)$$

on taking limit $s\to 0$ on both sides

$$\lim_{s\to 0} L\left\{\frac{dx(t)}{dt}\right\} = \lim_{s\to 0}\left[sX(s) - x(0)\right]$$

$$\lim_{s\to 0}\int_0^\infty \frac{dx(t)}{dt} e^{-st} dt = \lim_{s\to 0}\left[sX(s) - x(0)\right]$$

$$\int_0^\infty \frac{dx(t)}{dt}\left\{\lim_{s\to 0} e^{-st}\right\} dt = \left(\lim_{s\to 0} sX(s)\right) - x(0)$$

$$\int_0^\infty \frac{dx(t)}{dt}\, dt = \lim_{s\to 0} sX(s) - x(0)$$

$$\left[x(t)\right]_0^\infty = \lim_{s\to 0} sX(s) - x(0)$$

$$x(\infty) - \cancel{x(0)} = \lim_{s\to 0} sX(s) - \cancel{x(0)}$$

$$\therefore x(\infty) = \lim_{s\to 0} sX(s)$$

$$\boxed{\therefore \lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s)}$$

13) Convolution Theorem.

$$y(t) = x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau.$$

If $L\{x_1(t)\} = X_1(s)$ and $L\{x_2(t)\} = X_2(s)$

then $L\{x_1(t) * x_2(t)\} = X_1(s)\, X_2(s)$.

Proof: $X_1(s) = \int_{-\infty}^{\infty} x_1(t) e^{-st} dt$ —— ①

$X_2(s) = \int_{-\infty}^{\infty} x_2(t) e^{-st} dt$ —— ②

$$L\{x_1(t) * x_2(t)\} = \int_{-\infty}^{\infty}\{x_1(t) * x_2(t)\} e^{-st} dt$$

$$= \int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau\right\} e^{-st} dt$$

put $t-\tau = m$

$dt = dm$.

$$= \int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty} x_1(\tau)\, x_2(m)\, d\tau\right\} e^{-s(m+\tau)}\, dm$$

$$= \int_{-\infty}^{\infty} x_1(\tau)\, e^{-s\tau} d\tau \int_{-\infty}^{\infty} x_2(m) e^{-sm} dm$$

$$= X_1(s) \cdot X_2(s)$$

Scanned by CamScanner

① Determine the Laplace Transform of the following continuous time signals and their ROC. ⑦

a) $x(t) = A\,u(t)$ — Ans: A/s ROC → $\sigma > 0$

b) $x(t) = t\,u(t)$ — Ans: $1/s^2$ ROC → $\sigma > 0$

c) $x(t) = e^{-3t}u(t)$ → Ans: $\frac{1}{s+3}$ ROC → $\sigma > -3$

d) $x(t) = e^{-3t}u(-t)$ → Ans: $-\frac{1}{s+3}$ ROC → $\sigma < -3$

e) $x(t) = e^{-4|t|}$ → Ans: $\frac{1}{s+4} - \frac{1}{s-4}$ ROC → $-4 < \sigma < 4$.

2) Determine the Laplace Transform of the following signals.

a) $x(t) = \sin\omega_0 t\; u(t)$ → $X(s) = \frac{\omega_0}{s^2+\omega_0^2}$ ROC = $\sigma > 0$

b) $x(t) = \cos\omega_0 t\; u(t)$ → $X(s) = \frac{s}{s^2+\omega_0^2}$ ROC = $\sigma > 0$

c) $x(t) = \cosh\omega_0 t\; u(t)$ → $X(s) = \frac{\omega_0}{s^2-\omega_0^2}$ ROC = $\sigma > \omega_0$

d) $x(t) = \sinh\omega_0 t\; u(t)$ → $X(s) = \omega_0/s^2-\omega_0^2$ ROC = $\sigma > \omega_0$

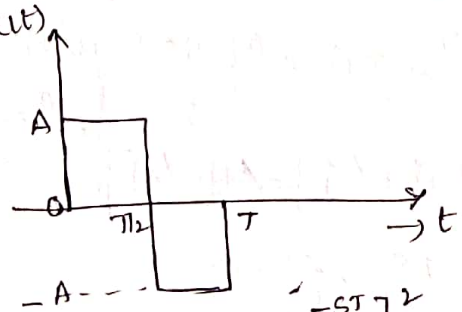
e) $x(t) = e^{-at}\sin\omega_0 t\; u(t)$ → $X(s) = \frac{\omega_0}{(s+a)^2+\omega_0^2}$ $\sigma > -a$

f) $x(t) = e^{-at}\cos\omega_0 t\; u(t)$ → $X(s) = \frac{s+a}{(s+a)^2+\omega_0^2}$ $\sigma > -a$

g) $x(t) = t e^{-at}u(t)$ → $X(s) = \frac{1}{(s+a)^2}$, ROC = $\sigma > -a$.

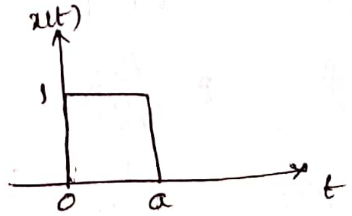
3) Determine L.T. of signals shown below

a)



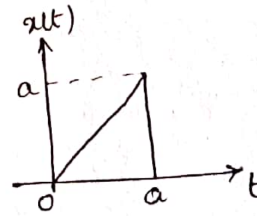
Ans: $\frac{A}{s}\left[1-e^{-\frac{sT}{2}}\right]^2$

b)



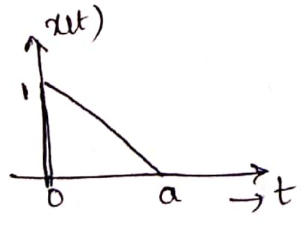
Ans: $\frac{1}{s}(1-e^{-as})$

c)



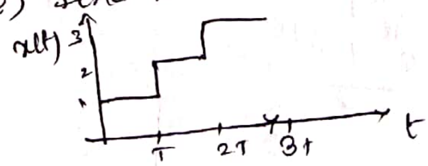
Ans: $\frac{1}{s^2}\left[1-e^{-as}(1+as)\right]$

d)



Ans: $\frac{1}{as^2}\left[e^{-as}+as-1\right]$

e) Find L.T. of stair case function



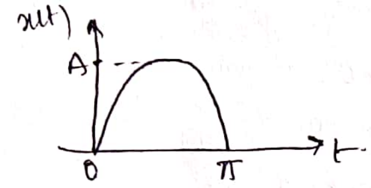
$x(t) = u(t) + u(t-2T) + u(t-3T) \ldots$

$L\{x(t)\} = \frac{1}{s} + e^{-T}\frac{1}{s} + \frac{1}{s}e^{-2T} + \ldots$

$= \frac{1}{s}\left[1 + e^{-T} + e^{-2T} + \ldots\right]$ $S_\infty = \frac{a}{1-r}$

$X(s) = \frac{1}{s(1-e^{-T})}$ //.

4) Determine the L.T. of Sine pulse.



Ans: $\frac{A}{s^2+1}\left(e^{-\pi s}+1\right)$.

Sol $x(t) = A\sin t, \quad 0<t<\pi$

$= 0, \quad \text{for } t>\pi$

$$L\{x(t)\} = X(s) = \int_0^\infty x(t)\, e^{-st}\,dt = \int_0^\pi A\sin t\, e^{-st}\,dt$$

$$= A\int_0^\pi \frac{e^{jt}-e^{-jt}}{2j}\, e^{-st}\,dt$$

$$= \frac{A}{2j}\int_0^\pi \left[e^{-(s-j)t} - e^{-(s+j)t}\right] dt$$

$$= \frac{A}{2j}\left[\frac{e^{-(s-j)t}}{-(s-j)} - \frac{e^{-(s+j)t}}{-(s+j)}\right]_0^\pi$$

$$= \frac{A}{2j}\left[\frac{e^{-(s+j)t}}{s+j} - \frac{e^{-(s-j)t}}{s-j}\right]_0^\pi$$

$$= \frac{A}{2j}\left[\frac{(s-j)\,e^{-(s+j)t} - (s+j)\,e^{-(s-j)t}}{s^2-j^2}\right]_0^\pi$$

$$= \frac{A}{2j}\left[\frac{(s-j)\,e^{-st}\,e^{-jt} - (s+j)\,e^{-st}\cdot e^{jt}}{s^2+1}\right]_0^\pi$$

$$= \frac{A}{2j}\left[\left\{\frac{(s-j)\,e^{-s\pi}\cdot e^{-j\pi} - (s+j)\,e^{-s\pi}\,e^{j\pi}}{s^2+1}\right\} - \left\{\frac{(s-j)\,e^{0} + (s+j)\,e^{0}}{s^2+1}\right\}\right]$$

$$= \frac{A}{2j(s^2+1)}\left[\left\{-(s-j)\,e^{-s\pi} + (s+j)\,e^{-s\pi}\right\} - (s-j) + (s+j)\right]$$

$e^{\pm j\pi} = \cos\pi \pm j\sin\pi = -1\pm 0 = -1$

$$= \frac{A}{2j(s^2+1)}\left[e^{-\pi s}\left[-\cancel{s}+j+\cancel{s}+j\right] - \cancel{s}+j+\cancel{s}+j\right]$$

$$= \frac{A}{2j(s^2+1)}\left[2j\,e^{-\pi s} + 2j\right]$$

$$= \frac{A}{s^2+1}\left(e^{-\pi s}+1\right)$$

1) $\sin\theta = \frac{e^{j\theta}-e^{-j\theta}}{2j}$

2) $\cos\theta = \frac{e^{j\theta}+e^{-j\theta}}{2}$

3) $\sinh\theta = \frac{e^{\theta}-e^{-\theta}}{2}$

4) $\cosh\theta = \frac{e^{\theta}+e^{-\theta}}{2}$

5) $e^{\pm j\theta} = \cos\theta \pm j\sin\theta$

Scanned by CamScanner

Solution of Differential Equations

① using L.T. determine the natural response of the system described by the equation

$$\frac{d^2y(t)}{dt} + 6\frac{dy(t)}{dt} + 5y(t) = \frac{dx(t)}{dt} + 4x(t) \quad ; \quad y(0) = 1 \; ; \; \left.\frac{dy(t)}{dt}\right|_{t=0} = -2$$

NOTE

1) The natural response is the response of the system due to initial values of output alone. Hence for Natural response the input $x(t)$ is considered as zero. Therefore the natural response is also called zero-input response, $y_{zi}(t)$

2) The forced response (zero-state response) is the response of the system due to input alone. For forced response, the system equation is solved for the given input with zero initial output. (But initial values of input should be considered).

3) Total Response $= y(t) = y_{zi}(t) + y_{zs}(t)$

$$\boxed{\frac{d^n y(t)}{dt^n} = s^n Y(s) - s^{n-1}y(0) - s^{n-2}y'(0) - s^{n-3}y''(0)}$$

Sol on substituting i/p $x(t) = 0$ and $y(t) = y_{zi}(t)$

$$\frac{d^2y_{zi}(t)}{dt} + \frac{6dy_{zi}(t)}{dt} + 5y_{zi}(t) = 0.$$

Taking L.T.

$$s^2Y(s) - sy(0) - y'(0) + 6[sY(s) - y(0)] + 5Y(s) = 0$$

$$s^2Y(s) - s(1) - (-2) + 6[sY(s) - 1] + 5Y(s) = 0.$$

$$s^2Y(s) - s + 2 + 6sY(s) - 6 + 5Y(s) = 0.$$

$$[s^2 + 6s + 5]\,Y(s) - s - 4 = 0$$

$$(s^2 + 6s + 5)\,Y(s) = s + 4$$

$$\therefore Y(s) = \frac{s+4}{s^2+6s+5} = \frac{s+4}{(s+1)(s+5)}$$

$$Y(s) = \frac{A}{s+1} + \frac{B}{s+5}$$

$$\left.\frac{A}{s+1}\right|_{s=-1} = \frac{s+4}{s+3} = \frac{-1+4}{-1+5} = 3/4 = 3/4$$

$$\left.\frac{B}{s+5}\right|_{s=-5} = \frac{s+4}{s+1} = \frac{-5+4}{-5+1} = \frac{-1}{-4} = 1/4.$$

$$\therefore Y(S) = \frac{3}{4}\frac{1}{S+1} + \frac{1}{4}\frac{1}{S+5}$$

$$L^{-1}[Y(S)] = y(t) = \frac{3}{4} \cdot e^{-t} u(t) + \frac{1}{4} e^{-5t} u(t)$$

$$= \frac{1}{4}\left[3e^{-t} + e^{-5t}\right] u(t)$$

2) using Laplace Transform determine the forced response of the system described by the eqn.

$$5\frac{dy(t)}{dt} + 10y(t) = 2x(t) \quad ; \quad \text{for the input }, x(t) = 2u(t)$$

Sol

$x(t) = 2u(t)$

$$\therefore X(S) = L\{2u(t)\} = \frac{2}{S}$$

$$5\frac{dy(t)}{dt} + 10y(t) = 2x(t)$$

Taking L.T.

$$5SY(S) + 10Y(S) = 2X(S).$$

$$5(S+2)Y(S) = 2X(S)$$

$$Y(S) = \frac{2}{5(S+2)} X(S)$$

$$= \frac{2}{5(S+2)} \cdot \frac{2}{S}$$

$$= \frac{4}{5}\frac{1}{S(S+2)}$$

$$= \frac{A}{S} + \frac{B}{S+2}.$$

$$A = \frac{1}{S(S+2)} \times S \Big|_{S=0} = \frac{1}{0+2} = \frac{1}{2}.$$

$$B = \frac{1}{S(S+2)}(S+2)\Big|_{S=-2} = \frac{1}{-2} = -1/2.$$

$$\therefore Y(S) = \frac{4}{5}\left[\frac{1}{2S} - \frac{1}{2(S+2)}\right]$$

$$L^{-1}[Y(S)] = y(t) = \frac{4}{5}\frac{1}{2}u(t) - \frac{4}{5}\frac{1}{2}e^{-2t}u(t)$$

$$= \frac{2}{5}(1 - e^{-2t})u(t)$$

Impulse Response and System Transfer function

Let $x(t)$ = i/p of LTI continuous time system

$y(t)$ = o/p / Response of " "

$h(t)$ = Impulse response (i.e. Response for Impulse i/p)

$$y(t) = x(t) * h(t)$$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

Taking L.T.

$$L\{y(t)\} = L\{x(t) * h(t)\}$$

$$Y(s) = X(s)\, H(s).$$

$$\therefore H(s) = \frac{Y(s)}{X(s)}.$$

$\therefore$ Impulse response, $h(t) = L^{-1}[H(s)] = L^{-1}\left[\frac{Y(s)}{X(s)}\right]$

Problems.

① Determine Impulse response $h(t)$ of the following system. Assume zero initial conditions.

a) $y(t) = x(t-t_0)$ → Ans: $\delta(t-t_0)$.

b) $\frac{d^2y(t)}{dt^2} + 4\frac{dy}{dt} + 3y(t) = \frac{dx(t)}{dt} + 2x(t)$ → Ans: $0.5(e^{-t}+e^{-3t})u(t)$

c) $\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = x(t)$ → Ans: $(e^{-t}-e^{-2t})u(t)$

② Find the Transfer function of the systems governed by the following Impulse response.

a) $h(t) = (2+t)e^{-3t}u(t)$ → Ans - $\frac{2s+7}{s^2+6s+9}$.

b) $h(t) = t^2u(t) - e^{-4t}u(t) + e^{-7t}u(t)$ → Ans: $\frac{-3s^3+2s^2+22s+56}{s^5+11s^4+28s^3}$

③ The unit step response of continuous time systems are given below. Determine the transfer function of the systems

a) $s(t) = u(t) + e^{-2t}u(t)$ → Ans $\frac{2s+2}{s+2}$

b) $s(t) = t^2u(t) + te^{-4t}u(t)$ → Ans: $\frac{s^3+2s^2+16s+32}{s^4+8s^3+16s^2}$

c) $s(t) = t\,u(t) + \sin t\, u(t)$ → Ans: $\frac{2s^2+1}{s^3+s}$.

4) The i/p and o/p Impulse response of continuous time systems are given below. Find o/p of the continuous time systems

a) $x(t) = \delta(t)$, $h(t) = e^{-at}u(t)$ Ans: $y(t) = e^{-at}u(t)$

b) $x(t) = e^{-2t}u(t)$, $h(t) = u(t)$ → Ans: $y(t) = \frac{1}{2}(1-e^{-2t})u(t)$

c) $x(t) = \cos 4t\, u(t) + \cos 7t\, u(t)$ and $h(t) = \delta(t-3)$ → Ans: $\cos 4(t-3)u(t-3) + \cos 7(t-3)u(t-3)$

Sol

$$X(S) = \frac{S}{S^2+4^2} + \frac{S}{S^2+7^2}$$

$$H(S) = e^{-3S}.L\{\delta(t)\} = e^{-3S}.1 = e^{-3S}.$$

$$H(S) = \frac{Y(S)}{X(S)} \Rightarrow Y(S) = H(S)X(S)$$

$$= e^{-3S}\left[\frac{S}{S^2+4^2} + \frac{S}{S^2+7^2}\right]$$

$$L^{-1}\{Y(S)\} = y(t) = L^{-1}\left[e^{-3S}\left[\frac{S}{S^2+4^2} + \frac{S}{S^2+7^2}\right]\right]$$

If $L\{x(t)\} = X(S)$
$L\{x(t-a)\} = e^{-as}X(S)$

$$= L^{-1}\left[\frac{S}{S^2+4^2} + \frac{S}{S^2+7^2}\right]_{t=t-3}.$$

$$= \left[\cos 4t\, u(t) + \cos 7t\, u(t)\right]_{t=t-3}$$

$$= \cos 4(t-3)u(t-3) + \cos 7(t-3)u(t-3)$$

$$= [\cos 4(t-3) + \cos 7(t-3)]u(t-3)$$

5) Perform convolution of the following causal signals using LT

a) $x_1(t) = 2u(t)$, $x_2(t) = u(t)$ → Ans: $2t\,u(t)$

b) $x_1(t) = e^{-2t}u(t)$, $x_2(t) = e^{-5t}u(t)$ → Ans: $\frac{1}{3}(e^{-2t} - e^{-5t})u(t)$

c) $x_1(t) = \cos t\, u(t)$, $x_2(t) = t\,u(t)$ → Ans: $u(t) - \cos t\, u(t)$.

6) Perform deconvolution operation to extract the signal $x_1(t)$

a) $x_1(t) * x_2(t) = 2t\,u(t)$; $x_2(t) = u(t)$

Sol: $x_3(t) = x_1(t) * x_2(t)$

$L\{x_3(t)\} = L\{x_1(t) * x_2(t)\}$

$X_3(S) = X_1(S)X_2(S)$ —— ①.

$= L\{2t\,u(t)\} = 2/S^2$

$X_2(t) = u(t)$

$X_2(S) = 1/S$.

from ① $X_1(S) = \frac{X_3(S)}{X_2(S)} = \frac{2/S^2}{1/S} = 2/S$. ∴ $x_1(t) = 2u(t)$

(b) $x_1(t) * x_2(t) = \frac{1}{3}(e^{-2t} - e^{-5t})u(t)$

$x_2(t) = e^{-5t}u(t)$

Ans: $x_1(t) = e^{-2t}u(t)$

Laplace Transform

* Pierre Simon de Laplace

* The Fourier transform does not exist for signals that are not absolutely integrable such as $e^{at}u(t)$ $a>0$, e^{-at} $-\infty < t < \infty$, $t\,u(t)$.

* The difficulty is resolved by generalized the F.T so that the signal $x(t)$ is expressed as sum of complex expon e^{st}, where $s = \sigma + j\omega$ and thus is not restricted to the imaginary axis only.

* The Laplace transform (L.T) can be applied to the analysis and investigation such as stability, causality, and frequency response.

* The Unilateral Laplace transform is a convenient tool for solving differential equations with initial conditions.

* Bilateral Laplace transform

$$\mathcal{L}[x(t)] = X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt$$

the inverse Laplace transform

$$\mathcal{L}^{-1}[X(s)] = x(t) = \frac{1}{2\pi j}\int_{\sigma - j\infty}^{\sigma + j\infty} X(s)\, e^{st}\, ds$$

ILT can be determined without directly evaluating but by using the technique of fraction expansion.

Relationship between L.T and F.T

consider a C.T signal. If L.T is

$$\mathcal{L}[x(t)] = X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

substituting $s = \sigma + j\omega$

$$\mathcal{L}[x(t)] = \int_{-\infty}^{\infty} x(t) e^{-(\sigma + j\omega)t} dt$$

$$= \int_{-\infty}^{\infty} [x(t) e^{-\sigma t}] e^{-j\omega t} dt$$

$$= \mathcal{F}[x(t) e^{-\sigma t}]$$

Thus, the L.T of $x(t)$ is the F.T of $x(t) e^{-\sigma t}$.

if $\sigma = 0$

$$\mathcal{L}[x(t)] = \mathcal{F}[x(t)] \Big|_{\sigma = 0}$$

Region of Convergence (ROC) for Laplace transform

The L.T is guaranteed to converge if $x(t) e^{-\sigma t}$ is absolutely integrable.

i.e $$\int_{-\infty}^{\infty} |x(t) e^{-\sigma t}| dt < \infty$$

This guarantees that $X(s)$ will be finite, since

$$\mathcal{L}[x(t)] = X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

$$= \int_{-\infty}^{\infty} x(t) e^{-(\sigma + j\omega)t} dt$$

$$|X(s)| = \left| \int_{-\infty}^{\infty} x(t)\, e^{-\sigma t}\, e^{-j\omega t}\, dt \right|$$

$$|X(s)| \leq \int_{-\infty}^{\infty} \left| x(t)\, e^{-\sigma t}\, e^{-j\omega t} \right| dt$$

$$|X(s)| \leq \int_{-\infty}^{\infty} \left| x(t)\, e^{-\sigma t} \right| dt$$

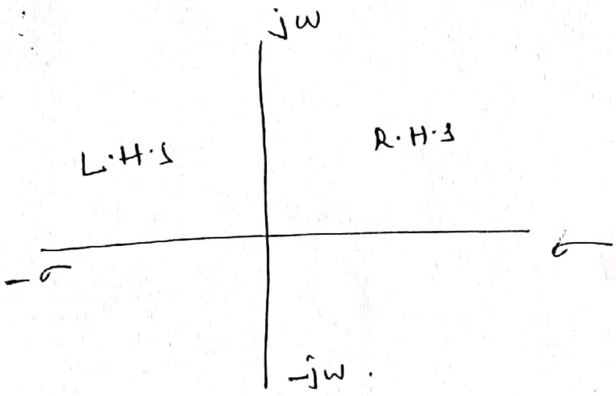
so if $\int_{-\infty}^{\infty} \left| x(t)\, e^{-\sigma t} \right| dt < \infty$ then $|X(s)| < \infty$

The range of $Re\{s\} = \sigma$ for which L.T converges is termed the region of convergence (ROC). That is, the ROC consists of those values of σ for which the F.T of $x(t)\, e^{-\sigma t}$ converges.

** L.T exists for some signals that does not have F.T.

* If the ROC for $X(s)$ include the $j\omega$ axis, $x(t)$ is Fourier transformable and $X(\omega)$ can be obtained by replacing s in $X(s)$ by $j\omega$.

S – PLANE :



POLES and ZEROS :

$$H(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0}$$

$$H(s) = \frac{N(s)}{D(s)} = K \frac{(s-z_1)(s-z_2)\cdots(s-z_{m-1})(s-z_m)}{(s-P_1)(s-P_2)\cdots(s-P_{n-1})(s-P_n)}$$

The z_i's are the roots of the equation $N(s)=0$ defined as zero's.

the P_i's are the roots of the equation $D(s)=0$. defined as the poles.

the coefficients of $N(s)$, $D(s)$ are real, and therefore the poles and zeros must either be purely real or apper in complex conjugate pairs.

the zero's in the s-plane are denoted with 'o'

and the pole's " " " " " x

PROPERTIES OF ~~the~~ ROC

① The ROC of $X(s)$ consists of strips $\parallel^{le}$ to the $j\omega$ axis in the s-plane

② If $X(s)$ is rational, then the ROC must not contain any poles.

③ If $x(t)$ is of finite duration and is absolutely (i.e $\int_{-\infty}^{\infty} |x(t)|\, dt < \infty$), the ROC is the entire s-plane.

④ If $x(t)$ is Right sided and of infinite duration (i.e $x(t) = 0$ for all $t < T_1$) then The ROC is the region in the s-plane to the right of the rightmost pole

⑤ If $x(t)$ is left sided and of infinite duration (i.e $x(t) = 0$ $\forall$ $t > T_2$) then ROC is the region in the s-plane to the left of the leftmost pole.

⑥ If $x(t)$ is two-sided and of infinite duration (i.e the signal is of infinite extent for both $t < 0$ and $t > 0$) then the ROC will consists of a strip in the s-plane.

Pb (1) Determine the L.T of the causal signal $x(t) = e^{-at} u(t)$ and depit the ROC, Poles, zero's in s-plane.

Sol:

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt = \int_{-\infty}^{\infty} e^{-at} u(t) e^{-st} dt$$

$$X(s) = \int_{0}^{\infty} e^{-at} e^{-st} dt$$

$$\boxed{\begin{array}{lll} u(t) & = 1 & t > 0 \\ & = 0 & t < 0 \end{array}}$$

$$= \int_{0}^{\infty} e^{-(s+a)t} dt = \frac{1}{-(s+a)} \left[e^{-(s+a)t} \right]_0^{\infty}$$

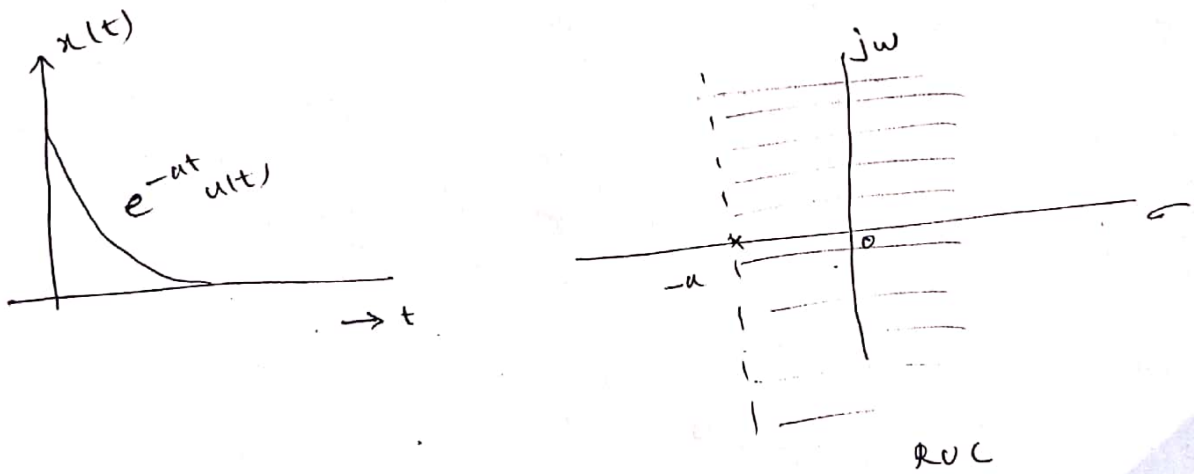
$s = \sigma + j\omega$

$$X(s) = -\frac{1}{\sigma + j\omega + a} e^{-(\sigma+a)t} e^{-j\omega t} \Big|_0^{\infty}$$

Now if $(\sigma + a) > 0$ or $\sigma > -a$, Then $e^{-(\sigma+a)t} \to 0$ as $t \to \infty$

$$X(s) = \frac{1}{-\sigma + j\omega + a} [0 - 1] \quad \sigma > -a$$

$$= \frac{1}{s+a} \quad R(s) > -a$$



$$\therefore \boxed{e^{-at} u(t) \longleftrightarrow \frac{1}{s+a} \quad R(s) > -a}$$

(2) $x(t) = -e^{-at}u(-t)$.

Sol:

$$X(s) = \int_{-\infty}^{\infty} x(t)\,e^{-st}\,dt = \int_{-\infty}^{\infty} -e^{-at}u(-t)\,e^{-st}\,dt$$

$$X(s) = -\int_{-\infty}^{0} e^{-at}e^{-st}\,dt \qquad \because u(-t) = \begin{cases} 0 & t>0 \\ 1 & t<0 \end{cases}$$

$$= -\int_{-\infty}^{0} e^{-(s+a)t}\,dt = \frac{1}{(s+a)}\,e^{-(s+a)t}\Big|_{-\infty}^{0}$$

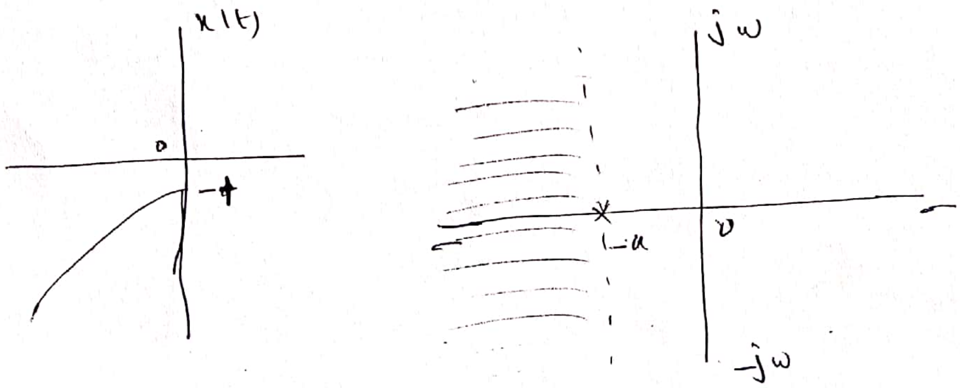
$s = \sigma + j\omega$.

$$X(s) = \frac{1}{\sigma + j\omega + a}\cdot e^{-(\sigma+a)t}\,e^{-j\omega t}\Big|_{-\infty}^{0}$$

Now if $(\sigma + a) < 0$ or $\sigma < -a$, then $e^{-(\sigma+a)t} \to 0$ as $t \to -\infty$

$$X(s) = \frac{1}{\sigma + j\omega + a}\,[1-0] \qquad \sigma < -a$$

$$= \frac{1}{(s+a)} \qquad R(s) < -a$$



$$\boxed{-e^{-at}u(-t) \longleftrightarrow \frac{1}{(s+a)} \qquad R(s) < -a}$$

(3) $$x(t) = e^{-2t}u(t) - e^{-3t}u(t).$$

$$L[x(t)] = L[e^{-2t}u(t)] - L[e^{-3t}u(t)]$$

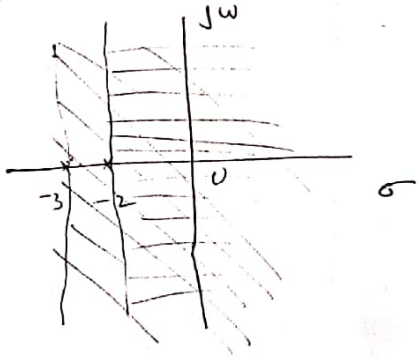
We have $$e^{-at}u(t) \longleftrightarrow \frac{1}{s+a} \qquad R(s) > -a$$

$$\therefore \quad L[e^{-2t}u(t)] \longleftrightarrow \frac{1}{s+2} \qquad R(s) > -2$$

$$L[e^{-3t}u(t)] \longleftrightarrow \frac{1}{s+3} \qquad R(s) > -3$$

$$X(s) = \frac{1}{s+2} - \frac{1}{s+3} \qquad R(s) > -2$$

$$= \frac{1}{s^2+5s+6}$$



(4) $$x(t) = -e^{-2t}u(-t) + e^{-3t}u(-t)$$

$$= L[-e^{-2t}u(-t)] - L[-e^{-3t}u(-t)]$$

Using $$-e^{-at}u(-t) \longleftrightarrow \frac{1}{s+a} \qquad R(s) < -a$$

$$-e^{-2t}u(-t) \longleftrightarrow \frac{1}{s+2} \qquad R(s) < -2$$

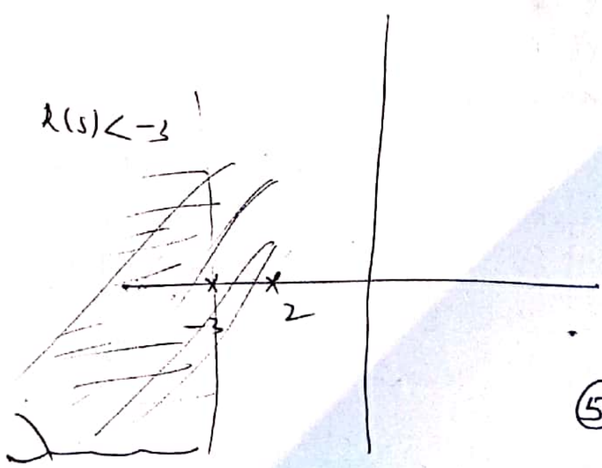
$$-e^{-3t}u(-t) \longleftrightarrow \frac{1}{s+3} \qquad R(s) < -3$$

$$X(s) = \frac{1}{s+2} - \frac{1}{s+3} \qquad R(s) < -3$$

$$= \frac{1}{s^2+5s+6}$$

ROC

$R(s) < -3$



(5) Determine L.T of the finite duration signal

$$x(t) = \begin{cases} e^{-at} & 0 < t < T \\ 0 & \text{otherwise} \end{cases}$$

Sol:

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt = \int_0^T e^{-at} e^{-st} dt$$

$$= \int_0^T e^{-(s+a)t} dt = \frac{1}{-(s+a)} e^{-(s+a)t} \Big]_0^T$$

$$= \frac{1}{s+a} \left[1 - e^{-(s+a)T}\right]$$

the ROC is entire s-plane.

At $s = -a$ both NR & DR are zero, to determine $X(s)$ at $s = -a$ we can use L' Hospital rule

$$\lim_{s \to -a} X(s) = \lim_{s \to -a} \left[\frac{\frac{d}{ds}\left(1 - e^{-(s+a)T}\right)}{\frac{d}{ds}(s+a)} \right]$$

$$X(-a) = \lim_{s \to -a} \frac{T \cdot e^{-(s+a)T}}{1}$$

$$X(-a) = T$$

⑦

$x(t) = \sin w_0 t \; u(t)$

$$= \frac{1}{2j}\left[e^{jw_0 t} u(t) - e^{-jw_0 t} u(t) \right]$$

$$L[x(t)] = \frac{1}{2j}\left(L\left[e^{jw_0 t} u(t) \right] - L\left[e^{-jw_0 t} u(t) \right] \right)$$

we have $e^{-at} u(t) \longleftrightarrow \frac{1}{s+a}$ $\quad R(s) > -a$

$$\therefore \quad e^{jw_0 t} u(t) \longleftrightarrow \frac{1}{s - jw_0} \quad R(s) > 0$$

($\because$ real part is zero in s)

$$e^{-jw_0 t} u(t) \longleftrightarrow \frac{1}{s + jw_0} \quad R(s) > 0$$

$$\therefore \quad X(s) = \frac{1}{2j}\left(\frac{1}{s - jw_0} - \frac{1}{s + jw_0} \right)$$

$$= \frac{1}{2j}\left(\frac{s + jw_0 - s + jw_0}{s^2 + w_0^2} \right) = \frac{w_0}{s^2 + w_0^2}$$

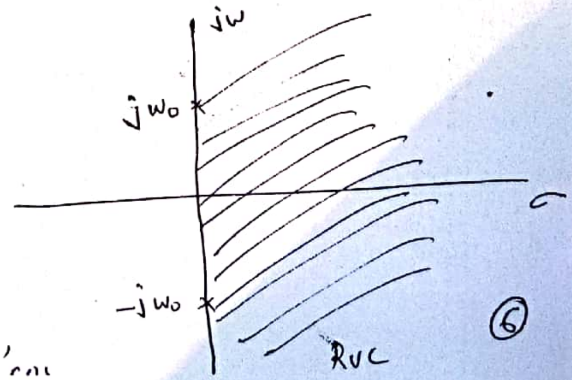
$$\boxed{\therefore \quad \sin w_0(t) \; u(t) \longleftrightarrow \frac{w_0}{s^2 + w_0^2} \qquad R(s) > 0}$$

⑧ $x(t) = \cos w_0 t \; u(t) = \frac{1}{2}\left[e^{jw_0 t} + e^{-jw_0 t} \right] u(t)$

from above

$$X(s) = \frac{1}{2}\left(\frac{1}{s - jw_0} + \frac{1}{s + jw_0} \right) = \frac{s}{s^2 + w_0^2} \qquad R(s) > 0.$$

$$\boxed{\therefore \quad \cos(w_0 t) \; u(t) \longleftrightarrow \frac{s}{s^2 + w_0^2} \qquad R(s) > 0}$$



⑥

For both sin, cos

(6) (a) a unit impluse function $x(t) = \delta(t)$

$$X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt = \int_{-\infty}^{\infty} \delta(t)\, e^{-st}\, dt = e^{-st}\Big|_{t=0} = 1$$

$$\boxed{\delta(t) \longleftrightarrow 1 \quad \text{ROC is entire s-plane}}$$

(b) a unit step function $x(t) = u(t)$

$$X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt = \int_{-\infty}^{\infty} u(t)\, e^{-st}\, dt = \int_{0}^{\infty} e^{-st}\, dt$$

$$= -\frac{1}{s} e^{-st}\Big]_0^{\infty} = -\frac{1}{s}[0-1] \quad R(s) > 0$$

$$\boxed{u(t) \longleftrightarrow \frac{1}{s} \quad R(s) > 0}$$

(c) a ~~xxx~~ unit ramp signal $x(t) = r(t) = t \cdot u(t)$

$$X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt = \int_{-\infty}^{\infty} t \cdot u(t)\, e^{-st}\, dt = \int_{0}^{\infty} t \cdot e^{-st}\, dt$$

$$= t \cdot \frac{d}{dt} e^{-st}\Big|_0^{\infty} +$$

$$= -\frac{1}{s} t \cdot e^{-st}\Big|_0^{\infty} + \frac{1}{s}\int_0^{\infty} e^{-st}\, dt$$

$$= -\frac{1}{s^2} e^{-st}\Big]_0^{\infty} = -\frac{1}{s^2}[0-1] \quad R(s) > 0$$

$$X(s) = \frac{1}{s^2} \quad R(s) > 0$$

$$\boxed{r(t) \longleftrightarrow \frac{1}{s^2} \quad R(s) > 0}$$

Scanned by CamScanner

(9)

$x(t) = \sinh(\omega_0 t)\, u(t)$

$= \frac{1}{2}\left[e^{\omega_0 t}u(t) - e^{-\omega_0 t}u(t)\right]$

$x(t) = \cosh \omega_0 t\, u(t)$

$= \frac{1}{2}\left[e^{\omega_0 t}u(t) + e^{-\omega_0 t}u(t)\right]$

- $e^{\omega_0 t}u(t) \longleftrightarrow \frac{1}{s-\omega_0} \quad R(s) > \omega_0$

$e^{-\omega_0 t}u(t) \longleftrightarrow \frac{1}{s+\omega_0} \quad R(s) > -\omega_0$

$X(s) = \frac{1}{2}\left[\frac{1}{s-\omega_0} - \frac{1}{s+\omega_0}\right]$

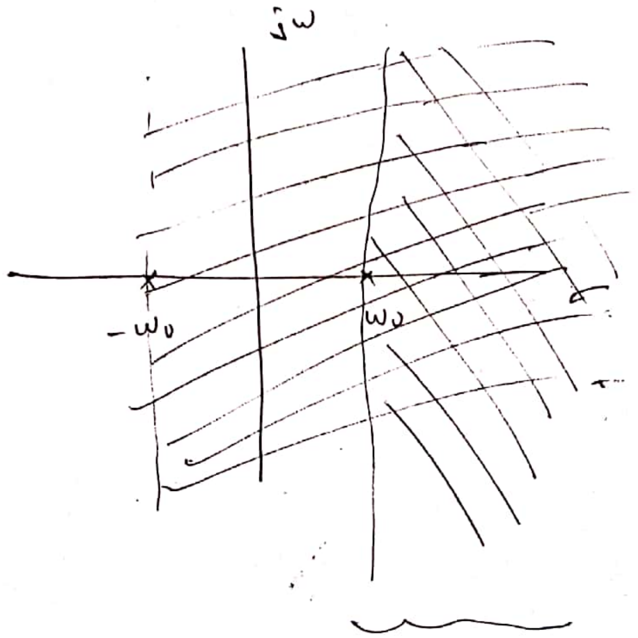
$= \frac{\omega_0}{s^2-\omega_0^2}$

$\boxed{\sinh \omega_0 t\, u(t) \longleftrightarrow \frac{\omega_0}{s^2-\omega_0^2} \quad R(s) > \omega_0}$

$X(s) = \frac{1}{2}\left(\frac{1}{s-\omega_0} + \frac{1}{s+\omega_0}\right)$

$X(s) = \frac{s}{s^2-\omega_0^2}$

$\boxed{\cosh \omega_0 t\, u(t) \longleftrightarrow \frac{s}{s^2-\omega_0^2} \quad R(s) > \omega_0}$



ROC of $\sinh \omega_0 t\ u(t)$
$\cosh \omega_0 t\ u(t)$.

(10) $x(t) = e^{-a|t|} \quad a>0$

$$= \begin{cases} e^{-at} & t \geq 0 \\ e^{at} & t < 0 \end{cases}$$

$$= e^{at} u(-t) + e^{-at} u(t)$$

we have

$$e^{-at} u(t) \longleftrightarrow \frac{1}{s+a} \qquad R(s) > -a$$

we have

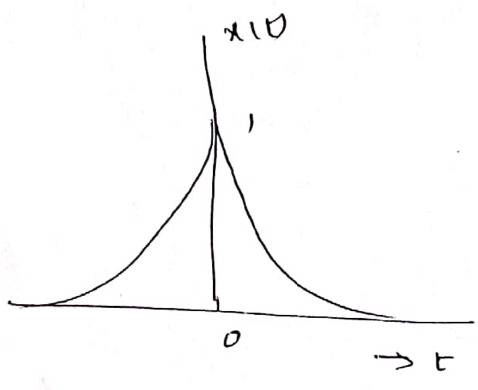
$$-e^{-at} u(t) \longleftrightarrow \frac{1}{s+a} \qquad R(s) < -a$$

so for $\quad e^{at} u(-t) \longleftrightarrow \frac{-1}{s-a} \qquad R(s) < a$

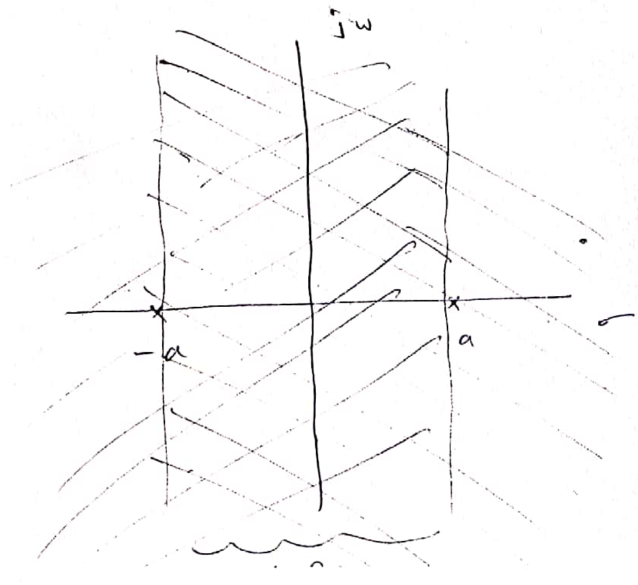
$$\therefore \; X(s) = \frac{1}{s+a} - \frac{1}{s-a} = \frac{-2a}{s^2-a^2}$$

The set of values of R(s) for which the L.T of both terms converge is $-a < R(s) < a$.

$$\boxed{X(s) \longleftrightarrow \frac{-2a}{s^2-a^2} \qquad -a < R(s) < a}$$



$e^{-a|t|} \quad a>0$



(11) $x(t) = e^{-5t}\, u(t-1)$

$$X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt = \int_{-\infty}^{\infty} e^{-5t}\, u(t-1)\, e^{-st}\, dt$$

$$u(t-1) = \begin{cases} 1 & t > 1 \\ 0 & t < 1 \end{cases}$$

$$= \int_{1}^{\infty} e^{-5t}\, e^{-st}\, dt = \int_{1}^{\infty} e^{-(s+5)t}\, dt$$

$$= \frac{e^{-(s+5)t}}{-(s+5)} \Big|_{1}^{\infty} = \frac{-1}{(s+5)} \left[0 - e^{-(s+5)} \right] \quad \boxed{R(s) > -5}$$

$$\boxed{X(s) \longleftrightarrow \frac{e^{-(s+5)}}{(s+5)} \qquad R(s) > -5}$$

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

$$\cosh\theta = \frac{e^{\theta} + e^{-\theta}}{2}$$

$$\sinh\theta = \frac{e^{\theta} - e^{-\theta}}{2}$$

$$e^{j\theta} = \cos\theta + j\sin\theta$$

PROPERTIES OF THE LAPLACE TRANSFORM

① Linearity

$x_1(t) \longleftrightarrow X_1(s)$ ROC R_1

$x_2(t) \longleftrightarrow X_2(s)$ ROC R_2

$$\boxed{a_1x_1(t) + a_2x_2(t) \longleftrightarrow a_1X_1(s) + a_2X_2(s) \quad \text{ROC} = R_1 \cap R_2}$$

② Time shifting

$x(t) \longleftrightarrow X(s)$ ROC $= R$

$$\boxed{x(t-t_0) \longleftrightarrow X(s)\,e^{-st_0} \quad \text{ROC}=R}$$

$$= \int_{-\alpha}^{\alpha} x(t-t_0)\, e^{-st}\, dt$$

$t_1 = t - t_0$

$$= \int_{-\alpha}^{\alpha} x(t_1)\, e^{-(t_1+t_0)s}\, dt$$

$$= e^{-st_0}\, X(s).$$

③ Shifting in S-Domain

$x(t) \longleftrightarrow X(s)$ ROC $= R$

$$\boxed{x(t)\,e^{s_0t} \longleftrightarrow X(s-s_0) \quad \text{ROC} = R + R\{s_0\}}$$

$$L[x(t)e^{s_0t}] = \int_{-\alpha}^{\alpha} x(t)e^{s_0t}e^{-st}\, dt$$

$$= \int_{-\alpha}^{\alpha} x(t)\, e^{-(s-s_0)t}\, dt$$

$$= X(s-s_0).$$

Note: If $X(s)$ has a pole or zero at $s=a$ then $X(s-s_0)$ has a pole or zero at $s-s_0=a$

i.e $s = a+s_0$

④ Time scaling

$x(t) \longleftrightarrow X(s)$ ROC $= R$

$x(at) \longleftrightarrow \frac{1}{|a|}X\left(\frac{s}{a}\right)$ ROC $= aR$

$$L[x(at)] = \int_{-\alpha}^{\alpha} x(at)\, e^{-st}\, dt$$

case 1: a +ve

let $t_1 = at$

$$= \frac{1}{a}\int_{-\alpha}^{\alpha} x(t_1)\, e^{-\frac{s}{a}t_1}\, dt_1$$

$$= \frac{1}{a}X\left(\frac{s}{a}\right)$$

case 2: a -ve

$$L[x(-at)] = \int_{-\alpha}^{\alpha} x(-at)\,e^{-st}\, dt$$

$t_1 = -at$

$$= -\frac{1}{a}\int_{+\alpha}^{-\alpha} x(t_1)\, e^{-\left(-\frac{s}{a}\right)t_1}\, dt_1$$

$$= \frac{1}{a}X\left(\frac{s}{-a}\right)$$

$$\boxed{x(at) \longleftrightarrow \frac{1}{|a|}X\left(\frac{s}{a}\right)}$$

(5) $x(t) \longleftrightarrow X(s)$ ROC = R

$$\frac{1}{|a|} x\left(\frac{t}{a}\right) \longleftrightarrow X(as) \quad ROC = \frac{R}{a}$$

case 1 a +ve

$$L\left[\frac{1}{a} x\left(\frac{t}{a}\right)\right] = \int_{-\alpha}^{\alpha} \frac{1}{a} x\left(\frac{t}{a}\right) e^{-st} dt$$

let $t_1 = \frac{t}{a}$ $dt = a\,dt_1$

$$= \int_{-\alpha}^{\alpha} x(t_1) e^{-ast_1} dt_1$$

$$= X(as).$$

case 2 a -ve

$$= \int_{-\alpha}^{\alpha} \frac{1}{-a} x\left(\frac{t}{-a}\right) e^{-st} dt$$

let $t_1 = \frac{t}{-a}$

$$= \int_{+\alpha}^{-\alpha} x(t_1) e^{-(as)t_1} dt_1$$

$$= -X(-as)$$

i.e $L\left(\frac{1}{a} x\left(\frac{t}{-a}\right)\right) = X(-as)$

$$\boxed{\therefore L\left[\frac{1}{|a|} x\left(\frac{t}{a}\right)\right] \longleftrightarrow X(as)}$$

Scaling in s Domain

(6) Time Reversal

$x(t) \longleftrightarrow X(s)$ ROC = R

$$\boxed{x(-t) \longleftrightarrow X(-s) \quad ROC = -R}$$

Put $a = -1$ in the above property.

(7) Differentiation in time Domain

$x(t) \longleftrightarrow X(s)$ ROC = R

$$\frac{dx(t)}{dt} \longleftrightarrow sX(s) \quad ROC = R$$

Proof

$$x(t) = L^{-1}[X(s)] = \frac{1}{2\pi j} \int_{\sigma - j\alpha}^{\sigma + j\alpha} X(s) e^{st} ds$$

Differentiate both side with t

$$\frac{d}{dt} x(t) = \frac{1}{2\pi j} \int_{\sigma - j\alpha}^{\sigma + j\alpha} [sX(s)] e^{st} ds$$

$$= L^{-1}[sX(s)]$$

$$\therefore \boxed{\frac{dx(t)}{dt} \longleftrightarrow sX(s) \quad R}$$

$$\boxed{\frac{d^n x(t)}{dt^n} \longleftrightarrow s^n X(s)}$$

8) Differentiation in s-Domain

If

$$x(t) \longleftrightarrow X(s) \qquad ROC = R$$

$$t\,x(t) \longleftrightarrow -\frac{dX(s)}{ds} \qquad ROC = R$$

Proof

$$X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt$$

Diff with s

$$\frac{dX(s)}{ds} = \int_{-\infty}^{\infty} [-t\,x(t)]\, e^{-st}\, dt$$

$$\frac{dX(s)}{ds} = L[-t\,x(t)]$$

$$\therefore \quad t\,x(t) \longleftrightarrow -\frac{dX(s)}{ds}$$

$$\boxed{t^n x(t) \longleftrightarrow (-1)^n \frac{d^n X(s)}{ds^n}}$$

9) Convolution Property

If

$$x_1(t) \longleftrightarrow X_1(s) \qquad ROC = R_1$$

$$x_2(t) \longleftrightarrow X_2(s) \qquad ROC = R_2$$

$$x_1(t) * x_2(t) \longleftrightarrow X_1(s) \cdot X_2(s)$$

$$ROC = R_1 \cap R_2$$

L.T

$$= \int_{-\infty}^{\infty} [x_1(t) * x_2(t)]\, e^{-st}\, dt$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau\; e^{-st}\, dt$$

$$= \int_{-\infty}^{\infty} x_1(\tau) \left(\int_{-\infty}^{\infty} x_2(t-\tau)\, e^{-st}\, dt \right) d\tau$$

using time shifting property. The bracketed term is $X_2(s)\, e^{-s\tau}$

$$= \int_{-\infty}^{\infty} x_1(\tau) \left[X_2(s)\, e^{-s\tau} \right] d\tau$$

$$= X_2(s) \int_{-\infty}^{\infty} x_1(\tau)\, e^{-s\tau}\, d\tau$$

$$= X_2(s) \cdot X_1(s)$$

$$\boxed{x_1(t) * x_2(t) \longleftrightarrow X_1(s) \cdot X_2(s) \quad R_1 \cap R_2}$$

10) Multiplication Property

If

$$x_1(t) \longleftrightarrow X_1(s) \qquad ROC = R_1$$

$$x_2(t) \longleftrightarrow X_2(s) \qquad ROC = R_2$$

$$x_1(t)\, x_2(t) \longleftrightarrow \frac{1}{2\pi j} \left[X_1(s) * X_2(s) \right]$$

$$= \frac{1}{2\pi j} \int_{\sigma_1 + j\infty}^{\sigma_1 + j\infty} X_1(s)\, X_2(s - s_1)\, ds, \qquad R_1 \cap R_2$$

11) Integration in the Time Domain

If

$$x(t) \longleftrightarrow X(s) \qquad ROC = R$$

then

$$\int_{-\infty}^{t} x(\tau)\, d\tau \longleftrightarrow \frac{1}{s} X(s)$$

$$ROC = R \cap \{R(s) > 0\}$$

Proof:

$$x(t) * u(t) = \int_{-\infty}^{\infty} x(\tau)\, u(t-\tau)\, d\tau$$

$$u(t-\tau) = \begin{cases} 1 & \tau < t \\ 0 & \tau > t \end{cases}$$

1 (5)

we have

$$x(t) * u(t) = \int_{-\alpha}^{t} x(\tau)\, d\tau$$

$$L\left[\int_{-\alpha}^{t} x(\tau)\, d\tau\right] = L\left[x(t) * u(t)\right]$$

$$= X(s) \cdot U(s) = \frac{1}{s} X(s)$$

$$\therefore \quad \boxed{\int_{-\alpha}^{t} x(\tau)\, d\tau \longleftrightarrow \frac{1}{s} X(s)}$$

(12) <u>Conjugate Property</u>

If $\quad x(t) \longleftrightarrow X(s)$

$$x^*(t) \longleftrightarrow X^*(s^*)$$

<u>Proof</u>: $L\left[x^*(t)\right] = \int_{-\alpha}^{\alpha} x^*(t)\, e^{-st}\, dt$

$$= \left[\int_{-\alpha}^{\alpha} x(t)\, e^{-s^* t}\, dt\right]^*$$

$$= X^*(s^*)$$

$\therefore$ If $x(t)$ is real $\quad x(t) = x^*(t)$

$$X(s) = X^*(s^*)$$

$$X^*(s) = X(s^*)$$

If $x(t)$ is real and if $X(s)$ has a pole or zero at $s = s_0$ then $X(s)$ also has a pole or zero at the complex conjugate point $s = s_0^*$

Determine the Laplace transform of the following signals.

(a) $x_1(t) = \delta(t+T) + \delta(t+T)$

$\delta(t) \longleftrightarrow 1$ ROC s plane.

using time shifting

$\delta(t+T) \longleftrightarrow e^{sT}$

$\delta(t-T) \longleftrightarrow e^{-sT}$

$X_1(s) = e^{sT} + e^{-sT}$ ROC = s plane

(b) $x_2(t) = u(t+T) - u(t-T)$

$u(t) \longleftrightarrow \frac{1}{s}$ $R(s) > 0$

using time shifting

$u(t+T) \longleftrightarrow \frac{1}{s} e^{sT}$

$u(t-T) \longleftrightarrow \frac{1}{s} e^{-sT}$

$X_2(s) = \frac{1}{s}\left[e^{sT} - e^{-sT}\right]$

$R(s) > 0$.

(c) $x_3(t) = r(t+T) - r(t-T)$

$r(t) \longleftrightarrow \frac{1}{s^2}$ ROC > 0

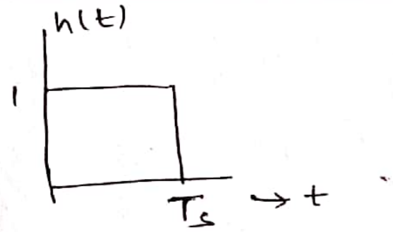
using time shifting

$r(t+T) \longleftrightarrow \frac{1}{s^2} e^{sT}$

$r(t-T) \longleftrightarrow \frac{1}{s^2} e^{-sT}$

$X_3(s) = \frac{1}{s^2}\left(e^{sT} - e^{-sT}\right)$ $R(s) > 0$

(d)



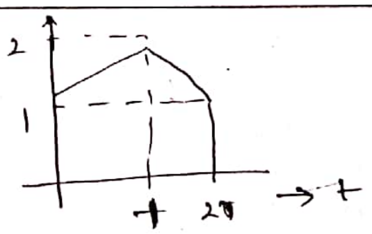
$h(t) = u(t) - u(t-T_s)$.

$H(s) = \frac{1}{s} - \frac{1}{s} e^{-T_s}$

$= \frac{1}{s}\left(1 - e^{-sT_s}\right)$

Finite duration ROC = s plane

(e)

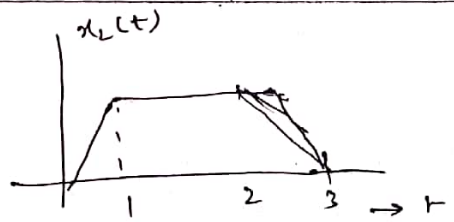


$x(t) = u(t) + r(t) - r(t-1) - r(t-1)$
$\quad \cancel{-r(t-1)} + r(t-2) - u(t-2)$

$X(s) = \frac{1}{s} + \frac{1}{s^2} - \frac{2}{s^2} e^{-s} + \frac{1}{s^2} e^{-2s}$
$\quad - \frac{1}{s} e^{-2s}$

$X(s) = \frac{1}{s}\left[1 - e^{-2s}\right] + \frac{1}{s^2}\left[1 - 2e^{-s} + e^{-2s}\right]$

(f)



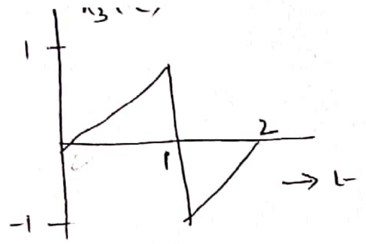
$x_2(t) = r(t) - r(t-1) - r(t-2) + r(t-3)$

$= \frac{1}{s^2} - \frac{1}{s^2} e^{-s} - \frac{1}{s^2} e^{-2s} + \frac{1}{s^2} e^{-3s}$

$= \frac{1}{s^2}\left(1 - e^{-s} - e^{-2s} + e^{-3s}\right]$

11

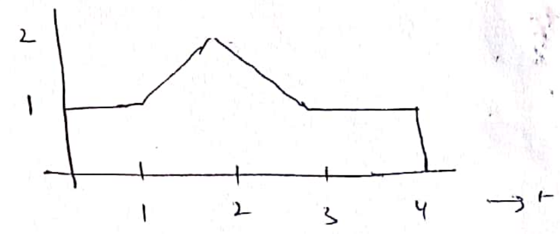
(7)



$x_3(t) = r(t) - 2u(t-1) - r(t-2)$

$X_3(s) = \frac{1}{s^2} - \frac{2}{s}e^{-s} - \frac{1}{s^2}e^{-2s}$.

(8)



$x_4(t) = u(t) + r(t-1) - 2r(t-2)$
$+ r(t-3) - u(t-4)$

$X_4(s) = \frac{1}{s} + \frac{1}{s^2}e^{-s} - \frac{2}{s^2}e^{-2s} + \frac{1}{s^2}e^{-3s} - \frac{1}{s}e^{-4s}$

(9) $g_1(t) = e^{-at}\cos(w_0 t)\,u(t)$.

let $g_1(t) = e^{-at}x(t)$

where $x(t) = \cos w_0 t\ u(t)$

$L.T\{x(t)\} = X(s) = \frac{s}{s^2+w_0^2}$ $R\{s\} > 0$

$L[g_1(t)] = L[e^{-at}x(t)]$

$G_1(s) = X(s+a)$ using s domain shifting

$= \frac{s+a}{(s+a)^2+w_0^2}$ $R\{s\} > -a$

(10) $g_2(t) = e^{-at}\sin(w_0 t)\,u(t)$.

$= e^{-at}x(t)$.

$X(s) = \frac{w_0}{s^2+w_0^2}$ $R\{s\} > 0$

$L[g_2(t)] = L[e^{-at}x(t)]$

$G_2(s) = X(s+a) = \frac{w_0}{(s+a)^2+w_0^2}$ $R\{s\} > -a$

Shifting in s-domain.

(11) $g_1(t) = \delta(2t)$

$= \frac{1}{2}\delta(t)$ $\because \delta(at) = \frac{1}{a}\delta(t)$

$G_1(s) = \frac{1}{2}L[\delta(t)] = 1/2$

(12) $g_2(t) = r(2t)$

let $x(t) = r(t)$

$X(s) = \frac{1}{s^2}$ $R(s) > 0$

$L[g_2(t)] = L[x(2t)]$

$= \frac{1}{2}X(\frac{s}{2})$

$= \frac{1}{2}\frac{1}{(s/2)^2}$

$= \frac{2}{s^2}$ $R\{s\} > 0$.

(13) $g_1(t) = \delta(2t-3)$

$\delta(t) \longleftrightarrow 1$

Time shift $\delta(t-3) \longleftrightarrow 1 \cdot e^{-3s}$

time scal $\delta(2t-3) \longleftrightarrow \frac{1}{2}e^{-3s/2}$

(14) $g(t) = u(2t-1)$

$u(t) \longleftrightarrow \frac{1}{s}$

$u(t-1) \longleftrightarrow \frac{1}{s} e^{-s}$

$u(2t-1) \longleftrightarrow \frac{1}{2} \frac{1}{s/2} e^{-s/2}$

$\longleftrightarrow \frac{1}{s} e^{-s/2}$

(15) $g(t) = r\left(\frac{1}{3}t - 2\right)$

$r(t) \longleftrightarrow \frac{1}{s^2}$

$r(t-2) \longleftrightarrow \frac{1}{s^2} e^{-2s}$

$r\left(\frac{1}{3}t-2\right) \longleftrightarrow \frac{1}{1/3} \frac{1}{(s3)^2} e^{-2 \times 3s}$

$r\left(\frac{1}{3}t-2\right) \longleftrightarrow \frac{1}{3s^2} e^{-6s}$

(16) $g(t) = u(-2t-1)$

$u(t) \longleftrightarrow \frac{1}{s}$

time shifting $u(t-1) \longleftrightarrow \frac{1}{s} e^{-s}$

time scaling $u(2t-1) \longleftrightarrow \frac{1}{2} \frac{1}{s/2} e^{-s/2}$

time reversal $u(-2t-1) \longleftrightarrow \frac{1}{(-s)} e^{-(-s)/2}$

$u(-2t-1) \longleftrightarrow \frac{-1}{s} e^{s/2}$

(17) $g(t) = r(-3t-2)$

$r(t) \longleftrightarrow 1/s^2$

time shifting $r(t-2) \longleftrightarrow \frac{1}{s^2} e^{-2s}$

time scaling $r(3t-2) \longleftrightarrow \frac{1}{3} \frac{1}{(s/3)^2} e^{-\frac{2s}{3}}$

$\longleftrightarrow \frac{1}{\cancel{3}} \frac{\cancel{9}\,3}{s^2} e^{-2s/3}$

time reversal $r(-3t-2) \longleftrightarrow \frac{3}{(-s)^2} e^{-2(-s)/3}$

$r(-3t-2) \longleftrightarrow \frac{3}{s^2} e^{2s/3}$

(18) $x(t) = \frac{d^2}{dt^2}\left(e^{-3(t-2)} u(t-2)\right)$

$e^{-3t} u(t) \longleftrightarrow \frac{1}{s+3} \quad R\{s\} > -3$

timing - shifting property.

$e^{-3(t-2)} u(t-2) \longleftrightarrow \frac{1}{s+3} e^{-2s} \quad R\{s\} > -$

using differentiation in time domain

$\frac{d^2}{dt^2} e^{-3(t-2)} u(t-2) \longleftrightarrow \frac{s^2}{s+3} e^{-2s}$

$R(s) > -3$

(19) $g(t) = t \cdot e^{-at} u(t)$

$= t \cdot x(t)$

$x(t) = e^{-at} u(t)$

$X(s) = \frac{1}{s+a} \quad R\{s\} > -a$

using differentiation in s-domain

$L[g_1(t)] = L[t\,x(t)]$

$G_1(s) = -\frac{dX(s)}{ds} = -\frac{d}{ds}\left(\frac{1}{s+a}\right) = \frac{1}{(s+a)^2}$

$R(s) > -a$

(20) $g(t) = t \cdot \cos w_0 t \; u(t)$

$= t \cdot x(s)$

$x(s) = \dfrac{s}{s^2 + w_0^2} \quad R(s) > 0$

$L[g(t)] = L[t \cdot x(t)]$

$= -\dfrac{d}{ds} X(s)$

$= -\dfrac{d}{ds} \dfrac{s}{(s^2 + w_0^2)}$

$= -\left[\dfrac{-s(2s) + (s^2 + w_0^2)}{(s^2 + w_0^2)^2}\right]$

$= \dfrac{s^2 - w_0^2}{(s^2 + w_0^2)^2} \quad R\{s\} > 0$

(21) Given

$y(t) = x_1(t-2) * x_2(-t+3)$

$x_1(t) = e^{-2t} u(t)$

$x_2(t) = e^{-3t} u(t)$

$x_1(t) = e^{-2t} u(t) \longleftrightarrow \dfrac{1}{s+2} \quad R(s) > -2$

$x_1(t-2) = \dfrac{1}{s+2} e^{-2s} \quad R(s) > -2$

$x_2(t) = e^{-3t} u(t) \longleftrightarrow \dfrac{1}{s+3} \quad R(s) > -3$

$x_2(t+3) \longleftrightarrow \dfrac{1}{s+3} e^{3s}$

$x_2(-t+3) \longleftrightarrow \dfrac{1}{-s+3} e^{-3s}$

$R(s) < +3$

$L[y(t)] = L[x_1(t-2)] \cdot L(x(-t+3))$

$Y(s) = \dfrac{e^{-2s}}{(s+2)} \cdot \dfrac{e^{-3s}}{-s+3} \quad -2 < R < 3$

$= \dfrac{e^{-5s}}{(s+2)(3-s)}$

$-2 < R < 3$

(22) $x(t) = e^{-t} \dfrac{d}{dt} \left(e^{-(t+1)} u(t+1)\right)$

$e^{-t} u(t) \longleftrightarrow \dfrac{1}{s+1} \quad R(s) > -1$

using time shifting

$e^{-(t+1)} u(t+1) \longleftrightarrow \dfrac{1}{s+1} e^{s} \quad R(s) > -1$

using time differentiation

$\dfrac{d}{dt}\left(e^{-(t+1)} u(t+1)\right) \longleftrightarrow \dfrac{s}{s+1} e^{s}$

Now using shifting the s-domain

$e^{-t} \dfrac{d}{dt} \left(e^{-(t+1)} u(t+1)\right)$

$\dfrac{s+1}{(s+1)+1} e^{(s+1)} \quad R(s) > -2$

$= \dfrac{s+1}{s+2} e^{(s+1)} \quad R(s) > -2$

(28) $g(t) = |t|e^{-2|t|}$

$= -t\,e^{2t}u(-t) + t\cdot e^{-2t}u(t)$

we know

$-e^{2t}u(-t) \longleftrightarrow \frac{1}{s-2} \quad R\{s\}<2$

$t[-e^{2t}u(-t)] \longleftrightarrow \frac{d}{ds}\left(\frac{1}{s-2}\right)$

$\longrightarrow \frac{1}{(s-2)^2} \quad R\{s\}<2$

and

$e^{-2t}u(t) \longleftrightarrow \frac{1}{s+2} \quad R(s) > -2$

$t\,e^{-2t}u(t) \longleftrightarrow -\frac{d}{ds}\left(\frac{1}{s+2}\right)$

$t\,e^{-2t}u(t) \longleftrightarrow \frac{1}{(s+2)^2} \quad R(s) > -2$

$\therefore X(s) = \frac{1}{(s-2)^2} + \frac{1}{(s+2)^2} \quad -2<R\{s\}<2$

$= \frac{2s^2+8}{(s^2-4)^2} \quad -2<R\{s\}<2$

(24) $x(t) = \delta(3t) + u(3t)$

$L[x(t)] = X(s) = \frac{1}{3} + \frac{1}{s}$

$= \frac{s+3}{3s} \quad R(s)>0$

(25) $x(t) = \cos^3(3t)\,u(t)$

$= \frac{1}{4}\cos(9t)u(t) + \frac{3}{4}\cos 3t\,u(t)$

$X(s)$

$= \frac{1}{4}\left[\frac{s}{s^2+81}\right] + \frac{3}{4}\frac{s}{s^2+9}$

$X(s) = \frac{1}{4}\left[\frac{s}{s^2+81} + \frac{3s}{s^2+9}\right]$

(31) Given that

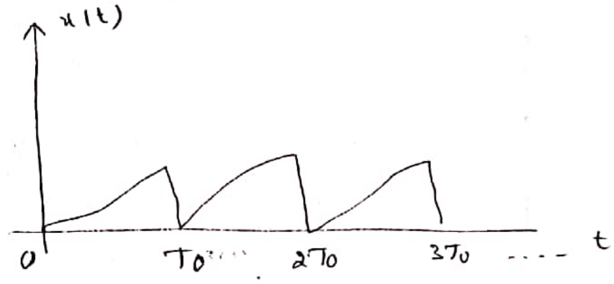
$x(t) = (1+0.5\sin t)\sin 1000t\ u(t)$

$= \sin 1000t + \frac{1}{2}\sin t\cdot\sin 1000t\ u(t)$

$= \left[\sin 1000t + \frac{1}{4}\cos 999t + \frac{1}{4}\cos 1001t\right] u(t)$

$= \frac{1000}{s^2+10^6} + \frac{1}{4}\frac{s}{s^2+999^2} - \frac{1}{4}\frac{s}{s^2+(1001)^2}$

LAPLACE TRANSFORM OF CAUSAL PERIODIC SIGNALS



let $x_1(t), x_2(t), x_3(t) \ldots$ be the signals representing 1st, 2nd, 3rd of causal periodic signal $x(t)$

$$x(t) = x_1(t) + x_2(t) + \ldots$$

$$= x_1(t) + x_1(t - T_0) + x_1(t - 2T_0) + \ldots$$

assume $x_1(t) \longleftrightarrow X_1(s)$ using time-shifting property.

$$X(s) = X_1(s) + X_1(s)e^{-sT_0} + X_1(s)e^{-2sT_0} + \ldots$$

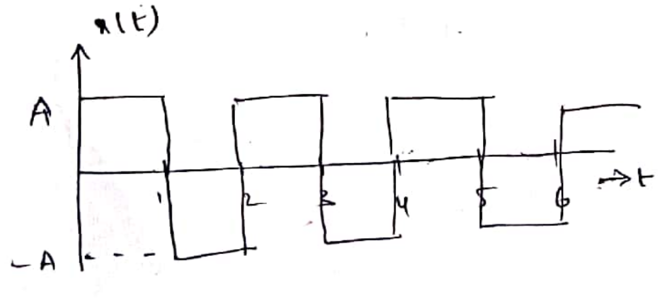
$$= X_1(s)\left[1 + e^{-sT_0} + e^{-s2T_0} + \ldots\right]$$

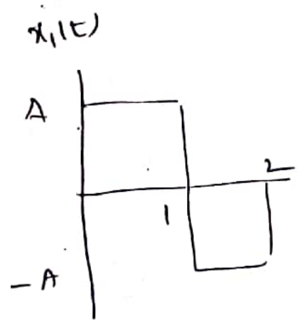
$$= X_1(s)\sum_{n=0}^{\infty} e^{-snT_0}$$

$$= X_1(s)\frac{1}{1-e^{-sT_0}}$$

$$\therefore \boxed{X(s) = \frac{X_1(s)}{1-e^{-sT_0}}}$$ causal periodic signal.

Pb: Find Laplace Transform of the square wave shown below.





we have

$$x_1(t) = A\,u(t) - 2A\,u(t-1) + A\,u(t-2)$$

Taking L.T of the above eq^n

$$X_1(s) = \frac{A}{s} - \frac{2A}{s}e^{-s} + \frac{A}{s}e^{-2s} = \frac{A}{s}\left[1 - 2e^{-s} + e^{-2s}\right]$$

$$= \frac{A}{s}(1-e^{-s})^2$$

Substitute $X_1(s)$ and $T_0 = 2$

$$X(s) = \frac{X_1(s)}{1-e^{-sT_0}} = \frac{A}{s}\frac{(1-e^{-s})^2}{1-e^{-2s}}$$

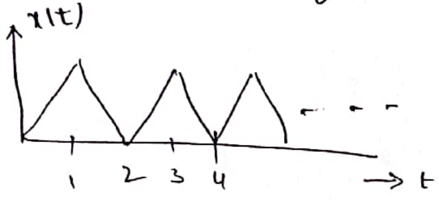
$$= \frac{A}{s}\frac{(1-e^{-s})(1-e^{-s})}{(1+e^{-s})(1-e^{-s})}$$

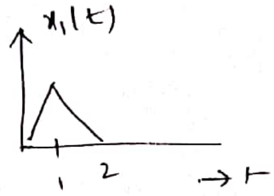
$$= \frac{A}{s}\frac{1-e^{-s}}{1+e^{-s}}$$

$$= \frac{A}{2}\frac{e^{-s/2}\left[e^{s/2} - e^{-s/2}\right]}{e^{-s/2}\left[e^{s/2} + e^{-s/2}\right]}$$

$$X(s) = \frac{A}{s}\tanh\left(\frac{s}{2}\right)$$

(2) L.T of triangular wave





$$x_1(t) = r(t) - 2r(t-1) + r(t-2)$$

$$L[x_1(t)] = X_1(s) = \frac{1}{s^2} - \frac{2}{s^2}e^{-s} + \frac{1}{s^2}e^{-2s}$$

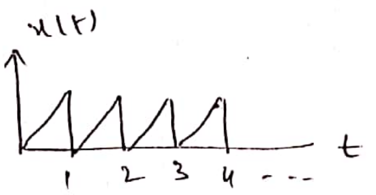
$$= \frac{1}{s^2}\left[1 - 2e^{-s} + e^{-2s}\right] = \frac{1}{s^2}(1-e^{-s})^2$$

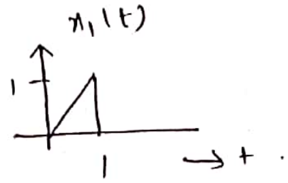
$T_0 = 2$

$$X(s) = \frac{X_1(s)}{1-e^{-sT_0}} = \frac{1}{s^2}\frac{(1-e^{-s})^2}{1-e^{-2s}}$$

$$= \frac{1}{s^2}\tanh(s/2).$$

(3) L.T of the sawtooth wave





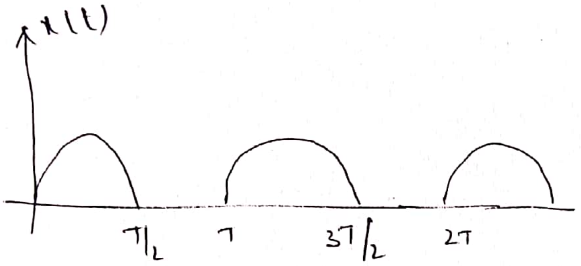
$$x_1(t) = r(t) - r(t-1) - u(t-1)$$

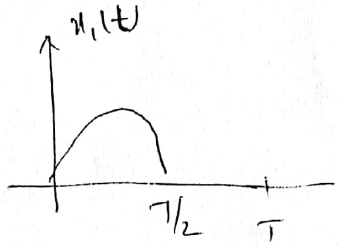
$$X_1(s) = \frac{1}{s^2} - \frac{1}{s^2}e^{-s} - \frac{1}{s}e^{-s} = \frac{1}{s^2}\left[1 - e^{-s} - se^{-s}\right]$$

$T_0 = 1$

$$X(s) = \frac{X_1(s)}{1-e^{-sT_0}} = \frac{1}{s^2}\frac{\left[1-e^{-s} - se^{-s}\right]}{1-e^{-s}}$$

(4) Find L.T of Half-wave rectification $\sin(\omega t)$





Sol:

$$x_1(t) = \begin{cases} A \sin\omega t & 0 \le t \le T/2 \\ 0 & T/2 \le t \le T \end{cases}$$

$$= A\sin(\omega t)\,u(t) + A\sin\left(\omega(t-T/2)\right)u(t-T/2)$$

Taking L.T on B.S

$$X_1(s) = A\frac{\omega}{s^2+\omega^2} + A\frac{\omega}{s^2+\omega^2}e^{-sT/2} = \frac{A\omega}{s^2+\omega^2}\left[1+e^{-sT/2}\right]$$

$T_0 = T$

$$X(s) = \frac{X_1(s)}{1-e^{-sT_0}}$$

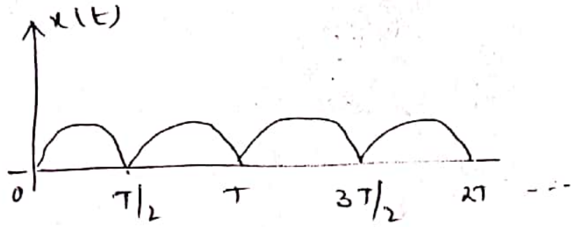
$$= \frac{A\omega}{s^2+\omega^2}\,\frac{1+e^{-sT/2}}{1-e^{-sT}}$$

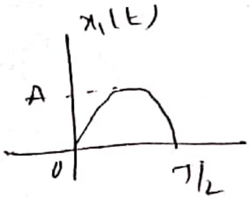
$$= \frac{A\omega}{s^2+\omega^2}\,\frac{1+e^{-sT/2}}{(1+e^{-sT/2})(1-e^{-sT/2})}$$

$$= \frac{A\omega}{s^2+\omega^2}\left(\frac{1}{1-e^{-sT/2}}\right)$$

$$\boxed{X(s) = \frac{A\omega}{s^2+\omega^2}\left(\frac{1}{1-e^{-sT/2}}\right) \quad \text{H.W.R}}$$

(5) Find the L.T of F.W.R





$$x_1(t) = A \sin wt \quad 0 \le t \le T/2 \qquad w = \frac{2\pi}{T}$$

$$= A \sin wt \; u(t) + A \sin(w(t - T/2)) \, u(t - T/2)$$

$$L[x_1(t)] = \frac{A w}{s^2 + w^2} + A \frac{w}{s^2 + w^2} e^{-sT/2}$$

$$= \frac{A w}{s^2 + w^2} \left[1 + e^{-sT/2} \right]$$

$T_0 = T/2$

$$X(s) = \frac{x(s)}{1 - e^{-sT_0}} = \frac{A w}{s^2 + w^2} \frac{1 + e^{-sT/2}}{1 - e^{-sT/2}}$$

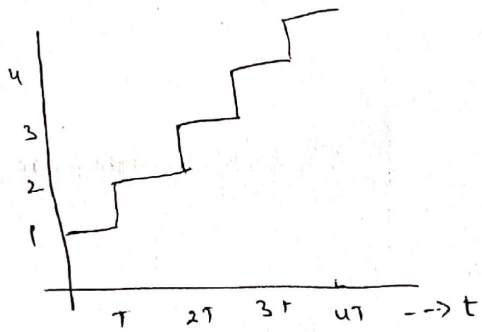
$$= \frac{A w}{s^2 + w^2} \frac{e^{-sT/4} \left(e^{sT/4} + e^{-sT/4} \right)}{e^{-sT/4} \left(e^{sT/4} - e^{-sT/4} \right)}$$

$$= \frac{A w}{s^2 + w^2} \coth\left(\frac{sT}{4} \right)$$

$$\boxed{\therefore \; X(s) = \frac{A w}{s^2 + w^2} \coth\left(\frac{sT}{4} \right)}$$

V/i)

⑥ Find the L.T of Stair case function.



$x(t) = u(t) + u(t-T) + u(t-2T) + \dots$

$X(s) = \frac{1}{s} + \frac{1}{s} e^{-T} + \frac{1}{s} e^{-2T} + \dots$

$= \frac{1}{s} (1 + e^{-T} + e^{-2T} + \dots)$

$$\boxed{X(s) = \frac{1}{s(1-e^{-sT})}}$$

Scanned by CamScanner

Inverse Laplace transform

(1) Find the I.L.T $X(s) = \frac{-3}{(s+2)(s-1)}$

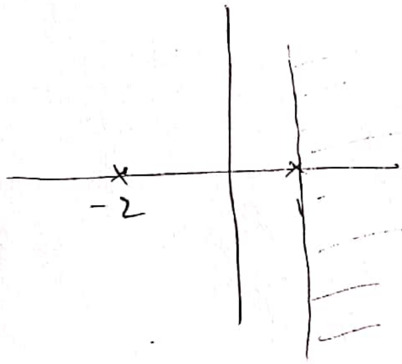
Partial fraction expansion of $X(s)$

$$X(s) = \frac{1}{s+2} - \frac{1}{s-1}$$

$X(s)$ has poles at -2 and 1.

(a) $R\{s\} > 1$, it is the right of the Right most pole,

so both poles corresponds to causal.

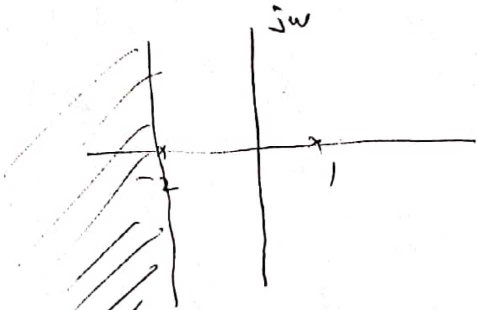


$\therefore e^{-2t}u(t) \longleftrightarrow \frac{1}{s+2}$

$e^{t}u(t) \longleftrightarrow \frac{1}{s-1}$

$x(t) = e^{-2t}u(t) - e^{t}u(t)$.

(b) $R(s) < -2$, it the left of the leftmost pole, so both poles corresponds to anticausal (left sided)



$\therefore -e^{-2t}u(-t) \longleftrightarrow \frac{1}{s+2}$

$-e^{t}u(-t) \longleftrightarrow \frac{1}{s-1}$

(c) $-2 < R[s] < 1$, ROC is a strip. The pole -2 lies to the right of this pole, so right sided

$\therefore e^{-2t}u(t) \longleftrightarrow \frac{1}{s+2}$

the pole $s=1$ is the left of this pole. so this corresponds to a anti causal (left sided) $-e^{t}u(-t) \longleftrightarrow \frac{1}{s-1}$

$\therefore x(t) = e^{-2t}u(t) + e^{t}u(-t)$.

1(c)

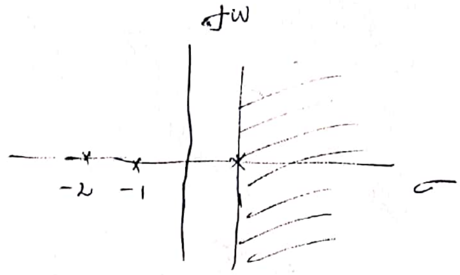
② $$X(s) = \frac{-5s-7}{(s+1)(s-1)(s+2)}$$

Partial fraction

$$X(s) = \frac{1}{s+1} - \frac{2}{s-1} + \frac{1}{s+2}$$

$X(s)$ has poles at $-1, 1, -2$.

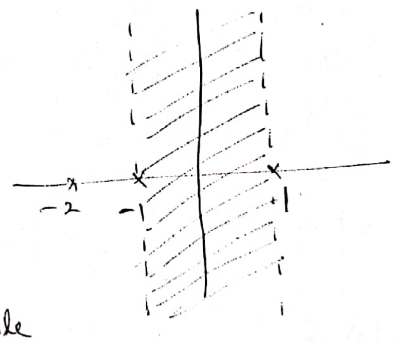
ⓐ $R\{s\} > 1$, right of rightmost pole



$e^{-t}u(t) \longleftrightarrow \frac{1}{s+1}$, $2e^{t}u(t) \longleftrightarrow \frac{2}{s-1}$

$e^{-2t}u(t) \longleftrightarrow \frac{1}{s+2}$ $\Rightarrow x(t) = e^{-t}u(t) - 2e^{t}u(t) + e^{-2t}u(t)$.

ⓑ $-1 < R(s) < 1$



pole $s=-1$ lies to the Right of this pole

$e^{-t}u(t) \longleftrightarrow \frac{1}{s+1}$

pole $s=1$ lies ROC to the left of this pole

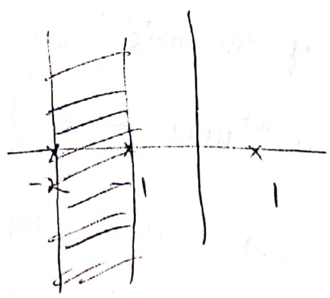
$-2e^{t}u(-t) \longleftrightarrow \frac{2}{s-1}$

pole $s=-2$, ROC is to Right of this pole

$e^{-2t}u(t) \longleftrightarrow \frac{1}{s+2}$

$x(t) = e^{-t}u(t) + 2e^{t}u(-t) + e^{-2t}u(t)$

ⓒ $-2 < R\{s\} < -1$



$-e^{-t}u(-t) \longleftrightarrow \frac{1}{s+1}$

$-2e^{t}u(-t) \longleftrightarrow \frac{2}{s-1}$

$e^{-2t}u(t) \longleftrightarrow \frac{1}{s+2}$

$x(t) = -e^{-t}u(-t) + 2e^{t}u(-t) + e^{-2t}u(t)$.

UNILATERAL (ONE-SIDED) LAPLACE TRANSFORM

$$\mathcal{L}_u(x(t)) = X(s) = \int_{0^-}^{\infty} x(t)\, e^{-st}\, dt .$$

* Two signals that differ for $t<0$, but that are identical for $t \geq 0$ will have different bilateral Laplace transform but indentical unilateral L-T.

* Any signal is identically zero for $t<0$ will have identical bilateral and unilateral transform.

Pb: Find (a) L_{Bi} (b) $\mathcal{L}_u$ for signal.

$$x(t) = e^{-a(t+1)} u(t+1)$$

Sol: we have

$$e^{-at} u(t) \longleftrightarrow \frac{1}{s+a} \quad R(s) > -a$$

time shifting

$$e^{-a(t+1)} u(t+1) \longleftrightarrow \frac{1}{s+a} e^{s} \quad R(s) > -a .$$

$$\boxed{\mathcal{L}_{Bi}\left[e^{-a(t+1)} u(t+1)\right] \longleftrightarrow \frac{e^{s}}{s+a} \quad R(s) > -a}$$

(b) $\mathcal{L}_u(x(t)) = \int_{0^-}^{\infty} x(t) e^{-st} dt = \int_{0^-}^{\infty} e^{-a(t+1)} u(t+1) e^{-st} dt$

$u(t+1) = 1 \quad t > -1$
$= 0 \quad t < 0$

$$= \int_{0^-}^{\infty} e^{-a(t+1)} e^{-st} dt$$

$$= e^{-a} \int_{0^-}^{\infty} e^{-(s+a)t} dt = \frac{e^{-a}}{s+a} \quad R\{s\} > -a$$

$$\boxed{\mathcal{L}_u\left(e^{-a(t+1)} u(t+1)\right) \longleftrightarrow \frac{e^{-a}}{s+a} \quad R(s) > -a}$$

* Relationship b/w Bilateral and Unilateral L.T.

$$\mathcal{L}[x(t)] = \mathcal{L}_u[x(t).u(t)] + \mathcal{L}_u[x(-t)u(t)]_{s\to -s}$$

$$= \mathcal{L}_u[x(t)] + \mathcal{L}_u(x(-t)]_{s\to -s}.$$

* Initial-value theorem :-

IVT allows us to determine the initial value $x(0^+)$ of $x(t)$ directly from $X(s)$.

If $x(t)=0$ $t<0$ and if contains no impulse or higher-order singularity at the origin the

$$x(0^+) = \lim_{s\to\infty} s\,X(s)$$

Proof:

$$\mathcal{L}_u\left[\frac{dx(t)}{dt}\right] = \int_{0^-}^{\infty} \frac{dx(t)}{dt} e^{-st}\,dt$$

$$s\,X(s) - x(0^-) = \int_{0^-}^{0^+} \frac{dx(t)}{dt} e^{-st}\,dt + \int_{0^+}^{\infty} \frac{dx(t)}{dt} e^{-st}\,dt$$

$$= x(t)\Big|_{0^-}^{0^+} + \int_{0^+}^{\infty} \frac{dx(t)}{dt} e^{-st}\,dt$$

$$s\,X(s) - x(0^-) = x(0^+) - x(0^-) + \int_{0^+} \frac{dx(t)}{dt} e^{-st}\,dt.$$

$$s\,X(s) = x(0^+) + \int_{0^+}^{\infty} \frac{dx(t)}{dt} e^{-st}\,dt$$

$$\lim_{s\to\infty} s\,X(s) = x(0^+) + \lim_{s\to\infty} \int_{0^+}^{\infty} \frac{dx(t)}{dt} e^{-st}\,dt$$

$$\Rightarrow \boxed{\lim_{s\to\infty} s\,X(s) = x(0^+)}$$

Scanned by CamScanner

Pb: Find the initial value of $X(s) = \frac{s+1}{s(s+2)}$ and verify the answer.

Sol: we have $x(0^+) = \lim_{s\to\infty} s\,X(s)$

$$= \lim_{s\to\infty} \frac{s+1}{s+2} = \lim_{s\to\infty} \frac{1+1/s}{1+2/s}$$

$$x(0^+) = 1$$

using partial fraction expr

$$X(s) = \frac{1/2}{s} + \frac{1/2}{s+2}$$

I-L-T

$$x(t) = \frac{1}{2}u(t) + \frac{1}{2}e^{-2t}u(t).$$

$$\therefore \quad x(t)\Big|_{t=0^+} = x(0^+) = \frac{1}{2} + \frac{1}{2} = 1.$$

* Initial value Theorem does not apply to rational function $X(s)$ in which the order of NR is greater than that of the DR polynomial.

Final-value Theorem :- $\lim_{t\to\infty} x(t) = x(\infty) = \lim_{s\to 0} s\,X(s)$

Proof:

$$\mathcal{L}\left[\frac{dx(t)}{dt}\right] = \int_{0^-}^{\infty} \frac{dx(t)}{dt} e^{-st}\,dt$$

using differentiation in the time domain

$$s\,X(s) - x(0^-) = \int_{0^-}^{\infty} \frac{dx(t)}{dt} e^{-st}\,dt$$

$$\lim_{s\to 0}\left[s\,X(s) - x(0^-)\right] = \lim_{s\to 0} \int_{0^-}^{\infty} \frac{dx(t)}{dt} e^{-st}\,dt$$

$$= x(t)\Big|_{0^-}^{\infty}$$

$$\lim_{s\to 0} sX(s) - x(0^-) = x(\infty) - x(0^-)$$

$$\Rightarrow \boxed{\lim_{s\to 0} sX(s) = x(\infty)}$$

The FVT is applicable only if poles of $X(s)$ are in the L.H.S splane, with at most a single pole at $s=0$.

Pb: The final value of $x(t) = [2 + \cancel{3}e^{-3t}]u(t)$ is $x(\infty) = 2$

prove that.

Sol: Given $x(t) = (2 + e^{-3t})u(t)$

L.T $\quad X(s) = \frac{2}{s} + \frac{1}{s+3}$.

FVT $\quad \lim_{t\to\infty} x(t) = x(\infty) = \lim_{s\to 0} sX(s)$

$$= \lim_{s\to 0} s\left[\frac{2}{s} + \frac{1}{s+3}\right]$$

$$= \lim_{s\to 0} 2 + \frac{s}{s+3} = 2.$$

Pb: Find IVT & FVT of $X(s) = \frac{10(2s+3)}{s(s^2+2s+5)}$

Sol:

IVT $\quad x(0^+) = \lim_{s\to\infty} sX(s) = \lim_{s\to\infty} \cancel{s}\, \frac{10(2s+3)}{\cancel{s}(s^2+2s+5)} = 0.$

FVT $\quad x(\infty) = \lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s)$

$$= \lim_{s\to 0} \cancel{s} \times \frac{10(2s+3)}{\cancel{s}(s^2+2s+5)} = \frac{10\times 3}{5}$$

$$= 6.$$

Scanned by CamScanner

$$\frac{dx(t)}{dt} \xleftrightarrow{L.T} s\,X(s) - x(0^-)$$

$$\frac{d^n x(t)}{dt^n} \xleftrightarrow{L.T} s^n X(s) - \sum_{K=1}^{n} s^{n-K} x^{(K-1)}(0^-)$$

Steps:

① For a given set of initial conditions, take the L.T of both sides of the diff equation to algebric equation in $Y(s)$.

② slove the algebric equation for $Y(s)$

③ Take the inverse Laplace transform to obtain $y(t)$.

Pb1: slove the second order diff equation

$$\frac{d^2 y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 6y(t) = \frac{dx(t)}{dt} + x(t)$$

for initial conditions $y(0^-) = 2$ and $\left.\frac{dy}{dt}\right|_{t=0^-} = \dot{y}(0^-) = 1$

$x(t) = e^{-4t} u(t)$.

Sol: If $x(t) = e^{-4t} u(t)$ then $x(0^-) = 0$

$$X(s) = \frac{1}{s+4}$$

Given

$$\frac{d^2 y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 6\,y(t) = \frac{dx(t)}{dt} + x(t)$$

Taking L.T on both side.

$$\left[s^2 Y(s) - sy(0^-) - \dot{y}(0^-)\right] + 5\left[sY(s) - y(0^-)\right] + 6Y(s)$$
$$= \left[sX(s) - x(0^-)\right] + X(s)$$

$$[s^2 Y(s) - 2s - 1] + 5[sY(s) - 2] + 6Y(s) = [sX(s) - 0] + X(s)$$

$$= (s^2+5s+6)Y(s) - (2s+11) = X(s)(s+1)$$

$$\Rightarrow (s^2+5s+6)Y(s) = +(2s+11) + (s+1) \times \frac{1}{(s+4)}$$

$$\Rightarrow Y(s) = \frac{2s^2+20s+45}{(s+4)(s+2)(s+3)}$$

$$= \frac{13/2}{s+2} - \frac{3}{s+3} - \frac{3/2}{s+4}$$

Taking inverse L.T

$$y(t) = \left[\frac{13}{2} e^{-2t} - 3e^{-3t} - \frac{3}{2} e^{-4t}\right] u(t)$$

(2) $$\frac{d^2 y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 6y(t) = \frac{dx(t)}{dt} + 6x(t)$$

$x(t) = u(t)$, $y(0^-) = 1$ $\dot{y}(0^-) = 2$.

Sol: $x(t) = u(t) \Rightarrow X(s) = \frac{1}{s}$; $x(0^-) = 0$.

L.T of the above equation

$$\left[s^2 Y(s) - sy(0^-) - \dot{y}(0^-)\right] + 5\left[sY(s) - y(0^-)\right] + 6Y(s)$$

$$= [sX(s) - x(0^-)] + 6X(s)$$

$$[s^2 Y(s) - s - 2] + 5[sY(s) - 1] + 6Y(s) = s(X(s) - 0] + 6X$$

$$(s^2+5s+6]Y(s) - (s+7) = X(s)(s+6)$$

$$Y(s) = \underbrace{\frac{(s+7)}{(s+3)(s+2)}}_{\text{initial condition term}} + \underbrace{\frac{s+6}{(s+2)(s+3)} \times \frac{1}{s}}_{\text{input term}}$$

using partial fraction.

$$Y(s) = \left[\frac{5}{s+2} - \frac{4}{s+3}\right] + \left[\frac{1}{s} - \frac{2}{s+2} + \frac{1}{s+3}\right]$$

Inverse Laplace transform.

$$y(t) = \left[5e^{-2t}u(t) - 4e^{-3t}u(t)\right] + \left[u(t) - 2e^{-2t}u(t) + e^{-3t}u(t)\right]$$

$$= \left[5e^{-2t} - 4e^{-3t}\right]u(t) + \left[1 - 2e^{-2t} + e^{-3t}\right]u(t).$$

zero input response (or) natural response

zero state response or forced response.

(3) $$\frac{d^3y}{dt^3} + 6\frac{d^2y}{dt^2} + 11\frac{dy}{dt} + 6y(t) = x(t)$$

(a) $x(t) = e^{-4t}u(t)$

Sol: $x(t) = e^{-4t}u(t) \Rightarrow X(s) \longleftrightarrow \frac{1}{s+4}.$

Taking L.T of above eq

$$s^3 Y(s) + 6s^2Y(s) + 11sY(s) + 6Y(s) = X(s)$$

$$Y(s)\left[s^3 + 6s^2 + 11s + 6\right] = \frac{1}{s+4}$$

$$Y(s) = \frac{1}{(s+4)(s+1)(s+2)(s+3)}$$

$$= \frac{1/6}{s+1} + \frac{1/2}{s+2} - \frac{1/2}{s+3} - \frac{1/6}{s+4}$$

$$y(t) = \frac{1}{6}e^{-t}u(t) + \frac{1}{2}e^{-2t}u(t) - \frac{1}{2}e^{-3t}u(t) - \frac{1}{6}e^{-4t}u(t)$$

(b) $x(t)=0$, $y(0^-)=1$ $\dot{y}(0^-)=-1$ $\ddot{y}(0^-)=1$

given

$$\frac{d^3y}{dt^3} + 6\frac{d^2y}{dt^2} + 11\frac{dy}{dt} + 6y(t) = x(t).$$

L.T

$$\left[s^3 y(s) - s^2 y(0^-) - s\dot{y}(0^-) - \ddot{y}(0^-)\right] + 6\left(s^2 y(s) - s y(0^-) - \dot{y}(0^-)\right)$$
$$+ 11\left(s y(s) - y(0^-)\right] + 6y(s) = x(s).$$

$$\left[s^3 y(s) - s^2 + s - 1\right] + 6\left[s^2 y(s) - s + 1\right] + 11\left[s y(s) - 1\right] + 6y(s) = 0$$

$$y(s)\left[s^3 + 6s^2 + 11s + 6\right] - s^2 - 5s - 6 = 0.$$

$$\Rightarrow \quad y(s) = \frac{s^2 + 5s + 6}{\left[s^3 + 6s^2 + 11s + 6\right]}$$

$$= \frac{(s+2)(s+3)}{(s+1)(s+2)(s+3)}$$

$$y(s) = \frac{1}{s+1}$$

$$\Rightarrow \quad y(t) = e^{-t}u(t)$$