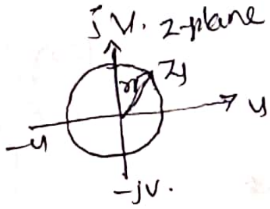
UNIT- III

IV Z-Transform

(1)

- Laplace Transforms are popularly used for analysis of continuous time signals and systems.
- Similarly Z-transform plays an important role in analysis and representation of discrete time signals and systems.
- The Z transform of $x(n)$ will convert the time domain signal $x(n)$ to Z-domain signal $X(Z)$, where the signal becomes a function of complex variable z.



The complex variable z is defined as

$$Z = u + jv = re^{j\omega}$$

where u = real part of z
v = Imaginary "
$r = \sqrt{u^2+v^2}$ = magnitude of z
$\omega = \tan^{-1}\frac{v}{u}$ = phase (or) Argument of z

Definition of Z-Transform

Let $x(n)$ = discrete time signal
$X(Z)$ = Z-Transform of $x(n)$

The Z-transform of a discrete time signal, $x(n)$ defined as

$$\boxed{X(Z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}}$$

where z is a complex variable

$$\boxed{Z\{x(n)\} = X(Z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}}$$ → Two sided Z-transfer.

$$\boxed{X(Z) = \sum_{n=0}^{\infty} x(n) z^{-n}}$$ → one sided Z-Transform.

- The computation of $X(Z)$ involves summation of infinite terms which are functions of z. Hence it is possible that the infinite series may not converge to finite value for certain values of z. For which values of z, $X(Z)$ can be computed is called ROC of $X(Z)$.

Inverse Z-Transform

$$\boxed{x(n) = Z^{-1}\{X(z)\} = \frac{1}{2\pi j}\oint_C X(z)\, z^{n-1}\, dz}$$

$$x(n) \underset{Z^{-1}}{\overset{Z}{\rightleftharpoons}} X(z)$$

Geometric Series

The z-transform of a discrete time signals involves convergence of geometric series. Hence the following two geometric series sum formula will be useful in evaluating z-transform.

1) Infinite geometric series sum formula

If C is a complex constant and $0 < |C| < 1$, then

$$\boxed{\sum_{n=0}^{\infty} C^n = \frac{1}{1-C}}$$ (C be strictly less than unity)

2) Finite geometric series sum formula

If C is a complex constant and

$$\boxed{\text{When } C \neq 1, \quad \sum_{n=0}^{N-1} C^n = \frac{1-C^N}{1-C} = \frac{C^N - 1}{C-1}}$$

$$\boxed{\text{When } C = 1, \quad \sum_{n=0}^{N-1} C^n = N}$$

(∵ It is valid for any values of 'C')

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Region of Convergence (ROC)

- The ROC of $X(Z)$ is the set of all values of Z, for which $X(Z)$ attains a finite value.
- The ROC for the following six types of signals
  i) Finite duration, right sided (causal) signal
  ii) " " , left " (Anti causal) signal
  iii) " " , two " (non causal) signal
  iv) Infinite duration, right sided (causal) signal
  v) " " , left " (Anti causal) "
  vi) " " , two " (non causal) signal.

i) Finite duration, right sided (causal) signal)

Let $x(n)$ be finite duration with N samples, defined in the range $0 \leq n \leq (N-1)$

$\therefore \; x(n) = \{x(0), x(1), x(2) \cdots x(N-1)\}$.

Now the z-transform of $x(n)$ is

$$X(Z) = \sum_{n=0}^{N-1} x(n) z^{-n}$$

$$= x(0) + x(1) z^{-1} + x(2) z^{-2} + \cdots + x(N-1) z^{-(N-1)}$$

$$X(Z) = x(0) + \frac{x(1)}{z} + \frac{x(2)}{z^2} + \cdots + \frac{x(N-1)}{z^{N-1}}$$

In the above summation, when $Z=0$ all the terms except the first term become infinite. Hence the $X(Z)$ exists for all values of z except $z=0$.

$\therefore$ ROC for finite duration, right sided (causal) signal is entire z-plane except $z=0$.

ii) Finite duration, left sided (anticausal) signal

Let $x(n)$ be finite duration with N-samples, defined in the range $-(N-1) \leq n \leq 0$

$\therefore x(n) = \{ -(N-1), \cdots, x(-2), x(-1), x(0) \}$

$$X(z) = \sum_{n=-(N-1)}^{0} x(n) z^{-n}$$

$$= x(-(N-1)) z^{+(N-1)} + \cdots + x(-2)z^{2} + x(-1)z^{1} + x(0)$$

In the above, when $z = \infty$, all the terms except the last term become infinite. Hence $X(z)$ exists for all values of z except $z = \infty$.

$\therefore$ ROC of $X(z)$ is entire z-plane except $z = \infty$.

iii) Finite duration, Two sided (non causal) signal

Let $x(n)$ be a finite duration with N-samples. defined in the range $-M \leq n \leq +M$ where $M = \frac{N-1}{2}$

$\therefore x(n) = \{ x(-M), \cdots, x(-2), x(-1), x(0), x(1), x(2) \cdots x(M) \}$

Now z-transform of $x(n)$ is

$$X(z) = \sum_{n=-M}^{+M} x(n) z^{-n}$$

$$= x(-M)z^{M} + \cdots + x(-2)z^{2} + x(-1)z^{1} + x(0) + x(1)z^{-1} + x(2)z^{-2} + \cdots + x(M)z^{-M}$$

$$= x(-M)z^{-M} + \cdots + x(-2)z^{2} + x(-1)z^{1} + x(0) + \frac{x(1)}{z} + \frac{x(2)}{z^{2}} + \cdots + \frac{x(M)}{z^{M}}$$

In the above, when $z = 0$, the terms with negative powers of z attain infinity and when $z = \infty$, the terms with positive power of z attain infinity. Hence $X(z)$ converges for all values of z, except $z = 0$ and $z = \infty$.

$\therefore$ ROC is entire z-plane, except $z = 0$ & $z = \infty$.

iv) <u>Infinite duration, right sided (Causal) Signal</u>

Let $x(n) = r_1^n$, $n \geq 0$

$$Z\{x(n)\} = X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

$$= \sum_{n=0}^{\infty} r_1^n \cdot z^{-n}$$

$$= \sum_{n=0}^{\infty} (r_1 z^{-1})^n$$

$\sum_{n=0}^{\infty} c^n = \frac{1}{1-c}$ where $0<|c|<1$

if $0<|r_1 z^{-1}|<1$ then $\sum_{n=0}^{\infty} (r_1 z^{-1})^n = \frac{1}{1-r_1 z^{-1}}$

$$\therefore X(z) = \frac{1}{1-r_1 z^{-1}} = \frac{1}{1-\frac{r_1}{z}} = \frac{z}{z-r_1}$$

Here the condition to be satisfied for the convergence of $X(z)$ is

$$0 < |r_1 z^{-1}| < 1$$

$$\therefore |r_1 z^{-1}| < 1$$

$$\frac{|r_1|}{|z|} < 1 \Rightarrow |z| > |r_1|$$

$\therefore$ ROC is exterior the circle.

v) <u>Infinite duration, left sided (anticausal) signal</u>

Let $x(n) = r_2^n$, $n \leq 0$

$$Z\{x(n)\} = X(z) = \sum_{n=-\infty}^{0} x(n)\, z^{-n}$$

$$= \sum_{n=-\infty}^{0} r_2^n z^{-n} = \sum_{n=0}^{\infty} r_2^{-n} z^n = \sum_{n=0}^{\infty} (r_2^{-1} z)^n$$

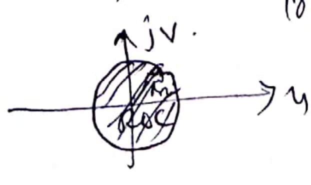
if $0<|r_2^{-1} z|<1$, then $\sum_{n=0}^{\infty} (r_2^{-1} z)^n = \frac{1}{1-r_2^{-1} z}$

$$\therefore X(z) = \frac{1}{1-r_2^{-1} z} = \frac{1}{1-\frac{z}{r_2}} = \frac{r_2}{r_2-z} = -\frac{r_2}{z-r_2}$$

condition for convergence of $X(z)$

$$0<|r_2^{-1} z|<1 \Rightarrow |r_2^{-1} z|<1 \Rightarrow \frac{|z|}{|r_2|}<1 \Rightarrow |z|<|r_2|$$

$\therefore$ ROC of $X(z)$ is interior of the circle.



vi) Infinite duration, Two sided (~~anti~~ non causal) signal.

Let $x(n) = r_1^n u(n) + r_2^n u(-n)$

$$Z\{x(n)\} = X(Z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=-\infty}^{0} x(n) z^{-n} + \sum_{n=0}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=-\infty}^{0} r_2^n z^{-n} + \sum_{n=0}^{\infty} r_1^n z^{-n}$$

$$= \sum_{n=0}^{\infty} r_2^{-n} z^{n} + \sum_{n=0}^{\infty} r_1^n z^{-n}$$

$$= \frac{1}{1 - r_2^{-1} z} + \frac{1}{1 - r_1 z^{-1}}$$ (using infinite Geometric series)

~~First term~~

first term converges if

$0 < |r_2^{-1} z| < 1$

$|r_2^{-1} z| < 1$

$\frac{|Z|}{|r_2|} < 1$

$|Z| < |r_2|$

Second term converges if,

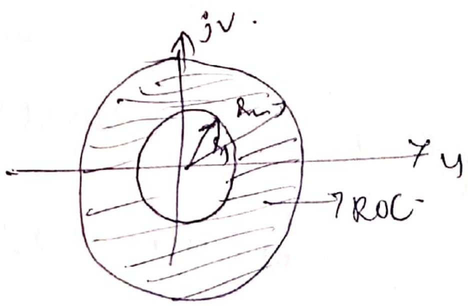
$0 < |r_1 z^{-1}| < 1$

$|r_1 z^{-1}| < 1$

$\frac{|r_1|}{|Z|} < 1$

$|Z| > |r_1|$

$\therefore$ ROC is between two circles of radius r_1 & r_2



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Properties of Z-Transforms

1) Linearity property

Let $Z\{x_1(n)\} = X_1(Z)$ and $Z\{x_2(n)\} = X_2(Z)$, where a_1 & a_2 are constants.

Then $Z\{a_1x_1(n) + a_2x_2(n)\} = a_1X_1(Z) + a_2X_2(Z)$

Proof:

$$X_1(Z) = Z\{x_1(n)\} = \sum_{n=-\infty}^{\infty} x_1(n) z^{-n}$$

$$X_2(Z) = Z\{x_2(n)\} = \sum_{n=-\infty}^{\infty} x_2(n) z^{-n}$$

$$Z\{a_1x_1(n) + a_2x_2(n)\} = \sum_{n=-\infty}^{\infty} \{a_1x_1(n) + a_2x_2(n)\} z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} a_1x_1(n) z^{-n} + \sum_{n=-\infty}^{\infty} a_2x_2(n) z^{-n}$$

$$= a_1X_1(Z) + a_2X_2(Z)$$

2) Shifting property

if $Z\{x(n)\} = X(Z)$

then $Z\{x(n-m)\} = z^{-m} X(Z)$

$Z\{x(n+m)\} = z^{m} X(Z)$

Proof: $X(z) = Z\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$

$$Z\{x(n-m)\} = \sum_{n=-\infty}^{\infty} x(n-m) z^{-n}.$$

Let $n-m = p$.

$n = p+m$.

$n = -\infty \Rightarrow p = -\infty$

$n = \infty \Rightarrow p = \infty$.

$$= \sum_{p=-\infty}^{\infty} x(p) z^{-(p+m)}$$

$$= \sum_{p=-\infty}^{\infty} x(p) z^{-p} \cdot z^{-m}$$

$$= z^{-m} X(z)$$

3) Multiplication by n (or differentiation in z-domain)

If $Z\{x(n)\} = X(z)$

then $Z\{n x(n)\} = -z \frac{d}{dz} X(z)$

Proof: $X(z) = Z\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$

$$Z\{n x(n)\} = \sum_{n=-\infty}^{\infty} n x(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} n x(n) z^{-n} \cdot z \cdot z^{-1}$$

$$= -z \sum_{n=-\infty}^{\infty} x(n) \left[-n z^{-n-1}\right]$$

$$= -z \sum_{n=-\infty}^{\infty} x(n) \left[\frac{d}{dz} z^{-n}\right]$$

$$= -z \frac{d}{dz} \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= -z \frac{d}{dz} X(z)$$

4) Multiplication by an exponential sequence, a^n (or scaling in z-domain)

If $Z\{x(n)\} = X(z)$

Then $Z\{a^n x(n)\} = X(a^{-1} z)$

Proof: $Z\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$

$$Z\{a^n x(n)\} = \sum_{n=-\infty}^{\infty} \{a^n x(n)\} z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x(n) \{a^{-1} z\}^{-n}$$

$$= X(a^{-1} z)$$

5) Time Reversal

If $Z\{x(n)\} = X(z)$

Then $Z\{x(-n)\} = X(z^{-1})$

Proof: $Z\{x(n)\} = X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$

$$Z\{x(-n)\} = \sum_{n=-\infty}^{\infty} x(-n) z^{-n}$$

let $-n = p$

$n \to \infty \Rightarrow p = -\infty$

$n \to -\infty \Rightarrow p = +\infty$

$$= \sum_{p=\infty}^{-\infty} x(p) z^{+p} = \sum_{p=-\infty}^{\infty} x(p) (z^{-1})^{-p} = X(z^{-1})$$

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6) Conjugation

If $Z\{x(n)\} = X(z)$

Then $Z\{x^*(n)\} = X^*(z^*)$

Proof: $Z\{x(n)\} = X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$

$$Z\{x^*(n)\} = \sum_{n=-\infty}^{\infty} x^*(n) z^{-n}$$

$$= \left[\sum_{n=-\infty}^{\infty} x(n) (z^*)^{-n}\right]^*$$

$$= [X(z^*)]^* = X^*(z^*)$$

7) Convolution Theorem

If $Z\{x_1(n)\} = X_1(z)$ and $Z\{x_2(n)\} = X_2(z)$

Then $Z\{x_1(n) * x_2(n)\} = X_1(z) X_2(z)$

Proof: $Z\{x_1(n)\} = X_1(z) = \sum_{n=-\infty}^{\infty} x_1(n) z^{-n}$ —— ①

$Z\{x_2(n)\} = X_2(z) = \sum_{n=-\infty}^{\infty} x_2(n) z^{-n}$ —— ②

$$Z\{x_1(n) * x_2(n)\} = \sum_{n=-\infty}^{\infty} \{x_1(n) * x_2(n)\} z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \left[\sum_{k=-\infty}^{\infty} x_1(k) x_2(n-k)\right] z^{-n}$$

$$= \sum_{k=-\infty}^{\infty} x_1(k) z^{-k} \sum_{n=-\infty}^{\infty} x_2(n-k) z^{-n} \cdot z^{k}$$ (multiply z^k & z^{-k})

$$= \sum_{k=-\infty}^{\infty} x_1(k) z^{-k} \sum_{p=-\infty}^{\infty} x_2(p) z^{-(n-k)}$$

let $n-k=p$

$n=-\infty \Rightarrow p=-\infty$

$n=\infty \Rightarrow p=\infty$

$$= \sum_{k=-\infty}^{\infty} x_1(k) z^{-k} \sum_{p=-\infty}^{\infty} x_2(p) z^{-p}$$

$$= \left[\sum_{k=-\infty}^{\infty} x_1(k) z^{-k}\right] \left[\sum_{p=-\infty}^{\infty} x_2(p) z^{-p}\right]$$

$$= X_1(z) X_2(z)$$

8) Correlation Property

If $Z\{x(n)\} = X(Z)$ and $Z\{y(n)\} = Y(Z)$

Then $Z\{r_{xy}(m)\} = X(Z)Y(Z^{-1})$

where $r_{xy}(m) = \sum_{n=-\infty}^{\infty} x(n)y(n-m)$

Proof:

$$X(Z) = Z\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n)Z^{-n}$$

$$Y(Z) = Z\{y(n)\} = \sum_{n=-\infty}^{\infty} y(n)Z^{-n}$$

$$Z\{r_{xy}(m)\} = \sum_{m=-\infty}^{\infty} r_{xy}(m)\, z^{-m}$$

$$= \sum_{m=-\infty}^{\infty}\left[\sum_{n=-\infty}^{\infty} x(n)y(n-m)\right]z^{-m}.$$

$$= \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} x(n)\, y(n-m)\, z^{-m}.z^{-n}.z^{n}$$

$$= \sum_{n=-\infty}^{\infty} x(n)z^{-n} \sum_{m=-\infty}^{\infty} y(n-m)z^{n-m}$$

Let $n-m=p$

$\therefore m = n-p$

when $m=-\infty \Rightarrow p\to\infty$

$m=\infty \Rightarrow p=-\infty.$

$$= \sum_{n=-\infty}^{\infty} x(n)z^{-n} \sum_{p=\infty}^{-\infty} y(p)z^{p}$$

$$= \sum_{n=-\infty}^{\infty} x(n)z^{-n} \sum_{p=-\infty}^{\infty} y(p)(z^{-1})^{-p}$$

$$= X(Z)Y(z^{-1})$$

9) Initial Value Theorem

Let $x(n)$ be an one sided signal defined in the range $0 \le n \le \infty$

If $Z\{x(n)\} = X(Z)$

Then initial value of $x(n)$ is given by

$$x(0) = \lim_{Z\to\infty} X(Z).$$

Proof.

$$X(Z) = \sum_{n=0}^{\infty} x(n)\, Z^{-n}$$

$$X(Z) = x(0) + x(1)Z^{-1} + x(2)Z^{-2} + \cdots$$

$$\therefore X(Z) = x(0) + \frac{x(1)}{Z} + \frac{x(2)}{Z^2} + \cdots$$

$$\lim_{Z\to\infty} X(Z) = \lim_{Z\to\infty}\left[x(0) + \frac{x(1)}{Z} + \frac{x(2)}{Z^2} + \cdots\right]$$

$$= x(0) + 0 + 0 + \cdots$$

$$\therefore \boxed{x(0) = \lim_{Z\to\infty} X(Z)}$$

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10) Final Value Theorem

Let $x(n)$ be one sided signal defined in the range $0 \le n \le \infty$

If $Z\{x(n)\} = X(z)$

then final value of $x(n)$ (i.e. $x(\infty)$) is given by

$$x(\infty) = \lim_{z\to 1} (1-z^{-1})\, X(z) \quad \text{(or)} \quad x(\infty) = \lim_{z\to 1} \left(\frac{z-1}{z}\right) X(z)$$

Proof:

$$Z\{x(n)\} = X(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$$

$$\therefore \underbrace{Z\{x(n-1) - x(n)\}}_{\text{RHS}} = \underbrace{\sum_{n=0}^{\infty} \{x(n-1) - x(n)\} z^{-n}}_{\text{LHS.}}$$

$$\text{RHS} = Z\{x(n-1) - x(n)\}$$

$$= Z\{x(n-1)\} - Z\{x(n)\} \qquad \text{(using Linearity property).}$$

$$= z^{-1} X(z) + x(-1) - X(z). \qquad \text{(using Shifting).}$$

$$= x(-1) - (1-z^{-1})\, X(z)$$

$$= \lim_{z\to 1} \left[x(-1) - (1-z^{-1}) X(z)\right]$$

$$= x(-1) - \lim_{z\to 1} (1-z^{-1}) X(z) \quad \text{--- (1).}$$

$$\text{LHS} := \sum_{n=0}^{\infty} [x(n-1) - x(n)] z^{-n}$$

$$= \lim_{z\to 1} \sum_{n=0}^{\infty} [x(n-1) - x(n)] z^{-n}.$$

$$= \sum_{n=0}^{\infty} [x(n-1) - x(n)] (1^{-n})$$

Take $\lim_{p\to\infty}$ & change summation index from 0 to p.

$$= \lim_{p\to\infty} \sum_{n=0}^{p} [x(n-1) - x(n)]$$

$$= \lim_{p\to\infty} \Big[x(-1) - x(0) + [x(0) - x(1)] + [x(1) - x(2)] + \dots + [x(p-2) - x(p-1)] + [x(p-1) - x(p)] \Big]$$

$$= \lim_{p\to\infty} [x(-1) - x(p)]$$

$$= x(-1) - x(\infty) \quad \text{--- (2)}$$

equate (1) & (2)

$$\cancel{x(-1)} - x(\infty) = \cancel{x(-1)} - \lim_{z\to 1} (1-z^{-1}) X(z).$$

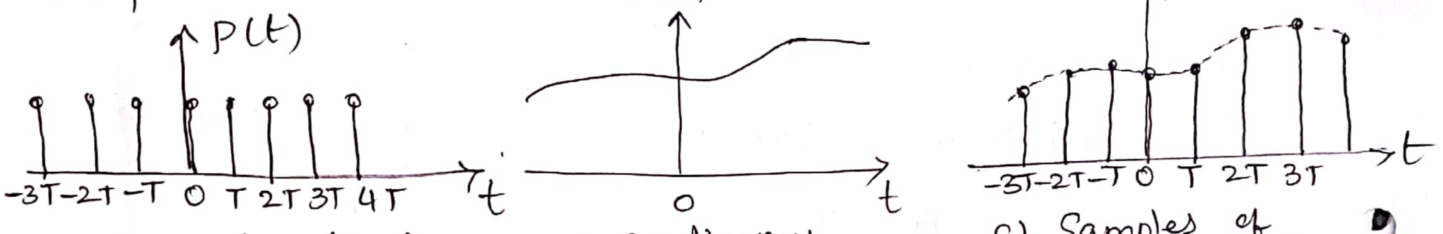
$$\therefore x(\infty) = \lim_{z\to 1} (1-z^{-1})\, X(z)$$

Relation between LT and ZT (Z plane & S-plane correspondance)

Consider a periodic impulse train $p(t)$ with period T. pulse train can be expressed as

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t-nT) \quad \text{——— (1)}$$

When a continuous signal $x(t)$ multiplied by impulse train $p(t)$, the product signal will have impulses.



a) Impulse train b) Continuous Signal. c) Samples of Continuous-time Signal.

$$\therefore x_p(t) = \sum_{n=-\infty}^{\infty} x(nT)\, \delta(t-nT)$$

where $x(nT)$ are Samples of $x(t)$ at $t=nT$

Taking LT to this

$$L\{x_p(t)\} = X_p(s) = L\left\{\sum_{n=-\infty}^{\infty} x(nT)\, \delta(t-nT)\right\}$$

$$= \sum_{n=-\infty}^{\infty} x(nT)\, L\{\delta(t-nT)\}$$

$$= \sum_{n=-\infty}^{\infty} x(nT)\, e^{-nsT} \cdot 1 \qquad \left[\because L[x(t-t_0)] = e^{-st_0} X(s)\right]$$

$$\boxed{X_p(s) = \sum_{n=-\infty}^{\infty} x(nT)\,(e^{sT})^{-n}}$$

Let $e^{sT} = z$

$$\therefore X_p(s) = \sum_{n=-\infty}^{\infty} x(nT)\, z^{-n} \quad \text{——— (2)}$$

The Z-T of $x(nT)$ is

$$X(z) = \sum_{n=-\infty}^{\infty} x(nT)\, z^{-n} \quad \text{——— (3)}$$

Comparing 2 & 3

$$\boxed{X(S) = X(z)\big|_{z=e^{ST}}}$$

$x(nT)$ = Sampled version of $x(t)$

∴ The ZT of discrete time signal can be obtained from LT of sampled version of $x(t)$ by choosing transformation $e^{ST} = z$. This transformation is also called Impulse Invariant transformation.

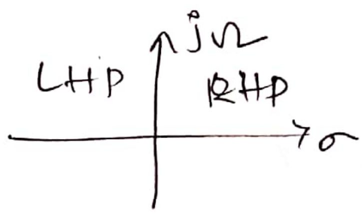
Relation between S-plane and z-plane

$$z = e^{ST}$$

but $S = \sigma + j\Omega$

$$\therefore\ z = e^{(\sigma + j\Omega)T}$$

$$z = e^{\sigma T} \cdot e^{j\Omega T}$$



$|z| = e^{\sigma T}$

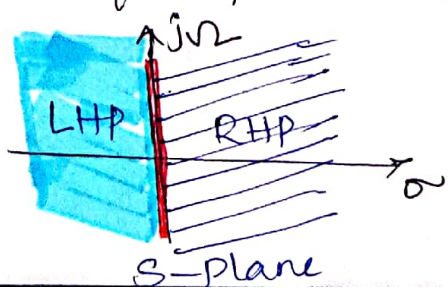
$\angle z = e^{j\Omega T}$

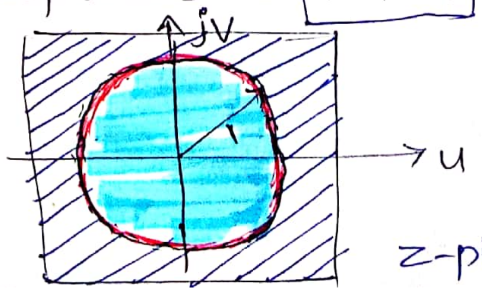
(i) If $\boxed{\sigma < 0}$ (i.e. σ is negative), then points s lies on Left Half (LHP) of s-plane. In this case $\boxed{|z| < 1}$. $\left(\text{i.e. } z = e^{-T} = \frac{1}{e^{T}} < 1\right)$

ii) If $\sigma = 0$ (The imaginary axis on s-plane).

$\boxed{\sigma = 0}$ corresponds to $\boxed{|z| = 1}$ $\{\because |z| = e^{0} = 1\}$

iii) If $\boxed{\sigma > 0}$ (i.e. σ is +ve), then points s lies on RHP of S-plane corresponds to $\boxed{|z| > 1}$.





ignals .

~~$\delta(n)$~~ $x(n)$	$X(z)$	ROC
$\delta(n)$	1	entire z-plane.
$u(n)$ or 1	$\frac{z}{z-1}$	$\lvert z\rvert > 1$
$a^n u(n)$	$z/z-a$	$\lvert z\rvert > \lvert a\rvert$
$u(-n)$	$1/1-z$	$\lvert z\rvert < 1$
$a^n u(-n)$	$-a/a-z$	$\lvert z\rvert < \lvert a\rvert$
$a^n u(-n-1)$	$-z/z-a$	$\lvert z\rvert < \lvert a\rvert$
$n a^n u(n)$	$\frac{az}{(z-a)^2}$	$\lvert z\rvert > \lvert a\rvert$
$n\,u(n)$	$z/(z-1)^2$	$\lvert z\rvert > 1$
$e^{-at}u(t) = e^{-anT}u(nT)$	$z/z-e^{-aT}$	$\lvert z\rvert > \lvert e^{-aT}\rvert$
$te^{-at}u(t) = nTe^{-anT}u(nT)$	$\frac{e^{-aT}Tz}{(z-e^{-aT})^2}$	$\lvert z\rvert > \lvert e^{-aT}\rvert$
$\sin w_0 t\ u(t)$	$\frac{z \sin w}{z^2-2z\cos w+1}$	$\lvert z\rvert > 1$
$\cos w_0 t\ u(t)$	$\frac{z(z-\cos w)}{z^2-2z\cos w+1}$	$\lvert z\rvert > 1$.

$= x(0)z^{-0} \Rightarrow X(z) = 1$ ROC $\Rightarrow$ for all values of 'z'

1. Determine the z-Transform & ROC of following signals.

i) $x(n) = \{\underset{\uparrow}{1}, 2, 3, 0\}$

$$Z\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=0}^{3} x(n) z^{-n}$$

$$= x(0) + x(1)z^{-1} + x(2)z^{-2} + x(3)z^{-3}$$

$$= 1 + 2z^{-1} + 3z^{-2} + 0 \cdot z^{-3}$$

$$X(z) = 1 + \frac{2}{z} + \frac{3}{z^2}$$

ROC ⇒ All values of z except 'z=0'

ii) $x(n) = \{1, 2, \underset{\uparrow}{3}, 4, 5\}$

$$Z\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=-2}^{2} x(n) z^{-n}$$

$$= x(-2)z^{2} + x(-1)z^{1} + x(0)z^{0} + x(1)z^{-1} + x(2)z^{-2}$$

$$= (1)z^2 + 2(z) + 3(1) + 4(z^{-1}) + 5(z^{-2})$$

$$X(z) = z^2 + 2z + 3 + \frac{4}{z} + \frac{5}{z^2}$$

ROC → All values except $z=0$ & ∞

iii) $x(n) = \delta(n)$ (Impulse)

$$\delta(n) = 1, \quad n = 0$$
$$= 0, \quad n \neq 0$$

$$Z\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= x(0) z^{-0} \Rightarrow X(z) = 1$$ ROC ⇒ for all values of 'z'

iv) $x(n) = u(n)$

$u(n) = 1 \ , \ n > 0$

$$Z\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=0}^{\infty} u(n) z^{-n}$$

$$= \sum_{n=0}^{\infty} (1) z^{-n}$$

$$= 1 + z^{-1} + z^{-2} + z^{-3} + \ldots\ldots \quad (G.P)$$

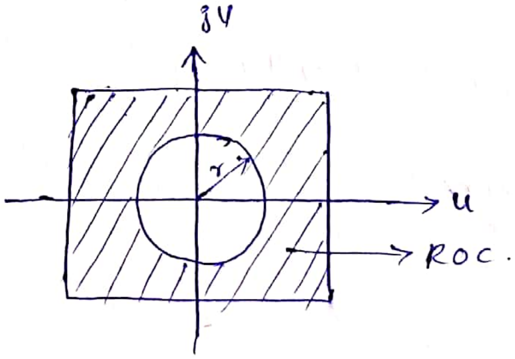
$$X(z) = \frac{1}{1 - z^{-1}}$$

$$X(z) = \frac{z}{z-1}$$

$$0 < |z^{-1}| < 1$$

$$\frac{1}{|z|} < 1$$

ROC $\rightarrow$ $|z| > 1$



V) $x(n) = 2^n u(n)$

$u(n) = 1 \ , \ n > 0$

$$z\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} 2^n z^{-n} u(n)$$

$$= \sum_{n=0}^{\infty} 2^n z^{-n}$$

$$= \sum_{n=0}^{\infty} (2z^{-1})^n$$

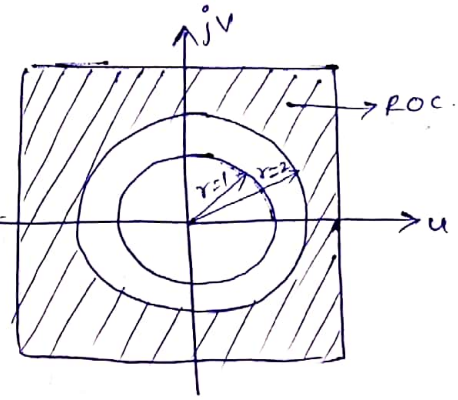
$$= \frac{1}{1 - 2z^{-1}}$$

$$0 < |2z^{-1}| < 1$$

$$0 < \left|\frac{2}{z}\right| < 1$$

$$\frac{2}{z} > 0 \qquad \frac{2}{z} < 1$$

$$\text{ROC} \Rightarrow \boxed{|z| > 2}$$



vi) $x(n) = u(-n)$

$$z\{x(n)\} = \sum_{n=-\infty}^{\infty} u(-n) z^{-n}$$

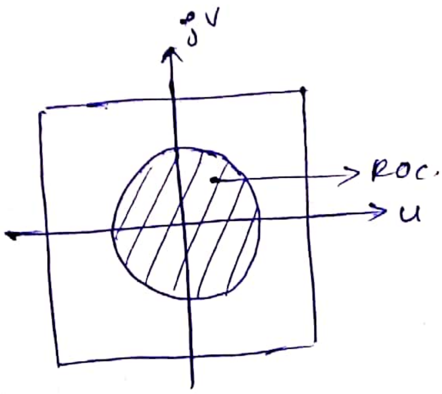
$$u(-n) = 1, \quad n \leq 0$$

$$= \sum_{n=-\infty}^{0} z^{-n}$$

$$= \sum_{-n=\infty}^{-n=0} z^{n}$$

$$= \sum_{n=0}^{\infty} (z)^{n}$$

$$X(z) = \frac{1}{1-z}$$



$$0 < |z| < 1$$

$$\text{ROC} \rightarrow \boxed{|z| < 1}$$

vii) $x(n) = 2^n u(-n-1)$

$$z\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$u(n) = 1, \quad n \geq 0$$
$$= 0, \text{ for } n < 0$$

$$u(-n-1) = 1, \text{ for } -n-1 \geq 0 \rightarrow -n \geq 1$$
$$n \leq -1$$

$u(-n-1) = 0$, for $-n-1 < 0 \Rightarrow -n < 1$

$n > -1$

$$Z\{x(n)\} = \sum_{n=-1}^{-\infty} 2^n z^{-n}$$

$$= \sum_{-n=-1}^{-n=-\infty} (2z^{-1})^{-n}$$

$$= \sum_{n=1}^{\infty} (2z^{-1})^{-n}$$

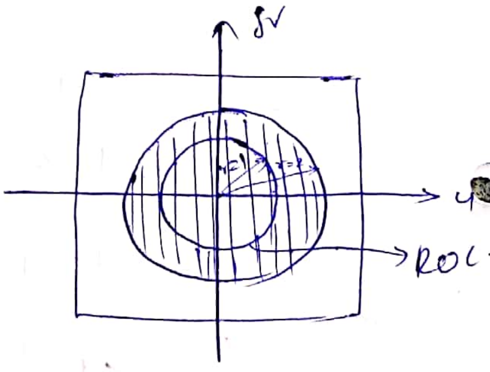
$$= \sum_{n=0}^{\infty} (2z^{-1})^{-n} - 1$$

$$= \sum_{n=0}^{\infty} (2^{-1}z)^{n} - 1$$

$$X(z) = \frac{1}{1 - 2^{-1}z} - 1$$

$$= \frac{2}{2-z} - 1 \Rightarrow \frac{2-2+z}{2-z}$$

$$= \frac{z}{2-z}$$



ROC $\Rightarrow$ $0 < |2^{-1}z| < 1$

$$0 < \left|\frac{z}{2}\right| < 1$$

$$\left|\frac{z}{2}\right| < 1$$

$$\boxed{|z| < 2}$$

viii) $x(n) = n u(n)$

$$Z\{x(n)\} = Z\{n u(n)\} = -z \frac{d}{dz}[Z(u(n))]$$

$$= -z \frac{d}{dz}\left[\frac{z}{z-1}\right]$$

$$= -z\left[\frac{(z-1)(1) - z(1)}{(z-1)^2}\right]$$

$$= -z\left[\frac{-1}{(z-1)^2}\right]$$

$$= \frac{z}{(z-1)^2}$$

2. Find the z-Transforms of $x(n) = n a^{n-1}$

$$z\{x(n)\} = z\{n a^{n-1}\}$$

$$x(n) = n a^{n-1}$$

$$= n a^n a^{-1}$$

$$= \frac{n a^n}{a}$$

$$z\{x(n)\} = \frac{1}{a} z\{\underset{n}{n}\,\underset{x(n)}{a^n}\}$$

$$= \frac{1}{a}\left[-z \frac{d}{dz} z\{a^n\}\right]$$

$$= \frac{1}{a}\left[-z \frac{d}{dz}\left(\frac{z}{z-a}\right)\right]$$

$$= \frac{1}{a}\left[-z\left(\frac{(z-a)(1)-(z)(1)}{(z-a)^2}\right)\right]$$

$$X(z) = \frac{1}{\cancel{a}}\left[\frac{\cancel{a} z}{(z-a)^2}\right] = \frac{z}{(z-a)^2}$$

Q:-- $x(t) = \sin \Omega_0 t \, u(t)$

$t \to nT$

$x(nT) = \sin \Omega_0 nT \, u(nT)$

$y(n) = \sin(\Omega_0 T) n \, u(n)$

$y(n) = \sin \omega n$

$$z\{y(n)\} = \sum_{n=-\infty}^{\infty} y(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \sin(\omega n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \left[\frac{e^{j\omega n} - e^{-j\omega n}}{2j} \right] z^{-n}$$

$$= \frac{1}{2j} \left[\sum_{n=-\infty}^{\infty} e^{j\omega n} z^{-n} - \sum_{n=-\infty}^{\infty} e^{-j\omega n} z^{-n} \right]$$

$$= \frac{1}{2j} \left[\sum_{n=0}^{\infty} (e^{j\omega} z^{-1})^n - \sum_{n=0}^{\infty} (e^{-j\omega} z^{-1})^n \right]$$

$$= \frac{1}{2j} \left[\frac{1}{1 - e^{j\omega} z^{-1}} - \frac{1}{1 - e^{-j\omega} z^{-1}} \right]$$

$$= \frac{1}{2j} \left[\frac{z}{z - e^{j\omega}} - \frac{z}{z - e^{-j\omega}} \right]$$

$$= \frac{1}{2j} \left[\frac{\cancel{z^2} - ze^{-j\omega} - \cancel{z^2} + ze^{j\omega}}{(z - e^{j\omega})(z - e^{-j\omega})} \right]$$

$$= \frac{z}{2j} \left[\frac{e^{j\omega} - e^{-j\omega}}{z^2 - ze^{-j\omega} - ze^{j\omega} + 1} \right]$$

$$= \frac{z \sin \omega}{z^2 - 2z\cos\omega + 1} = \frac{z \sin \Omega_0 T}{z^2 - 2z\cos\Omega_0 T + 1}$$

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$$ROC :- \quad 0 < |e^{j\omega} z^{-1}| < 1 \qquad 0 < |e^{-j\omega} z^{-1}| < 1$$

$$|z| > e^{j\omega} \qquad |z| > e^{-j\omega}$$

Q:- $x(t) = \cos \Omega_0 t$

$t \rightarrow nT$

$x(nT) = \cos \Omega_0 nT \, u(nT)$

$y(n) = \cos(\Omega_0 T) n \, u(n)$

$y(n) = \cos \omega n$

$$Z\{y(n)\} = \sum_{n=-\infty}^{\infty} y(n) z^{-n}$$

$$= \sum_{n=0}^{\infty} \cos(\omega n) z^{-n}$$

$$= \sum_{n=0}^{\infty} \left[\frac{e^{j\omega n} + e^{-j\omega n}}{2} \right] z^{-n}$$

$$= \frac{1}{2} \left[\sum_{n=0}^{\infty} e^{j\omega n} z^{-n} + \sum_{n=0}^{\infty} e^{-j\omega n} z^{-n} \right]$$

$$= \frac{1}{2} \left[\sum_{n=0}^{\infty} (e^{j\omega} z^{-1})^n + \sum_{n=0}^{\infty} (e^{-j\omega} z^{-1})^n \right]$$

$$= \frac{1}{2} \left[\frac{1}{1 - e^{j\omega} z^{-1}} + \frac{1}{1 - e^{-j\omega} z^{-1}} \right]$$

$$= \frac{1}{2} \left[\frac{z}{z - e^{j\omega}} + \frac{z}{z - e^{-j\omega}} \right]$$

$$= \frac{1}{2} \left[\frac{2z^2 - 2z\cos\omega}{z^2 - 2z\cos\omega + 1} \right]$$

$$= \frac{1}{\cancel{2}} \left[\frac{\cancel{2}(z^2 - z\cos\Omega_0 T)}{z^2 - 2z\cos\Omega_0 T + 1} \right] = \frac{z^2 - z\cos\Omega_0 T}{z^2 - 2z\cos\Omega_0 T + 1}$$

ROC :- $0 < |e^{j\omega} z^{-1}| < 1$ $\qquad$ $0 < |e^{-j\omega} z^{-1}| < 1$

$$\left|\frac{e^{j\omega}}{z}\right| < 1 \qquad \left|\frac{e^{-j\omega}}{z}\right| < 1$$

$$|z| > e^{j\omega} \qquad |z| > e^{-j\omega}$$

Q:- $x(t) = e^{-at} \sin \Omega_0 t$

$$x(nT) = e^{-anT} \sin \Omega_0 nT$$

$$= e^{-anT} \sin (\Omega_0 T) n$$

$$\boxed{y(n) = e^{-anT} \sin \omega n}$$

$$Z\{y(n)\} = \sum_{n=-\infty}^{\infty} y(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} e^{-anT} \sin \omega n \, z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} e^{-anT} \left(\frac{e^{j\omega n} - e^{-j\omega n}}{2j}\right) z^{-n}$$

$$= \frac{1}{2j}\left[\sum_{n=0}^{\infty} \left(e^{-aT} e^{j\omega} z^{-1}\right)^n - \sum_{n=0}^{\infty} \left(e^{-aT} e^{-j\omega} z^{-1}\right)^n\right]$$

$$= \frac{1}{2j}\left[\frac{1}{1 - e^{-aT} e^{j\omega} z^{-1}} - \frac{1}{1 - e^{-aT} e^{-j\omega} z^{-1}}\right]$$

$$= \frac{1}{2j}\left[\frac{z}{z - e^{-aT} e^{j\omega}} - \frac{z}{z - e^{-aT} e^{-j\omega}}\right]$$

$$= \frac{1}{2j}\left[\frac{\cancel{z^2} - z e^{-aT} e^{-j\omega} - \cancel{z^2} + z e^{-aT} e^{j\omega}}{z^2 - z e^{-aT} e^{-j\omega} - z e^{-aT} e^{j\omega} + e^{-2aT}}\right]$$

$$= \frac{1}{2j}\left[\frac{z(e^{-aT} e^{j\omega} - e^{-aT} e^{-j\omega})}{z^2 - z e^{-aT}(e^{j\omega} + e^{-j\omega}) + e^{-2aT}}\right]$$

$$= \frac{1}{2j}\left[\frac{z e^{-aT}\left[e^{j\omega} - e^{-j\omega}\right]}{z^2 - z e^{-aT}\left(e^{j\omega} + e^{-j\omega}\right) + e^{-2aT}}\right]$$

$\Rightarrow$ Multiply N & D by e^{2aT}

$$= \frac{z e^{aT} \sin\omega}{z^2 e^{2aT} - 2 z e^{aT}(\cos\omega) + 1}$$

Q: $x(t) = e^{-at}\cos\Omega_0 t$.

$x(nT) = e^{-anT}\cos\Omega_0 nT$

$= e^{-anT}\cos(\Omega_0 T)n$

$$\boxed{y(n) = e^{-anT}\cos\omega n.}$$

$$z\{y(n)\} = \sum_{n=-\infty}^{\infty} y(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} e^{-anT}\cos\omega n \, z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} e^{-anT}\left(\frac{e^{j\omega n} + e^{-j\omega n}}{2}\right) z^{-n}$$

$$= \frac{1}{2}\left[\sum_{n=0}^{\infty}\left(e^{-aT}e^{j\omega}z^{-1}\right)^n + \sum_{n=0}^{\infty}\left(e^{-aT}e^{-j\omega}z^{-1}\right)^n\right]$$

$$= \frac{1}{2}\left[\frac{1}{1 - e^{-aT}e^{j\omega}z^{-1}} + \frac{1}{1 - e^{-aT}e^{-j\omega}z^{-1}}\right]$$

$$= \frac{1}{2}\left[\frac{z}{z - e^{-aT}e^{j\omega}} + \frac{z}{z - e^{-aT}e^{-j\omega}}\right]$$

$$= \frac{1}{2}\left[\frac{z^2 - z e^{-aT}e^{-j\omega} + z^2 - z e^{-aT}e^{j\omega}}{z^2 - z e^{-aT}e^{-j\omega} - z e^{-aT}e^{j\omega} + e^{-2aT}}\right]$$

$$\frac{1}{2}\left[\frac{2z^2 - ze^{-aT}(e^{j\omega} + e^{-j\omega})}{z^2 - ze^{-aT}(e^{j\omega} + e^{-j\omega}) + e^{-2aT}}\right]$$

$$\frac{1}{\cancel{2}}\left[\frac{\cancel{2}(z^2 - ze^{-aT}\cos\omega)}{z^2 - 2ze^{-aT}\cos\omega t + e^{-2aT}}\right]$$

$\Rightarrow$ Multiply and $\div$ by e^{2aT}

$$= \frac{z^2 e^{2aT} - ze^{aT}\cos\omega}{z^2 e^{2aT} - 2ze^{aT}\cos\omega t + 1}$$

$$= \frac{ze^{aT}\left[ze^{aT} - \cos\omega\right]}{z^2 e^{2aT} - 2ze^{aT}\cos\omega + 1}$$

Find the initial & final values of

Q:- $X(z) = \dfrac{1}{1 - z^{-2}}$

Initial value:

$$x(0) = \lim_{z\to\infty} X(z)$$

$$= \lim_{z\to\infty} \frac{1}{1 - z^{-2}}$$

$$= \lim_{z\to\infty} \frac{1}{1 - \frac{1}{z^2}}$$

$$= \frac{1}{1-0} = 1,$$

Final value:-

$$x(\infty) = \lim_{z\to 1} (1 - z^{-1})\, X(z)$$

$$= \lim_{z\to 1} \left(1 - \frac{1}{z}\right)\left(\frac{1}{1 - \frac{1}{z^2}}\right)$$

$$= \lim_{z\to 1} \left(\frac{z-1}{z}\right)\left(\frac{z^2}{z^2-1}\right)$$

$$= \lim_{z\to 1} \frac{\cancel{(z-1)}}{\cancel{z}} \cdot \frac{z^{\cancel{2}}}{\cancel{(z-1)}(z+1)}$$

$$= \lim_{z\to 1} \frac{z}{(z+1)}$$

$$= \frac{1}{1+1} = \frac{1}{2}$$

Q:- $X(z) = \dfrac{1}{1+2z^{-1}-3z^{-2}}$

Initial value:-

$$x(0) = \lim_{z\to\infty} x(z)$$

$$= \lim_{z\to\infty} \frac{1}{1+2z^{-1}-3z^{-2}}$$

$$= \lim_{z\to\infty} \frac{1}{1+\frac{2}{z}-\frac{3}{z^2}}$$

$$= \frac{1}{1} = 1$$

final value:-

$$x(\infty) = \lim_{z\to 1} (1-z^{-1})\, x(z)$$

$$= \lim_{z\to 1} \left(1-\frac{1}{z}\right)\left(\frac{1}{1+2z^{-1}-3z^{-2}}\right)$$

$$= \lim_{z\to 1} \left(\frac{z-1}{z}\right)\left(\frac{z^2}{z^2+2z-3}\right)$$

$$= \lim_{z\to 1} \left(\frac{\cancel{z-1}}{\cancel{z}}\right) \cdot \frac{z^{\cancel{2}}}{\cancel{(z-1)}(z+3)}$$

$$= \lim_{z \to 1} \frac{z}{z+3}$$

$$= \frac{1}{1+3} = \frac{1}{4}$$

→ Inverse z-Transforms.

$$x(n) \underset{IZT}{\overset{ZT}{\rightleftharpoons}} X(z)$$

1. Residue Method
2. Partial fraction Method
3. Long division Method (Power series method)

Q:- Determine the inverse z-Transforms of the function

$$X(z) = \frac{3+2z^{-1}+z^{-2}}{1-3z^{-1}+2z^{-2}}$$

i) Partial fraction Method :-

$$X(z) = \frac{\cancel{z^{-2}}[3z^2+2z+1]}{\cancel{z^{-2}}[z^2-3z+2]}$$

$$\frac{X(z)}{z} = \frac{3z^2+2z+1}{z(z-1)(z-2)}$$

$$\frac{X(z)}{z} = \frac{A}{z} + \frac{B}{z-1} + \frac{C}{z-2}$$

$$A = \Big|_{z=0} = \frac{1}{2}$$

$$B = \Big|_{z=1} = -6$$

$$C = \Big|_{z=2} = 17/2$$

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$$\frac{X(z)}{z} = \frac{1/2}{z} + \frac{-6}{z-1} + \frac{17/2}{z-2}$$

$$X(z) = \frac{1}{2}\frac{z}{z} - 6\frac{z}{z-1} + \frac{17}{2}\frac{z}{z-2}$$

$$z^{-1}\{x(z)\} = \frac{1}{2}z^{-1}\{1\} - 6z^{-1}\left\{\frac{z}{z-1}\right\} + \frac{17}{2}z^{-1}\left\{\frac{z}{z-2}\right\}$$

$$x(n) = \frac{1}{2}\delta(n) - 6u(n) + \frac{17}{2}2^n u(n) \qquad \left[\because \delta(n) = 1, n=0 \atop = 0, \text{else}\right]$$

$n=0 \rightarrow \frac{1}{2}\delta(0) - 6u(0) + \frac{17}{2}2^0 u(0)$

$= \frac{1}{2}(1) - 6(1) + \frac{17}{2}$

$= 3$

$n=1 \Rightarrow \frac{1}{2}\delta(1) - 6u(1) + \frac{17}{2}2u(1)$

$= 0 - 6 + 17$

$= 11$

$n=2 \Rightarrow \frac{1}{2}\delta(2) - 6u(2) + \frac{17}{\cancel{2}}(\cancel{4}^{2})u(2)$

$= -6 + 34$

$= 28$

$n=3 \Rightarrow \frac{1}{2}\delta(3) - 6u(3) + \frac{17}{\cancel{2}}(\cancel{8}^{4})u(3)$

$= -6 + 68$

$= 62$

$\therefore x(n) = \{3, 11, 28, 62, 130, \ldots\}$

(ii) Power series method:-

$$X(z) = \frac{3 + 2z^{-1} + z^{-2}}{1 - 3z^{-1} + 2z^{-2}}$$

$$1-3z^{-1}+2z^{-2}\,\big)\; 3+2z^{-1}+z^{-2} \;\big(\, 3+11z^{-1}+28z^{-2}+62z^{-3}+130z^{-4}+\dots$$

$$\begin{array}{r}
3+2z^{-1}+z^{-2} \\
3-9z^{-1}+6z^{-2} \\
-\quad + \quad - \\
\hline
11z^{-1}-5z^{-2} \\
11z^{-1}-33z^{-2}+22z^{-3} \\
-\quad + \quad - \\
\hline
28z^{-2}-22z^{-3} \\
28z^{-2}-84z^{-3}+56z^{-4} \\
-\quad + \quad - \\
\hline
62z^{-3}-56z^{-4} \\
62z^{-3}-186z^{-4}+124z^{-5} \\
-\quad + \quad - \\
\hline
130z^{-4}-124z^{-5}
\end{array}$$

$$X(z) = 3+11z^{-1}+28z^{-2}+62z^{-3}+130z^{-4}+\dots \quad \text{—} ①$$

$$Z\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

$$= \dots\dots + x(-2)z^{2}+x(-1)z^{1}+x(0)+x(1)z^{-1}+x(2)z^{-2}+ x(3)z^{-3}+x(4)z^{-4}+\dots \quad \text{—} ②$$

compare ① & ②

we get, $x(-2)=0$

$x(-1)=0$

$x(0)=3$

$x(1)=11$

$x(2)=28$

$x(3)=62$

$x(4)=130$

$$x(n) = \{\underset{\substack{\uparrow \\ n=0}}{3}, 11, 28, 62, 130, \dots\}$$

Q:- Determine the Inverse Z-Transform of

$$X(z) = \frac{1}{1-1.5z^{-1}+0.5z^{-2}}$$

if i) ROC $|z| > 1.0$

ii) ROC $|z| < 0.5$

iii) ROC $1 < |z| < 0.5$.

$$X(z) = \frac{1}{1-\frac{3}{2}z^{-1}+\frac{1}{2}z^{-2}}$$

i) ROC $|z| > 1.0$

$$1-1.5z^{-1}+0.5z^{-2} \,\big)\, 1 \,\big(\, 1+1.5z^{-1}+1.75z^{-2}+1.875z^{-3}+1.937z^{-4}+\ldots$$

$$\begin{array}{l}
1-1.5z^{-1}+0.5z^{-2} \\
\underline{-\quad +\qquad -} \\
1.5z^{-1}-0.5z^{-2} \\
1.5z^{-1}-2.25z^{-2}+0.75z^{-3} \\
\underline{-\quad +\qquad -} \\
1.75z^{-2}-0.75z^{-3} \\
1.75z^{-2}-2.625z^{-3}+0.875z^{-4} \\
\hline
1.875z^{-3}-0.875z^{-4} \\
1.875z^{-3}-2.812z^{-4}+0.937z^{-5} \\
\hline
1.937z^{-4}-0.937z^{-5}
\end{array}$$

$$X(z) = 1+1.5z^{-1}+1.75z^{-2}+1.875z^{-3}+1.937z^{-4}+\ldots \quad -(1)$$

$$Z\{x(n)\} = \sum_{n=0}^{\infty} x(n)\, z^{-n}$$

$$= x(0)+x(1)z^{-1}+x(2)z^{-2}+x(3)z^{-3}+x(4)z^{-4}+\ldots \quad -(2)$$

compare (1) & (2)

$$\therefore x(n) = \{\underset{\uparrow}{1}, 1.5, 1.75, 1.87, 1.93, \ldots\}$$

ii) ROC $|z| < 0.5$.

$$
\begin{array}{r|l}
0.5z^{-2} - 1.5z^{-1} + 1 & 1 \quad (2z^2 + 6z^3 + 14z^4 + 30z^5 + 62z^6 + \ldots \\
 & \underline{\cancel{1} - 3z + 2z^2} \\
 & \quad 3z - 2z^2 \\
 & \quad \underline{\cancel{3z} - 9z^2 + 6z^3} \\
 & \qquad 7z^2 - 6z^3 \\
 & \qquad \underline{\cancel{7z^2} - 21z^3 + 14z^4} \\
 & \qquad\quad 15z^3 - 14z^4 \\
 & \qquad\quad \underline{\cancel{15}z^3 - 45z^4 + 30z^5} \\
 & \qquad\qquad 31z^4 - 30z^5
\end{array}
$$

$$X(z) = 2z^2 + 6z^3 + 14z^4 + 30z^5 + 62z^6 + \ldots \quad —(1)$$

$$Z\{x(n)\} = \sum_{n=-\infty}^{0} x(n) z^{-n}$$

$$= \ldots\ldots + x(-6)z^6 + x(-5)z^5 + x(-4)z^4 + x(-3)z^3 + x(-2)z^2 + x(-1)z +$$

$$x(0) + \ldots\ldots \quad —(2)$$

$$\therefore x(n) = \{\ldots\ldots 62, 30, 14, 6, 2, 0, \underset{\uparrow\, n=0}{0}\}$$

iii) ROC $1 < |z| < 0.5 \longrightarrow$ No solution.

Q. Determine Inverse z-Transform of $X(z) = \dfrac{1}{1 - 0.8z^{-1} + 0.12z^{-2}}$

if i) ROC $|z| > 0.6$

ii) ROC $|z| < 0.2$

iii) ROC $0.2 < |z| < 0.6$.

i) ROC $|z| > 0.6$

$$X(z) = \frac{1}{1 - 0.8z^{-1} + 0.12z^{-2}}$$

$$= \frac{z^2}{z^2 - 0.8z + 0.12}$$

$$= \frac{z^2}{(z-0.6)(z-0.2)}$$

$$\frac{X(z)}{z} = \frac{z}{(z-0.6)(z-0.2)}$$

$$= \frac{A}{z-0.6} + \frac{B}{z-0.2}$$

$$A = \Big|_{z=0.6} = \frac{0.6}{0.6-0.2} = \frac{0.6}{0.4} = \frac{3}{2}$$

$$B = \Big|_{z=0.2} = \frac{0.2}{-0.4} = \frac{-1}{2}$$

$$\therefore \frac{X(z)}{z} = \frac{3}{2}\frac{1}{(z-0.6)} - \frac{1}{2}\left(\frac{1}{z-0.2}\right)$$

$$X(z) = \frac{3}{2}\left(\frac{z}{z-0.6}\right) - \frac{1}{2}\left(\frac{z}{z-0.2}\right)$$

→ Apply Inverse z-Transform.

$$x(n) = \frac{3}{2}(0.6)^n u(n) - \frac{1}{2}(0.2)^n u(n)$$

ii) $|z| < 0.2$

$$x(n) = -\frac{3}{2}(0.6)^n u(-n-1) + \frac{1}{2}(0.2)^n u(-n-1)$$

$\because a^n u(n) \Rightarrow \frac{z}{z-a}, |z| > a$

$a^n u(-n-1) = \frac{-z}{z-a}, |z| < a$

iii) $0.2 < |z| < 0.6$

$$x(n) = -\frac{3}{2}(0.6)^n u(-n-1) - \frac{1}{2}(0.2)^n u(n)$$

Transfer function:

$$x(n) \longrightarrow \boxed{\begin{matrix}\text{System} \\ h(n)\end{matrix}} \longrightarrow y(n)$$

Input (Excitation) — output (Response)

$$y(n) = x(n) * h(n)$$

$$Z\{y(n)\} = Z\{x(n) * h(n)\}$$

$$Y(z) = X(z) \cdot H(z)$$

$$\boxed{H(z) = \frac{Y(z)}{X(z)}}$$

Solution of difference equations:

$$x(n) \longrightarrow \text{System} \longrightarrow y(n)$$

$$\begin{matrix} \downarrow & & \downarrow \\ x(n-m) & & y(n-m) \end{matrix}$$

$$Z\{x(n-m)\} = z^{-m} X(z)$$

$$Z\{y(n-m)\} = z^{-m} Y(z)$$

Q: Determine the impulse Response $h(n)$ of the system. Described by the second order difference equation. Plot impulse sequence

$$y(n) - 3y(n-1) - 4y(n-2) = x(n) + 2x(n-1)$$

$$Z\{y(n) - 3y(n-1) - 4y(n-2)\} = Z\{x(n) + 2x(n-1)\}$$

$$Y(z) - 3z^{-1}Y(z) - 4z^{-2}Y(z) = X(z) + 2z^{-1}X(z)$$

$$Y(z)\left[1 - 3z^{-1} - 4z^{-2}\right] = X(z)\left[1 + 2z^{-1}\right]$$

Transfer function $H(z) = \frac{Y(z)}{X(z)} = \frac{1+2z^{-1}}{1-3z^{-1}-4z^{-2}}$

Apply, Inverse Z-Transforms.

$$H(z) = \left[\frac{1+2z^{-1}}{1-3z^{-1}-4z^{-2}}\right]$$

$$\frac{H(z)}{z} = \frac{\cancel{z^{-1}}[z+2]}{z\,\cancel{z^{-2}}[z^2-3z-4]}$$

$$\frac{H(z)}{z} = \frac{z+2}{(z-4)(z+1)}$$

$$\frac{H(z)}{z} = \frac{A}{z-4} + \frac{B}{z+1}$$

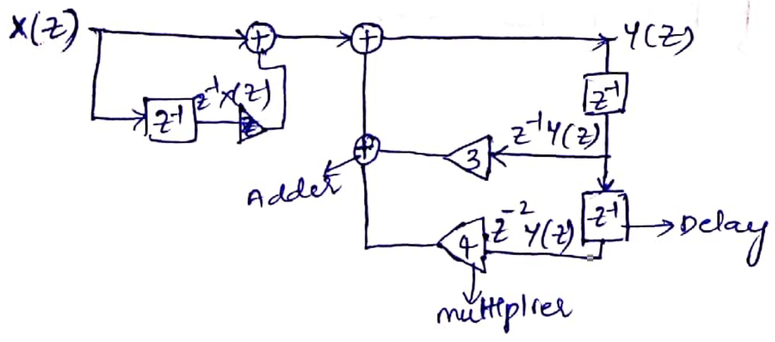
$$A = \Big|_{z=4} = \frac{6}{5}$$

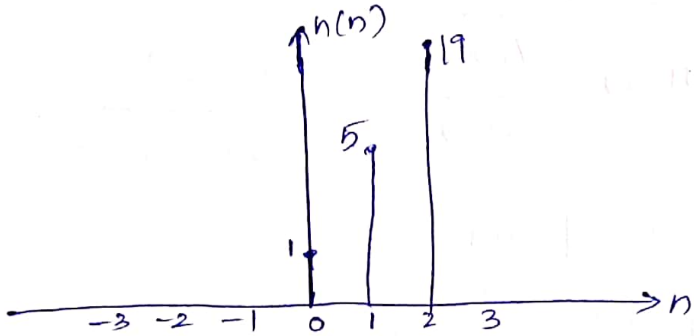
$$B = \Big|_{z=-1} = \frac{1}{-5}$$

$$H(z) = \frac{6}{5}\left[\frac{z}{z-4}\right] - \frac{1}{5}\left[\frac{z}{z+1}\right]$$

$$Z^{-1}[H(z)] = \frac{6}{5} Z^{-1}\left[\frac{z}{z-4}\right] - \frac{1}{5} Z^{-1}\left[\frac{z}{z+1}\right]$$

$$h(n) = \frac{6}{5}(4)^n u(n) - \frac{1}{5}(-1)^n u(n)$$





Obtain and sketch the shift Invariant system described by

$y(n) = 0.4x(n) + x(n-1) + 0.6x(n-2) + x(n-3) + 0.4x(n-4)$

Apply z-Transform.

$Y(z) = 0.4X(z) + z^{-1}X(z) + 0.6z^{-2}X(z) + z^{-3}X(z) + 0.4z^{-4}X(z)$

$Y(z) = X(z)\left[0.4 + z^{-1} + 0.6z^{-2} + z^{-3} + 0.4z^{-4}\right]$

$H(z) = \dfrac{Y(z)}{X(z)} = 0.4 + z^{-1} + 0.6z^{-2} + z^{-3} + 0.4z^{-4}$ —— (1)

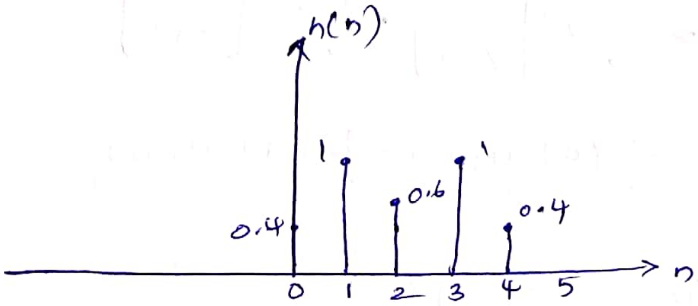
$Z\{h(n)\} = H(z) = \sum_{n=0}^{4} h(n)z^{-n}$

$= h(0) + h(1)z^{-1} + h(2)z^{-2} + h(3)z^{-3} + h(4)z^{-4}$ —— (2)

comparing (1) & (2)

$h(n) = \{\underset{\uparrow}{0.4}, 1, 0.6, 0.4\}$

$n=0$



Find the response of the time Invariant system if the impulse response $h(n) = \{1, 2, 1, -1\}$ to an input signal $x(n) = \{1, 2, 3, 1\}$

$y(n) = x(n) * h(n)$

Apply z-transforms

$Y(z) = X(z) \cdot H(z)$

$x(n) = \{\underset{\substack{\uparrow \\ n=0}}{1}, 2, 3, 1\}$

$Z\{x(n)\} = \sum_{n=0}^{3} x(n) z^{-n}$

$X(z) = x(0) + x(1) z^{-1} + x(2) z^{-2} + x(3) z^{-3}$

$X(z) = 1 + 2z^{-1} + 3z^{-2} + z^{-3}$ —— (1)

$h(n) = \{\underset{\substack{\uparrow \\ n=0}}{1}, 2, 1, -1\}$

$Z\{h(n)\} = \sum_{n=0}^{3} h(n) z^{-n}$

$= h(0) + h(1) z^{-1} + h(2) z^{-2} + h(3) z^{-3}$

$H(z) = 1 + 2z^{-1} + z^{-2} - z^{-3}$

$Y(z) = X(z) \cdot H(z)$.

$= (1 + 2z^{-1} + 3z^{-2} + z^{-3})(1 + 2z^{-1} + z^{-2} - z^{-3})$

$= 1 + 2z^{-1} + z^{-2} - z^{-3} + 2z^{-1} + 4z^{-2} + 2z^{-3} - 2z^{-4} + 3z^{-2} + 6z^{-3} +$
$3z^{-4} - 3z^{-5} + z^{-3} + 2z^{-4} + z^{-5} - z^{-6}$

$= 1 + z^{-1}[2+2] + z^{-2}[1+4+3] + z^{-3}[-1+2+6+1] + z^{-4}[-2+3+2]$
$+ z^{-5}[-3+1] + z^{-6}[-1]$

$Y(z) = 1 + 4z^{-1} + 8z^{-2} + 8z^{-3} + 3z^{-4} - 2z^{-5} - z^{-6}$ —— (2)

$Z\{y(n)\} = Y(z) = \sum_{n=0}^{6} y(n) z^{-n}$

$= y(0) + y(1) z^{-1} + y(2) z^{-2} + y(3) z^{-3} + y(4) z^{-4} + y(5) z^{-5} +$
$y(6) z^{-6}$ —— (3)

Compare ② & ③

$y(0) = 1$

$y(1) = 4$

$y(2) = 8$

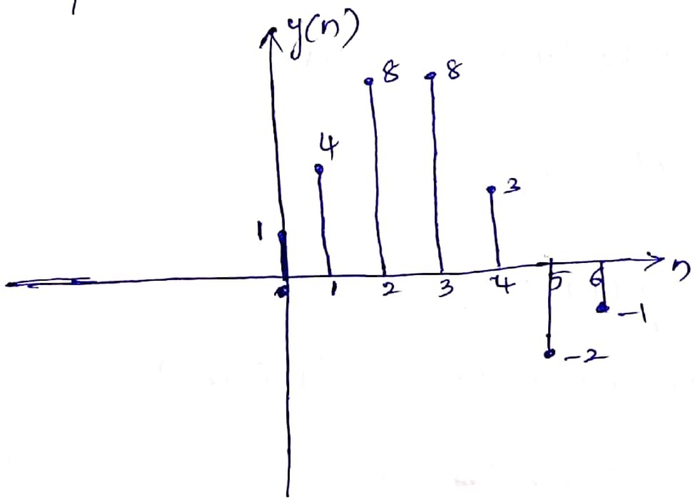
$y(3) = 8$

$y(4) = 3$

$y(5) = -2$

$y(6) = -1$

$$y(n) = \{\underset{\uparrow}{1}, 4, 8, 8, 3, -2, -1\}$$



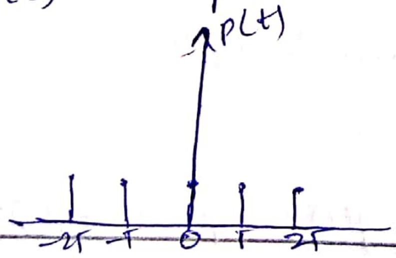
*** Sampling :**

Converting a time domain signal into discrete time domain signal is called sampling.

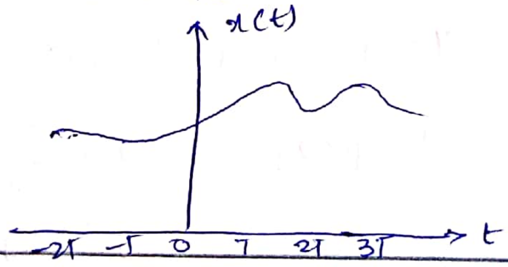
Relationship b/w z-Transform and Laplace transform (z-plane & s-plane)

1. Let us consider a periodic impulse train $P(t)$ with period 'T' and pulse train can be expressed as $P(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$ — ①

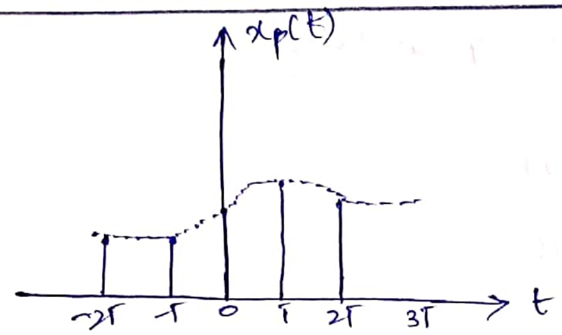
2. When a continuous time signal $x(t)$ multiplied by impulse train $P(t)$ the product signal will have



a) Impulse train



b) continuous signal



c) samples of continuous time signals

$$\therefore x_p(t) = \sum_{n=-\infty}^{\infty} p(t)\, x(t)$$

$$= \sum_{n=-\infty}^{\infty} x(nT)\, \delta(t-nT)$$

take LT

$$L\{x_p(t)\} = L\left\{\sum_{n=-\infty}^{\infty} x(nT)\, \delta(t-nT)\right\}$$

$$\boxed{\because L\{x(t-t_0)\} = e^{-st_0} X(s)}$$

$$X_p(s) = \sum_{n=-\infty}^{\infty} x(nT)\, L\{\delta(t-nT)\}$$

$$= \sum_{n=-\infty}^{\infty} x(nT)\, e^{-snT} L\{\delta(t)\}$$

$$= \sum_{n=-\infty}^{\infty} x(nT)\, e^{-nsT} (1)$$

$$x_p(s) = \sum_{n=-\infty}^{\infty} x(nT)\, e^{-nsT}$$

put $e^{sT} = z$

$$x_p(s) = \sum_{n=-\infty}^{\infty} x(nT)\, z^{-n} \quad \text{——} \; (2)$$

$$z\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

$$z\{x(nT)\} = \sum_{n=-\infty}^{\infty} x(nT)\, z^{-n} \quad \text{——} \; (3)$$

Comparing (2) & (3)

$$\boxed{x_p(s) = X(z)\Big|_{z=e^{sT}}}$$

Relation b/w z-plane & s-plane

$$z = e^{sT}$$

$$z = e^{(\sigma + j\Omega)T}$$

$$z = e^{\sigma T} \cdot e^{j\Omega T}$$

$$|z| = e^{\sigma T}$$

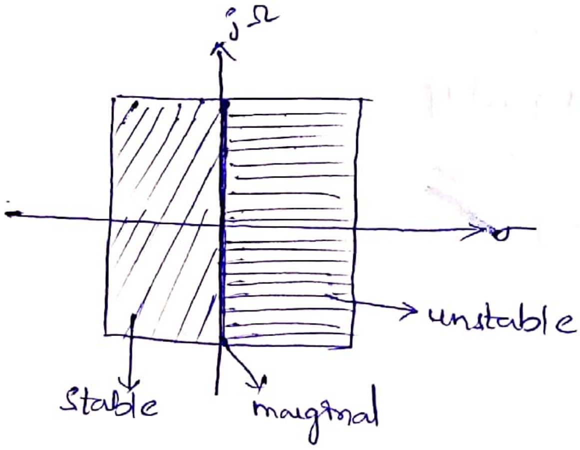
i) $\sigma > 0$ i.e σ +ve $\Rightarrow |z| =$ increasing

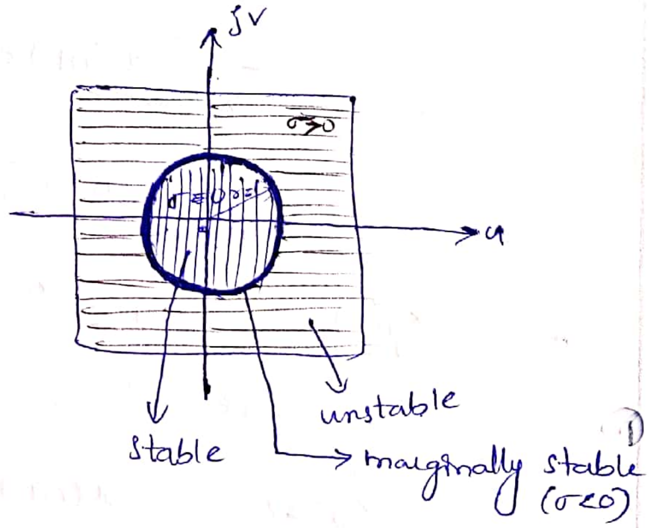
$|z| > 1$

ii) $\sigma < 0$ i.e σ −ve $\Rightarrow |z| =$ decreasing

$|z| < 1$

iii) $\sigma = 0$ ie $\sigma = 0$ $\Rightarrow |z| = 1$





UNIT-III (Z-TRANSFORMS)

1)

a) Find the z-transform and sketch the ROC for the following sequences.

i) $x(n) = \left(\frac{1}{2}\right)^n U(-n)$

ii) $x(n) = \left(\frac{1}{2}\right)^{n-1} U(n-1)$

b) A causal LTI system is described by the difference equation.

$y(n) = y(n-1) + y(n-2) + x(n-1)$.

Find the unit sample response of the system.

2)

(a) Using the power series expansion technique, find the inverse z-transform of $X(z) = \frac{z}{2z^2 - 3z + 1}, \ |z| < \frac{1}{2}$.

(b) Solve the following difference equation with the given initial conditions :

$y(n) - 3y(n-1) = x(n)$, with $x(n) = 4U(n)$, $y(-1) = 1$.

3)

a) Find the inverse z-transform of $(1 - z^{-1})^{-2}$.

b) A linear discrete time system is given by $y(n) + 0.95y(n-1) = 0.05x(n)$

i) Find the impulse response of the system

ii) Find the response of the system if $x(n) = 0.5^n u(n)$.

4)

a) Determine the z-transform of $x(n) = (\cos^2 \omega n)u(n)$.

b) Using partial fraction expansion method obtain the inverse z-transform of

$x(z) = \frac{6z^3 + 2z^2 - z}{z^3 - z^2 - z + 1}$.

5) (a) Find the inverse z-transform of $x(t) = \frac{z+1}{(z+0.2)(z-0.6)}$

(b) For the system given by the difference equation, draw the canonical form realization diagram.
$y(n) + 0.5y(n-1) + 2y(n-2) + 3y(n-3) + 0.8y(n-4) = 3x(n) + 5x(n-2)$.

6) ~~16~~. (a) State and prove sampling theorem.

(b) Realize the system with 2 delays $y(n) = y(n-1) + 2x(n) + 8x(n-1)$.

7) a) Determine the inverse z-transform of

$$X(z) = \frac{1}{1-1.5z^{-1}+0.5z^{-2}} \quad \text{ROC}: |z| < 0.5.$$

Using long division method.

b) Determine the inverse z-transform for the following function :

$$X(z) = \frac{(z+1)(z+5)}{(z+2)(z+3)(z+6)} \text{ for } |z| \leq 3.$$

8) a) Find inverse Z-transform of $X(Z) = \frac{1+Z^{-1}}{1-(\frac{1}{3})Z^{-1}}$ if ROC $|Z| > 1/3$.

b) Consider a causal discrete-time system whose output y(n) and input x(n) are related by $y(n) - \frac{5}{6}y(n-1) + \frac{1}{6}y(n-2) = x(n)$

i)Find its transfer function H(z).

ii) Find its impulse response h(n).

9) a) Find inverse Z-transform of $X(Z) = \frac{1}{[1-1.5Z^{-1}+0.5Z^{-2}]}$

if a) ROC $|Z| > 1$ b) ROC $|Z| < 0.5$ c) ROC $0.5 < |Z| < 1$

b) A causal system is represented by the following difference equation. Find its transfer function.

$$y(n) - \frac{1}{3}y(n-1) = x(n) + \frac{1}{4}x(n-1)$$