UNIT- V

Discrete time Signals and Systems

* Analog signal or Continuous time signal :-

It is one in which the time and amplitude both are in continuous.

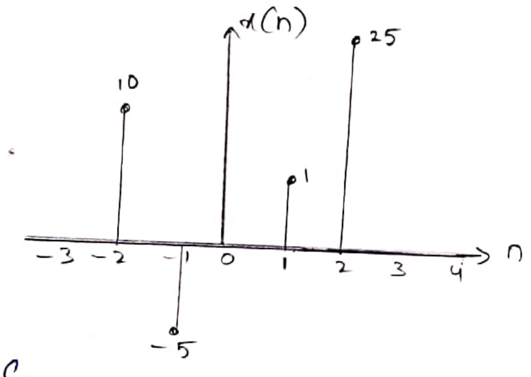
* A discrete time signal is one which discrete in time and continous in amplitude.

Representation of discrete signal :-

1) Functional representation :-

$x(n) = 10$, for $n = -2$
$= -5$, for $n = -1$
$= 0$, for $n = 0$
$= 1$, for $n = 1$
$= 25$, for $n = 2$

2) Graphical representation :-



3) Sequence

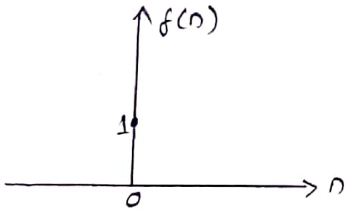
$x(n) = \{10, -5, \underset{\uparrow}{0}, 1, 25\}$

4) Tabular

n	-2	-1	0	1	2
$x(n)$	10	-5	0	1	25

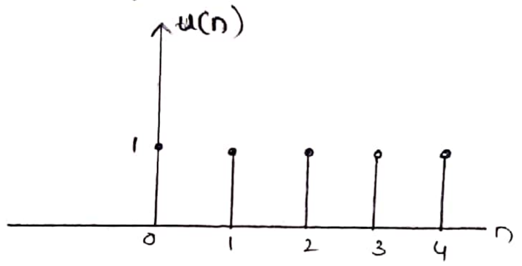
→ Standard discrete time Signal (DTS)

1) Impulse signal



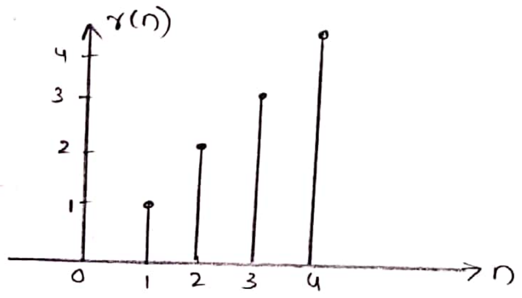
$$\delta(n) = 1, \text{ for } n \neq 0$$
$$\quad\;\; 0, \text{ for } n \neq 0$$

2) Step Signal



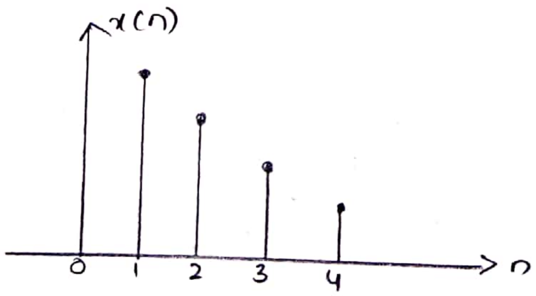
$$u(n) = 1, \text{ for } n > 0$$
$$\quad\;\; 0, \text{ for } n < 0$$

3) Ramp Signal :-



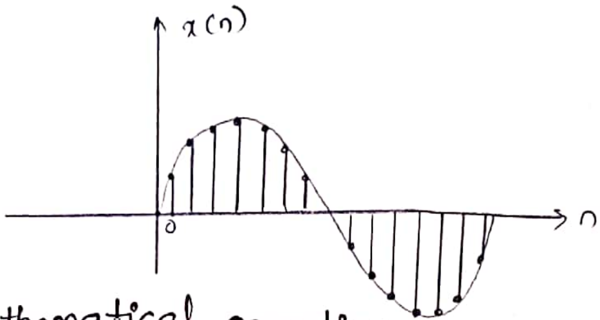
$$r(n) = n, \text{ for } n \geq 0$$
$$\quad\;\; 0, \text{ for } n < 0$$

4) Exponentially decayed signal.



$$x(n) = \bar{a}^{n}, \text{ for } n \geq 0$$
$$\quad\;\; 0, \text{ for } n < 0$$

5> Sinusoidal Signal



* Mathematical operations on DTS :-

Scaling — Amplitude scaling
— Time saling

1> Amplitude scaling

Let $x(n) \rightarrow A\,x(n)$

Ex:- $x(n) = 2$, for $n=-1$
5, for $n=0$
-3, for $n=1$
1, for $n=2$
2, for $n=3$

$x_1(n) = 5\,x(n)$

case i) if $A>1 \Rightarrow n=-1 \Rightarrow x_1(-1) = 5x(-1) = 5(2) = 10$

$n=0 \Rightarrow x_1(0) = 5x(0) = 5(5) = 25$

$n=1 \Rightarrow x_1(1) = 5x(1) = 5(-3) = -15$

$n=2 \Rightarrow x_1(2) = 5x(2) = 5(1) = 5$

$n=3 \Rightarrow x_1(3) = 5x(3) = 5(2) = 10$

Amplitude Increases.

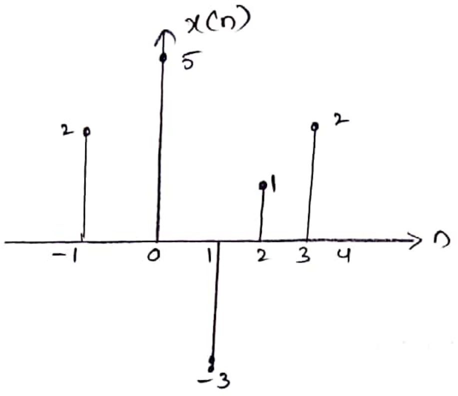
case ii) $A<1$.

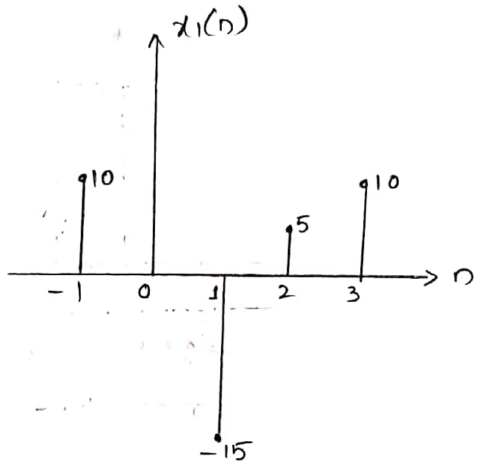
$x_2(n) = 0.2\,x(n)$

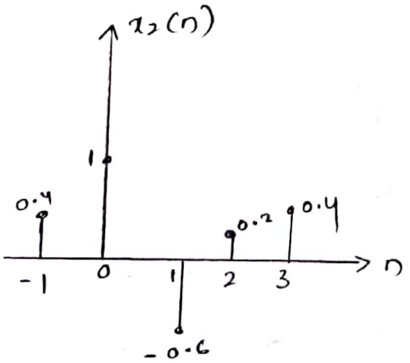
$n=-1 \Rightarrow x_1(-1) = 0.2x(-1)$
$= 0.2(2) = 0.4$

$n=0 \Rightarrow x_1(0) = 1$
$x_1(1) = 0.6$
$x_1(2) = 0.2$
$x_1(3) = 0.4$

Amplitude decreases.


x(n)
5
2
1
2
-1
0
1
2
3
4
n
-3


x1(n)
10
10
5
-1
0
1
2
3
n
-15


x2(n)
1
0.4
0.2
0.4
-1
0
1
2
3
n
-0.6

2) Time Scaling :-

$$x(n) \xrightarrow{TS} x(An)$$

case i) if $n>1 = x_1(n) = x(5n)$

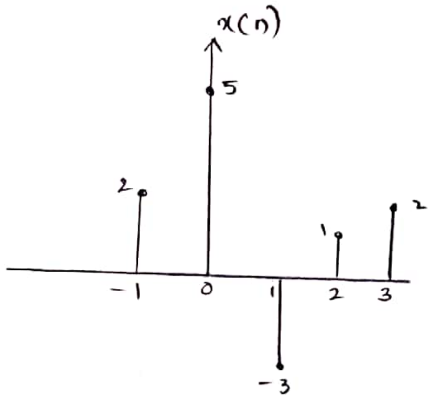
$$x_1(-1) = x(5(-1)) = x(-5) = 0$$

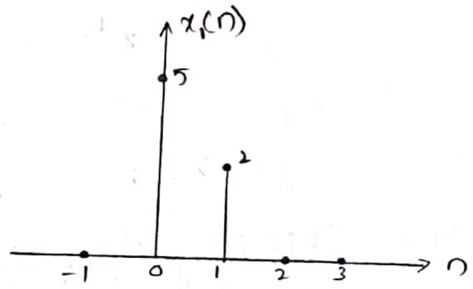
$$x_1(0) = x(5(0)) = x(0) = 5$$

$$x_1(1) = x(5(1)) = 0$$

$$x_1(2) = x(5(2)) = 0$$

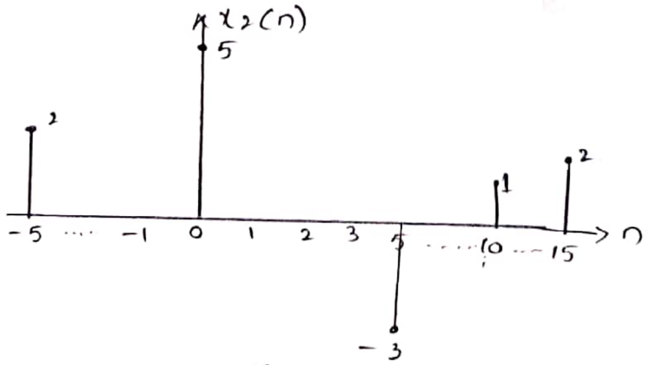
$$x_1(3) = x(5(3)) = x(15) = 0$$





Signal get's Compressed.

case ii) If $A<1 \Rightarrow x_2(n) = x(0.2n)$

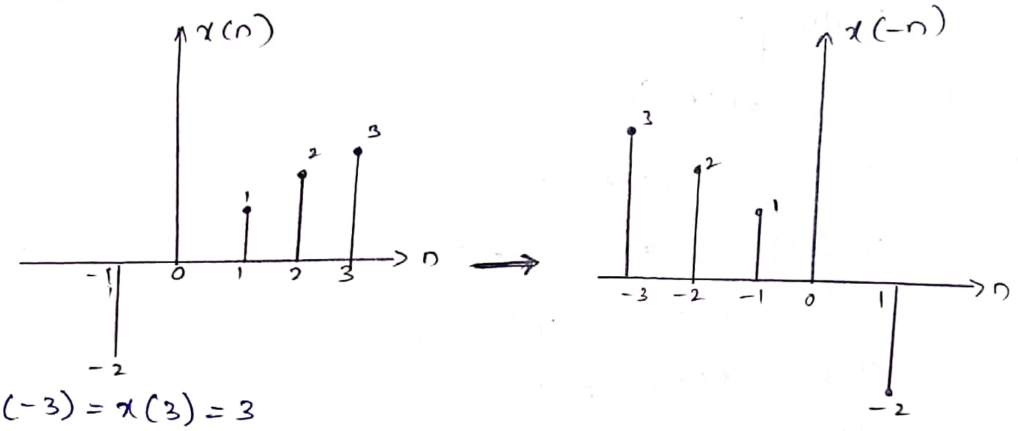


Signal gets Expanded.

→ **Folding (or) reflection:-**

$$n \longrightarrow -n$$

$$x(n) \rightarrow x(-n) = y_1(n)$$



$n = -3 \Rightarrow y_1(-3) = x(3) = 3$

$n = -2 \Rightarrow y_1(-2) = x(2) = 2$

$n = -1 \Rightarrow y_1(-1) = x(1) = 1$

$n = 0 \Rightarrow y_1(0) = x(0) = 0$

$n = 1 \Rightarrow y_1(1) = x(-1) = -2$

$n = 2 \Rightarrow y_1(2) = x(-2) = 0.$

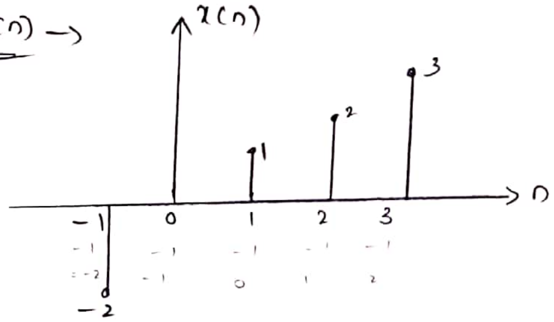
* Time Shifting Operation

Let $x(n) \rightarrow x(n-m)$ delayed shifting

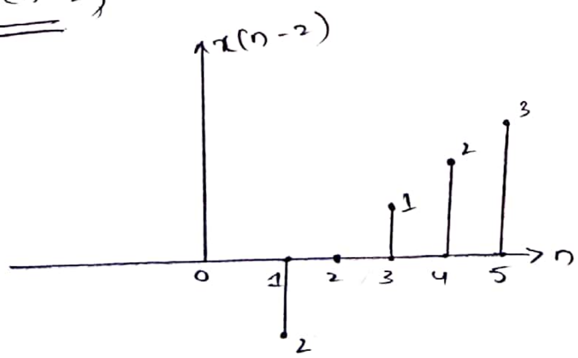
$x(n+m)$ Advanced.

1) Find $x(n-2)$ & $x(n+1)$

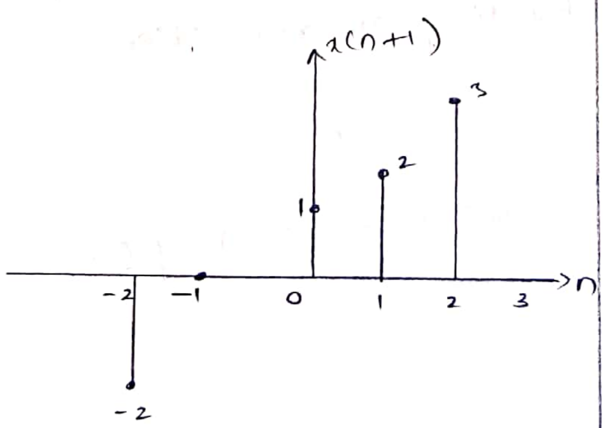
$x(n) \rightarrow$



$x(n-2)$



$x(n+1)$



Addition Operation:-

Let $x_1(n)$ & $x_2(n)$ are two signals

then $y(n) = x_1(n) + x_2(n)$

$$x_1(n) = \{-2, \underset{\substack{\uparrow \\ n=0}}{0}, 1, 2, 3\}$$

$$x_2(n) = \{\underset{\substack{\uparrow \\ n=0}}{1}, 2, 3, 4\}$$

$$\therefore y(n) = \{-2, \underset{\substack{\uparrow \\ n=0}}{1}, 3, 5, 7\}.$$

Multiplication Operation:-

Let $x_1(n)$ & $x_2(n)$ are two signals:

then $y(n) = x_1(n) \cdot x_2(n)$

$$x_1(n) = \{-2, \underset{\substack{\uparrow \\ n=0}}{0}, 1, 2, 3\}$$

$$x_2(n) = \{\underset{\substack{\uparrow \\ n=0}}{1}, 2, 3, 4\}$$

$$y(n) = \{0, \underset{\substack{\uparrow \\ n=0}}{0}, 2, 6, 12\}$$

* CLASSIFICATION OF DISCRETE TIME SIGNALS :-

i) Deterministic and non-deterministic signals

A signal $x(n)$ that can be expressed in mathematical form is called deterministic signal.

Ex:- Impulse, step, ramp, exponential etc.

A signal can't be expressed by mathematical representation is called non-deterministic signal

Ex:- noise, Amplifier etc

2) Periodic and Aperiodic Signal :-

A signal is said to be periodic if it satisfies

$$x(n) = x(n+N)$$

$$x(n) \neq x(n+N) \longrightarrow \text{Aperiodic}$$

where 'N' is fundamental period.

→ All periodic signals are power signals

→ All Aperiodic signals are energy signals.

3) Even (Symmetric) & odd (Asymmetric) signals

A signal is said to be Even if it satisfies the condition

$$x(n) = x(-n)$$

A signal is said to be odd signal if it satisfies the condition

$$x(n) = -x(-n)$$

* Even part of signal $= x_e(n) = \frac{1}{2}[x(n) + x(-n)]$

* odd part of signal $= x_o(n) = \frac{1}{2}[x(n) - x(-n)]$

4) Causal and Non-Causal Signal :-

A signal is said to be Causal signal if it is defined only for $n \geq 0$.

Non - Causal :-

A signal is defined either for $n \leq 0$ and $n > 0$

Anti Causal :-

A non-Causal signal which is defined for only $n \leq 0$ is called anti Causal.

5) Energy and power Signal :-

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2$$

$$P = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2$$

→ For Energy signals the energy to be finite i.e., $0 < E < \infty$ and power to be zero.

→ For power signals the energy to be Infinite and the power to be finite i.e., $0 < P < \infty$.

* Whether the following signals are periodic or not. If periodic find fundamental period.

1) $x(n) = \sin\left(\frac{6\pi}{7}n + 1\right)$

$x(n) = x(n+N)$

$$x(n+N) = \sin\left\{\frac{6\pi}{7}(n+N)+1\right\}$$

$$= \sin\left[\frac{6\pi}{7}n + 1 + \frac{6\pi}{7}N\right]$$

2π

$$= \sin\left(\frac{6\pi}{7}n+1\right)$$

where $\frac{6\pi}{7}N = 2\pi m$ where $m = 1, 2, 3$ are integers

$$N = \frac{7}{3}m$$

$$= \frac{7}{\cancel{3}} \times \cancel{3}$$

$$\boxed{N = 7}$$

∴ periodic

ii) $x(n) = \cos\left[\frac{n}{8} - \pi\right]$

$x(n+N) = \cos\left[\left(\frac{n+N}{8}\right) - \pi\right]$

$= \cos\left[\frac{n}{8} - \pi + \frac{N}{8}\right]$ 2π

$\frac{N}{8} = 2\pi m$

$N = 16\pi m$

where $m = 1, 2, 3$ integers

$N = 16\pi(1)$

$\boxed{N = 16\pi}$

$\therefore$ Not periodic

iii) $x(n) = e^{j7\pi n}$

$x(n+N) = e^{j7\pi(n+N)}$

$= e^{j7\pi n + j7\pi N}$

$= e^{j7\pi n} \cdot e^{j7\pi N}$

$= x(n)$ $\longrightarrow e^{j7\pi N} = 1$

$\cos(7\pi N) + j(\sin(7\pi N)) = 1$

$7\pi N = m2\pi$

$N = \frac{2}{7} m$

$\boxed{N = 2}$

$\therefore$ Periodic

2) Determine the following signals are even or odd if not find even and odd part of the signals.

i) $x(n) = a^n$

$x(-n) = a^{-n}$ $\quad\quad$ $-x(-n) = -a^{-n}$

$x(n) \neq x(-n)$ $\quad\quad$ $x(n) \neq -x(-n)$

Not even $\quad\quad$ not odd

→ Even part of signal

$$x_e(n) = \frac{1}{2}[x(n) + x(-n)]$$

$$= \frac{1}{2}[a^n + a^{-n}]$$

→ odd part of signal

$$x_o(n) = \frac{1}{2}[x(n) - x(-n)]$$

$$= \frac{1}{2}(a^n - a^{-n})$$

ii) $x(n) = \{\underset{\uparrow}{1}, 2, 3, -1\}$

$n=0$

$x(-n) = \{-1, 3, 2, \underset{\uparrow}{1}\}$

$n=0$

$-x(-n) = \{1, -3, -2, \underset{\uparrow}{-1}\}$

$n=0$

$x(n) = x(-n) \rightarrow$ Not even

$x(n) \neq x(-n) \rightarrow$ not odd

Even part of signal

$$x_e(n) = \frac{1}{2}[x(n) + x(-n)]$$

$$= \frac{1}{2}[-1, 3, 2, \underset{\uparrow}{2}, 2, 3, -1]$$

$$= \{0.5, 1.5, 1, \underset{\uparrow}{1}, 1, 1.5, -0.5\}$$

$$x_o(n) = \frac{1}{2}[x(n) - x(-n)]$$

$$= [0.5, -1.5, -1, \underset{\uparrow}{1}, 1.5, 1, -0.5)$$

iii) $x(n) = \frac{2}{a^n}$

$x(-n) = \frac{2}{a^{-n}} = 2a^n$

$-x(-n) = -\frac{2}{a^{-n}} = -2a^n$

$x(n) \neq x(-n) \rightarrow$ not even

$x(n) = -x(-n) \rightarrow$ not odd

Even $x_e(n) = \frac{1}{2}[x(n) + x(-n)]$

$= \frac{1}{2}[2a^{-n} + 2a^n] = a^{-n} + a^n$

$x_o(n) = \frac{1}{2}[2a^{-n} - 2a^n] = a^{-n} - a^n$

3) Determine whether the following signals are Energy or power signals.

i) $x(n) = \left(\frac{1}{3}\right)^n u(n)$

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2$$

$$E = \sum_{n=-\infty}^{\infty} \left|\left(\frac{1}{3}\right)^n u(n)\right|^2$$

$u(n) = 1$ for $n > 0$
0, for $n < 0$.

$$= \sum_{n=0}^{\infty} \left|\left(\frac{1}{3}\right)^n\right|^2$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^{2n} \Rightarrow \sum_{n=0}^{\infty} \left(\frac{1}{9}\right)^n = \frac{1}{1-\frac{1}{9}} = \frac{9}{8}$$

$$P = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} \left|\left(\frac{1}{3}\right)^n u(n)\right|^2$$

$\therefore \sum_{n=0}^{\infty} c^n = \frac{1}{1-c}$, $0<|c|<1$

$\sum_{n=0}^{N-1} c^n = \frac{1-c^N}{1-c}$

$$= \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} \left|\left(\frac{1}{3}\right)^n u(n)\right|^2$$

$$= \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=0}^{N} \left|\left(\frac{1}{3}\right)^n\right|^2$$

$$= \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=0}^{N} \left(\frac{1}{9}\right)^n$$

$$\lim_{N\to\infty} \frac{1}{2N+1}\left[\frac{\left(\frac{1}{9}\right)^{N}-1}{\frac{1}{9}-1}\right]$$

$$= \lim_{N\to\infty} \frac{1}{N\left[2+\frac{1}{N}\right]}\left[\frac{\left(\frac{1}{9}\right)^{N}-1}{\frac{1}{9}-1}\right]$$

$$= \frac{0}{2}\left[\frac{\infty}{\frac{1}{9}-1}\right] = \frac{0}{(\ )} = 0$$

$$P = 0$$

$\therefore$ It is Energy signal.

ii) $x(n) = u(n)$

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2$$

$$= \sum_{n=-\infty}^{\infty} |u(n)|^2$$

$$= \sum_{n=0}^{\infty} (1)^2 = 1+1+1+1+\cdots\cdots\infty$$

$$E = \infty$$

$$P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2$$

$$= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |1|^2$$

$$= \lim_{N\to\infty} \frac{1}{2N+1} [1+1+1+\cdots\cdots N]$$

$$= \lim_{N\to\infty} \frac{1}{2N+1} [N+1]$$

$$= \lim_{N\to\infty} \frac{1}{\cancel{N}\left[2+\frac{1}{N}\right]} \cancel{N}\left[1+\frac{1}{N}\right] \Rightarrow \lim_{N\to\infty} \frac{1}{2+\frac{1}{N}}$$

$$= \frac{1}{2} \text{ watts}$$

$\therefore$ It is power signal.

iii) $x(n) = \left(\frac{3}{8}\right)^n u(n)$

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2$$

$$E = \sum_{n=-\infty}^{\infty} \left|\left(\frac{3}{8}\right)^n u(n)\right|^2$$

$$= \sum_{n=0}^{\infty} \left|\left(\frac{3}{8}\right)^n\right|^2$$

$$= \sum_{n=0}^{\infty} \left(\frac{3}{8}\right)^{2n} \Rightarrow \sum_{n=0}^{\infty} \left|\frac{9}{64}\right|^n$$

$$= \frac{1}{1-\frac{9}{64}} = \frac{64}{64-9} = \frac{64}{55} = 1.16$$

$$\boxed{E = 1.16}$$

$$P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2$$

$$= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} \left|\left(\frac{3}{9}\right)^n u(n)\right|^2$$

$$= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=0}^{N} \left|\left(\frac{3}{9}\right)^n\right|^2$$

$$= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=0}^{N} \left(\frac{9}{64}\right)^n$$

$$= \lim_{N\to\infty} \frac{1}{2N+1} \left[\frac{\left(\frac{9}{64}\right)^{N-1}}{\frac{9}{64}-1}\right]$$

$$= \lim_{N\to\infty} \frac{1}{N\left[2+\frac{1}{N}\right]} \left[\frac{\left(\frac{9}{64}\right)^{N}-1}{\frac{9}{64}-1}\right]$$

$$\boxed{P = 0}$$

$\therefore$ It is a Energy signal.

* CLASSIFICATION OF DISCRETE TIME SYSTEMS :-

1) Static & Dynamic System :-

↓ (memory less)   ↓ (memory).

→ A static system is one if it output depends only on present

→ A system is said to be dynamic if it output depends on past input and future.

Ex:- $y(n) = \sum_{m=0}^{\infty} x(n-m)$

$y(n) = x(n) + x(n-1) + x(n-2) + \ldots\ldots$

$$\underset{\text{present}}{y(2)} = \underset{\text{present past}}{x(2)} + x(1) + \underset{\text{past}}{x(0)}$$

∴ It is a Dynamic system

2) Time Variant and Time Invariant System :-

$$x(n) \longrightarrow \underset{\text{Delay}}{\boxed{z^{-m}}} \xrightarrow{x(n-m)} \underset{\text{system}}{\boxed{H}} \longrightarrow y_1(n)$$

$$x(n) \longrightarrow \underset{\text{system}}{\boxed{H}} \xrightarrow{y(n)} \underset{\text{Delay}}{\boxed{z^{-m}}} \longrightarrow y_2(n)$$

$y_1(n) = y_2(n) = \text{TIV}$

$y_1(n) \neq y_2(n) = \text{TV}$

Ex:-

i) $y(n) = x(n) + x(n-1)$

$$x(n) \longrightarrow \underset{\text{Delay}}{\boxed{z^{m}}} \xrightarrow{x(n-m)} \boxed{\text{system}} \longrightarrow y_1(n) = x(n-m) + x(n-m-1)$$

$$x(n) \longrightarrow \boxed{\text{system}} \xrightarrow{y(n)} \underset{\text{Delay}}{\boxed{z^{-m}}} \longrightarrow y_2(n) = x(n-m) + x(n-m-1)$$

$= x(n) + x(n-1)$

$y_1(n) = y_2(n) \rightarrow \text{TIV}$

ii) $y(n) = n\,x(n)$

$$x(n) \longrightarrow \underset{\text{Delay}}{\boxed{z^{m}}} \xrightarrow{x(n-m)} \boxed{\text{system}} \longrightarrow y_1(n) = (n-m)\,x(n-m)$$

$$x(n) \longrightarrow \boxed{\text{system}} \xrightarrow[= n x(n)]{y(n)} \underset{\text{Delay}}{\boxed{z^{m}}} \longrightarrow y_2(n) = n\,x(n-m)$$

$y_1(n) \neq y_2(n) \Rightarrow TV$

iii) $y(n) = x^{2}(n)$

$$x(n) \longrightarrow \underset{\text{Delay}}{\boxed{z^{m}}} \xrightarrow{x(n-m)} \boxed{\text{system}} \longrightarrow y_1(n) = x^{2}(n-m)$$

$$x(n) \longrightarrow \boxed{\text{system}} \xrightarrow[= x^{2}(n)]{y(n)} \underset{\text{Delay}}{\boxed{z^{m}}} \longrightarrow y_2(n) = x^{2}(n-m)$$

$y_1(n) = y_2(n) = TIV$

iv $y(n) = x(n^{2})$

$$x(n) \longrightarrow \underset{\text{Delay}}{\boxed{z^{m}}} \xrightarrow{x(n-m)} \boxed{\text{system}} \longrightarrow y_1(n) = x(n-m)^{2}$$

$$x(n) \longrightarrow \boxed{\text{system}} \xrightarrow[= x(n^{2})]{y(n)} \underset{\text{Delay}}{\boxed{z^{m}}} \longrightarrow y_2(n) = x(n^{2}-m)$$

$y_1(n) \neq y_2(n) \Rightarrow TV$

v) $y(n) = e^{x(n)}$

$$x(n) \longrightarrow \underset{\text{Delay}}{\boxed{z^{m}}} \xrightarrow{x(n-m)} \boxed{\text{system}} \longrightarrow y_1(n) = e^{x(n-m)}$$

$$x(n) \longrightarrow \boxed{\text{system}} \xrightarrow[= e^{x(n)}]{y(n)} \underset{\text{Delay}}{\boxed{z^{m}}} \longrightarrow y_2(n) = e^{x(n-m)}$$

$y_1(n) = y_2(n) = TIV$

vi) $y(n) = x(-n)$

$$x(n) \longrightarrow \underset{\text{Delay}}{\boxed{z^{m}}} \xrightarrow{x(n-m)} \boxed{\text{system}} \longrightarrow y_1(n) = x(-(n-m))$$

$$x(n) \longrightarrow \boxed{\text{system}} \xrightarrow[= x(-n)]{y(n)} \underset{\text{Delay}}{\boxed{z^{m}}} \longrightarrow y_2(n) = x(-n-m)$$

$y_1(n) \neq y_2(n)$
$= TV$

vii) $y(n) = n x^{r}(n)$

$x(n) \rightarrow \boxed{\overset{\text{Delay}}{z^{m}}} \xrightarrow{x(n-m)} \boxed{\text{system}} \rightarrow y_1(n) = (n-m) x^{r}(n-m)$

$x(n) \rightarrow \boxed{\text{system}} \xrightarrow[= n x^{r}(n)]{y(n)} \boxed{\overset{\text{Delay}}{z^{m}}} \rightarrow y_2(n) = n x^{r}(n-m)$

$y_1(n) \neq y_2(n)$

TV

viii) $y(n) = x(n) + c$

$x(n) \rightarrow \boxed{\overset{\text{Delay}}{z^{m}}} \xrightarrow{x(n-m)} \boxed{\text{system}} \rightarrow y_1(n) = x(n-m) + c.$

$x(n) \rightarrow \boxed{\text{system}} \xrightarrow[= x(n)+c]{y(n)} \boxed{\overset{\text{Delay}}{z^{m}}} \rightarrow y_2(n) = x(n-m) + c$

$y_1(n) = y_2(n)$

TIV

ix) $\sum_{k=0}^{m} b_1 x(n-k) - \sum_{k=1}^{N} a_k y(n-k)$

$x(n) \rightarrow \boxed{\overset{\text{Delay}}{z^{m}}} \xrightarrow{x(n-m)} \boxed{\text{system}} \rightarrow y_1(n) = \sum_{k=0}^{M} b_1 x((n-m)-k) - \sum_{k=1}^{N} a_k y((n-m)-k)$

$x(n) \rightarrow \boxed{\text{system}} \xrightarrow{y(n)} \boxed{\overset{\text{Delay}}{z^{m}}} \rightarrow y_2(n) = \sum_{k=0}^{M} b_1 x(n-k-m) - \sum_{k=1}^{N} a_k y(n-k-m)$

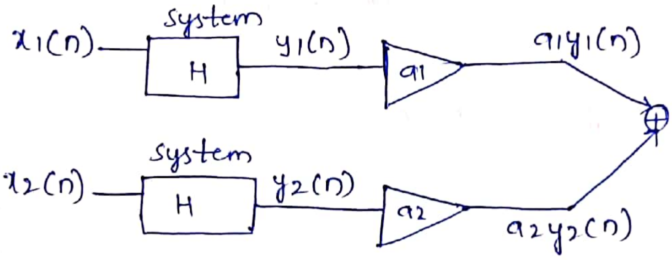
$= \sum_{k=0}^{m} b_1 x(n-k) - \sum_{k=1}^{N} a_k y(n-k)$

$y_1(n) = y_2(n)$

TIV

3> Linear and Non-linear Systems:-

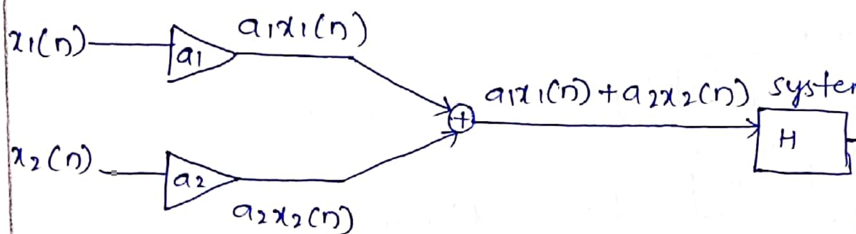
Let $x_1(n)$ & $x_2(n)$ are two systems



$y'(n) = a_1y_1(n) + a_2y_2(n)$

or

$a_1H[y_1(n)] + a_2H[y_2(n)]$

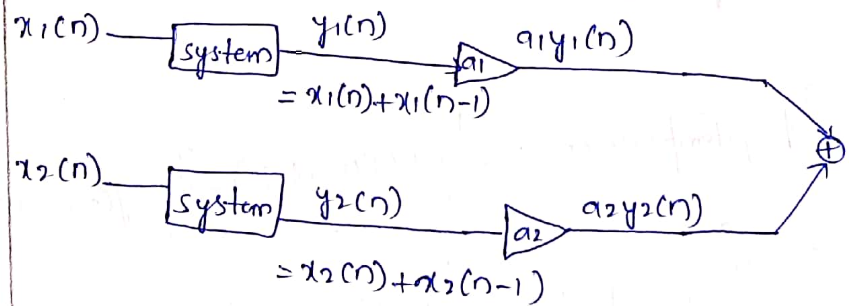


$y''(n) = H[a_1x_1(n) + a_2x_2(n)]$

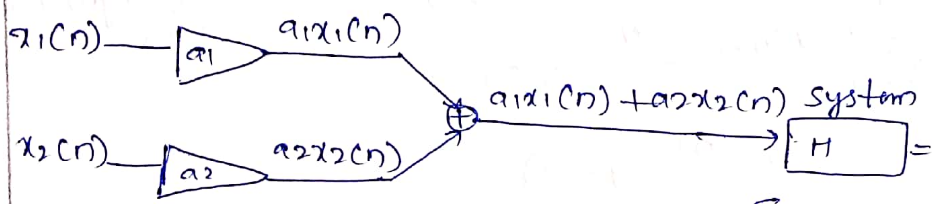
$y'(n) = y''(n) \rightarrow$ linear system

$y'(n) \neq y''(n) \rightarrow$ Non-linear system.

1) $y(n) = x(n) + x(n-1)$



$y'(n) = a_1[x_1(n) + x_1(n-1)]$
$+ a_2[x_2(n) + x_2(n-1)]$



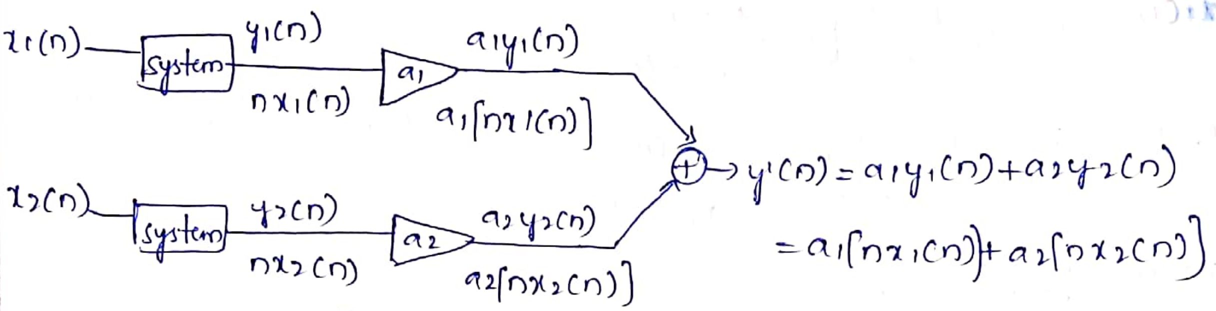
$= y''(n) =$

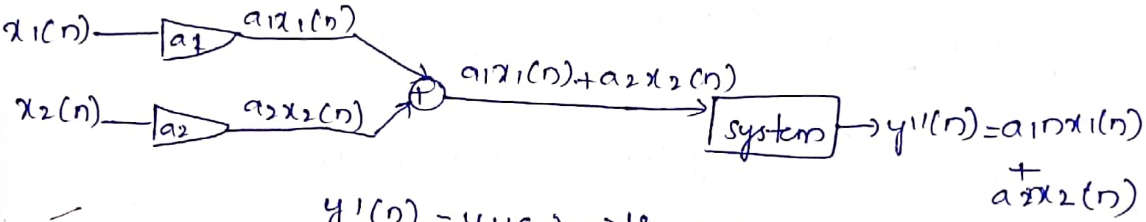
$\{a_1x_1(n) + a_2x_2(n)\} + \{a_1x_1(n-1)$
$+ a_2x_2(n-1)\}$

$y'(n) = y''(n)$

$\therefore$ linear

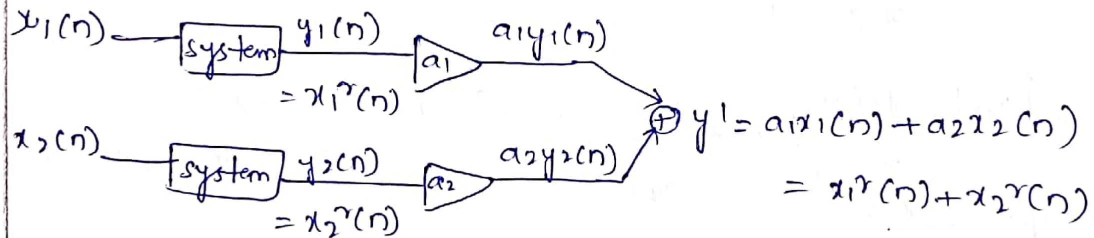
2) $y(n) = n x(n)$

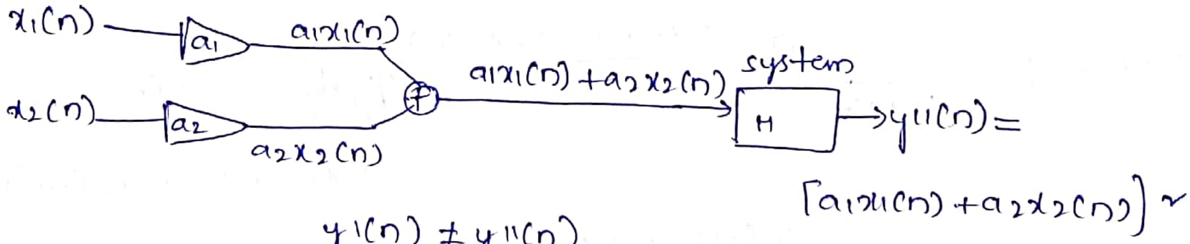




$y'(n) = y''(n) \rightarrow$ linear

3) $y(n) = x^2(n)$

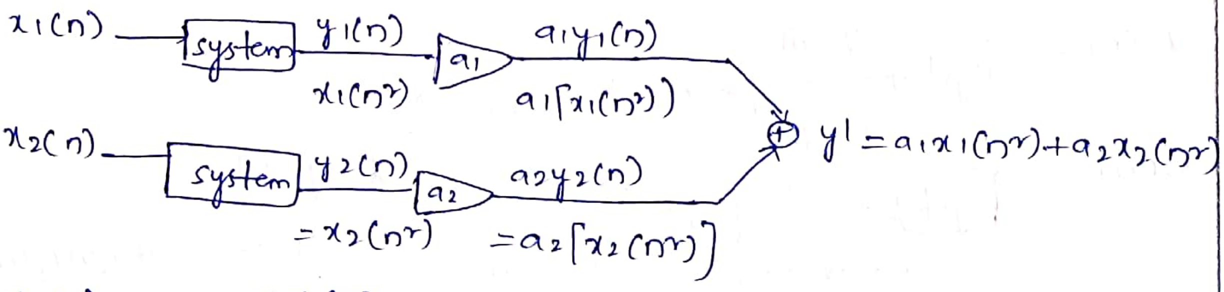


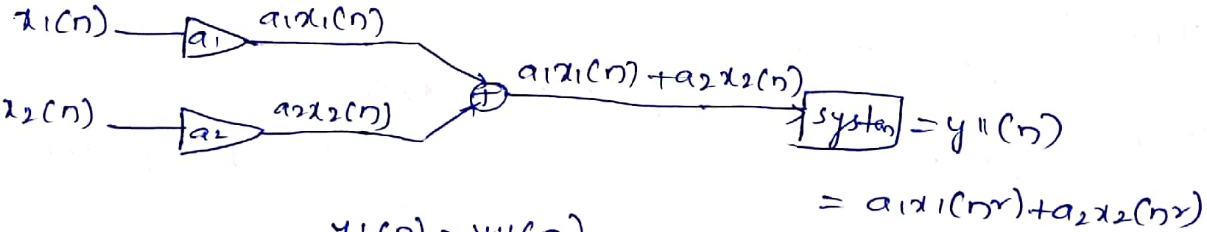


$y'(n) \neq y''(n)$

$\therefore$ Non-linear

4) $y(n) = x(n^2)$

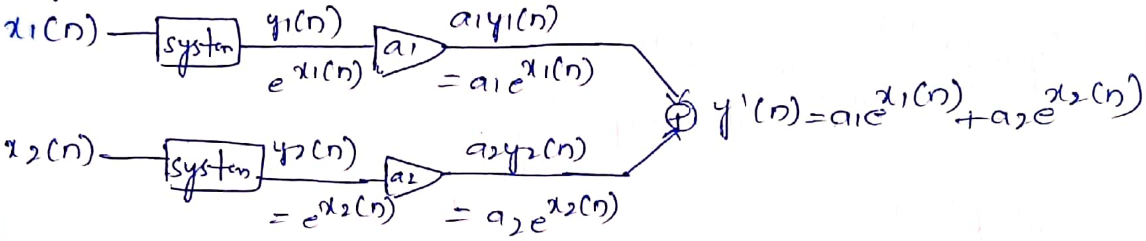


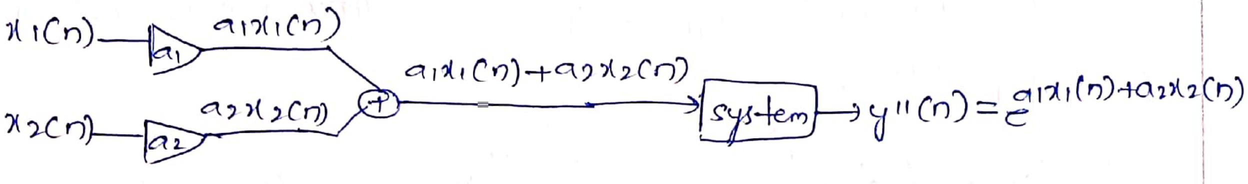


$y'(n) = y''(n)$

$\therefore$ Linear.

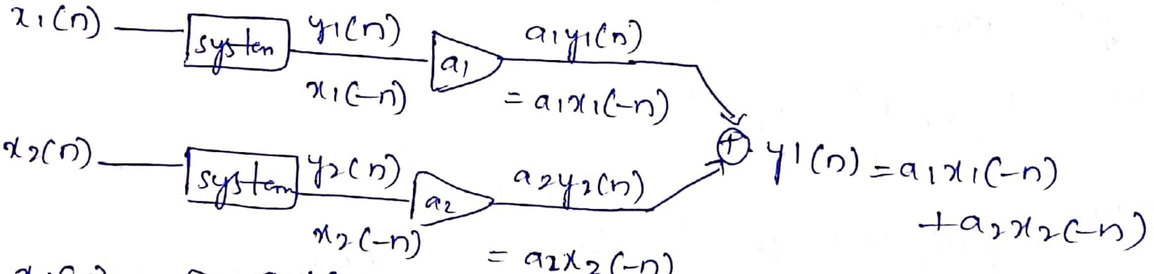
5> $y(n) = e^{x(n)}$

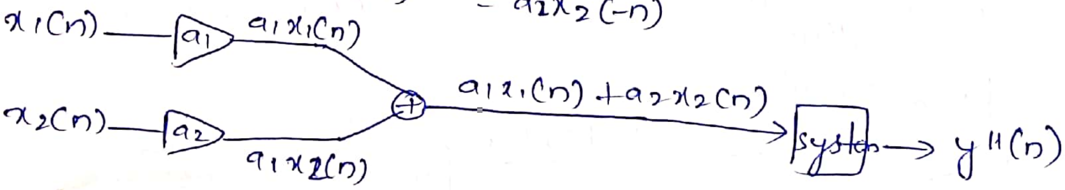




$y'(n) \neq y''(n) \rightarrow$ non linear.

6> $y(n) = x(-n)$

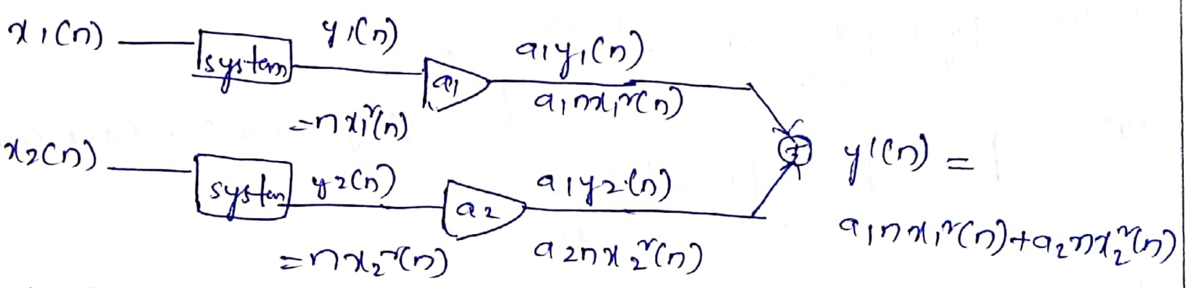


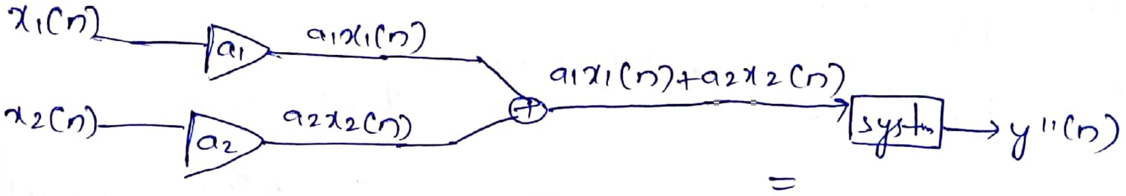


$= a_1 x_1(-n) + a_2 x_2(-n)$

$y'(n) = y''(n)$ $\therefore$ linear

7> $y(n) = n x^2(n)$





$= n[a_1 x_1^2(n) + a_2 x_2^2(n)]$

$y'(n) \neq y''(n)$ $\therefore$ Non linear

4) Causal and Non - Causal system:-

A system is said to be Causal if it output of system at any time n depends only on the present input, past input but does not depends on turture inputs and outputs.

→ If the system output at any time 'n' depends on furture input and output then the system is called non - Causal.

Ex:- 1) $y(n) = x(n) + x(n-1)$

$n = 0 \Rightarrow y(0) = x(0) + x(-1)$

$n = 1 \Rightarrow y(1) = x(1) + x(0)$

$n = -1 \Rightarrow y(-1) = x(-1) + x(-2)$

The system is Causal.

2) $y(n) = n x(n)$

$n = 0 \Rightarrow y(0) = 0$

$n = 1 \Rightarrow y(1) = x(1)$

$n = -1 \Rightarrow y(-1) = -x(-1)$

The system is Causal

3) $y(n) = \sum_{n=-\infty}^{\infty} x(n)$

$n = 0 \Rightarrow y(0) = \sum_{n=-\infty}^{\infty} x(0)$

$n = 1 \Rightarrow y(1) = \sum_{n=-\infty}^{\infty} x(1)$

$n = -1 \Rightarrow y(-1) = \sum_{n=-\infty}^{\infty} x(-1)$

The system is Causal

5) Stable and Unstable system:-

For every bounded input the output response is bounded then system is said to be stable system. or "BIBO" stable.

$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty$$

→ Test the stability of the following systems.

1) $y(n) = \cos x(n)$

The system is bounded in $\{-1, 1\}$

Hence it is stable.

2) $y(n) = x(n)$

case i) $n=0 \Rightarrow y(0) = 0 \rightarrow$ stable

$n = \infty \Rightarrow y(\infty) = \infty \rightarrow$ Unstable

→ Determine the range of values of a and b for the stability of LTI system with impulse response.

$h(n) = b^n$, for $n < 0$

a^n, for $n > 0$

We know that $\sum_{n=-\infty}^{\infty} |h(n)| < \infty$

$$= \sum_{n=-\infty}^{-1} |h(n)| + \sum_{n=0}^{\infty} |h(n)| \Rightarrow \sum_{n=-\infty}^{-1} b^n + \sum_{n=0}^{\infty} a^n$$

$$= \sum_{n=1}^{\infty} b^{-n} + \sum_{n=0}^{\infty} a^n$$

$$= \sum_{n=1}^{\infty} (b^{-1})^n + \sum_{n=0}^{\infty} a^n$$

$$= \sum_{n=0}^{\infty} (b^{-1})^n - 1 + \frac{1}{1-a}$$

$0 < |b^{-1}| < 1$

$$= \frac{1}{1-b^{-1}} - 1 + \frac{1}{1-a}$$

$\left|\frac{1}{b}\right| < 1$ | $0 < |a| < 1$

$\boxed{b > 1}$ | $\boxed{|a| < 1}$

Sampling And Sampling theorem:-

Sampling is a process of converting a time domain signal into discrete signal

$$x(t) \rightarrow x(nTs) — x(n)$$

Statement:-

A continuous time signal that can be represented in its samples and can be recovered back. When the sampling frequency (f_s) is greater than or equal to twice the highest frequency component of its sample

$$\therefore \boxed{f_s \geq 2 f_m}$$

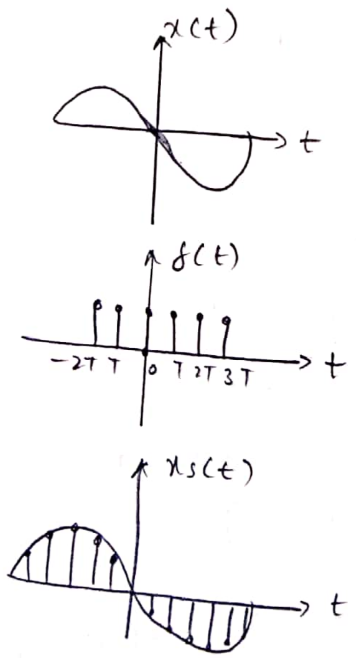
Ex:- If we have multiples of frequency

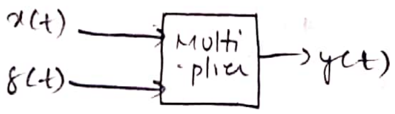
$$x(t) = \cos 2\pi t + \cos 4\pi t + \cos 6\pi t$$

$\omega_1 = 2\pi$	$\omega_2 = 4\pi$	$\omega_3 = 6\pi$	$[2\pi f_1 = 2\pi]$
$f_1 = 1$	$f_2 = 2$	$f_3 = 3$	$f_1 = 1$

$$f_m = \max(f_1, f_2, f_3)$$
$$= 3 Hz$$

$$f_s = 2(3) = 6 Hz \Rightarrow T_s = \frac{1}{f_s} = \frac{1}{6} sec$$





$$y(t) = x(t)\cdot \delta(t) \rightarrow ①$$

F.S of $\delta(t)$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\omega_s t + b_n \sin n\omega_s t) \rightarrow ②$$

$\rightarrow$ $$a_0 = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\,dt$$

$$= \frac{2}{T_s}\int_{-T/2}^{T/2} \delta(t)\,dt = \frac{2}{T_s}\delta(0) = \frac{2}{T_s}$$

$\rightarrow$ $$a_n = \frac{2}{T_s}\int_{-T/2}^{T/2} \overset{\delta(t)}{x(t)} \cos n\omega_s t\,dt$$

$$= \frac{2}{T_s}\delta(0)\cos(0) = \frac{2}{T_s}$$

$\rightarrow$ $$b_n = \frac{2}{T_s}\int_{-T/2}^{T/2} \delta(t)\sin n\omega_s t\,dt = 0$$

Sub in ②

$$\delta(t) = \frac{1}{T_s} + \sum_{n=1}^{\infty} \frac{2}{T_s}\cos n\omega_s t$$

$\therefore$ eq'n ①

$$y(t) = x(t)\cdot\left[\frac{1}{T_s} + \sum_{n=1}^{\infty}\frac{2}{T_s}\cos n\omega_s t\right]$$

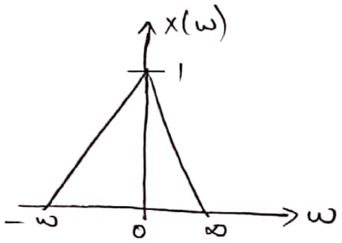
$$y(t) = \frac{1}{T_s}\left[x(t) + \sum_{n=1}^{\infty} x(t)\,2\cos n\omega_s t\right]$$

$$y(t) = \frac{1}{T_s}\left[x(t) + x(t)2\cos\omega_s t + x(t)2\cos 2\omega_s + x(t)2\cos 3\omega_s t + \cdots\right]$$

$$= \frac{1}{T_s}\left[x(t) + x(t)e^{-j\omega_s t} + x(t)e^{j\omega_s t} + x(t)e^{-j2\omega_s t} + x(t)e^{j2\omega_s t} + \cdots\right]$$

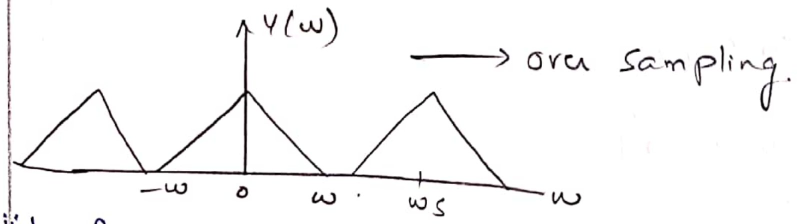
$\therefore Y(\omega) = \frac{1}{T_s}\left[X(\omega) + X(\omega+\omega_s) + X(\omega-\omega_s) + X(\omega+2\omega_s) + X(\omega-2\omega_s) + \dots \right.$

$$\boxed{Y(\omega) = \frac{1}{T_s} \sum_{n=-\infty}^{\infty} X(\omega - n\omega_s)}$$

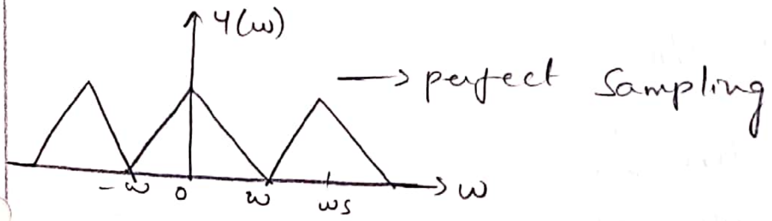


$f_s \geq 2f_m$

i) $f_s > 2f_m$



ii) $f_s = 2f_m$



iii) $f_s < 2f_m$

Types of Sampling

1) Impluse

2) Natural

3) Flat top

1) Find the nyquist sampling period of $\left(\frac{\sin at}{\pi t}\right)^2$

Nyquist rate = $f_s = 2f_m$

" interval = $T_s = \frac{1}{f_s} = \frac{1}{2f_m}$

Sol:- $\left(\frac{\sin at}{\pi t}\right)^2$

$= \frac{2}{2}\left(\frac{\sin at}{\pi t}\right)\left(\frac{\sin at}{\pi t}\right)$

$= \frac{1}{2\pi^2 t^2}\left[\cos(0) - \cos(2at)\right]$

$= \frac{1}{2\pi^2 t^2}\left[1 - \cos(2at)\right]$

$\omega = 2a$

$2\pi f_m = 2a$

$f_m = a/\pi$

$T_s = \frac{1}{2f_m} = \frac{\pi}{2a}$

2) $x(t) = 1 + \cos 2000\pi t + \sin 4000\pi t$

$\omega_1 = 2000\pi$

$2\pi f_1 = 2000\pi$

$f_1 = 100\,Hz$

$\omega_2 = 4000\pi$

$2\pi f_2 = 4000\pi$

$f_2 = 2000\,Hz$

$f_m = \text{Max}[f_1, f_2] = 2000\,Hz$

Nyquist rate $= f_s = 2f_m = 2 \times 2000 = 4000\,Hz$

" interval $= T_s = \frac{1}{2f_m} = \frac{1}{4000}$ sec//.

DISCRETE FOURIER SERIES

- A periodic discrete time signal with fundamental period N can be decomposed into N harmonically related frequency components. The summation of frequency components gives the Fourier series representation of periodic discrete time signal, in which the discrete time signal is represented as a function of frequency, ω. The Fourier series of discrete time signal is called Discrete time Fourier Series (DTFS)

- By letting the fundamental period N to infinity, and this Fourier method of representing non periodic discrete time signals as a function of discrete time frequency (ω) is called Fourier Transform of discrete time signals or Discrete Time Fourier Transform (DTFT).

- Fourier Series representation can be obtained only for periodic discrete time signals.

- Fourier Transform technique can be applied to both periodic and non periodic signals to obtain the frequency domain representation of discrete time signals.

Fourier Series of Discrete Time Signals (Discrete Time Fourier Series)

The Fourier Series of discrete time periodic signal $x(n)$ with periodicity N is defined as

$$\boxed{x(n) = \sum_{k=0}^{N-1} C_k e^{\frac{j2\pi kn}{N}} = \sum_{k=0}^{N-1} C_k e^{j\omega_0 kn} = \sum_{k=0}^{N-1} C_k e^{j\omega_k n}}$$

where C_k = Fourier Coefficients

ω_0 = fundamental frequency of $x(n)$

$\omega_k = \frac{2\pi k}{N}$ = k^{th} harmonic frequency of $x(n)$

$C_k e^{j\omega_k n}$ = k^{th} harmonic component of $x(n)$.

The Fourier Coefficients C_k for $K = 0, 1, 2 \ldots N-1$ can be evaluated by using below eqn.

$$\boxed{C_k = \frac{1}{N} \sum_{n=0}^{N-1} x(n) e^{\frac{-j2\pi kn}{N}}, \text{ for } k = 0, 1, 2, 3, \ldots, N-1}$$

	CTS	DTS
Periodic	CTFS	DTFS
Aperiodic	CTFT (or) FT	DTFT

Discrete Time Fourier Transform (DTFT)

Let $x(n)$ = Discrete time signal

$X(e^{j\omega})$ = FT of $x(n)$

$\therefore$ FT of $x(n)$ is defined as

$$\boxed{DTFT\{x(n)\} = X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}}$$

Frequency Spectrum

$$X(e^{j\omega}) = X_r(e^{j\omega}) + jX_i(e^{j\omega})$$

magnitude spectrum $= |X(e^{j\omega})| = \sqrt{X_r^2(e^{j\omega}) + X_i^2(e^{j\omega})}$

phase spectrum $= \angle X(e^{j\omega}) = \tan^{-1}\left[\frac{X_i(e^{j\omega})}{X_r(e^{j\omega})}\right]$

Inverse DTFT:

$$x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega \quad , \text{ for } n = -\infty \text{ to } \infty.$$

$$x(n) \underset{IFT}{\overset{FT}{\rightleftharpoons}} X(e^{j\omega})$$

$f_s \to$ frequency of pulse train

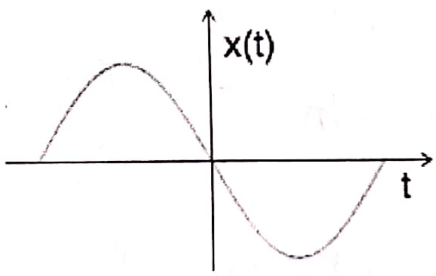
SIGNALS SAMPLING THEOREM

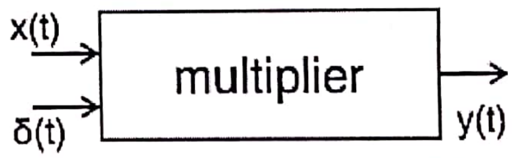
Statement: A continuous time signal can be represented in its samples and can be recovered back when sampling frequency f_s is greater than or equal to the twice the highest frequency component of message signal. i. e.

$$f_s \geq 2 f_m.$$

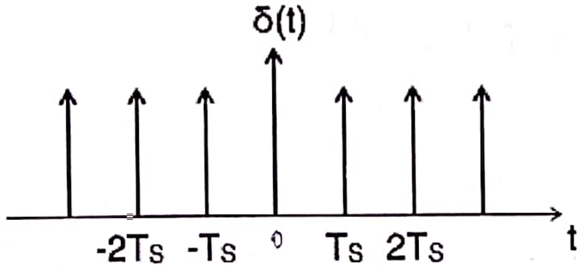
Proof: Consider a continuous time signal xt. The spectrum of xt is a band limited to f_m Hz i.e. the spectrum of xt is zero for $|\omega| > \omega_m$.

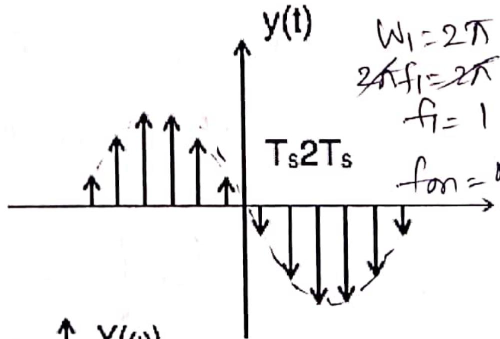
Sampling of input signal xt can be obtained by multiplying xt with an impulse train δt of period T_s. The output of multiplier is a discrete signal called sampled signal which is represented with yt in the following diagrams:





$x(t) = \sin 2\pi t + \sin 3\pi t + \sin 4\pi t$





$W_1 = 2\pi$ | $W_2 = 3\pi$ | $W_3 = 4\pi$

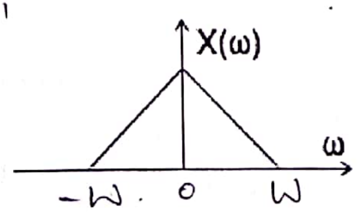
$2\pi f_1 = 2\pi$

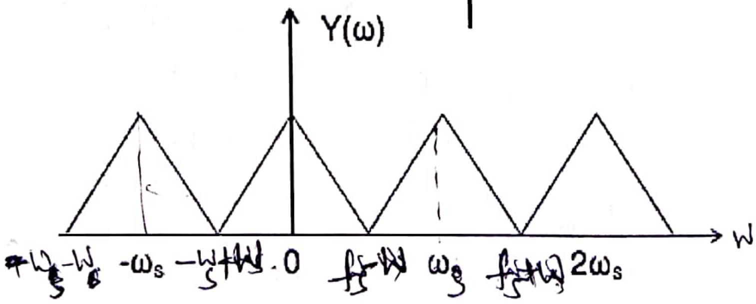
$f_1 = 1$ | $f_2 = 1.5$ | $f_3 = 2$

$f_m = \max(f_1, f_2, f_3)$

$= 2$

$X(\omega) = 0, \ \omega > \omega_m$



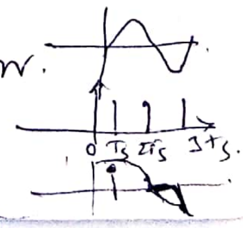


Here, you can observe that the sampled signal takes the period of impulse. The process of sampling can be explained by the following mathematical expression:

$f_s \to$ Nyquist rate.

$T_s \uparrow \ \downarrow f_s < 2 f_m$.

$T_s \downarrow \ \uparrow f_s > 2 f_m$



Nyquist rate

$f_s \geq 2f_m \Rightarrow \boxed{f_s = 2f_m}$

Nyquist ~~rate~~ Interval $= T_s = \frac{1}{2f_m}$.

① . Find Nyquist Sampling interval of $\left(\frac{\sin at}{\pi}\right)^2$

a) $2a/\pi$
b) π/a
c) a/π
d) $\pi/2a$

Sol $x(t) = \left(\frac{\sin^2 at}{\pi}\right)^2$

$W = 2a \Rightarrow W = 2a$

$2\pi f_m = 2a$

$f_m = a/\pi$

$\Rightarrow T_s = \frac{1}{2f_m}$

$\boxed{T_s = \frac{\pi}{2a}}$

② . Det. Nyquist rate & Nyquist ~~rate~~ interval.

a) $x(t) = 1 + \cos 2000\pi t + \sin 4000\pi t$

$W_{m1} = 2000\pi \Rightarrow f_{m1} = 1000$

$f_{m2} = 2000.$

$f_m\ max = (2000\ Hz \Rightarrow$ $T_s = \frac{1}{2f_m} = \frac{1}{4000} = 0.25ms$

$f_s = 2f_m = 4000\ Hz$

b). $x(t) = \frac{\sin 20t}{\pi t} + \frac{\sin 40t}{\pi t}$

$W_{m1} = 20 \Rightarrow f_{m1} = 10/\pi$

$W_{m2} = 40 \Rightarrow f_{m2} = \frac{20}{\pi}$.

$f_{max} = \frac{20}{\pi} Hz = f_m$

$f_s = 2f_m = \frac{40}{\pi} Hz.$

$T_s = \frac{\pi}{40}$ sec 0.0785 sec

c) $x(t) = \sin^2(100\pi t)$

$W = 200\pi$

$f_m = 200$

$W_m = 2(100\pi) \Rightarrow 2\pi f_m = 200\pi$

$f_m = 100$. $f_s = 2f_m = 200\ Hz$

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Sampled signal $y(t) = x(t).\,\delta(t)$ ...... (1)

The trigonometric Fourier series representation of δt is given by

$\delta(t) = a_0 + \Sigma_{n=1}^{\infty}(a_n \cos n\omega_s t + b_n \sin n\omega_s t)$ ...... (2)

Where $a_0 = \frac{1}{T_s}\int_{\frac{-T}{2}}^{\frac{T}{2}} \delta(t)dt = \frac{1}{T_s}\delta(0) = \frac{1}{T_s}$

$$a_n = \frac{2}{T_s}\int_{\frac{-T}{2}}^{\frac{T}{2}} \delta(t)\cos n\omega_s\, dt = \frac{2}{T_2}\delta(0)\cos n\omega_s 0 = \frac{2}{T}$$

$$b_n = \frac{2}{T_s}\int_{\frac{-T}{2}}^{\frac{T}{2}} \delta(t)\sin n\omega_s t\, dt = \frac{2}{T_s}\delta(0)\sin n\omega_s 0 = 0$$

Substitute above values in equation 2.

$\therefore\ \delta(t) = \frac{1}{T_s} + \Sigma_{n=1}^{\infty}(\frac{2}{T_s}\cos n\omega_s t + 0)$

Substitute δt in equation 1.

$\rightarrow y(t) = x(t).\,\delta(t)$

$$= x(t)[\frac{1}{T_s} + \Sigma_{n=1}^{\infty}(\frac{2}{T_s}\cos n\omega_s t)]$$

$$= \frac{1}{T_s}[x(t) + 2\Sigma_{n=1}^{\infty}(\cos n\omega_s t)x(t)]$$

$$y(t) = \frac{1}{T_s}[x(t) + 2\cos\omega_s t.\,x(t) + 2\cos 2\omega_s t.\,x(t) + 2\cos 3\omega_s t.\,x(t) \ldots\ldots]$$

$\cos w_0 t = \frac{e^{-jw_0 t} + e^{jw_0 t}}{2}$

$F\{e^{-jw_0 t}x(t) + e^{jw_0 t}x(t)\}$

$X(w+w_0) + X(w-w_0).$

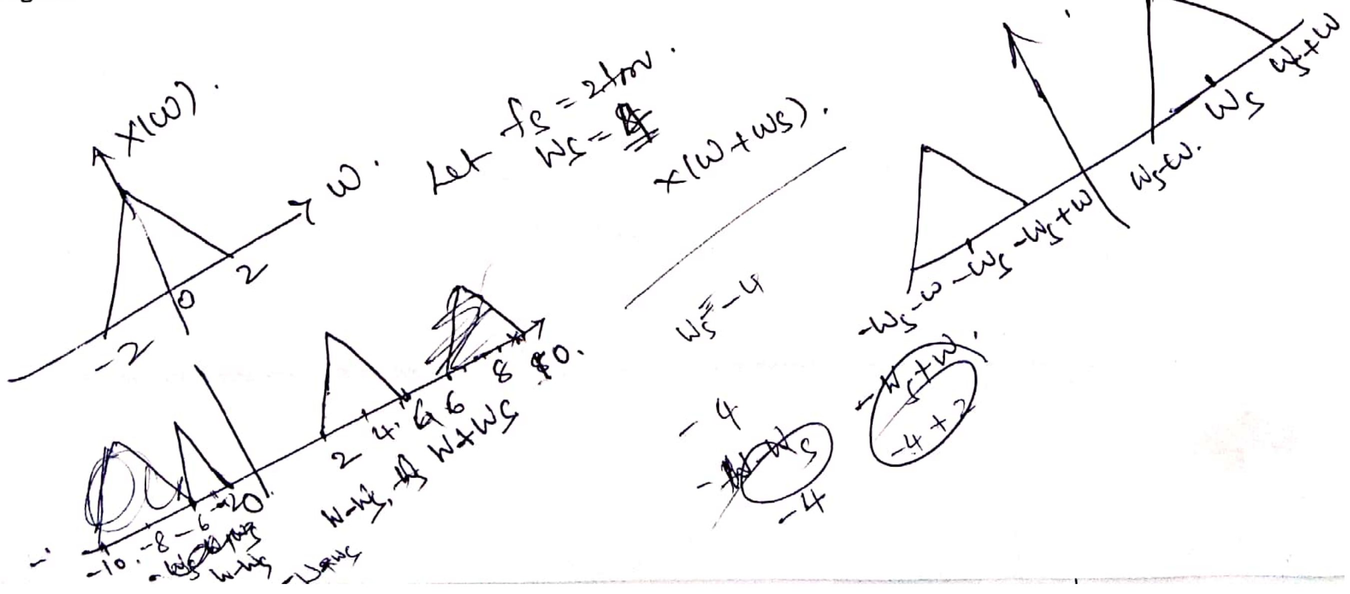
Take Fourier transform on both sides.

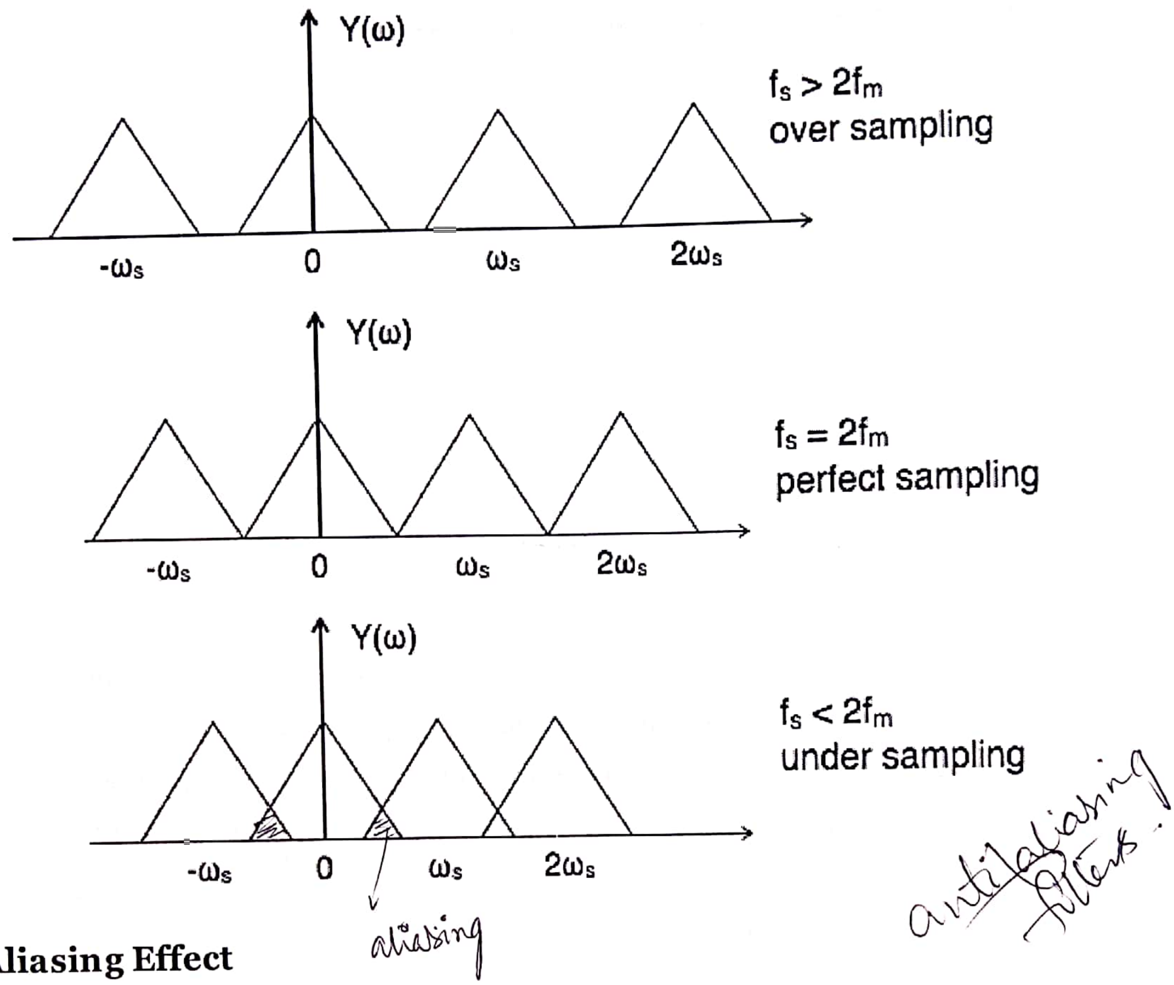
$Y(\omega) = \frac{1}{T_s}[X(\omega) + X(\omega - \omega_s) + X(\omega + \omega_s) + X(\omega - 2\omega_s) + X(\omega + 2\omega_s) + \ldots]$

$\therefore\ Y(\omega) = \frac{1}{T_s}\Sigma_{n=-\infty}^{\infty} X(\omega - n\omega_s)$ $\quad$ *where* $n = 0, \pm1, \pm2, \ldots$

To reconstruct x(t) you must recover input signal spectrum $X\omega$ from sampled signal spectrum $Y\omega$ which is possible when there is no overlapping between the cycles of $Y\omega$.

Possibility of sampled frequency spectrum with different conditions is given by the following diagrams:





Aliasing Effect

The overlapped region in case of under sampling represents aliasing effect, which can be removed by

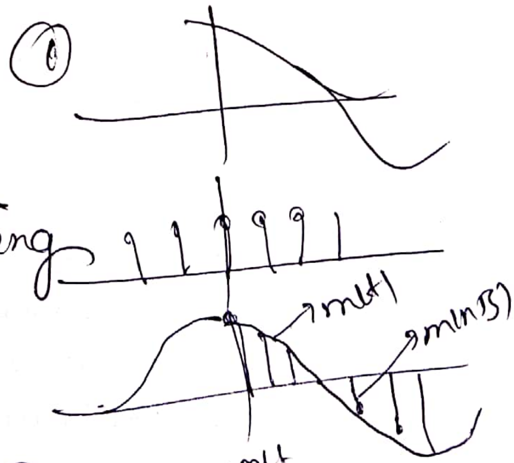
- considering $f_s > 2f_m$
- By using anti aliasing filters.

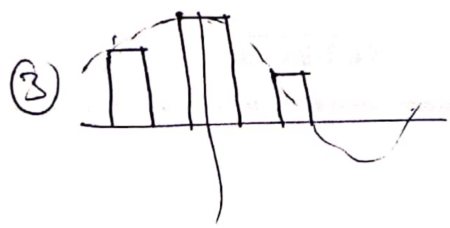
Sampling Techniques

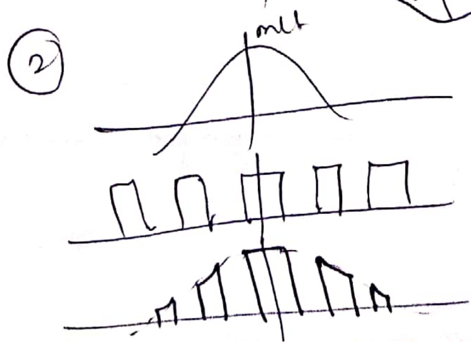
① Ideal (or) Impulse Sampling

② Natural Sampling

③ Flat top







DISCRETE FOURIER TRANSFORM (DFT)

- The drawback of in DTFT is that the frequency domain representation of a discrete time signals obtained using DTFT will be a continuous function of 'ω'. and so, it can not be processed by digital system.

- The DFT has been developed to convert a continuous function of ω to a discrete function of 'ω'.

- The DFT of a discrete time signal is obtained by sampling the DTFT. of the signal at uniform frequency intervals and the no. of samples should be sufficient to avoid aliasing of frequency spectrum.

- The samples of DTFT are represented as a function of integer K, and so the DFT is a sequence of complex numbers represented as X(K) for K=0,1,2... etc.

- Since X(K) is a sequence consisting complex numbers.
- The plot of magnitude Vs K is called Magnitude spectrum and plot of phase versus K is called phase spectrum. In general these plots are called frequency spectrum.

- The drawback in DFT is that, the computation of each sample of DFT involves a large number of calculations and when large number of samples are required, the number of calculations will further increase.

- In order to overcome this drawback, a no. of methods or algorithms have been developed to reduce number of calculations. The various methods developed to compute DFT with reduced no. of calculations are collectively called Fast Fourier Transform (FFT)

1) DIT FFT   2) DIF FFT.

Development of DFT from DTFT

- The frequency domain representation of a discrete time signal obtained using discrete time Fourier transform (DTFT) will be a continuous and periodic function of ω, with periodicity of 2π.

Inorder to obtain discrete function of ω, the DTFT can be sampled at sufficient number of frequency intervals.

Let $X(e^{j\omega})$ be discrete be discrete Fourier Transform of the discrete time signal $x(n)$. The discrete Fourier Transform (DFT) of $x(n)$ is obtained by sampling one period of the discrete time Fourier Transform $X(e^{j\omega})$ at a finite number of frequency points.

- The frequency domain sampling is conventionally performed at N equally spaced frequency points in the period, $0 \leq \omega \leq 2\pi$. The sampling frequency points are denoted as ω_K and they given by

$$\omega_K = \frac{2\pi K}{N} \quad \text{for } K = 0, 1, 2 \ldots, N-1$$

Now, the DFT is a sequence consisting of N-samples of DTFT. Let the samples are denoted by $X(K)$ for $K = 0, 1, 2 \ldots N-1$

Therefore Sampling of $X(e^{j\omega})$ is mathematically expressed as

$$X(K) = X(e^{j\omega})\Big|_{\omega = \frac{2\pi K}{N}} \quad ; \text{ for } K = 0, 1, 2 \ldots N-1$$

- The DFT Sequence starts at $K=0$, corresponding to $\omega = 0$ but does not include $K=N$, corresponding to $\omega = 2\pi$ (Since sample at $\omega = 0$ same at $\omega = 2\pi$)

- Generally DFT is defined along with number of samples and is called N-point DFT.

- The no. of samples N for a finite duration sequence $x(n)$ of length L should be such that, $N \geq L$ in order to avoid aliasing of frequency spectrum.

* DEFINITION OF DFT:

The N-point DFT of $x(n)$, where $N \geq L$, is defined as

$$X(K) = \sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi Kn}{N}} \quad ; \text{for } K = 0, 1, 2, \ldots N-1$$

where $x(n)$ = Discrete time signal of length L

$X(K)$ = DFT of $x(n)$.

* FREQUENCY SPECTRUM USING DFT:

The $X(K)$ is a discrete function of discrete time frequency ω, and so it is called discrete frequency spectrum (or signal spectrum) of the discrete time signal $x(n)$

The $X(K)$ is a complex valued function of K and so it can be expressed in rectangular form as,

$$X(K) = X_r(K) + jX_i(K)$$

where, $X_r(K)$ = Real part of $X(K)$

$X_i(K)$ = Imaginary part of $X(K)$.

Magnitude :

$$|X(K)|^2 = X_r^2(K) + X_i^2(K)$$

$$|X(K)| = \sqrt{X_r^2(K) + X_i^2(K)}$$

phase :

$$\angle X(K) = \tan^{-1}\left[\frac{X_i(K)}{X_r(K)}\right]$$

* INVERSE DFT:

The inverse discrete fourier transform of the sequence $X(K)$ of length N is defined as

$$x(n) = \frac{1}{N} \sum_{K=0}^{N-1} X(K)\, e^{\frac{j2\pi Kn}{N}} \quad ; \text{ for } n = 0, 1, ---, N-1$$

where, $x(n)$ = Discrete time signal

$X(K)$ = N-point DFT of $x(n)$

we also refer to $x(n)$ and $X(K)$ as a DFT pair and the relation is expressed as,

$$x(n) \underset{DFT^{-1}}{\overset{DFT}{\rightleftarrows}} X(K)$$

* PROPERTIES OF DFT:-

1. Linearity: Let, $DFT\{x_1(n)\} = X_1(K)$

$DFT\{x_2(n)\} = X_2(K)$ then

$$DFT\{a_1x_1(n) + a_2x_2(n)\} = a_1X_1(K) + a_2X_2(K)$$

Proof:
$$X_1(K) = DFT\{x_1(n)\} = \sum_{n=0}^{N-1} x_1(n)\, e^{\frac{-j2\pi Kn}{N}}$$

$$X_2(K) = DFT\{x_2(n)\} = \sum_{n=0}^{N-1} x_2(n)\, e^{\frac{-j2\pi Kn}{N}}$$

$$DFT\{a_1x_1(n) + a_2x_2(n)\} = \sum_{n=0}^{N-1} \left[a_1x_1(n) + a_2x_2(n)\right] e^{\frac{-j2\pi Kn}{N}}$$

$$= a_1 \sum_{n=0}^{N-1} x_1(n) e^{\frac{-j2\pi Kn}{N}} + a_2 \sum_{n=0}^{N-1} x_2(n)\, e^{\frac{-j2\pi Kn}{N}}$$

$$\boxed{DFT\{a_1x_1(n)+a_2x_2(n)\} = a_1X_1(K)+a_2X_2(K)}$$

2. Periodicity:

$$x(n+N) = x(n) \; ; \text{for all } 'n'$$

$$X(K+N) = X(K) \; ; \text{for all } 'K'$$

Proof:

$$X(K+N) = \sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi n(K+N)}{N}}$$

$$= \sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi Kn}{N}} \cdot e^{\frac{-j2\pi nN}{N}}$$

$$= \sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi Kn}{N}}\, e^{-j2\pi n} = \sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi Kn}{N}}$$

$$\left[\because e^{-j2\pi n} = 1\right]$$

$$\boxed{X(K+N) = X(K)}$$

3. Circular time shift

$$DFT\{x(n)\} = X(K) \text{ then } DFT\{x((n-m))_N\} = X(K)\, e^{\frac{-j2\pi Km}{N}}$$

Proof: $$DFT\{x((n-m))_N\} = \sum_{n=0}^{N-1} x((n-m))_N\, e^{\frac{-j2\pi Kn}{N}}$$

Let $p = n-m$

$n = p+m$

$$= \sum_{p=0}^{N-1} x(p)\, e^{\frac{-j2\pi K(p+m)}{N}}$$

$$= \sum_{p=0}^{N-1} x(p)\, e^{\frac{-j2\pi Kp}{N}}\, e^{\frac{-j2\pi Km}{N}}$$

$$= \left[\sum_{p=0}^{N-1} x(p)\, e^{\frac{-j2\pi Kp}{N}}\right] e^{\frac{-j2\pi Kn}{N}}$$

$$\boxed{DFT\{x((n-m))_N\} = X(K)\, e^{-j2\pi Kn/N}}$$

4. Time reversal

$DFT\{x(n)\} = X(K)$ then $DFT\{x(N-n)\} = X(N-K)$

Proof: $$DFT\{x(N-n)\} = \sum_{n=0}^{N-1} x(N-n)\, e^{\frac{-j2\pi Kn}{N}}$$

Let $m = N-n$

$n = N-m$

$$= \sum_{m=0}^{N-1} x(m)\, e^{\frac{-j2\pi K(N-m)}{N}}$$

$$= \sum_{m=0}^{N-1} x(m)\, e^{\frac{-j2\pi KN}{N}} \cdot e^{\frac{j2\pi Km}{N}}$$

$$= \sum_{m=0}^{N-1} x(m)\, e^{\frac{j2\pi Km}{N}} \cdot e^{-j2\pi K}$$

$$= \sum_{m=0}^{N-1} x(m)\, e^{\frac{j2\pi Km}{N}} \qquad [\because e^{-j2\pi K} = 1]$$

$$= \sum_{m=0}^{N-1} x(m)\, e^{\frac{j2\pi Km}{N}}\, e^{-j2\pi m} \qquad [e^{-j2\pi m} = 1]$$

$$= \sum_{m=0}^{N-1} x(m)\, e^{\frac{j2\pi Km}{N}}\, e^{\frac{-j2\pi mN}{N}} = \sum_{m=0}^{N-1} x(m)\, e^{\frac{-j2\pi m(N-K)}{N}}$$

$$\boxed{DFT\{x(N-n)\} = X(N-K)}$$

5. Conjugation

$$DFT\{x(n)\} = X(K), \text{ then } DFT\{x^*(n)\} = X^*(N-K)$$

Proof: $$DFT\{x^*(n)\} = \sum_{n=0}^{N-1} x^*(n)\, e^{\frac{-j2\pi kn}{N}}$$

$$= \left[\sum_{n=0}^{N-1} x(n)\, e^{\frac{j2\pi kn}{N}}\right]^*$$

$$= \left[\sum_{n=0}^{N-1} x(n)\, e^{\frac{j2\pi kn}{N}}\, e^{-j2\pi n}\right]^* \quad [\because e^{-j2\pi n} = 1]$$

$$= \left[\sum_{n=0}^{N-1} x(n)\, e^{\frac{j2\pi kn}{N}}\, e^{\frac{-j2\pi Nn}{N}}\right]^*$$

$$= \left[\sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi n(N-K)}{N}}\right]^* = [X(N-K)]^*$$

$$\boxed{DFT\{x^*(n)\} = X^*(N-K)}$$

6. Circular frequency shift:

$$DFT\{x(n)\} = X(K) \text{ then } DFT\left\{x(n)\, e^{\frac{j2\pi mn}{N}}\right\} = X((K-m))_N$$

Proof: $$DFT\left\{x(n)\, e^{\frac{j2\pi mn}{N}}\right\} = \sum_{n=0}^{N-1} x(n)\, e^{\frac{j2\pi mn}{N}}\, e^{\frac{-j2\pi kn}{N}}$$

$$= \sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi(k-m)n}{N}}$$

$$\boxed{DFT\left\{x(n)\, e^{\frac{j2\pi mn}{N}}\right\} = X(K-m)_N}$$

7. Circular convolution.

$DFT\{x_1(n)\} = X_1(K)$, $DFT\{x_2(n)\} = X_2(K)$ then

$$DFT\{x_1(n) \circledast x_2(n)\} = X_1(K) X_2(K)$$

Proof: Let $x_1(n)$ and $x_2(n)$ be N-point sequences

Let $n=m$

$$X_1(K) = \sum_{n=0}^{N-1} x_1(n) e^{\frac{-j2\pi nK}{N}} = \sum_{m=0}^{N-1} x_1(m) e^{\frac{-j2\pi mK}{N}} \quad ; K=0,1,2,\ldots N-1$$

$$X_2(K) = \sum_{n=0}^{N-1} x_2(n) e^{\frac{-j2\pi nK}{N}} \qquad \text{Let } n=p$$

$$= \sum_{p=0}^{N-1} x_2(p) e^{\frac{-j2\pi pK}{N}} \quad ; K=0,1,2,\ldots N-1$$

Consider the product $X_1(K) X_2(K)$.

The inverse DFT of the product is given by

$$DFT^{-1}\{X_1(K) X_2(K)\} = \frac{1}{N} \sum_{K=0}^{N-1} X_1(K) X_2(K) e^{\frac{j2\pi Kn}{N}}$$

$$= \frac{1}{N} \sum_{K=0}^{N-1} \left[\sum_{m=0}^{N-1} x_1(m) e^{\frac{-j2\pi mK}{N}} \right] \left[\sum_{p=0}^{N-1} x_2(p) e^{\frac{-j2\pi pK}{N}} \right] e^{\frac{j2\pi nK}{N}}$$

$$= \frac{1}{N} \sum_{m=0}^{N-1} x_1(m) \sum_{p=0}^{N-1} x_2(p) \sum_{K=0}^{N-1} e^{\frac{j2\pi K(n-m-p)}{N}}$$

Let $n-m-p = qN$, where q is an integer

$$\therefore \sum_{K=0}^{N-1} e^{\frac{j2\pi K(n-m-p)}{N}} = \sum_{K=0}^{N-1} e^{\frac{j2\pi KqN}{N}} = \sum_{K=0}^{N-1} \left(e^{j2\pi q}\right)^K = \sum_{K=0}^{N-1} 1^K = N$$

$$\left[\because e^{j2\pi q} = 1\right]$$

since $n-m-p=qN$, $p=n-m-qN$

$$\sum_{p=0}^{N-1} x_2(p) = \sum_{m=0}^{N-1} x_2(n-m-qN) = \sum_{m=0}^{N-1} x_2(n-m, \text{mod } N)$$

$$= \sum_{m=0}^{N-1} x_2((n-m))_N$$

$$DFT^{-1}\{X_1(K)X_2(K)\} = \frac{1}{N}\sum_{m=0}^{N-1} x_1(m) \sum_{m=0}^{N-1} x_2((n-m))_N N$$

$$= \sum_{m=0}^{N-1} x_1(m)\, x_2((n-m))_N$$

$$= x_1(n) \circledast x_2(n)$$

$$\boxed{\therefore X_1(K)X_2(K) = DFT\{x_1(n) \circledast x_2(n)\}}$$

8. Circular Correlation

$DFT\{x(n)\} = X(K)$ and $DFT\{y(n)\} = Y(K)$ then

$$DFT\{\bar{r}_{xy}(m)\} = DFT\left\{\sum_{n=0}^{N-1} x(n)\, y^*((n-m))_N\right\} = X(K)Y^*(K)$$

Proof: $$X(K) = \sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi Kn}{N}}$$

Let $n=m$

$$= \sum_{m=0}^{N-1} x(m) e^{\frac{-j2\pi Km}{N}} \quad ; K=0,1,2,\ldots N-1$$

$$Y(K) = \sum_{n=0}^{N-1} y(n)\, e^{\frac{-j2\pi nK}{N}}$$

let $n=p$

$$= \sum_{p=0}^{N-1} y(p)\, e^{\frac{-j2\pi pk}{N}} \quad ; k = 0, 1, 2, \ldots, N-1$$

$$DFT^{-1}\{X(k)Y^*(k)\} = \frac{1}{N} \sum_{k=0}^{N-1} X(k)\, Y^*(k)\, e^{\frac{j2\pi nk}{N}}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} \left[\sum_{m=0}^{N-1} x(m)\, e^{\frac{-j2\pi mk}{N}} \right] \left[\sum_{p=0}^{N-1} y(p)\, e^{\frac{-j2\pi pk}{N}} \right]^* e^{\frac{j2\pi nk}{N}}$$

$$= \frac{1}{N} \sum_{m=0}^{N-1} x(m) \sum_{p=0}^{N-1} y^*(p) \sum_{k=0}^{N-1} e^{\frac{j2\pi k(n-m+p)}{N}}$$

Let $n-m+p = qN$, where q is an integer

$$\therefore \sum_{k=0}^{N-1} e^{\frac{j2\pi k(n-m+p)}{N}} = \sum_{k=0}^{N-1} e^{\frac{j2\pi kqN}{N}} = \sum_{k=0}^{N-1} \left(e^{j2\pi q}\right)^k = \sum_{k=0}^{N-1} 1^k = N$$

Since $n-m+p = qN$, $p = n-m+qN$

$$\sum_{p=0}^{N-1} y^*(p) = \sum_{m=0}^{N-1} y^*(n-m+qN) = \sum_{m=0}^{N-1} y^*(n-m, \text{mod } N)$$

$$= \sum_{m=0}^{N-1} y^*((n-m))_N$$

$$DFT^{-1}\{X(k)Y^*(k)\} = \frac{1}{N} \sum_{m=0}^{N-1} x(m) \sum_{m=0}^{N-1} y^*((n-m))_N\, N$$

$$= \sum_{m=0}^{N-1} x(m)\, y^*((n-m))_N = \bar{r}_{xy}(m)$$

$$\boxed{\therefore X(k)Y^*(k) = DFT\{\bar{r}_{xy}(m)\}}$$

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9. Parseval's relation

$DFT\{x_1(n)\} = X_1(K)$ and $DFT\{x_2(n)\} = X_2(K)$ then

$$\sum_{n=0}^{N-1} x_1(n)\, x_2^*(n) = \frac{1}{N}\sum_{K=0}^{N-1} X_1(K)\, X_2^*(K)$$

Proof: By definition of DFT, $X_1(K) = \sum_{n=0}^{N-1} x_1(n)\, e^{\frac{-j2\pi nk}{N}}$

By definition of IDFT, $x_2(n) = \frac{1}{N}\sum_{K=0}^{N-1} X_2(K)\, e^{\frac{j2\pi nk}{N}}$

$$\frac{1}{N}\sum_{K=0}^{N-1} X_1(K)X_2^*(K) = \frac{1}{N}\sum_{K=0}^{N-1}\left[\sum_{n=0}^{N-1} x_1(n)\, e^{\frac{-j2\pi Kn}{N}}\right] X_2^*(K)$$

$$= \sum_{n=0}^{N-1} x_1(n)\left[\frac{1}{N}\sum_{K=0}^{N-1} X_2^*(K)\, e^{\frac{-j2\pi Kn}{N}}\right]$$

$$= \sum_{n=0}^{N-1} x_1(n)\left[\frac{1}{N}\sum_{K=0}^{N-1} X_2(K)\, e^{\frac{j2\pi Kn}{N}}\right]^*$$

$$\boxed{\frac{1}{N}\sum_{K=0}^{N-1} X_1(K)X_2^*(K) = \sum_{n=0}^{N-1} x_1(n)\, x_2^*(n)}$$

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