

### IV Z-Transform

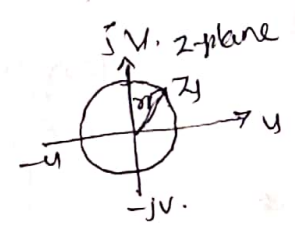
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- Laplace Transforms are popularly used for analysis of continuous time signals and systems.
- Similarly Z-transform plays an important role in analysis and representation of discrete time signals and systems.
- The Z transform of  $x(n)$  will convert the time domain signal  $x(n)$  to z-domain signal  $X(z)$ , where the signal becomes a function of complex variable  $z$ .

The complex variable  $z$  is defined as

$$z = u + jv = r e^{j\omega}$$

where  $u = \text{real part of } z$   
 $v = \text{Imaginary part of } z$



$$r = \sqrt{u^2 + v^2} = \text{magnitude of } z$$

$$\omega = \tan^{-1} \frac{v}{u} = \text{Phase (or) Argument of } z$$

### Definition of Z-Transform

Let  $x(n)$  = discrete time signal  
 $X(z)$  = Z-transform of  $x(n)$

The Z-transform of a discrete time signal,  $x(n)$  defined as

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

where  $z$  is a complex variable

$$\mathcal{Z}\{x(n)\} = X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

→ two sided Z-transform.

$$X(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$$

→ one sided Z-transform.

- The computation of  $X(z)$  involves summation of infinite terms which are functions of  $z$ . Hence it is possible that the infinite series may not converge to finite value for certain values of  $z$ . For which values of  $z$ ,  $X(z)$  can be computed is called ROC of  $X(z)$ .

## Inverse Z-Transform

$$x(n) = \mathcal{Z}^{-1}\{X(z)\} = \frac{1}{2\pi j} \oint_C X(z) z^{n-1} dz$$

$$x(n) \xrightleftharpoons[\mathcal{Z}^{-1}]{\mathcal{Z}} X(z)$$

## Geometric Series

The Z-transform of a discrete time signals involves convergence of geometric series. Hence the following two geometric series sum formula will be useful in evaluating Z-transform.

1) Infinite geometric<sup>series</sup> sum formula

If  $c$  is a complex constant and  $0 < |c| < 1$ , then

$$\sum_{n=0}^{\infty} c^n = \frac{1}{1-c}$$

( $c$  be strictly less than unity)

2) Finite geometric series sum formula

If  $c$  is a complex constant and

$$\text{When } c \neq 1, \sum_{n=0}^{N-1} c^n = \frac{1-c^N}{1-c} = \frac{c^N-1}{c-1}$$

$$\text{When } c=1, \sum_{n=0}^{N-1} c^n = N$$

( $\because$  It is valid for any values of  $c$ )

## Region of Convergence (ROC)

- The ROC of  $X(z)$  is the set of all values of  $z$ , for which  $X(z)$  attains a finite value.

- The ROC for the following six types of signals

- i) Finite duration, right sided (causal) signal
- ii) " " , left " (Anticausal) signal
- iii) " " , two " (non causal) signal
- iv) Infinite duration, right sided (causal) signal
- v) " " , left " (Anticausal) "
- vi) " " , two " (non causal) signal .

### i) Finite duration, right sided (causal) signal

Let  $x(n)$  be finite duration with  $N$  samples, defined in the range  $0 \leq n \leq (N-1)$

$$\therefore x(n) = \{ x(0), x(1), x(2), \dots, x(N-1) \}$$

Now the  $z$ -transform of  $x(n)$  is

$$X(z) = \sum_{n=0}^{N-1} x(n) z^{-n}$$

$$= x(0) + x(1)z^{-1} + x(2)z^{-2} + \dots + x(N-1)z^{-(N-1)}$$

$$X(z) = x(0) + \frac{x(1)}{z} + \frac{x(2)}{z^2} + \dots + \frac{x(N-1)}{z^{N-1}}$$

In the above summation, when  $z=0$  all the terms except the first term become infinite. Hence the  $X(z)$  exists for all values of  $z$  except  $z=0$ .

$\therefore$  ROC for finite duration, right sided (causal) signal is entire  $z$ -plane except  $z=0$ .



ii) Finite duration, left sided (noncausal) signal

Let  $x(n)$  be finite duration with  $N$ -Samples, defined in the range  $-(N-1) \leq n \leq 0$

$$\therefore x(n) = \{x(-N+1), \dots, x(-2), x(-1), x(0)\}$$

$$X(z) = \sum_{n=-(N-1)}^0 x(n) z^{-n}$$

$$= x(-N+1)z^{+(N-1)} + \dots + x(-2)z^2 + x(-1)z^1 + x(0)$$

In the above, when  $z = \infty$ , all the terms except the last term become infinite. Hence  $X(z)$  exists for all values of  $z$  except  $z = \infty$ .

$\therefore$  ROC of  $X(z)$  is entire  $z$ -plane except  $z = \infty$ .

iii) Finite duration, Two sided (noncausal) signal

Let  $x(n)$  be a finite duration with  $N$ -Samples, defined in the range  $-M \leq n \leq +M$  where  $M = \frac{N-1}{2}$

$$\therefore x(n) = \{x(-M), \dots, x(-2), x(-1), x(0), x(1), x(2), \dots, x(M)\}$$

Now  $z$ -transform of  $x(n)$  is

$$X(z) = \sum_{n=-M}^{+M} x(n) z^{-n}$$

$$= x(-M)z^M + \dots + x(-2)z^2 + x(-1)z^1 + x(0) + x(1)z^{-1} + x(2)z^{-2} + \dots + x(M)z^{-M}$$

$$= x(-M)z^M + \dots + x(-2)z^2 + x(-1)z^1 + x(0) + \frac{x(1)}{z} + \frac{x(2)}{z^2} + \dots + \frac{x(M)}{z^M}$$

In the above, when  $z = 0$ , the terms with negative powers of  $z$  attain infinity and when  $z = \infty$ , the terms with positive power of  $z$  attain infinity. Hence  $X(z)$  converges for all values of  $z$ , except  $z = 0$  and  $z = \infty$ .

$\therefore$  ROC is entire  $z$ -plane, except  $z = 0$  &  $z = \infty$ .

iv) Infinite duration, right sided (causal) signal

Let  $x(n) = r^n, n \geq 0$

$$Z\{x(n)\} = X(Z) = \sum_{n=-\infty}^{\infty} x(n) Z^{-n}$$
$$= \sum_{n=0}^{\infty} r^n \cdot Z^{-n}$$
$$= \sum_{n=0}^{\infty} (r, Z^{-1})^n$$

$$\sum_{n=0}^{\infty} c^n = \frac{1}{1-c}$$
  
where  $0 < |c| < 1$

if  $0 < |r, Z^{-1}| < 1$  then  $\sum_{n=0}^{\infty} (r, Z^{-1})^n = \frac{1}{1-r, Z^{-1}}$

$$\therefore X(Z) = \frac{1}{1-r, Z^{-1}} = \frac{1}{1-\frac{r}{Z}} = \frac{Z}{Z-r}$$

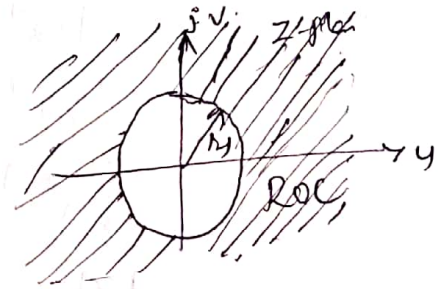
Here the condition to be satisfied for the convergence of  $X(Z)$  is

$0 < |r, Z^{-1}| < 1$

$\therefore |r, Z^{-1}| < 1$

$\frac{|r|}{|Z|} < 1 \Rightarrow |Z| > |r|$

$\therefore$  ROC is exterior the circle.



v) Infinite duration, left sided (anti-causal) signal

Let  $x(n) = r_2^n, n \leq 0$

$$Z\{x(n)\} = X(Z) = \sum_{n=-\infty}^0 x(n) Z^{-n}$$
$$= \sum_{n=0}^{\infty} r_2^n Z^{-n} = \sum_{n=0}^{\infty} r_2^{-n} Z^n = \sum_{n=0}^{\infty} (r_2^{-1} Z)^n$$

if  $0 < |r_2^{-1} Z| < 1$ , then  $\sum_{n=0}^{\infty} (r_2^{-1} Z)^n = \frac{1}{1-r_2^{-1} Z}$

$$\therefore X(Z) = \frac{1}{1-r_2^{-1} Z} = \frac{1}{1-\frac{r_2}{Z}} = \frac{Z}{Z-r_2} = \frac{r_2}{r_2-Z} = -\frac{r_2}{Z-r_2}$$

Condition for convergence of  $X(Z)$

$0 < |r_2^{-1} Z| < 1 \Rightarrow |r_2^{-1} Z| < 1 \Rightarrow \frac{|Z|}{|r_2|} < 1 \Rightarrow |Z| < |r_2|$

$\therefore$  ROC of  $X(Z)$  is interior of the circle.



vii) Infinite duration, two sided (non-causal) signal.

$$\text{Let } x(n) = r_1^n u(n) + r_2^n u(-n)$$

$$\text{ZT of } x(n) = X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} r_1^n z^{-n} + \sum_{n=0}^{\infty} r_2^n z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} r_2^{-n} z^n + \sum_{n=0}^{\infty} r_1^n z^{-n}$$

$$= \sum_{n=0}^{\infty} r_2^{-n} z^n + \sum_{n=0}^{\infty} r_1^n z^{-n}$$

$$= \frac{1}{1 - r_2^{-1} z} + \frac{1}{1 - r_1 z^{-1}}$$

(Using infinite Geometric S.E.S.)

~~First term~~

First term converges if

$$0 < |r_2^{-1} z| < 1$$

$$|r_2^{-1} z| < 1$$

$$|z| < |r_2|$$

$$|z| < |r_2|$$

$$|z| < |r_2|$$

Second term converges if,

$$0 < |r_1 z^{-1}| < 1$$

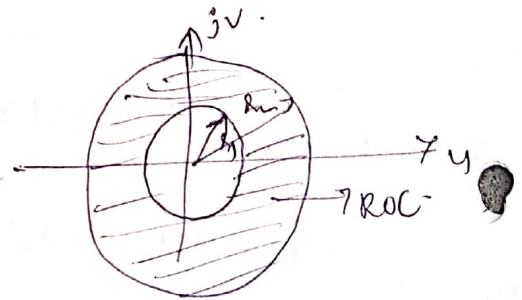
$$|r_1 z^{-1}| < 1$$

$$\frac{|r_1|}{|z|} < 1$$

$$|z| > |r_1|$$

$$|z| > |r_1|$$

∴ ROC is between two circles of radius  $|r_1|$  &  $|r_2|$



# Properties of z-Transforms

## 1) Linearity property

Let  $z\{x_1(n)\} = X_1(z)$  and  $z\{x_2(n)\} = X_2(z)$   
 Then  $z\{a_1x_1(n) + a_2x_2(n)\} = a_1X_1(z) + a_2X_2(z)$ , where  $a_1$  &  $a_2$  are constants.

Proof:

$$X_1(z) = z\{x_1(n)\} = \sum_{n=-\infty}^{\infty} x_1(n)z^{-n}$$

$$X_2(z) = z\{x_2(n)\} = \sum_{n=-\infty}^{\infty} x_2(n)z^{-n}$$

$$z\{a_1x_1(n) + a_2x_2(n)\} = \sum_{n=-\infty}^{\infty} \{a_1x_1(n) + a_2x_2(n)\}z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} a_1x_1(n)z^{-n} + \sum_{n=-\infty}^{\infty} a_2x_2(n)z^{-n}$$

$$= a_1X_1(z) + a_2X_2(z)$$


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## 2) Shifting property

if  $z\{x(n)\} = X(z)$   
 Then  $z\{x(n-m)\} = z^{-m}X(z)$   
 $z\{x(n+m)\} = z^mX(z)$

Proof:

$$X(z) = z\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n)z^{-n}$$

$$z\{x(n-m)\} = \sum_{n=-\infty}^{\infty} x(n-m)z^{-n}$$

Let  $n-m = p$   
 $n = p+m$   
 $n = -\infty \Rightarrow p = -\infty$   
 $n = \infty \Rightarrow p = \infty$

$$= \sum_{p=-\infty}^{\infty} x(p)z^{-(p+m)}$$

$$= \sum_{p=-\infty}^{\infty} x(p)z^{-p} \cdot z^{-m}$$

$$= z^{-m}X(z)$$


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### 3) Multiplication by n (or differentiation in z-domain)

If  $Z\{x(n)\} = X(z)$

then  $Z\{n x(n)\} = -z \frac{d}{dz} X(z)$

Proof:  $X(z) = Z\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$

$$Z\{n x(n)\} = \sum_{n=-\infty}^{\infty} n x(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} n x(n) z^{-n} \cdot z \cdot z^{-1}$$

$$= -z \sum_{n=-\infty}^{\infty} x(n) [-n z^{-n}]$$

$$= -z \sum_{n=-\infty}^{\infty} x(n) \left[ \frac{d}{dz} z^{-n} \right]$$

$$= -z \frac{d}{dz} \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= -z \frac{d}{dz} X(z)$$

### 4) Multiplication by an exponential sequence, $a^n$ (or scaling in z-domain)

If  $Z\{x(n)\} = X(z)$

then  $Z\{a^n x(n)\} = X(a^{-1}z)$

Proof:  $Z\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$

$$Z\{a^n x(n)\} = \sum_{n=-\infty}^{\infty} \{a^n x(n)\} z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x(n) \{a^n z^{-n}\}$$

$$= X(a^{-1}z)$$

### 5) Time Reversal

If  $Z\{x(n)\} = X(z)$

then  $Z\{x(-n)\} = X(z^{-1})$

Proof:  $Z\{x(n)\} = X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$

$$Z\{x(-n)\} = \sum_{n=-\infty}^{\infty} x(-n) z^{-n}$$

$$= \sum_{p=-\infty}^{\infty} x(p) z^{-p} = \sum_{p=-\infty}^{\infty} x(p) (z^{-1})^{-p} = X(z^{-1})$$

Let  $-n = p$   
 $n \rightarrow \infty \Rightarrow p \rightarrow -\infty$   
 $n \rightarrow -\infty \Rightarrow p \rightarrow \infty$



6) Conjugation

If  $Z\{x(n)\} = X(Z)$   
 Then  $Z\{x^*(n)\} = X^*(Z^*)$

Proof:  $Z\{x(n)\} = X(Z) = \sum_{n=-\infty}^{\infty} x(n)Z^{-n}$   
 $Z\{x^*(n)\} = \sum_{n=-\infty}^{\infty} x^*(n)Z^{-n}$   
 $= \left[ \sum_{n=-\infty}^{\infty} x(n)(Z^*)^{-n} \right]^*$   
 $= [X(Z^*)]^* = X^*(Z^*)$

7) Convolution Theorem

If  $Z\{x_1(n)\} = X_1(Z)$  and  $Z\{x_2(n)\} = X_2(Z)$

Then  $Z\{x_1(n) * x_2(n)\} = X_1(Z) X_2(Z)$

Proof:  $Z\{x_1(n)\} = X_1(Z) = \sum_{n=-\infty}^{\infty} x_1(n)Z^{-n}$  — ①

$Z\{x_2(n)\} = X_2(Z) = \sum_{n=-\infty}^{\infty} x_2(n)Z^{-n}$  — ②

$Z\{x_1(n) * x_2(n)\} = \sum_{n=-\infty}^{\infty} \{x_1(n) * x_2(n)\} Z^{-n}$   
 $= \sum_{n=-\infty}^{\infty} \left[ \sum_{k=-\infty}^{\infty} x_1(k) x_2(n-k) \right] Z^{-n}$   
 $= \sum_{k=-\infty}^{\infty} x_1(k) Z^{-k} \sum_{n=-\infty}^{\infty} x_2(n-k) Z^{-n} \cdot Z^k$  (multiply  $Z^k Z^k$ )  
 $= \sum_{k=-\infty}^{\infty} x_1(k) Z^{-k} \sum_{p=-\infty}^{\infty} x_2(p) Z^{-(n-k)}$  let  $n-k=p$   
 $n=\infty \Rightarrow p=\infty$   
 $n=-\infty \Rightarrow p=-\infty$   
 $= \sum_{k=-\infty}^{\infty} x_1(k) Z^{-k} \sum_{p=-\infty}^{\infty} x_2(p) Z^{-p}$   
 $= \left[ \sum_{k=-\infty}^{\infty} x_1(k) Z^{-k} \right] \left[ \sum_{p=-\infty}^{\infty} x_2(p) Z^{-p} \right]$   
 $= X_1(Z) X_2(Z)$

8) Correlation Property

If  $Z\{x(n)\} = X(Z)$  and  $Z\{y(n)\} = Y(Z)$   
 Then  $Z\{x(n)y(n)\} = X(Z)Y(Z^{-1})$

where  $x(n)y(n) = \sum_{n=-\infty}^{\infty} x(n)y(n-m)$

Proof:

$$X(Z) = Z\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n)Z^{-n}$$

$$Y(Z) = Z\{y(n)\} = \sum_{n=-\infty}^{\infty} y(n)Z^{-n}$$

$$Z\{x(n)y(n)\} = \sum_{m=-\infty}^{\infty} x(n)y(n-m)Z^{-m}$$

$$= \sum_{m=-\infty}^{\infty} \left[ \sum_{n=-\infty}^{\infty} x(n)y(n-m) \right] Z^{-m}$$

$$= \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} x(n)y(n-m) Z^{-m} \cdot Z^n \cdot Z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x(n)Z^{-n} \sum_{m=-\infty}^{\infty} y(n-m)Z^{-m}$$

Let  $n-m=p$

$\therefore m=n-p$

When  $m=-\infty \Rightarrow p \rightarrow \infty$   
 $m=\infty \Rightarrow p \rightarrow -\infty$

$$= \sum_{n=-\infty}^{\infty} x(n)Z^{-n} \sum_{p=-\infty}^{\infty} y(p)Z^p$$

$$= \sum_{n=-\infty}^{\infty} x(n)Z^{-n} \sum_{p=-\infty}^{\infty} y(p)(Z^{-1})^p$$

$$= X(Z)Y(Z^{-1})$$

9) Initial Value Theorem

Let  $x(n)$  be an one sided signal defined in the range  $0 \leq n < \infty$

If  $Z\{x(n)\} = X(Z)$

Then initial value of  $x(n)$  is given by

$$x(0) = \lim_{Z \rightarrow \infty} X(Z)$$

Proof:

$$X(Z) = \sum_{n=0}^{\infty} x(n)Z^{-n}$$

$$X(Z) = x(0) + x(1)Z^{-1} + x(2)Z^{-2} + \dots$$

$$\therefore X(Z) = x(0) + \frac{x(1)}{Z} + \frac{x(2)}{Z^2} + \dots$$

$$\lim_{Z \rightarrow \infty} X(Z) = \lim_{Z \rightarrow \infty} \left[ x(0) + \frac{x(1)}{Z} + \frac{x(2)}{Z^2} + \dots \right]$$

$$= x(0) + 0 + 0 + \dots$$

$$\therefore x(0) = \lim_{Z \rightarrow \infty} X(Z)$$

10) Final Value Theorem

Let  $x(n)$  be one sided signal defined in the range  $0 \leq n < \infty$

If  $Z\{x(n)\} = X(Z)$

Then final value of  $x(n)$  (i.e.  $x(\infty)$ ) is given by

$$x(\infty) = \lim_{z \rightarrow 1} (1-z^{-1}) X(z) \quad (\text{or}) \quad x(\infty) = \lim_{z \rightarrow 1} \left( \frac{z-1}{z} \right) X(z)$$

Proof:

$$Z\{x(n)\} = X(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$$

$$\therefore Z\{x(n-1) - x(n)\} = \sum_{n=0}^{\infty} \{x(n-1) - x(n)\} z^{-n}$$

RHS LHS.

$$\text{RHS} = Z\{x(n-1) - x(n)\}$$

$$= Z\{x(n-1)\} - Z\{x(n)\} \quad (\text{Using linearity property}).$$

$$= z^{-1} X(z) + X(-1) - X(z) \quad (\text{Using shift}).$$

$$= X(-1) - (1-z^{-1}) X(z)$$

$$= \lim_{z \rightarrow 1} [X(-1) - (1-z^{-1}) X(z)]$$

$$= X(-1) - \lim_{z \rightarrow 1} (1-z^{-1}) X(z) \quad \text{--- (1)}$$

$$\text{LHS} = \sum_{n=0}^{\infty} [x(n-1) - x(n)] z^{-n}$$

$$= \lim_{z \rightarrow 1} \sum_{n=0}^{\infty} [x(n-1) - x(n)] z^{-n}$$

$$= \sum_{n=0}^{\infty} [x(n-1) - x(n)] (1)^{-n}$$

Take  $\lim_{p \rightarrow \infty}$  & change summation index from 0 to p.

$$= \lim_{p \rightarrow \infty} \sum_{n=0}^p [x(n-1) - x(n)]$$

$$= \lim_{p \rightarrow \infty} [x(-1) - x(0) + [x(0) - x(1)] + [x(1) - x(2)] + \dots$$

$$\dots + [x(p-2) - x(p-1)] + [x(p-1) - x(p)]]$$

$$= \lim_{p \rightarrow \infty} [x(-1) - x(p)]$$

$$= x(-1) - x(\infty) \quad \text{--- (2)}$$

equating (1) & (2)

$$x(-1) - x(\infty) = x(-1) - \lim_{z \rightarrow 1} (1-z^{-1}) X(z).$$

$$\therefore x(\infty) = \lim_{z \rightarrow 1} (1-z^{-1}) X(z)$$

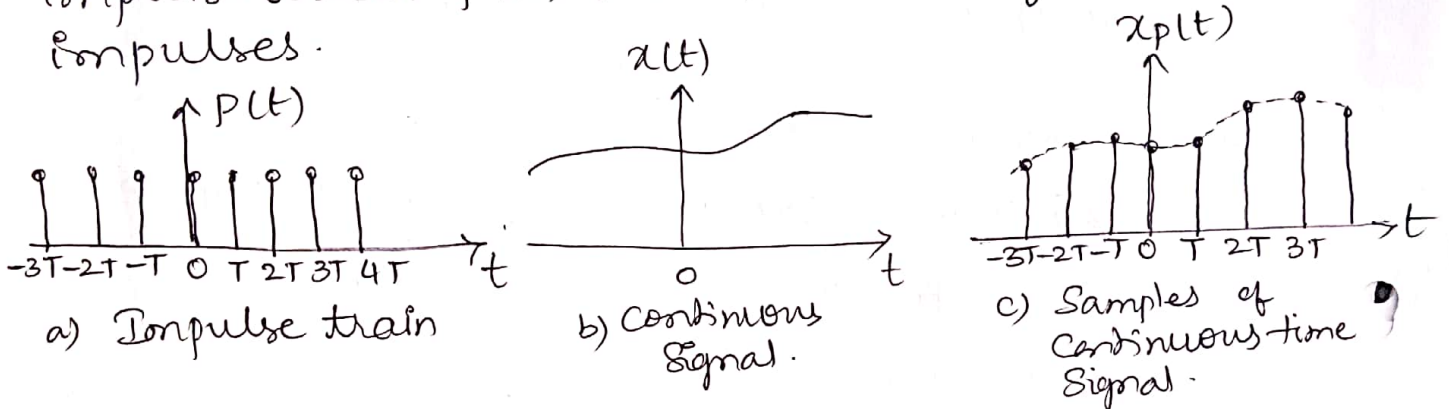


# Relation between LT and ZT (Z plane & S-plane correspondence with P(t))

Consider a periodic impulse train  $P(t)$  with period  $T$ . pulse train can be expressed as

$$P(t) = \sum_{n=-\infty}^{\infty} \delta(t-nT) \quad \text{--- (1)}$$

When a continuous signal  $x(t)$  multiplied by impulse train  $P(t)$ , the product signal will have impulses.



$$\therefore x_p(t) = \sum_{n=-\infty}^{\infty} x(nT) \delta(t-nT)$$

where  $x(nT)$  are samples of  $x(t)$  at  $t=nT$

Taking LT to this

$$L\{x_p(t)\} = X_p(s) = L\left\{\sum_{n=-\infty}^{\infty} x(nT) \delta(t-nT)\right\}$$

$$= \sum_{n=-\infty}^{\infty} x(nT) L\{\delta(t-nT)\}$$

$$= \sum_{n=-\infty}^{\infty} x(nT) e^{-nTs} \quad \left[ \because L\{x(t-t_0)\} = e^{-st_0} X(s) \right]$$

$$X_p(s) = \sum_{n=-\infty}^{\infty} x(nT) (e^{sT})^{-n}$$

let  $e^{sT} = z$

$$\therefore X_p(s) = \sum_{n=-\infty}^{\infty} x(nT) z^{-n} \quad \text{--- (2)}$$

The Z-T of  $x(nT)$  is

$$X(z) = \sum_{n=-\infty}^{\infty} x(nT) z^{-n} \quad \text{--- (3)}$$



Comparing ② & ③

⑦

$$X(s) = X(z) \Big|_{z=e^{sT}} \quad x(nT) = \text{Sampled version of } x(t)$$

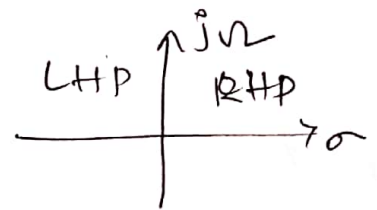
∴ The ZT of discrete time signal can be obtained from LT of sampled version of  $x(t)$  by choosing transformation  $e^{sT} = z$ . This transformation is also called Impulse Invariant transformation.

### Relation between S-plane and Z-plane

$$z = e^{sT} \quad \text{but } s = \sigma + j\omega$$

$$\therefore z = e^{(\sigma + j\omega)T}$$

$$z = e^{\sigma T} \cdot e^{j\omega T}$$

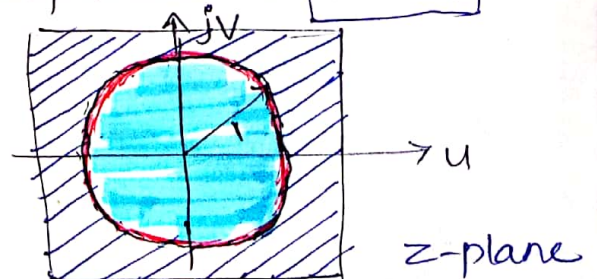
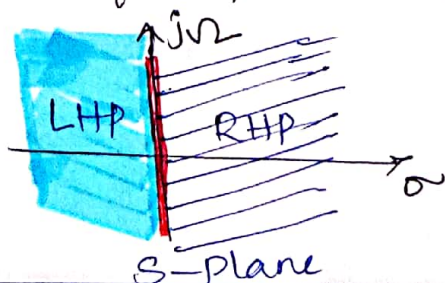


$$|z| = e^{\sigma T} \\ \angle z = e^{j\omega T}$$

(i) If  $\sigma < 0$  (i.e.  $\sigma$  is negative), then points  $s$  lies on Left Half (LHP) of  $s$ -plane. In this case  $|z| < 1$ . (i.e.  $z = e^{-T} = \frac{1}{e^T} < 1$ )

(ii) If  $\sigma = 0$  (The imaginary axis on  $s$ -plane),  $\sigma = 0$  corresponds to  $|z| = 1$  {∵  $|z| = e^0 = 1$ }

(iii) If  $\sigma > 0$  (i.e.  $\sigma$  is +ve), then points  $s$  lies on RHP of  $s$ -plane corresponds to  $|z| > 1$ .



<del><math>x(n)</math></del> $x(n)$	$X(z)$	ROC	signals
$\delta(n)$	1	entire z-plane.	
$u(n)$	$\frac{z}{z-1}$	$ z  > 1$	
$a^n u(n)$	$\frac{z}{z-a}$	$ z  >  a $	
$u(-n)$	<del><math>\frac{z}{z-1}</math></del> $\frac{1}{1-z}$	$ z  < 1$	
$a^n u(-n)$	$-\frac{a}{a-z}$	$ z  <  a $	
$a^n u(-n-1)$	$-\frac{z}{z-a}$	$ z  <  a $	
$na^n u(n)$	$\frac{az}{(z-a)^2}$	$ z  >  a $	
$nu(n)$	$\frac{z}{(z-1)^2}$	$ z  > 1$	
$e^{-at} u(t) = e^{-an} u(n)$	$\frac{z}{z-e^{-aT}}$	$ z  > e^{-aT}$	
$t e^{-at} u(t) = nT e^{-an} u(n)$	$\frac{-aT z}{(z-e^{-aT})^2}$	$ z  > e^{-aT}$	
$\sin \omega t u(t)$	$\frac{z \sin \omega T}{z^2 - 2z \cos \omega T + 1}$	$ z  > 1$	
$\cos \omega t u(t)$	$\frac{z(z - \cos \omega T)}{z^2 - 2z \cos \omega T + 1}$	$ z  > 1$	

$= x(0)z^0 \Rightarrow x(z) = 1$  ROC  $\Rightarrow$  for all values of 'z'

1. Determine the z-Transform & Roc of following signals

i)  $x(n) = \{1, 2, 3, 0\}$   
 $\uparrow$

$$\begin{aligned} Z\{x(n)\} &= \sum_{n=-\infty}^{\infty} x(n)z^{-n} \\ &= \sum_{n=0}^3 x(n)z^{-n} \\ &= x(0) + x(1)z^{-1} + x(2)z^{-2} + x(3)z^{-3} \\ &= 1 + 2z^{-1} + 3z^{-2} + 0 \cdot z^{-3} \end{aligned}$$

$$X(z) = 1 + \frac{2}{z} + \frac{3}{z^2}$$

Roc  $\Rightarrow$  All values of  $z$  except ' $z=0$ '

ii)  $x(n) = \{1, 2, 3, 4, 5\}$   
 $\uparrow$

$$\begin{aligned} Z\{x(n)\} &= \sum_{n=-\infty}^{\infty} x(n)z^{-n} \\ &= \sum_{n=-2}^2 x(n)z^{-n} \\ &= x(-2)z^2 + x(-1)z^1 + x(0)z^0 + x(1)z^{-1} + x(2)z^{-2} \\ &= (1)z^2 + 2(z) + 3(1) + 4(z^{-1}) + 5(z^{-2}) \end{aligned}$$

$$X(z) = z^2 + 2z + 3 + \frac{4}{z} + \frac{5}{z^2}$$

Roc  $\rightarrow$  All values except  $z=0$  &  $\infty$

iii)  $x(n) = \delta(n)$  (Impulse)

$$\begin{aligned} \delta(n) &= 1, n=0 \\ &= 0, n \neq 0 \end{aligned}$$

$$\begin{aligned} Z\{x(n)\} &= \sum_{n=-\infty}^{\infty} x(n)z^{-n} \\ &= \sum_{n=-\infty}^{\infty} x(n)z^{-n} \\ &= x(0)z^0 \end{aligned}$$

$$\Rightarrow X(z) = 1 \quad \text{Roc} \Rightarrow \text{for all values of } 'z'$$

$$\text{iv) } x(n) = u(n)$$

$$u(n) = 1, n > 0$$

$$Z\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n)z^{-n}$$

$$= \sum_{n=0}^{\infty} u(n)z^{-n}$$

$$= \sum_{n=0}^{\infty} (1)z^{-n}$$

$$= 1 + z^{-1} + z^{-2} + z^{-3} + \dots \quad (\text{G.P.})$$

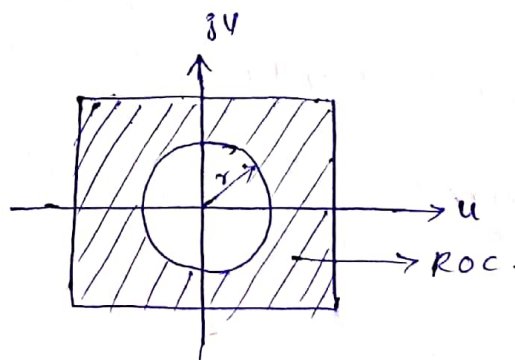
$$X(z) = \frac{1}{1 - z^{-1}}$$

$$X(z) = \frac{z}{z-1}$$

$$0 < |z^{-1}| < 1$$

$$\frac{1}{|z|} < 1$$

$$\text{Roc} \rightarrow |z| > 1$$



$$\text{v) } x(n) = 2^n u(n)$$

$$u(n) = 1, n > 0$$

$$Z\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n)z^{-n}$$

$$= \sum_{n=0}^{\infty} 2^n z^{-n} u(n)$$

$$= \sum_{n=0}^{\infty} 2^n z^{-n}$$

$$= \sum_{n=0}^{\infty} (2z^{-1})^n$$

$$= \frac{1}{1 - 2z^{-1}}$$

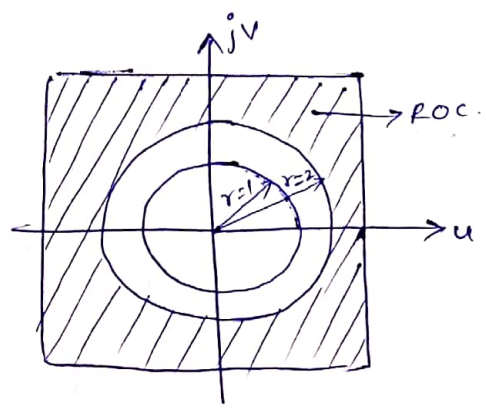


$$0 < |2z^{-1}| < 1$$

$$0 < \left| \frac{2}{z} \right| < 1$$

$$\frac{2}{z} > 0 \quad \frac{2}{z} < 1$$

$$\text{ROC} \Rightarrow \boxed{|z| > 2}$$



v i)  $x(n) = u(-n)$

$$z\{x(n)\} = \sum_{n=-\infty}^{\infty} u(-n)z^{-n}$$

$$u(-n) = 1, n \leq 0$$

$$= \sum_{n=-\infty}^0 z^{-n}$$

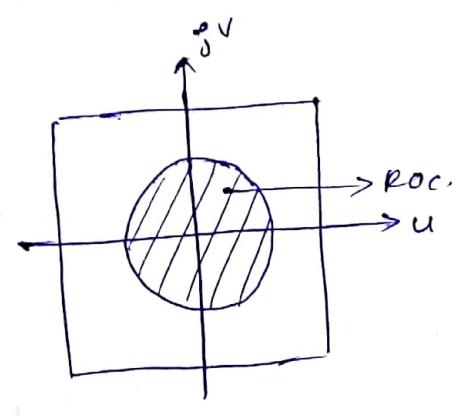
$$= \sum_{n=\infty}^{-n=0} z^n$$

$$= \sum_{n=0}^{\infty} (z)^n$$

$$X(z) = \frac{1}{1-z}$$

$$0 < |z| < 1$$

$$\text{ROC} \rightarrow \boxed{|z| < 1}$$



v ii)  $x(n) = 2^n u(-n-1)$

$$z\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n)z^{-n}$$

$$u(n) = 1, n \geq 0$$

$$= 0, \text{ for } n < 0$$

$$u(-n-1) = 1, \text{ for } -n-1 \geq 0 \rightarrow -n \geq 1$$

$$n \leq -1$$

$$u(-n-1) = 0, \text{ for } -n-1 < 0 \Rightarrow -n < 1$$

$$n > -1$$

$$Z\{x(n)\} = \sum_{n=-1}^{-\infty} 2^n z^{-n}$$

$$= \sum_{-n=-1}^{-n=-\infty} (2z^{-1})^{-n}$$

$$= \sum_{n=1}^{\infty} (2z^{-1})^{-n}$$

$$= \sum_{n=0}^{\infty} (2z^{-1})^n - 1$$

$$= \sum_{n=0}^{\infty} (2^{-1}z)^n - 1$$

$$X(z) = \frac{1}{1 - 2^{-1}z} - 1$$

$$= \frac{2}{2-z} - 1 \Rightarrow \frac{2-2+z}{2-z}$$

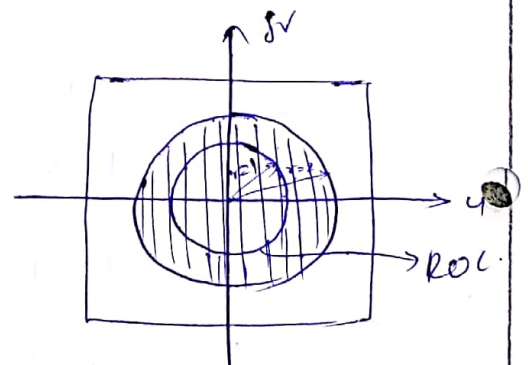
$$= \frac{z}{2-z}$$

$$\text{ROC} \Rightarrow 0 < |2^{-1}z| < 1$$

$$0 < \left| \frac{z}{2} \right| < 1$$

$$\left| \frac{z}{2} \right| < 1$$

$$\boxed{|z| < 2}$$



$$\text{Vii) } x(n) = nu(n)$$

$$z\{x(n)\} = z\{nu(n)\} = -z \frac{d}{dz} [z\{u(n)\}]$$

$$= -z \frac{d}{dz} \left[ \frac{z}{z-1} \right]$$

$$= -z \left[ \frac{(z-1)(1) - z(1)}{(z-1)^2} \right]$$

$$= -z \left[ \frac{-1}{(z-1)^2} \right]$$

$$= \frac{z}{(z-1)^2}$$

2. Find the z-Transforms of  $x(n) = na^{n-1}$

$$z\{x(n)\} = z\{na^{n-1}\}$$

$$x(n) = na^{n-1}$$

$$= na^n a^{-1}$$

$$= \frac{na^n}{a}$$

$$z\{x(n)\} = \frac{1}{a} z\{na^n\}$$

$$= \frac{1}{a} \left[ -z \frac{d}{dz} z\{a^n\} \right]$$

$$= \frac{1}{a} \left[ -z \frac{d}{dz} \left( \frac{z}{z-a} \right) \right]$$

$$= \frac{1}{a} \left[ -z \left( \frac{(z-a)(1) - (z)(1)}{(z-a)^2} \right) \right]$$

$$x(z) = \frac{1}{a} \left[ \frac{az}{(z-a)^2} \right] = \frac{z}{(z-a)^2}$$

$$Q:- x(t) = \sin \Omega_0 t u(t)$$

$$t \rightarrow nT$$

$$x(nT) = \sin \Omega_0 nT u(nT)$$

$$y(n) = \sin(\Omega_0 T) n u(n)$$

$$y(n) = \sin \omega n$$

$$z\{y(n)\} = \sum_{n=-\infty}^{\infty} y(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \sin(\omega n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \left[ \frac{e^{j\omega n} - e^{-j\omega n}}{2j} \right] z^{-n}$$

$$= \frac{1}{2j} \left[ \sum_{n=-\infty}^{\infty} e^{j\omega n} z^{-n} - \sum_{n=-\infty}^{\infty} e^{-j\omega n} z^{-n} \right]$$

$$= \frac{1}{2j} \left[ \sum_{n=0}^{\infty} (e^{j\omega} z^{-1})^n - \sum_{n=0}^{\infty} (e^{-j\omega} z^{-1})^n \right]$$

$$= \frac{1}{2j} \left[ \frac{1}{1 - e^{j\omega} z^{-1}} - \frac{1}{1 - e^{-j\omega} z^{-1}} \right]$$

$$= \frac{1}{2j} \left[ \frac{z}{z - e^{j\omega}} - \frac{z}{z - e^{-j\omega}} \right]$$

$$= \frac{1}{2j} \left[ \frac{z^2 - ze^{j\omega} - z^2 + ze^{-j\omega}}{(z - e^{j\omega})(z - e^{-j\omega})} \right]$$

$$= \frac{z}{2j} \left[ \frac{e^{j\omega} - e^{-j\omega}}{z^2 - ze^{j\omega} - ze^{-j\omega} + 1} \right]$$

$$= \frac{z \sin \omega}{z^2 - 2z \cos \omega + 1} = \frac{z \sin \Omega_0 T}{z^2 - 2z \cos \Omega_0 T + 1}$$



$$\text{ROC: } 0 < |e^{j\omega} z^{-1}| < 1$$

$$|z| > e^{j\omega}$$

$$0 < |e^{j\omega} z^{-1}| < 1$$

$$|z| > e^{-j\omega}$$

Q:-  $x(t) = \cos \Omega_0 t$

$t \rightarrow nT$

$x(nT) = \cos \Omega_0 nT u(nT)$

$y(n) = \cos(\Omega_0 T) n u(n)$

$y(n) = \cos \omega n$

$$Z\{y(n)\} = \sum_{n=-\infty}^{\infty} y(n) z^{-n}$$

$$= \sum_{n=0}^{\infty} \cos(\omega n) z^{-n}$$

$$= \sum_{n=0}^{\infty} \left[ \frac{e^{j\omega n} + e^{-j\omega n}}{2} \right] z^{-n}$$

$$= \frac{1}{2} \left[ \sum_{n=0}^{\infty} e^{j\omega n} z^{-n} + \sum_{n=0}^{\infty} e^{-j\omega n} z^{-n} \right]$$

$$= \frac{1}{2} \left[ \sum_{n=0}^{\infty} (e^{j\omega} z^{-1})^n + \sum_{n=0}^{\infty} (e^{-j\omega} z^{-1})^n \right]$$

$$= \frac{1}{2} \left[ \frac{1}{1 - e^{j\omega} z^{-1}} + \frac{1}{1 - e^{-j\omega} z^{-1}} \right]$$

$$= \frac{1}{2} \left[ \frac{z}{z - e^{j\omega}} + \frac{z}{z - e^{-j\omega}} \right]$$

$$= \frac{1}{2} \left[ \frac{2z^2 - 2z \cos \omega}{z^2 - 2z \cos \omega + 1} \right]$$

$$= \frac{1}{2} \left[ \frac{2(z^2 - z \cos \Omega_0 T)}{z^2 - 2z \cos \Omega_0 T + 1} \right] = \frac{z^2 - z \cos \Omega_0 T}{z^2 - 2z \cos \Omega_0 T + 1}$$

$$\text{ROC} \rightarrow 0 < |e^{j\omega} z^{-1}| < 1$$

$$0 < |e^{-j\omega} z^{-1}| < 1$$

$$\left| \frac{e^{j\omega}}{z} \right| < 1$$

$$\left| \frac{e^{-j\omega}}{z} \right| < 1$$

$$|z| > e^{j\omega}$$

$$|z| > e^{-j\omega}$$

$$Q:- x(t) = e^{-at} \sin \Omega_0 t$$

$$x(nT) = e^{-anT} \sin \Omega_0 nT$$

$$= e^{-anT} \sin(\Omega_0 T)n$$

$$y(n) = e^{-anT} \sin \omega n$$

$$Z\{y(n)\} = \sum_{n=-\infty}^{\infty} y(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} e^{-anT} \sin \omega n z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} e^{-anT} \left( \frac{e^{j\omega n} - e^{-j\omega n}}{2j} \right) z^{-n}$$

$$= \frac{1}{2j} \left[ \sum_{n=0}^{\infty} (e^{-aT} e^{j\omega} z^{-1})^n - \sum_{n=0}^{\infty} (e^{-aT} e^{-j\omega} z^{-1})^n \right]$$

$$= \frac{1}{2j} \left[ \frac{1}{1 - e^{-aT} e^{j\omega} z^{-1}} - \frac{1}{1 - e^{-aT} e^{-j\omega} z^{-1}} \right]$$

$$= \frac{1}{2j} \left[ \frac{z}{z - e^{-aT} e^{j\omega}} - \frac{z}{z - e^{-aT} e^{-j\omega}} \right]$$

$$= \frac{1}{2j} \left[ \frac{z^2 - z e^{-aT} e^{j\omega} - z^2 + z e^{-aT} e^{-j\omega}}{z^2 - z e^{-aT} e^{j\omega} - z e^{-aT} e^{-j\omega} + e^{-2aT}} \right]$$

$$= \frac{1}{2j} \left[ \frac{z(e^{-aT} e^{j\omega} - e^{-aT} e^{-j\omega})}{z^2 - z e^{-aT} (e^{j\omega} + e^{-j\omega}) + e^{-2aT}} \right]$$

$$= \frac{1}{2j} \left[ \frac{ze^{-aT} [e^{j\omega} - e^{-j\omega}]}{z^2 - ze^{-aT} (e^{j\omega} + e^{-j\omega}) + e^{-2aT}} \right]$$

⇒ multiply  $\frac{0}{0}$  by  $e^{2aT}$

$$= \frac{ze^{aT} \sin \omega}{z^2 e^{2aT} - 2ze^{aT} (\cos \omega) + 1}$$

Q:  $x(t) = e^{-at} \cos \omega_0 t$

$$x(nT) = e^{-anT} \cos \omega_0 nT$$

$$= e^{-anT} \cos(\omega_0 T)n$$

$$y(n) = e^{-anT} \cos \omega n$$

$$Z\{y(n)\} = \sum_{n=-\infty}^{\infty} y(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} e^{-anT} \cos \omega n z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} e^{-anT} \left( \frac{e^{j\omega n} + e^{-j\omega n}}{2} \right) z^{-n}$$

$$= \frac{1}{2} \left[ \sum_{n=0}^{\infty} (e^{-aT} e^{j\omega} z^{-1})^n + \sum_{n=0}^{\infty} (e^{-aT} e^{-j\omega} z^{-1})^n \right]$$

$$= \frac{1}{2} \left[ \frac{1}{1 - e^{-aT} e^{j\omega} z^{-1}} + \frac{1}{1 - e^{-aT} e^{-j\omega} z^{-1}} \right]$$

$$= \frac{1}{2} \left[ \frac{z}{z - e^{-aT} e^{j\omega}} + \frac{z}{z - e^{-aT} e^{-j\omega}} \right]$$

$$= \frac{1}{2} \left[ \frac{z^2 - ze^{-aT} e^{-j\omega} + z^2 - ze^{-aT} e^{j\omega}}{z^2 - ze^{-aT} e^{j\omega} - ze^{-aT} e^{-j\omega} + e^{-2aT}} \right]$$

$$\frac{1}{2} \left[ \frac{2z^2 - ze^{-aT}(e^{j\omega} + e^{-j\omega})}{z^2 - ze^{-aT}(e^{j\omega} + e^{-j\omega}) + e^{-2aT}} \right]$$

$$\frac{1}{2} \left[ \frac{2(z^2 - ze^{-aT} \cos \omega)}{z^2 - 2ze^{-aT} \cos \omega + e^{-2aT}} \right]$$

⇒ Multiply num & den by  $e^{2aT}$

$$= \frac{z^2 e^{2aT} - ze^{aT} \cos \omega}{z^2 e^{2aT} - 2ze^{aT} \cos \omega + 1}$$

$$= \frac{ze^{aT} [ze^{aT} - \cos \omega]}{z^2 e^{2aT} - 2ze^{aT} \cos \omega + 1}$$

Find the initial & final values of  $f$

Q:-  $X(z) = \frac{1}{1-z^{-2}}$

Initial value:

$$x(0) = \lim_{z \rightarrow \infty} X(z)$$

$$= \lim_{z \rightarrow \infty} \frac{1}{1-z^{-2}}$$

$$= \lim_{z \rightarrow \infty} \frac{1}{1-\frac{1}{z^2}}$$

$$= \frac{1}{1-0} = 1$$

Final value:

$$x(\infty) = \lim_{z \rightarrow 1} (1-z^{-1}) X(z)$$

$$= \lim_{z \rightarrow 1} \left(1 - \frac{1}{z}\right) \left(\frac{1}{1-\frac{1}{z^2}}\right)$$



$$= \lim_{z \rightarrow 1} \left( \frac{z-1}{z} \right) \left( \frac{z^2}{z^2-1} \right)$$

$$= \lim_{z \rightarrow 1} \frac{(z-1) \cdot z^2}{z(z-1)(z+1)}$$

$$= \lim_{z \rightarrow 1} \frac{z}{z+1}$$

$$= \frac{1}{1+1} = \frac{1}{2}$$

Q:-  $X(z) = \frac{1}{1+2z^{-1}-3z^{-2}}$

Initial value:-

$$\begin{aligned} x(0) &= \lim_{z \rightarrow \infty} X(z) \\ &= \lim_{z \rightarrow \infty} \frac{1}{1+2z^{-1}-3z^{-2}} \\ &= \lim_{z \rightarrow \infty} \frac{1}{1+\frac{2}{z}-\frac{3}{z^2}} \\ &= \frac{1}{1} = 1 \end{aligned}$$

final value

$$\begin{aligned} x(\infty) &= \lim_{z \rightarrow 1} (1-z^{-1}) X(z) \\ &= \lim_{z \rightarrow 1} \left( 1 - \frac{1}{z} \right) \left( \frac{1}{1+\frac{2}{z}-\frac{3}{z^2}} \right) \\ &= \lim_{z \rightarrow 1} \left( \frac{z-1}{z} \right) \left( \frac{z^2}{z^2+2z-3} \right) \\ &= \lim_{z \rightarrow 1} \left( \frac{z-1}{z} \right) \cdot \frac{z^2}{(z-1)(z+3)} \end{aligned}$$

$$= \lim_{z \rightarrow 1} \frac{z}{z+3}$$

$$= \frac{1}{1+3} = \frac{1}{4}$$

→ Inverse z-Transforms.

$$x(n) \xleftrightarrow{\substack{ZT \\ \leftarrow IZT}} X(z)$$

1. Residue Method
2. Partial fraction Method
3. Long division Method (power series method)

Q. Determine the inverse z-transforms of the function

$$X(z) = \frac{3 + 2z^{-1} + z^{-2}}{1 - 3z^{-1} + 2z^{-2}}$$

i) Partial fraction Method:-

$$X(z) = \frac{z^2 [3z^2 + 2z + 1]}{z^2 [z^2 - 3z + 2]}$$

$$\frac{X(z)}{z} = \frac{3z^2 + 2z + 1}{z(z-1)(z-2)}$$

$$\frac{X(z)}{z} = \frac{A}{z} + \frac{B}{z-1} + \frac{C}{z-2}$$

$$A = \left|_{z=0} = \frac{1}{2}\right.$$

$$B = \left|_{z=1} = -6\right.$$

$$C = \left|_{z=2} = \frac{17}{2}\right.$$

$$\frac{x(z)}{z} = \frac{1/2}{z} + \frac{-6}{z-1} + \frac{17/2}{z-2}$$

$$x(z) = \frac{1}{2} \frac{z}{z} + 6 \frac{z}{z-1} + \frac{17}{2} \frac{z}{z-2}$$

$$z^{-1} \{x(z)\} = \frac{1}{2} z^{-1} \{1\} - 6 z^{-1} \left\{ \frac{z}{z-1} \right\} + \frac{17}{2} z^{-1} \left\{ \frac{z}{z-2} \right\}$$

$$x(n) = \frac{1}{2} \delta(n) - 6u(n) + \frac{17}{2} 2^n u(n)$$

$[\because \delta(n) = 1, n=0$   
 $= 0, \text{ else}]$

$$n=0 \Rightarrow \frac{1}{2} \delta(0) - 6u(0) + \frac{17}{2} 2^0 u(0)$$

$$= \frac{1}{2} (1) - 6(1) + \frac{17}{2}$$

$$= 3$$

$$n=1 \Rightarrow \frac{1}{2} \delta(1) - 6u(1) + \frac{17}{2} 2u(1)$$

$$= 0 - 6 + 17$$

$$= 11$$

$$n=2 \Rightarrow \frac{1}{2} \delta(2) - 6u(2) + \frac{17}{2} (4) u(2)$$

$$= -6 + 34$$

$$= 28$$

$$n=3 \Rightarrow \frac{1}{2} \delta(3) - 6u(3) + \frac{17}{2} (8) u(3)$$

$$= -6 + 68$$

$$= 62$$

$$\therefore x(n) = \{3, 11, 28, 62, 130, \dots\}$$

(c) Power series method:-

$$x(z) = \frac{3 + 2z^{-1} + z^{-2}}{1 - 3z^{-1} + 2z^{-2}}$$

$$\begin{array}{r}
 1 - 3z^{-1} + 2z^{-2} \Big) 3 + 2z^{-1} + z^{-2} \left( 3 + 11z^{-1} + 28z^{-2} + 62z^{-3} + 130z^{-4} + \dots \right) \\
 \underline{3 - 9z^{-1} + 6z^{-2}} \\
 11z^{-1} - 5z^{-2} \\
 \underline{11z^{-1} - 33z^{-2} + 22z^{-3}} \\
 28z^{-2} - 22z^{-3} \\
 \underline{28z^{-2} - 84z^{-3} + 56z^{-4}} \\
 62z^{-3} - 56z^{-4} \\
 \underline{62z^{-3} - 186z^{-4} + 124z^{-5}} \\
 130z^{-4} - 124z^{-5}
 \end{array}$$

$$X(z) = 3 + 11z^{-1} + 28z^{-2} + 62z^{-3} + 130z^{-4} + \dots \quad \text{--- (1)}$$

$$Z\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n)z^{-n}$$

$$\begin{aligned}
 &= \dots + x(-2)z^2 + x(-1)z^1 + x(0) + x(1)z^{-1} + x(2)z^{-2} + \\
 &\quad x(3)z^{-3} + x(4)z^{-4} + \dots \quad \text{--- (2)}
 \end{aligned}$$

compare (1) & (2)

we get,  $x(-2) = 0$

$x(-1) = 0$

$x(0) = 3$

$x(1) = 11$

$x(2) = 28$

$x(3) = 62$

$x(4) = 130$

$$x(n) = \left\{ \underset{\substack{\uparrow \\ n=0}}{3}, 11, 28, 62, 130, \dots \right\}$$



Q:- Determine the Inverse z-Transform of

$$X(z) = \frac{1}{1 - 1.5z^{-1} + 0.5z^{-2}}$$

- i) ROC  $|z| > 1.0$
- ii) ROC  $|z| < 0.5$
- iii) ROC  $1 < |z| < 0.5$

$$X(z) = \frac{1}{1 - \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2}}$$

i) ROC  $|z| > 1.0$

$$\begin{array}{r}
 1 - 1.5z^{-1} + 0.5z^{-2} \quad | \quad (1 + 1.5z^{-1} + 1.75z^{-2} + 1.875z^{-3} + 1.937z^{-4} + \dots) \\
 \underline{+} \\
 1 - 1.5z^{-1} + 0.5z^{-2} \\
 \underline{+} \\
 1.5z^{-1} - 0.5z^{-2} \\
 \underline{+} \\
 1.5z^{-1} - 0.25z^{-2} + 0.75z^{-3} \\
 \underline{+} \\
 1.75z^{-2} - 0.75z^{-3} \\
 \underline{+} \\
 1.75z^{-2} - 2.625z^{-3} + 0.875z^{-4} \\
 \underline{+} \\
 1.875z^{-3} - 0.875z^{-4} \\
 \underline{+} \\
 1.875z^{-3} - 2.812z^{-4} + 0.937z^{-5} \\
 \underline{+} \\
 1.937z^{-4} - 0.937z^{-5}
 \end{array}$$

$$X(z) = 1 + 1.5z^{-1} + 1.75z^{-2} + 1.875z^{-3} + 1.937z^{-4} + \dots \quad \text{--- (1)}$$

$$z\{x(n)\} = \sum_{n=0}^{\infty} x(n) z^{-n}$$

$$= x(0) + x(1)z^{-1} + x(2)z^{-2} + x(3)z^{-3} + x(4)z^{-4} + \dots \quad \text{--- (2)}$$

compare (1) & (2)

$$\therefore x(n) = \{1, 1.5, 1.75, 1.87, 1.93, \dots\}$$

i) ROC  $|z| < 0.5$ .

$$\begin{array}{r}
 0.5z^{-2} - 1.5z^{-1} + 1 \quad | \quad (2z^2 + 6z^3 + 14z^4 + 30z^5 + 62z^6 + \dots) \\
 \hline
 \phantom{0.5z^{-2} - 1.5z^{-1} + 1} + 3z + 2z^2 \\
 \hline
 \phantom{0.5z^{-2} - 1.5z^{-1} + 1} \phantom{+ 3z} - 2z^2 \\
 \phantom{0.5z^{-2} - 1.5z^{-1} + 1} \phantom{+ 3z} \phantom{- 2z^2} + 6z^3 \\
 \hline
 \phantom{0.5z^{-2} - 1.5z^{-1} + 1} \phantom{+ 3z} \phantom{- 2z^2} \phantom{+ 6z^3} - 6z^3 \\
 \phantom{0.5z^{-2} - 1.5z^{-1} + 1} \phantom{+ 3z} \phantom{- 2z^2} \phantom{+ 6z^3} \phantom{- 6z^3} + 14z^4 \\
 \hline
 \phantom{0.5z^{-2} - 1.5z^{-1} + 1} \phantom{+ 3z} \phantom{- 2z^2} \phantom{+ 6z^3} \phantom{- 6z^3} \phantom{+ 14z^4} - 14z^4 \\
 \phantom{0.5z^{-2} - 1.5z^{-1} + 1} \phantom{+ 3z} \phantom{- 2z^2} \phantom{+ 6z^3} \phantom{- 6z^3} \phantom{+ 14z^4} \phantom{- 14z^4} + 30z^5 \\
 \hline
 \phantom{0.5z^{-2} - 1.5z^{-1} + 1} \phantom{+ 3z} \phantom{- 2z^2} \phantom{+ 6z^3} \phantom{- 6z^3} \phantom{+ 14z^4} \phantom{- 14z^4} \phantom{+ 30z^5} + 30z^5 \\
 \phantom{0.5z^{-2} - 1.5z^{-1} + 1} \phantom{+ 3z} \phantom{- 2z^2} \phantom{+ 6z^3} \phantom{- 6z^3} \phantom{+ 14z^4} \phantom{- 14z^4} \phantom{+ 30z^5} \phantom{+ 30z^5} + 62z^6 + \dots
 \end{array}$$

$$X(z) = 2z^2 + 6z^3 + 14z^4 + 30z^5 + 62z^6 + \dots \quad \text{--- (1)}$$

$$z\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$\begin{aligned}
 &= \dots + x(-6)z^6 + x(-5)z^5 + x(-4)z^4 + x(-3)z^3 + x(-2)z^2 + x(-1)z + \\
 &\quad x(0) + \dots \quad \text{--- (2)}
 \end{aligned}$$

$$\therefore x(n) = \{ \dots, 62, 30, 14, 6, 2, 0, 0 \}$$

$\uparrow$   
 $n=0$

ii) ROC  $1 < |z| < 0.5 \rightarrow$  NO solution.

Q. Determine Inverse z-transform of  $X(z) = \frac{1}{1 - 0.8z^{-1} + 0.12z^{-2}}$

If i) ROC  $|z| > 0.6$

ii) ROC  $|z| < 0.2$

iii) ROC  $0.2 < |z| < 0.6$ .

1) ROC  $|z| > 0.6$

$$X(z) = \frac{1}{1 - 0.8z^{-1} + 0.12z^{-2}}$$

$$= \frac{z^2}{z^2 - 0.8z + 0.12}$$

$$= \frac{z^2}{(z - 0.6)(z - 0.2)}$$

$$\frac{X(z)}{z} = \frac{z}{(z - 0.6)(z - 0.2)}$$

$$= \frac{A}{z - 0.6} + \frac{B}{z - 0.2}$$

$$A = \left. \frac{z}{z - 0.6} \right|_{z=0.6} = \frac{0.6}{0.6 - 0.2} = \frac{0.6}{0.4} = \frac{3}{2}$$

$$B = \left. \frac{z}{z - 0.2} \right|_{z=0.2} = \frac{0.2}{-0.4} = -\frac{1}{2}$$

$$\therefore \frac{X(z)}{z} = \frac{3}{2} \frac{1}{(z - 0.6)} - \frac{1}{2} \left( \frac{1}{z - 0.2} \right)$$

$$X(z) = \frac{3}{2} \left( \frac{z}{z - 0.6} \right) - \frac{1}{2} \left( \frac{z}{z - 0.2} \right)$$

→ Apply Inverse z-Transform.

$$x(n) = \frac{3}{2} (0.6)^n u(n) - \frac{1}{2} (0.2)^n u(n)$$

(i)  $|z| < 0.2$

$$x(n) = -\frac{3}{2} (0.6)^n u(-n-1) + \frac{1}{2} (0.2)^n u(-n-1)$$

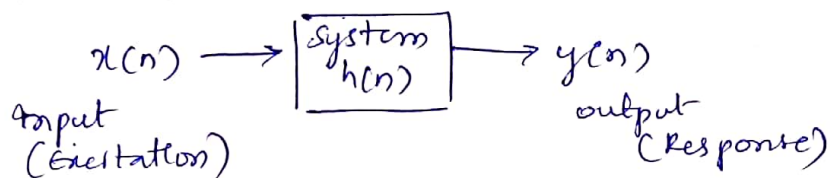
(ii)  $0.2 < |z| < 0.6$

$$x(n) = \frac{3}{2} (0.6)^n u(-n-1) - \frac{1}{2} (0.2)^n u(n)$$

$$\therefore a^n u(n) \Rightarrow \frac{z}{z-a}, |z| > a$$

$$a^n u(-n-1) = \frac{-z}{z-a}, |z| < a$$

Transfer function:



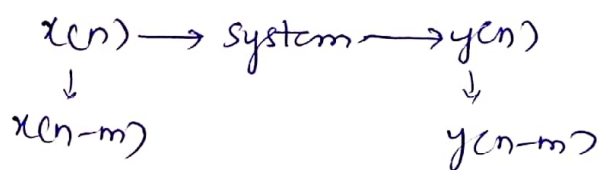
$$y(n) = x(n) * h(n)$$

$$z\{y(n)\} = z\{x(n) * h(n)\}$$

$$Y(z) = X(z) \cdot H(z)$$

$$\boxed{H(z) = \frac{Y(z)}{X(z)}}$$

Solution of difference equations:



$$z\{x(n-m)\} = z^{-m} X(z)$$

$$z\{y(n-m)\} = z^{-m} Y(z)$$

Q: Determine the impulse response  $h(n)$  of the system, described by the second order difference equation. Plot impulse sequence

$$y(n) - 3y(n-1) - 4y(n-2) = x(n) + 2x(n-1)$$

$$z\{y(n) - 3y(n-1) - 4y(n-2)\} = z\{x(n) + 2x(n-1)\}$$

$$Y(z) - 3z^{-1}Y(z) - 4z^{-2}Y(z) = X(z) + 2z^{-1}X(z)$$

$$Y(z) [1 - 3z^{-1} - 4z^{-2}] = X(z) [1 + 2z^{-1}]$$



Transfer function  $H(z) = \frac{Y(z)}{X(z)} = \frac{1 + 2z^{-1}}{1 - 3z^{-1} - 4z^{-2}}$

Apply, Inverse z-Transforms.

$$H(z) = \left[ \frac{1 + 2z^{-1}}{1 - 3z^{-1} - 4z^{-2}} \right]$$

$$\frac{H(z)}{z} = \frac{z^1 [z + 2]}{z z^{-2} [z^2 - 3z - 4]}$$

$$\frac{H(z)}{z} = \frac{z + 2}{(z - 4)(z + 1)}$$

$$\frac{H(z)}{z} = \frac{A}{z - 4} + \frac{B}{z + 1}$$

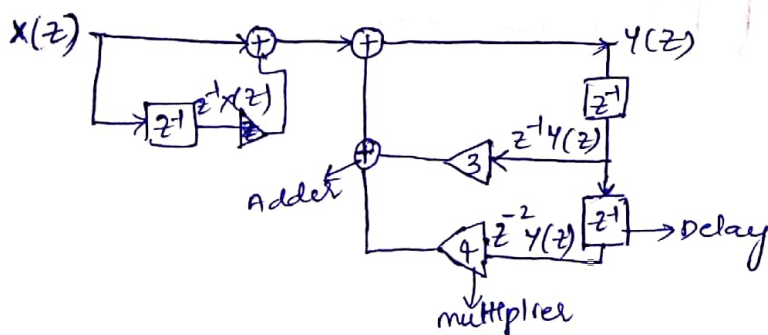
$$A = \left|_{z=4} = \frac{6}{5}$$

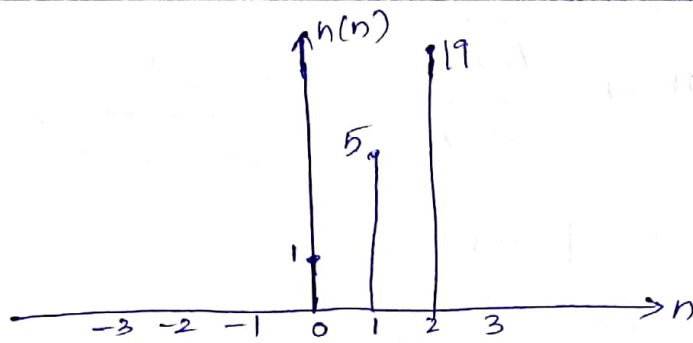
$$B = \left|_{z=-1} = \frac{1}{-5}$$

$$H(z) = \frac{6}{5} \left[ \frac{z}{z - 4} \right] - \frac{1}{5} \left[ \frac{z}{z + 1} \right]$$

$$z^{-1} [H(z)] = \frac{6}{5} z^{-1} \left[ \frac{z}{z - 4} \right] - \frac{1}{5} z^{-1} \left[ \frac{z}{z + 1} \right]$$

$$h(n) = \frac{6}{5} (4)^n u(n) - \frac{1}{5} (-1)^n u(n),$$





Obtain and sketch the shift Invariant system described by  
 $y(n] = 0.4x(n) + x(n-1) + 0.6x(n-2) + x(n-3) + 0.4x(n-4)$

Apply z-Transform.

$$Y(z) = 0.4X(z) + z^{-1}X(z) + 0.6z^{-2}X(z) + z^{-3}X(z) + 0.4z^{-4}X(z)$$

$$Y(z) = X(z) [0.4 + z^{-1} + 0.6z^{-2} + z^{-3} + 0.4z^{-4}]$$

$$H(z) = \frac{Y(z)}{X(z)} = 0.4 + z^{-1} + 0.6z^{-2} + z^{-3} + 0.4z^{-4} \quad \text{--- (1)}$$

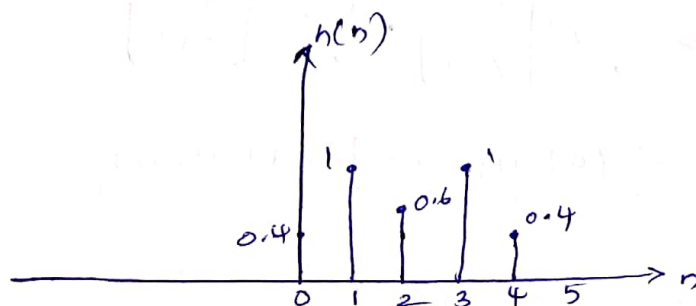
$$z\{h(n)\} = H(z) = \sum_{n=0}^4 h(n) z^{-n}$$

$$= h(0) + h(1)z^{-1} + h(2)z^{-2} + h(3)z^{-3} + h(4)z^{-4} \quad \text{--- (2)}$$

Comparing (1) & (2)

$$h(n) = \{0.4, 1, 0.6, 1, 0.4\}$$

↑  
n=0



Find the response of the time Invariant system if the impulse response  $h(n) = \{1, 2, 1, -1\}$  to an input signal  $x(n) = \{1, 2, 3, 1\}$

$$y(n) = x(n) * h(n)$$

Apply z-Transforms

$$Y(z) = X(z) \cdot H(z)$$

$$x(n) = \{1, 2, 3, 1\}$$

$$Z\{x(n)\} = \sum_{n=0}^3 x(n) z^{-n}$$

$$X(z) = x(0) + x(1)z^{-1} + x(2)z^{-2} + x(3)z^{-3}$$

$$X(z) = 1 + 2z^{-1} + 3z^{-2} + z^{-3} \quad \text{--- (1)}$$

$$h(n) = \{1, 2, 1, -1\}$$

$$Z\{h(n)\} = \sum_{n=0}^3 h(n) z^{-n}$$

$$= h(0) + h(1)z^{-1} + h(2)z^{-2} + h(3)z^{-3}$$

$$H(z) = 1 + 2z^{-1} + z^{-2} - z^{-3}$$

$$Y(z) = X(z) \cdot H(z)$$

$$= (1 + 2z^{-1} + 3z^{-2} + z^{-3})(1 + 2z^{-1} + z^{-2} - z^{-3})$$

$$= 1 + 2z^{-1} + z^{-2} - z^{-3} + 2z^{-1} + 4z^{-2} + 2z^{-3} - 2z^{-4} + 3z^{-2} + 6z^{-3} + 3z^{-4} - 3z^{-5} + z^{-3} + 2z^{-4} + z^{-5} - z^{-6}$$

$$= 1 + z^{-1}[2+2] + z^{-2}[1+4+3] + z^{-3}[-1+2+6+1] + z^{-4}[-2+3+2] + z^{-5}[-3+1] + z^{-6}[-1]$$

$$Y(z) = 1 + 4z^{-1} + 8z^{-2} + 8z^{-3} + 3z^{-4} - 2z^{-5} - z^{-6} \quad \text{--- (2)}$$

$$Z\{y(n)\} = Y(z) = \sum_{n=0}^6 y(n) z^{-n}$$

$$= y(0) + y(1)z^{-1} + y(2)z^{-2} + y(3)z^{-3} + y(4)z^{-4} + y(5)z^{-5} + y(6)z^{-6} \quad \text{--- (3)}$$

Compare (2) & (3)

$$y(0) = 1$$

$$y(1) = 4$$

$$y(2) = 8$$

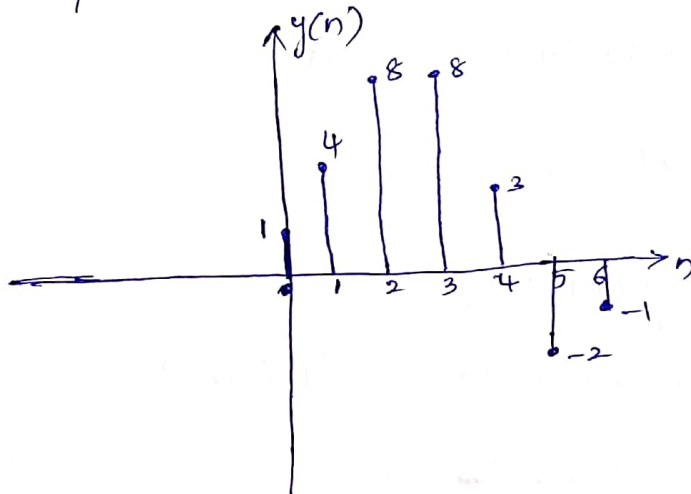
$$y(3) = 8$$

$$y(4) = 3$$

$$y(5) = -2$$

$$y(6) = -1$$

$$y(n) = \{1, 4, 8, 8, 3, -2, -1\}$$



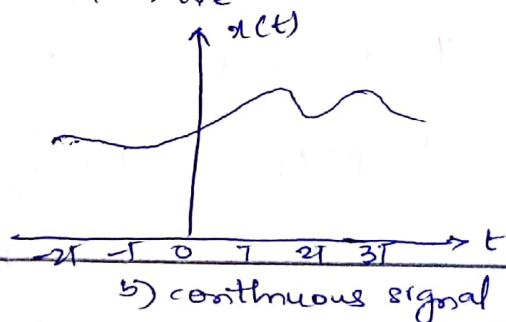
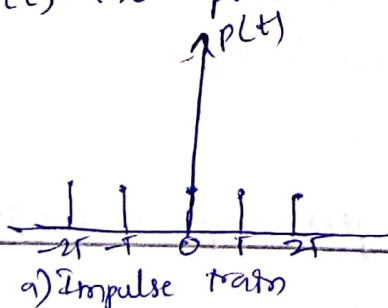
\* Sampling :

converting a time domain signal into discrete time domain signal is called sampling.

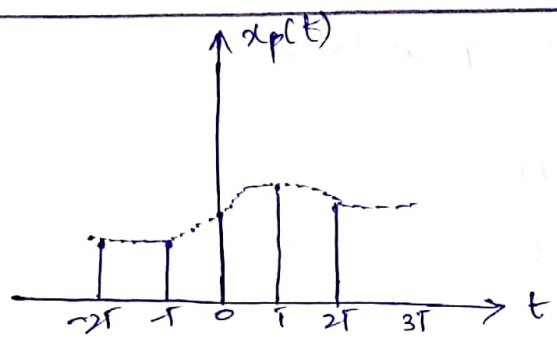
Relationship b/w z-transform and Laplace transform (z-plane & s-plane)

1. Let us consider a periodic impulse train  $p(t)$  with period  $T$  and pulse train can be expressed as  $p(t) = \sum_{n=-\infty}^{\infty} \delta(t-nT)$  — (1)

2. When a continuous time signal  $x(t)$  multiplied by impulse train  $p(t)$  the product signal will have







c) samples of continuous time signals

$$\begin{aligned} \therefore x_p(t) &= \sum_{n=-\infty}^{\infty} p(t) x(nT) \\ &= \sum_{n=-\infty}^{\infty} x(nT) \delta(t-nT) \end{aligned}$$

take LT

$$L\{x_p(t)\} = L\left\{\sum_{n=-\infty}^{\infty} x(nT) \delta(t-nT)\right\}$$

$$X_p(s) = \sum_{n=-\infty}^{\infty} x(nT) L\{\delta(t-nT)\}$$

$$= \sum_{n=-\infty}^{\infty} x(nT) e^{-snT} L\{\delta(t)\}$$

$$= \sum_{n=-\infty}^{\infty} x(nT) e^{-nsT} \quad (1)$$

$$X_p(s) = \sum_{n=-\infty}^{\infty} x(nT) e^{-nsT}$$

put  $e^{sT} = z$

$$X_p(s) = \sum_{n=-\infty}^{\infty} x(nT) z^{-n} \quad \text{--- (2)}$$

$$Z\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$Z\{x(nT)\} = \sum_{n=-\infty}^{\infty} x(nT) z^{-n} \quad \text{--- (3)}$$

Comparing (2) & (3)

$$X_p(s) = X(z) \Big|_{z=e^{sT}}$$

$$\therefore L\{x(t-t_0)\} = e^{-st_0} X(s)$$

Relation b/w z-plane & s-plane

$$z = e^{sT}$$

$$z = e^{(\sigma + j\omega)T}$$

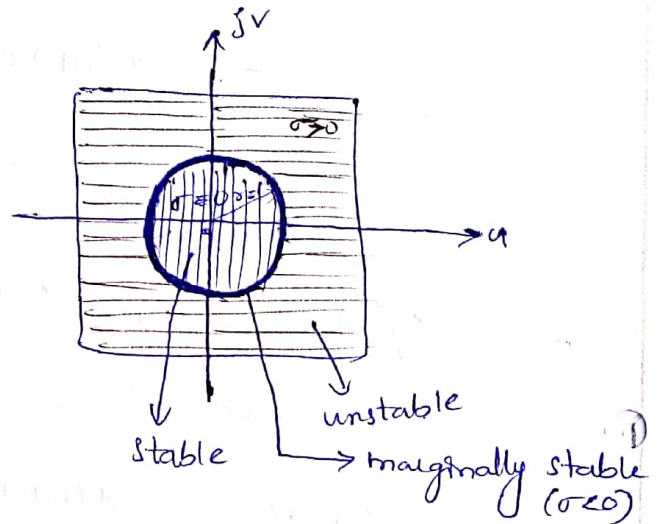
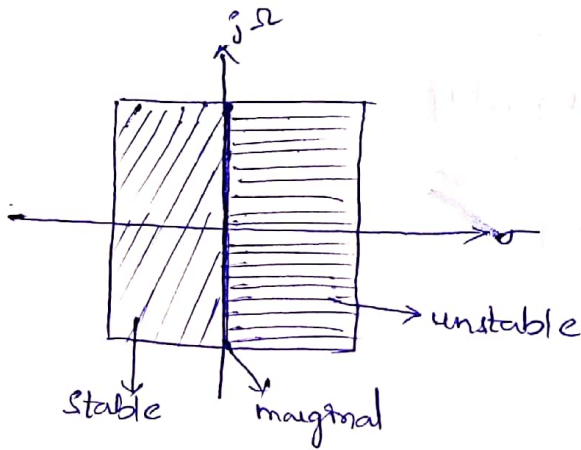
$$z = e^{\sigma T} \cdot e^{j\omega T}$$

$$|z| = e^{\sigma T}$$

i)  $\sigma > 0$  i.e.  $\sigma$  +ve  $\Rightarrow |z| = \text{increasing}$   
 $|z| > 1$

ii)  $\sigma < 0$  i.e.  $\sigma$  -ve  $\Rightarrow |z| = \text{decreasing}$   
 $|z| < 1$

iii)  $\sigma = 0$  i.e.  $\sigma = 0 \Rightarrow |z| = 1$



### UNIT-III (Z-TRANSFORMS)

1)

a) Find the z-transform and sketch the ROC for the following sequences.

i)  $x(n) = \left(\frac{1}{2}\right)^n U(-n)$

ii)  $x(n) = \left(\frac{1}{2}\right)^{n-1} U(n-1)$

b) A causal LTI system is described by the difference equation.

$$y(n] = y(n - 1) + y(n - 2) + x(n - 1).$$

Find the unit sample response of the system.

2)

(a) Using the power series expansion technique, find the inverse z-transform of

$$X(z) = \frac{z}{2z^2 - 3z + 1}, \quad |z| < \frac{1}{2}.$$

(b) Solve the following difference equation with the given initial conditions :

$$y(n) - 3y(n - 1) = x(n), \text{ with } x(n) = 4U(n), y(-1) = 1.$$

3)

a) Find the inverse z-transform of  $(1 - z^{-1})^{-2}$ .

b) A linear discrete time system is given by  $y(n] + 0.95y(n - 1) = 0.05x(n)$

i) Find the impulse response of the system

ii) Find the response of the system if  $x(n) = 0.5^n u(n)$ .

4)

a) Determine the z-transform of  $x(n) = (\cos^2 \omega n)u(n)$ .

b) Using partial fraction expansion method obtain the inverse z-transform of

$$x(z) = \frac{6z^3 + 2z^2 - z}{z^3 - z^2 - z + 1}.$$

- 5) (a) Find the inverse z-transform of  $x(t) = \frac{z+1}{(z+0.2)(z-0.6)}$   
 (b) For the system given by the difference equation, draw the canonical form realization diagram.  
 $y(n) + 0.5y(n-1) + 2y(n-2) + 3y(n-3) + 0.8y(n-4) = 3x(n) + 5x(n-2)$ .

- 6) (a) State and prove sampling theorem.  
 (b) Realize the system with 2 delays  $y(n) = y(n-1) + 2x(n) + 8x(n-1)$ .

- 7) a) Determine the inverse z-transform of

$$X(z) = \frac{1}{1 - 1.5z^{-1} + 0.5z^{-2}} \quad \text{ROC: } |z| < 0.5$$

Using long division method.

- b) Determine the inverse z-transform for the following function:

$$X(z) = \frac{(z+1)(z+5)}{(z+2)(z+3)(z+6)} \quad \text{for } |z| \leq 3$$

- 8) a) Find inverse Z-transform of  $X(Z) = \frac{1+z^{-1}}{1-(\frac{1}{3})z^{-1}}$  if ROC  $|Z| > 1/3$ .

- b) Consider a causal discrete-time system whose output  $y(n]$  and input  $x(n]$  are related by  $y(n) - \frac{5}{6}y(n-1) + \frac{1}{6}y(n-2) = x(n)$

i) Find its transfer function  $H(z)$ .

ii) Find its impulse response  $h(n)$ .

- 9) a) Find inverse Z-transform of  $X(Z) = \frac{1}{[1 - 1.5Z^{-1} + 0.5Z^{-2}]}$

if a) ROC  $|Z| > 1$  b) ROC  $|Z| < 0.5$  c) ROC  $0.5 < |Z| < 1$

- b) A causal system is represented by the following difference equation. Find its transfer function.

$$y(n) - \frac{1}{3}y(n-1) = x(n) + \frac{1}{4}x(n-1)$$