UNIT : 4

4

CHAPTER

Partial Differential Equations

4.0 INTRODUCTION

A differential equation which involves partial derivatives is called a partial differential equation.

Therefore a partial differential equation contains one dependent variable and more than one independent variables. In the case of two independent variables, z is considered as dependent variable and x, y as independent variables. And in the case of three independent variables, u is considered as dependent variable and x, y, z as independent variables.

In this chapter we shall study the formation and solution of partial differential equations.

Standard Notation :

If z is a function of two independent variables x and y, then we shall use the following notation for the partial derivatives of z.

$$\frac{\partial z}{\partial x} = p, \frac{\partial z}{\partial y} = q, \frac{\partial^2 z}{\partial x^2} = r, \frac{\partial^2 z}{\partial x \partial y} = x, \frac{\partial^2 z}{\partial y^2} = t$$

The order of a partial differential equation is the order of the highest order derivative appearing in the partial differential equation.

The degree of a partial differential equation is the degree of the highest order derivative appearing in a given equation after removing the radical sign.

Examples :

1. $$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = z$$

Here z is dependent variable and x, y are independent variables and this equation is of first order and first degree.

2. $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

Here u is dependent variable and x, y, z are independent variables and this equation is of second order and first degree.

4.1 FORMATION OF PARTIAL DIFFERENTIAL EQUATIONS

A partial differential equation can be formed by two methods :

1. By elimination of arbitrary constants
2. By elimination of arbitrary functions.

Elimination of Arbitrary Constants

If the number of arbitrary constants to be eliminated is equal to the number of independent variables, we get a partial differential equation of first order. If the number of arbitrary constants to be eliminated is greater than the number of independent variables, we get a partial differential equation of higher order.

Solved Problems

Problem 1: Form partial differential equations from the following equations by eliminating the arbitrary constants *a* and *b*.

(i) $z = ax + by + ab$ (ii) $z = ax^2 + by^2$ (iii) $z = (x + a)(y + b)$.

Solution. (i) Given $z = ax + by + ab$ ...(1)

Here the number of arbitrary constants is equal to the number of independent variables, hence we get a partial differential equation of first order.

Differentiating (1) partially w.r.t., x

$$p = \frac{\partial z}{\partial x} = a \quad ...(2)$$

Differentiating (1) partially w.r.t., y

$$q = \frac{\partial z}{\partial y} = b \quad ...(3)$$

Substituting (2) and (3) in (1), we get

$$\therefore \quad Z = px + qy + pq.$$

which is the required partial differential equation of the first order.

(ii) Given equation is

$$Z = ax^2 + by^2 \quad ...(1)$$

Differentiating (1) partially w.r.t., x

$$p = \frac{\partial z}{\partial x} = 2ax \quad ...(2)$$

Differentiating (1) partially w.r.t., y

$$q = \frac{\partial z}{\partial y} = 2by \quad ...(3)$$

From (2), $a = \dfrac{p}{2x}$

From (3), $b = \dfrac{q}{2y}$

Substituting these values in (1), we have

$$z = \frac{p}{2x}(x^2) + \frac{q}{2y}(y^2)$$

$\therefore$ $2Z = px + qy$

which is the required differential equation.

(iii) Given equation is

$$Z = (x + a)(y + b) \quad \text{...(1)}$$

Differentiating (1) w.r.t., x

$$p = \frac{\partial z}{\partial x} = 1(y + b) \quad \text{...(2)}$$

Differentiating (1) partially w.r.t., y

$$q = \frac{\partial z}{\partial y} = 1(x + a) \quad \text{...(3)}$$

From (2), $p = y + b$

From (3), $q = x + a$

Substituting in (1), we have

$$Z = pq$$

which is the required differential equation.

Problem 2: Form partial differential equations by eliminating arbitrary constants from the following*:*

(i) $Z = (x^2 + a)(y^2 + b)$ (ii) $(x - a)^2 + (y - b)^2 = z$

(iii) $Z = xy + y\sqrt{x^2 - a^2 + b^2}$.

Solution. (*i*) Given equation is

$$Z = (x^2 + a)(y^2 + b) \quad \text{...(1)}$$

Differentiating (1) partially w.r.t., x

$$\frac{\partial z}{\partial x} = p = 2x\left(y^2 + b\right) \quad \text{...(2)}$$

Differentiating (1) partially w.r.t., y

$$\frac{\partial z}{\partial y} = q = (x^2 + a)(2y) \qquad ...(3)$$

Multiplying (2) and (3), we get

$$pq = 4xy\,(x^2 + a)\,(y^2 + b)$$

$$pq = 4xyz$$

which is the required differential equation.

(ii) Given equation is

$$(x - a)^2 + (y - b)^2 = z \qquad ...(1)$$

Differentiating (1) partially w.r.t., x

$$2\,(x - a) = p = \frac{\partial z}{\partial x} \qquad ...(2)$$

Differentiating (1) partially w.r.t., y

$$2\,(y - b) = q = \frac{\partial z}{\partial y} \qquad ...(3)$$

From (2), $\quad x - a = \dfrac{p}{2}$

From (3), $\quad y - b = \dfrac{q}{2}$

Substituting these values in (1),

$$Z = \frac{p^2}{4} + \frac{q^2}{4}$$

$4Z = p^2 + q^2$, which is the required solution.

(iii) Given equation is

$$Z = xy + y\sqrt{x^2 - a^2 + b^2} \qquad ...(1)$$

Differentiating (1) w.r.t., x

$$p = y + y \cdot \frac{1}{2\sqrt{x^2 - a^2 + b^2}}\,(2x)$$

$$p - y = \frac{xy}{\sqrt{x^2 - a^2 + b^2}}$$

$$\sqrt{x^2 - a^2 + b^2} = \frac{xy}{p - y} \qquad \text{...(2)}$$

Differentiating (1) w.r.t., y

$$q = x + \sqrt{x^2 - a^2 + b^2}$$

$$q - x = \sqrt{x^2 - a^2 + b^2} \qquad \text{...(3)}$$

From (2) and (3),

$$\frac{xy}{p - y} = q - x$$

$$\Rightarrow \qquad (p - y)\,(q - x) = xy$$

$$\Rightarrow \qquad px + qy = pq$$

which is the required differential equation.

Problem 3: Form the partial differential equation by eliminating *a, b, c* from

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

Solution. Given equation is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \qquad \text{...(1)}$$

Here the number of arbitrary constants (3) is greater than the number of independent variables (2) and hence we get a partial differential equation of second order.

Differentiating (1) partially w.r.t., 'x'

$$\frac{2x}{a^2} + \frac{2y}{c^2}\frac{\partial z}{\partial x} = 0$$

$$\Rightarrow c^2 x + a^2 z \frac{\partial z}{\partial x} = 0 \qquad \text{...(2)}$$

Differentiating (1) partially w.r.t., y

$$\frac{2y}{b^2} + \frac{2z}{c^2} \cdot \frac{\partial z}{\partial y} = 0$$

$$\Rightarrow c^2 y + b^2 z \frac{\partial z}{\partial y} = 0 \qquad ...(3)$$

Again differentiating (2) partially w.r.t., x, we have

$$c^2 + a^2 \left(\frac{\partial z}{\partial x}\right)^2 + a^2 z \frac{\partial^2 z}{\partial x^2} = 0$$

$$\Rightarrow \frac{c^2}{a^2} + \left(\frac{\partial z}{\partial x}\right)^2 + z \frac{\partial^2 z}{\partial x^2} = 0 \qquad ...(4)$$

From (2), $\frac{c^2}{a^2} = -\frac{z}{x}\frac{\partial z}{\partial x}$

Substitute this value in (4), we have

$$-\frac{z}{x}\frac{\partial z}{\partial x} + \left(\frac{\partial z}{\partial x}\right)^2 + z \frac{\partial^2 z}{\partial x^2} = 0$$

$$\Rightarrow xz \frac{\partial^2 z}{\partial x^2} + \left(\frac{\partial z}{\partial x}\right)^2 - z \frac{\partial z}{\partial x} = 0$$

which is the required differential equation of second order.

Problem 4. Form the PDE by eliminating arbitrary constants from (1) $z=ax+by+a^2+b^2$ (OU Dec 2011)

Sol. Given equation $z=ax+by+a^2+b^2$ (1)

Differentiating equation (1) w.r.t to x partially

We get $\frac{\partial z}{\partial x} = a$ (2)

differentiating equation (1) w.r.t to y partially

We get $\frac{\partial z}{\partial y} = b$ (3)

Substitute (2), (3) in (1)

$$\Rightarrow z = \frac{\partial z}{\partial x} x + \frac{\partial z}{\partial y} y + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2$$

(or) $z = px + qy + p^2 + q^2$ where $p = \frac{\partial z}{\partial x}, q = \frac{\partial z}{\partial y}$.

which is required P.D.E.

2. z = a(x + y)b. **(OU Dec 2015)**

Sol. Given equation z = a(x + y)b. (1)

differentiating equation (1) w.r.t to x and y partially

We get $\frac{\partial z}{\partial x} = p = a$ (2), $\frac{\partial z}{\partial y} = q = a$ (3)

from equation (2) and (3)

$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial y}$ (or) p = q (or) p - q = 0.

which is required P.D.E.

3. 2z = (ax + y)^2 + b **(OU June 2012)**

Sol. Given equation 2z = (ax + y)^2 + b (1)

differentiating equation (1) w.r.t to x and y

which is required P.D.E

We get $2\frac{\partial z}{\partial x} = 2(ax+y)a \Rightarrow \frac{\partial z}{\partial x} = a(ax + y)$ (2)

$2\frac{\partial z}{\partial y} = 2(ax+y) \Rightarrow \frac{\partial z}{\partial y} = (ax + y)$ (3)

From (2) and (3) $\frac{\partial z}{\partial x} = a\frac{\partial z}{\partial y}$ (or) p = aq

$$\Rightarrow a = \left(\frac{p}{q}\right) \quad (4)$$

from (3) and (4) $q = \frac{p}{q}x + y$

or $q^2 = px + qy$

which is required P.D.E

Problem 5: Form the partial differential equation by eliminating arbitrary constants a, b from $Z = a \log \left[\frac{b(y-1)}{1-x}\right]$

Solution. Given equation is

$$Z = a \log \left[\frac{b(y-1)}{1-x}\right] \qquad \text{...(1)}$$

$$= a\,[\log b\,(y-1) - \log (1-x)]$$

$$= a\,[\log b + \log (y-1) - \log (1-x)]$$

$$\therefore \qquad Z = a \log b + a \log (y-1) - a \log (1-x) \qquad \text{...(2)}$$

Differentiating (2) partially w.r.t., x

$$p = \frac{\partial z}{\partial x} = -a\left[\frac{1}{1-x}\right](-1) \qquad \text{...(3)}$$

Differentiating (2) partially w.r.t., y

$$q = \frac{\partial z}{\partial y} = a\left[\frac{1}{y-1}\right](1)$$

$$\Rightarrow \frac{\partial z}{\partial y}(y-1) = a \qquad \text{...(4)}$$

From (3) and (4), we get

$$\frac{\partial z}{\partial y}(y-1) = (1-x)\frac{\partial z}{\partial x}$$

$$\Rightarrow \qquad p + q = px + qy$$

which is the required differential equation.

Problem 6: Form the differential equation of all spheres with centres on the xy-plane and of given radius r.

Or

Form the partial differential equation by eliminating the arbitrary constants from

$$(x-h)^2 + (y-k)^2 + z^2 = r^2$$

Solution. Equation of given sphere is $(x-h)^2 + (y-k)^2 + z^2 = r^2$ ...(1)

where h and k are arbitrary constants.

Differentiating (1) partially w.r.t., x

$$2\,(x-h) + 2z\frac{\partial z}{\partial x} = 0$$

$$\Rightarrow x - h = -zp \qquad \text{...(2)}$$

Differentiating (1) w.r.t., y

$$2(y-k) + 2z\frac{\partial z}{\partial y} = 0$$

$\Rightarrow \quad y - k = -zq$ ...(3)

Substituting (2) and (3) in (1),

$$(-zp)^2 + (-zq)^2 + z^2 = r^2$$

$\Rightarrow \quad z^2(p^2 + q^2 + 1) = r^2$

which is the required partial differential equation.

EXERCISE

1. Form the differential equation by eliminating arbitrary constants from the following :

(*i*) $Z = ax + by + \frac{a}{b} - b$ (*ii*) $Z = (x + a)(y + b)$

(*iii*) $Z = ax + by + a^2 + b^2$ (*iv*) $Z = a + b(x + y)$

(*v*) $2Z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ (*vi*) $Z = ax + a^2y^2 + b$

(*vii*) $(x - a)^2 + (y - b)^2 = z^2 \cot^2 a$ (*viii*) $ax + by + cz = 1$.

2. Form the differential equation of all planes whose *x*-intercept is always equal to the *y*-intercept.

[Hint. Equation of plane is $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

Since *x* and *y* intercept are equal, equation becomes] $\frac{x+y}{a} + \frac{z}{c} = 1$

3. Form the differential equation of all planes passing through the origin.

ANSWER

1. (i) $Z = px + qy - \frac{p}{q} - q$ (*ii*) $Z = pq$ (iii) $Z = px + qy + p^2 + q^2$

(iv) $p = q$ (v) $2Z = px + qy$ (*vi*) $q = 2p^2y$

(vii) $p^2 + q^2 = \tan^2 a$ (viii) $r = 0$ or $q = 0$ or $t = 0$.

2. $p = q$

3. $z = px + qy$.

4.1.1 ELIMINATION OF ARBITRARY FUNCTIONS

In this case the order of the partial differential equation is same as the number of arbitrary functions present in the given relation.

SOLVED PROBLEMS

Problem 1. Form the differential equations by eliminating the function "*f*" from the following:

(i) $z = x^n f\left(\frac{y}{x}\right)$ (ii) $z = y^2 + 2f\left(\frac{1}{x} + \log y\right)$

(iii) $z = f(x^2 - y^2)$ (*iv*) $z = f\left(\frac{xy}{z}\right)$

Solution. (i) Given equation is

$$z = x^n f\left(\frac{y}{x}\right) \quad \ldots(1)$$

Differentiating (1) partially w.r.t., x

$$\frac{\partial z}{\partial x} = p = x^n f'\left(\frac{y}{x}\right)\left(-\frac{y}{x^2}\right) + f\left(\frac{y}{x}\right) n x^{n-1}$$

$$\Rightarrow \quad px = -x^{n-1} y \; f'\left(\frac{y}{x}\right)\left(\frac{1}{x}\right) \quad \ldots(2)$$

Differentiating (1) partially w.r.t., y

$$\frac{\partial z}{\partial y} = q = x^n \; f'\left(\frac{y}{x}\right)\left(\frac{1}{x}\right)$$

$$\Rightarrow \quad q = x^{n-1} \; f'\left(\frac{y}{x}\right)$$

$$\Rightarrow \quad qy = x^{n-1} y \; f'\left(\frac{y}{x}\right) \quad \ldots(3)$$

Adding (2) and (3), we have

$$px + qy = n\,x^n\; f\left(\frac{y}{x}\right)$$

$\Rightarrow$ $$px + qy = nz \qquad \text{[From (1)]}$$

which is the required differential equation.

(ii) Given equation is,

$$z = y^2 + 2f\left(\frac{1}{x} + \log y\right) \qquad \ldots(1)$$

Differentiating (1) w.r.t., x

$$\frac{\partial z}{\partial x} = p = 2\,f'\left(\frac{1}{x} + \log y\right)\left(-\frac{1}{x^2}\right)$$

$\Rightarrow$ $$p = \frac{2}{x^2}\,f'\left(\frac{1}{x} + \log y\right) \qquad \ldots(2)$$

Differentiating (1) w.r.t., y

$$\frac{\partial z}{\partial y} = q = 2y + f'\left(\frac{1}{x} + \log y\right).\frac{1}{y}$$

$\Rightarrow$ $$q - 2y = \frac{2}{y}\;f'\left(\frac{1}{x} + \log y\right) \qquad \ldots(3)$$

Dividing (2) by (3),

$$\frac{p}{q - 2y} = \frac{-\frac{2}{x^2}}{\frac{2}{y}} = -\frac{y}{x^2}$$

$\Rightarrow$ $$px^2 = -\,qy + 2y^2$$

$\Rightarrow$ $$px^2 + qy = 2y^2$$

which is the required differential equation.

(iii) Given equation is

$$z = f\,(x^2 - y^2) \qquad \ldots(1)$$

Differentiating (1) partially w.r.t., x

$$\frac{\partial z}{\partial x} = p = \;f'\,(x^2 - y^2)\;(2x) \qquad \ldots(2)$$

Differentiating (1) partially w.r.t., y

$$\frac{\partial z}{\partial y} = q = -\,2y\ f'\ (x^2 - y^2) \qquad \ldots(3)$$

Dividing (2) by (3),

$$\frac{p}{q} = -\,\frac{x}{y}\ \Rightarrow py + qx = 0$$

which is the required differential equation.

(iv) Given equation is

$$z = f\left(\frac{xy}{z}\right) \qquad \ldots(1)$$

Differentiating (1) w.r.t., x

$$\frac{\partial z}{\partial x} = p = f'\left(x^2 - y^2\right)(2x)$$

$$\Rightarrow \qquad p = f'\left(\frac{xy}{z}\right).y\left[\frac{z(1) - x.p}{z^2}\right] \qquad \ldots(2)$$

Differentiating (1) w.r.t., y

$$\frac{\partial z}{\partial y} = q = \frac{x}{z^2}\ (z - qy)\ f'\left(\frac{xy}{z}\right) \qquad \ldots(3)$$

Dividing (2) by (3),

$$\frac{p}{q} = \frac{y(z - px)}{x(z - qy)}$$

$$\Rightarrow \qquad p\ zx - pq\ xy = q\ yz - pq\ xy$$

$$\Rightarrow \qquad z\ (px - qy) = 0$$

$$\Rightarrow \qquad px = qy$$

which is the required differential equation.

Problem 2: Form the partial differential equation by eliminating the arbitrary function from $z = xy + f\,(x^2 + y^2)$

Solution. Given $z = xy + f\,(x^2 + y^2)$ ...(1)

Differentiating (1) partially w.r.t., x

$$\frac{\partial z}{\partial x} = p = (1)\, y + f'(x^2 + y^2)\,(2x)$$

$$\Rightarrow \quad \frac{p}{2x} = \frac{y}{2x} + f'(x^2 + y^2)$$

$$\Rightarrow \quad \frac{p}{2x} - \frac{y}{2x} = f'(x^2 + y^2) \qquad \text{...(2)}$$

Differentiating (1) partially w.r.t., y

$$\frac{\partial z}{\partial y} = q = x\,(1) + f'(x^2 + y^2)\,(2y)$$

$$\Rightarrow \quad \frac{q}{2y} = \frac{x}{2y} + f\,¢\,(x^2 + y^2)$$

$$\Rightarrow \quad \frac{q}{2y} - \frac{x}{2y} = f\,¢\,(x^2 + y^2) \qquad \text{...(3)}$$

From (2) and (3), we have

$$\frac{p}{2x} - \frac{y}{2x} = \frac{q}{2y} - \frac{x}{2y}$$

$$\Rightarrow \quad \frac{p-y}{2x} = \frac{q-x}{2y}$$

$$\Rightarrow \quad py - y^2 = qx - x^2$$

$$\Rightarrow \quad py - qx = y^2 - x^2$$

which is the required differential equation.

Problem 3: Form the partial differential equation by eliminating the arbitrary function '*f*' from $x + y + z = f(x^2 - y^2 + z^2)$(OU Dec 2017)

Sol. given, $x + y + z = f(x^2 - y^2 + z^2)$ (1)

differentiating (1) partially w.r.t to x and y,

we have

$$1 + p = f'(x^2 - y^2 + z^2).(2x + 2pz) \qquad (2)$$

and $1 + q = f'(x^2 - y^2 + z^2).(-2y + 2qz)$ (3)

from equation (2) & (3)

$$\frac{1+p}{1+q}=\frac{2x+2pz}{-2y+2qz}$$

$$\Rightarrow \quad \frac{1+p}{1+q}=\frac{x+pz}{-y+zq}$$

$$(1+p)(-y+zq)=(x+pz)(1+q)$$

$$-y-yp+zq+zpq=x+xq+pz+pqz$$

it can be written as

$\Rightarrow$p(z + y) + q(x - z) = –(x + y)

$\Rightarrow$ which is the required P.D.E

Problem 4: Form a P.D.E by eliminating arbitrary function ϕ from $\phi(x+y+z, xyz)=0$

Sol. Given $\phi(x+y+z, xyz)=0$ (1)

let $u=x+y+z$ and $v=xyz$ (2)

So it is in the form of $\phi(u,v)=0$ (3)

We know the equation of P.D.E for (3) is the form of

$$Pp+Qq=R \tag{4}$$

where $P=\frac{\partial(u,v)}{\partial(y,z)},\quad Q=\frac{\partial(u,v)}{\partial(z,x)},\ R=\frac{\partial(u,v)}{\partial(x,y)}$

Now from (2)

$$\frac{\partial u}{\partial x}=1,\ \frac{\partial u}{\partial y}=1,\ \frac{\partial u}{\partial z}=1,\qquad \frac{\partial v}{\partial x}=yz,\ \frac{\partial v}{\partial z}=xy,\ \frac{\partial v}{\partial y}=xz$$

So that $P = \frac{\partial(u,v)}{\partial(y,z)} = \begin{vmatrix} \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ xz & \text{xy} \end{vmatrix} = xy - xz$

$$Q = \frac{\partial(u,v)}{\partial(z,x)} = \begin{vmatrix} \frac{\partial u}{\partial z} & \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial z} & \frac{\partial v}{\partial x} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ xy & \text{yz} \end{vmatrix} = yz - xy$$

$$R = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ yz & xz \end{vmatrix} = xz - yz$$

Substitute P, Q, R valves in $Pp+Qq=R$

$\Rightarrow (xy - xz)p + (yz - xy)q = (xz - yz)$

Problem 5: Eliminate the arbitrary function f and g to obtain a partial differential equation from $z = f(x^2 - y) + g(x^2 + y)$

Sol. Given $z = f(x^2 - y) + g(x^2 + y)$ (1)

Differentiating (1) partially w.r.t x and y, we have

$$\frac{\partial z}{\partial x} = f'(x^2 - y)(2x) + g'(x^2 + y)2x \quad (2)$$

$$\frac{\partial z}{\partial y} = f'(x^2 - y)(-1) + g'(x^2 + y)(1) \quad (3)$$

Again differentiating (2) Partially w.r.t. 'x'

$$\Rightarrow \frac{\partial^2 z}{\partial x^2} = f'(x^2 - y)(2) + 2xf''(x^2 - y)(2x) + g'(x^2 + y)2 + 2xg''(x^2 + y)$$

$$\Rightarrow \frac{\partial^2 z}{\partial x^2} = 2[f'(x^2 - y) + g'(x^2 + y)] + 4x^2[f''(x^2 - y) + g''(x^2 + y)] \quad (4)$$

Again differeniating (3) partially w.r.t. 'y'

$$\frac{\partial^2 z}{\partial y^2} = f''(x^2 - y) + g''(x^2 + y)] \qquad (5)$$

from (2), (4) and (5) ,we have

$$\frac{\partial^2 z}{\partial x^2} = 2\left[\frac{1}{2x}\frac{\partial z}{\partial x}\right] + 4x^2\frac{\partial^2 z}{\partial y^2}$$

$$\Rightarrow x\frac{\partial^2 z}{\partial x^2} = \frac{\partial z}{\partial x} + 4x^3\frac{\partial^2 z}{\partial y^2}$$

which is required P.D.E

Problem 6: Form the partial differential equation by eliminating the arbitrary functions from the following

(a) z = f (x + it) + g (x – it), where i = . $\sqrt{-1}$

(b) z = y f (x) + x g (y)

Solution. (*a*) Given

$$z = f(x + it) + g(x - it) \qquad ...(1)$$

Differentiating (1) twice partially w.r.t., *x* and *t*, we have

$$\frac{\partial z}{\partial x} = f'(x + it) + g'(x - it)$$

$$\frac{\partial^2 z}{\partial x^2} = f''(x + it) + g''(x - it) \qquad ...(2)$$

$$\frac{\partial z}{\partial t} = i\ f'(x + it) - i\ g'(x - it)$$

$$\frac{\partial^2 z}{\partial t^2} = i^2\ f''(x + it) + i^2\ g''(x - it)$$

$$\Rightarrow \quad \frac{\partial^2 z}{\partial t^2} = -f''(x + it) - g''(x - it) \qquad ...(3)$$

Adding (2) and (3), we have

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial t^2} = 0$$

which is a partial differential equation of second order.

(*b*) Given equation is,

$$z = y\, f(x) + x\, g(y) \qquad \text{...(1)}$$

Differentiating (1) w.r.t., *x*

$$\frac{\partial z}{\partial x} = p = y\, f'(x) + g(y) \qquad \text{...(2)}$$

Differentiating (1) w.r.t., *y*

$$\frac{\partial z}{\partial y} = q = f(x) + x\, g'(y) \qquad \text{...(3)}$$

Differentiating (3) w.r.t., *x*

$$\frac{\partial^2 z}{\partial x . \partial y} = s = f'(x) + g'(y) \qquad \text{...(4)}$$

Multiplying (2) by *x* and (3) by *y* and adding,

$$px + qy = xy\,\{ f'(x) + g'(y)\} + y\, f(x) + x\, g(y)$$

$$\Rightarrow \qquad px + qy = xy\, s + z$$

which is the required differential equations.

EXERCISE 2

Form the partial differential equations by eliminating the arbitrary functions from the following :

1. $z = f\left(\dfrac{y}{x}\right)$

2. $z = f_1(x)\, f_2(y)$

3. $z = f(x + ay) + g(x - ay)$

4. $z = f(x + 4t) + g(x - 4t)$

5. $f(x^2 + y^2, z - xy) = 0$

6. $f(xy + z^2, x + y + z) = 0$

7. $z = x + y + f(xy)$

8. $z = f(x^2 + y^2 + z^2)$

9. $xyz = f(x^2 + y^2 + z^2)$.

ANSWER

1. $px + qy = 0$

2. $z\dfrac{\partial^2 z}{\partial x \partial y} - \dfrac{\partial z}{\partial x}.\dfrac{\partial z}{\partial y} = 0$

3. $\dfrac{\partial^2 z}{\partial y^2} = a^2 \dfrac{\partial^2 z}{\partial x^2}$

4. $16\dfrac{\partial^2 z}{\partial x^2} - \dfrac{\partial^2 z}{\partial t^2} = 0$

5. $py - ax = y^2 - x^2$

6. $p(x - 2z) + q(2z - y) = y - x$

7. $px - qy = x - y$

8. $py = qx$

9. $x(y^2 - z^2)p + y(z^2 - x^2)q = (x^2 - y^2)z$.

4.2 SOLUTION OF PARTIAL DIFFERENTIAL EQUATION

In previous section, we have seen that partial differential equations can be formed either by elimination of arbitrary constants or by elimination of arbitrary functions.

A solution or integral of a differential equation is a relation between the variables, by means of which and the derivatives obtained there from, the equation is satisfied. The main difference between ordinary and partial differential equation is that, in the case of ordinary, we get only one solution whereas in the case of partial differential equation we get more than one type of solutions.

Complete Integral: Any solution of a partial differential equation in which the number of arbitrary constants is equal to the number of independent variables is called the complete integral or complete solution.

For example, $z = ax + by$, where a and b are constants is the complete integral of the partial differential equation $z = px + qy$.

Particular Integral : Any solution obtained from the complete integral by giving particular values to the arbitrary constants is called a particular integral.

Singular Integral :

Let $f(x, y, z, a, b) = 0$ be the complete integral of f $(x, y, z, p, q) = 0$.

Then the solution obtained by eliminating a and b from the equations

$$f(x, y, z, a, b) = 0 \qquad \text{...(1)}$$

$$\frac{\partial f}{\partial a} = 0 \qquad \text{...(2)}$$

$$\frac{\partial f}{\partial b} = 0 \qquad \text{...(3)}$$

is called the singular integral of the partial differential equation.

General Integral :

Let $$f(x, y, z, a, b) = 0 \qquad \text{...(1)}$$

be the complete integral of

$$\phi(x, y, z, p, q) = 0 \qquad \text{...(2)}$$

In (1), put $b = g(a)$, we get

$$f(x, y, z, a, g(a)) = 0 \qquad \text{...(3)}$$

Differentiating (3) w.r.t., 'a', we get

$$\frac{\partial f}{\partial a} + \frac{\partial f}{\partial g} g'(a) = 0 \qquad \text{...(4)}$$

The solution obtained by eliminating '*a*' from (3) and (4) is known as general integral of the partial differential equation f $(x, y, z, p, q) = 0$.

Note. We observe that partial differential equations may have solutions which are different from the complete integral or singular integral. For example, $z = y f\left(\frac{y}{x}\right)$ is also a solution of $z = px + qy$. This solution is different from the complete integral $z = ax + by$.

4.3 FIRST ORDER LINEAR PARTIAL DIFFERENTIAL EQUATIONS

A differential equation involving first order partial derivatives *p* and *q* only is called a partial differential equation of the first order. If *p* and *q* both occur in the first degree only and are not multiplied together, then it is called a linear partial differential equation of the first order.

4.4 LAGRANGE'S LINEAR EQUATION

The partial differential equation of the form

$$Pp + Qq = R \qquad \text{...(1)}$$

where P, Q and R are functions of *x*, *y*, *z* or constants and $p = \frac{\partial z}{\partial x}$, $q = \frac{\partial z}{\partial y}$ is the standard form of a linear partial differential equation of the first order and is called **'LAGRANGE'S Linear equation'.**

Solution of Lagrange's Linear Equation

Consider Lagrange's linear equation of the form

$$Pp + Qq = R \qquad \text{...(1)}$$

where P, Q, R are the functions of *x*, *y*, *z* and $p = \frac{\partial z}{\partial x}$ and $q = \frac{\partial z}{\partial y}$.

This form of equation is obtained by eliminating an arbitrary function f from

$$\phi(u, v) = 0 \qquad \text{...(2)}$$

where *u*, *v* are functions of *x*, *y*, *z*.

Differentiating (2) partially w.r.t. *x* and *y*

$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z} p\right) + \frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z} p\right) = 0 \qquad \text{...(3)}$$

and $\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}q\right)+\frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}q\right) = 0$ ...(4)

...(7)

Eliminating $\frac{\partial \phi}{\partial u}$ and $\frac{\partial \phi}{\partial v}$ from (3) and (4), we get

$$\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}p\right)\left(\frac{\partial v}{\partial y}+\frac{\partial v}{\partial z}q\right)-\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}p\right)\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}q\right) = 0 \quad ...(5)$$

or $\left(\frac{\partial u}{\partial y},\frac{\partial v}{\partial z}-\frac{\partial u}{\partial z},\frac{\partial v}{\partial y}\right)p+\left(\frac{\partial u}{\partial z},\frac{\partial v}{\partial x}-\frac{\partial u}{\partial x},\frac{\partial v}{\partial z}\right)q=\frac{\partial u}{\partial x},\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y},\frac{\partial v}{\partial x}$ ...(6)

which is same as equation (1) with

$$\left.\begin{aligned} P &= \frac{\partial u}{\partial y}.\frac{\partial v}{\partial z}-\frac{\partial u}{\partial z}.\frac{\partial v}{\partial y} \\ Q &= \frac{\partial u}{\partial z}.\frac{\partial v}{\partial x}-\frac{\partial u}{\partial x}.\frac{\partial v}{\partial z} \\ R &= \frac{\partial u}{\partial x}.\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}.\frac{\partial v}{\partial x} \end{aligned}\right\} \quad ...(7)$$

To determine *u*, *v* from P, Q, R, suppose that *u* = *a* and *v* = *b*, where *a*, *b* are constants, so that

$$\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy+\frac{\partial u}{\partial z}dz = du = 0 \quad ...(8)$$

and $$\frac{\partial v}{\partial x}dx+\frac{\partial v}{\partial y}dy+\frac{\partial v}{\partial z}dz = dv = 0 \quad ...(9)$$

Solving (8) and (9) by the method of cross multiplication, we get

$$\frac{dx}{\frac{\partial u}{\partial y}.\frac{\partial v}{\partial z}-\frac{\partial u}{\partial z}.\frac{\partial v}{\partial y}}=\frac{dy}{\frac{\partial u}{\partial z}.\frac{\partial v}{\partial x}-\frac{\partial u}{\partial x}.\frac{\partial v}{\partial z}}=\frac{dz}{\frac{\partial u}{\partial x}.\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}.\frac{\partial v}{\partial x}} \quad ...(10)$$

From (7) and (10), we have

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \quad ...(11)$$

The solutions of these equations are *u* = *a* and *v* = *b*. Thus determining *u*, *v* from the simultaneous equations (11), we have the solution of the partial differential equation

$$Pp + Qq = R \text{ as } f(u, v) = 0 \text{ or } u = f(v).$$

Note. Equations (11) are called 'Lagrange's auxiliary equations or subsidiary equations.

Working Rule :

To solve the equation $Pp + Qq = R$

1. Form the auxiliary equations

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

2. Solve the auxiliary equations by the method of grouping or by the method of multipliers or both to get two independent solutions $u = a$ and $v = b$, where a, b are constants.

3. Then $f(u, v) = 0$ or $u = f(v)$ is the general solution of the equation $Pp + Qq = R$.

4.4.1 METHOD OF GROUPING

Consider the Lagrange's equation

$$Pp + Qq = R \qquad \text{...(1)}$$

Auxiliary equations of (1) are given by

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \qquad \text{...(2)}$$

By taking any two of the above terms for example taking first and second terms and then by direct integration give rise solution $u(x, y) = a$, where a is constant. Similarly by taking another two terms, for example first and last terms and then by direct integration give rise solution $v(x, z) = b$, where b is constant.

$\therefore$ Solution of the equation (1) is

$$f(u, v) = 0 \text{ or } u = f(v).$$

Working Rule :

To solve the equation $Pp + Qq = R$ by the method of grouping,

1. Form the auxiliary equations

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

2. Solve any two equations to get two independent solutions from auxiliary equations. Let the solutions be $u = c_1$ and $v = c_2$.

3. Complete solution is$f(u, v) = 0$ or $u = f(v)$.

SOLVED PROBLEMS

Problem 1. *Solve* $px + qy = z$.

Solution. Given equation is

$$px + qy = z \qquad ...(1)$$

This equation is in the form

$$Pp + Qq = R$$

where P = x, Q = y and R = z.

The Lagrange's Auxiliary equations are

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z} \qquad ...(2)$$

Consider first two terms of (2),

$$\frac{dx}{x} = \frac{dy}{y}$$

by taking integration on both sides

$$\log x = \log y + \log c_1$$

$$\Rightarrow \quad \log x - \log y = \log c_1$$

$$\Rightarrow \quad \log\left(\frac{x}{y}\right) = \log c_1$$

$$\frac{x}{y} = c_1 \qquad ...(3)$$

Now consider second and third terms

$$\frac{dy}{y} = \frac{dz}{z}$$

$$\Rightarrow \quad \log y = \log z + \log c_2$$

$$\Rightarrow \quad \log y - \log z = \log c_2$$

$$\Rightarrow \quad \log\left(\frac{y}{z}\right) = \log c_2$$

$$\Rightarrow \quad \frac{y}{z} = c_2 \qquad ...(4)$$

The complete solution is

$$\phi\left(\frac{x}{y}, \frac{y}{z}\right) = 0 \text{ or } \frac{x}{y} = f\left(\frac{y}{z}\right)$$

Problem 5. Solve *p sin x + q cos y = tan z.*

Solution. Given equation is

$$p \sin x + q \cos y = \tan z \qquad \text{...(1)}$$

This equation is in the form of

$$Pp + Qq = R$$

where P = sin x, Q = cos y, R = tan z.

Lagrange's Auxiliary equations are

$$\frac{dx}{\sin x} = \frac{dy}{\cos y} = \frac{dz}{\tan z} \qquad \text{...(2)}$$

Consider first two terms of equation (2),

$$\frac{dx}{\sin x} = \frac{dy}{\cos y} \Rightarrow \operatorname{cosec} x\, dx = \sec y\, dy$$

Integrating

$$\log(\operatorname{cosec} x - \cot x) = \log(\sec y + \tan y) + \log c_1$$

$$\Rightarrow \quad \log\left(\frac{\operatorname{cos}ec\, x - \cot x}{\sec y + \tan y}\right) = \log c_1$$

$$\Rightarrow \quad \frac{\operatorname{cosec} x - \cot x}{\sec y + \tan y} = c_1 \qquad \text{...(3)}$$

Now consider last two terms of equation (2),

$$\frac{dy}{\cos y} = \frac{dz}{\tan z} \Rightarrow \sec y\, dy = \cot z\, dz$$

Integrating

$$\log(\sec y + \tan y) = \log \sin z + \log c_2$$

$$\Rightarrow \quad \frac{\sec y + \tan y}{\sin z} = c_2 \qquad \text{...(4)}$$

From (3) and (4), the general solution is

$$\phi\left(\frac{\operatorname{cos}ec\, x - \cot x}{\sec y + \tan y}, \frac{\sec y + \tan y}{\sin z}\right) = 0$$

Problem 6. *Solve* $(x^2 - yz)\, p + (y^2 - zx)\, q = z^2 - xy$

Solution. Given equation is

$$(x^2 - yz)\, p + (y^2 - zx)\, q = z^2 - xy \qquad \text{...(1)}$$

This equation is in the form of

$$Pp + Qq = R$$

where $P = x^2 - yz$, $Q = y^2 - zx$, $R = z^2 - xy$.

Lagrange's Auxiliary equations are

$$\frac{dx}{x^2 - yz} = \frac{dy}{y^2 - zx} = \frac{dz}{z^2 - xy} \quad \ldots(2)$$

From (2),

$$\frac{dx - dy}{(x - y)(x + y + z)} = \frac{dy - dz}{(y - z)(x + y + z)} = \frac{dz - dx}{(z - x)(x + y + z)}$$

Consider, first two terms, we have

$$\frac{dx - dy}{x - y} = \frac{dy - dz}{y - z}$$

Integrating $\log (x - y) = \log (y - z) + \log c_1$

$$\Rightarrow \quad \frac{x - y}{y - z} = c_1 \quad \ldots(3)$$

Now consider last two terms, we have

$$\frac{dy - dz}{y - z} = \frac{dz - dx}{z - x} \qquad \phi$$

Integrating $\log (y - z) = \log (z - x) + \log c_2$

$$\Rightarrow \quad \frac{y - z}{z - x} = c_2 \quad \ldots(4)$$

$\therefore$ From (3) and (4), solution of (1) is

$$f\left(\frac{x - y}{y - z}, \frac{y - z}{z - x}\right) = 0 \text{ or}$$

$$\frac{x - y}{y - z} = f\left(\frac{y - z}{z - x}\right).$$

Problem 4: Solve $\quad \boldsymbol{pz - qz = z^2 + (x + y)^2}$.

Solution. Given equation is

$$pz - qz = z^2 + (x + y)^2 \quad \ldots(1)$$

This equation is in the form of

$$Pp + Qv = R$$

where P = z, Q = $-z$, R = $z^2 + (x + y)^2$.

Lagrange's Auxiliary equations are

$$\frac{dx}{z} = \frac{dy}{-z} = \frac{dz}{z^2 + (x+y)^2} \qquad ...(2)$$

Consider first two terms of equation (2),

$$\frac{dx}{z} = \frac{dy}{-z} \Rightarrow dx + dy = 0$$

Integrating $\qquad x + y = c_1 \qquad ...(3)$

Now consider first and third terms of equation (2),

$$\frac{dx}{z} = \frac{dz}{z^2 + (x+y)^2}$$

$$\Rightarrow \qquad dx = \frac{z\,dz}{z^2 + c_1^2} \qquad \text{(since } x + y = c_1\text{)}$$

$$\Rightarrow \qquad \frac{2z\,dz}{z^2 + c_1^2} = 2\,dx.$$

Integrating,

$$\log (z^2 + c_1^2) = 2x + c_2$$

$$\Rightarrow \qquad \log [z^2 + (x + y)^2] - 2x = c_2 \qquad ...(4)$$

$\therefore$ From (3) and (4), the general solution of (1) is

$\phi\ [x + y, \log (x^2 + y^2 + z^2 + 2xy) - 2x] = 0$

or $\quad x + y = f\,[\log (x^2 + y^2 + z^2 + 2xy) - 2x]$

EXERCISE 3

Solve :

1. $(a - x)\,p + (b - y)\,q = c - z$

2. $\dfrac{y^2 z}{x}\,p + xz\,q = y^2$

3. $yz\,p + xz\,q = xy$

4. $px^2 + qy^2 = z^2$

5. $p \tan x + q \tan y = \tan z$

6. $p\sqrt{x} + q\sqrt{y} = \sqrt{z}$

7. $px - qy = -z$

8. $xp + yq = 3z$

9. $p - q = \log (x + y)$

10. $x^2 p + y^2 q = (x + y)\,z$

ANSWER

1. $\phi\left(\dfrac{b-y}{a-x},\dfrac{c-z}{b-y}\right)=0$

2. $\phi\,(x^3-y^3,\,x^2-z^2)=0.$

3. $\phi\,(x^2-y^2,\,y^2-z^2)=0.$

4. $\phi\left(\dfrac{1}{x}-\dfrac{1}{y},\dfrac{1}{y}-\dfrac{1}{z}\right)=0$

5. $\phi\left(\dfrac{\sin x}{\sin y},\dfrac{\sin y}{\sin z}\right)=0$

6. $\phi\left(\sqrt{x}-\sqrt{y},\sqrt{y}-\sqrt{z}\right)=0$

7. $\phi\left(xy,\dfrac{y}{z}\right)=0$

8. $\phi\left(\dfrac{x}{y},\dfrac{x^3}{z}\right)=0$

9. $\phi\left(x+y,x-\dfrac{z}{\log(x+y)}\right)=0$

10. $\phi\left(\dfrac{1}{x}-\dfrac{1}{y},\dfrac{xy}{z}\right)=0$

4.4.2 METHOD OF MULTIPLIERS

Consider Lagrange's equation of the form

$$\text{P}p + \text{Q}q = \text{R} \qquad \text{...(1)}$$

Let the Auxiliary equations be

$$\frac{dx}{\text{P}} = \frac{dy}{\text{Q}} = \frac{dz}{\text{R}}$$

By an algebraic principle we can choose multipliers l, m, n which are functions of x, y, z or constants in such a way that

$$\text{Each ratio} = \frac{ldx+mdy+ndz}{lP+mQ+nR}$$

These l, m, n are so choosen that the

$l\,\text{P} + m\,\text{Q} + n\,\text{R} = 0$, then we have

$$l\,dx + m\,dy + n\,dz = 0 \qquad \text{...(2)}$$

which is the exact differential of the denominator $l\,\text{P} + m\,\text{Q} + n\,\text{R}$.

Solving the differential equation (2), we get

$$u\,(x, y, z) = c_1 \qquad \text{...(3)}$$

Similarly, choose another set of multiplier's l', m', n' such that

$$\text{Each ratio} = \frac{l'dx+m'dy+n'dz}{l'P+m'Q+n'R}$$

then $$l'dx + m'dy + n'dz = 0$$

taking integration, we get

$$v\,(x, y, z) = c_2 \qquad \text{...(4)}$$

$\therefore$ From (3) and (4), solution of equation (1) is,

$$\phi\,(u, v) = 0 \text{ or } u = f\,(v).$$

Note 1. The multipliers l, m, n are known as Lagrange's multipliers.

2. Both the methods *i.e.*, grouping and multipliers can be used to solve the problem.

Solved Problems

Problem 1: Solve $x^2\,(y - z)\,p + y^2\,(z - x)\,q = z^2\,(x - y)$.

Solution. Given equation is

$$x^2\,(y - z)\,p + y^2\,(z - x)\,q = z^2\,(x - y) \qquad \text{...(1)}$$

This equation is in the form of Lagrange's equation

$$\text{P}p + \text{Q}q = \text{R}$$

where $\text{P} = x^2\,(y - z)$, $\text{Q} = y^2\,(z - x)$, $\text{R} = z^2\,(x - y)$

Lagrange's auxiliary or subsidiary equations are

$$\frac{dx}{x^2(y-z)} = \frac{dy}{y^2(z-x)} = \frac{dz}{z^2(x-y)} \qquad \text{...(2)}$$

Using $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ as multipliers,

$$\text{Each ratio} = \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{\frac{1}{x}[x^2(y-z)] + \frac{1}{y}[y^2(z-x)] + \frac{1}{z}[z^2(x-y)]}$$

$$\therefore \qquad \frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0$$

Integrating

$$\log x + \log y + \log z = \log c_1$$

$$\Rightarrow \qquad xyz = c_1 \qquad \text{...(3)}$$

Again using $\frac{1}{x^2}, \frac{1}{y^2}, \frac{1}{z^2}$ as multipliers,

$$\text{Each ratio} = \frac{\frac{1}{x^2}dx + \frac{1}{y^2}dy + \frac{1}{z^2}dz}{\frac{1}{x}[x^2(y-z)] + \frac{1}{y}[y^2(z-x)] + \frac{1}{z}[z^2(x-y)]}$$

$$\therefore \quad \frac{1}{x^2}dx + \frac{1}{y^2}dy + \frac{1}{z^2}dz = 0$$

integrating
$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = c_2 \qquad \ldots(4)$$

$\therefore$ From (3) and (4) complete solution of (1) is

$$\phi\left(xyz, \frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) = 0$$

or
$$xyz = f\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right).$$

Problem 2: *Solve* $x(y^2 - z^2)p + y(z^2 - x^2)q = z(x^2 - y^2)$.

Solution. Given equation is

$$x(y^2 - z^2)p + y(z^2 - x^2)q = z(x^2 - y^2) \qquad \ldots(1)$$

This equation is in the form of

$$Pp + Qq = R$$

where $P = x(y^2 - z^2)$, $Q = y(z^2 - x^2)$, $R = z(x^2 - y^2)$.

Lagrange's Auxiliary equations are

$$\frac{dx}{x(y^2 - z^2)} = \frac{dy}{y(z^2 - x^2)} = \frac{dz}{z(x^2 - y^2)} \qquad \ldots(2)$$

Using x, y, z as Lagrange's multipliers,

$$\text{Each ratio} = \frac{xdx + ydy + zdz}{x[x(y^2 - z^2)] + y[y(z^2 - x^2)] + z[z(x^2 - y^2)]}$$

$$\therefore \quad x\,dx + y\,dy + z\,dz = 0$$

Integrating
$$x^2 + y^2 + z^2 = c_1 \qquad \ldots(3)$$

Again using $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ as multipliers,

$$\text{Each ratio} = \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{0}$$

$$\therefore \qquad \frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0$$

Integrating

$$\log x + \log y + \log z = \log c_2$$

$$\Rightarrow \qquad xyz = c_2 \qquad \ldots(4)$$

$\therefore$ From (3) and (4) solution of equation (1) is

$$\phi\ (x^2 + y^2 + z^2, xyz) = 0 \text{ or}$$

$$x^2 + y^2 + z^2 = f\,(xyz).$$

Problem 3: Solve $(y - z)\ p + (x - y)\ q = z - x.$

Solution. Given equation is

$$(y - z)\ p + (x - y)\ q = z - x \qquad \ldots(1)$$

This equation is in the form of

$$\text{P}p + \text{Q}q = \text{R}$$

where $\text{P} = y - z$, $\text{Q} = x - y$, $\text{R} = z - x$.

Lagrange's Auxiliary equations are

$$\frac{dx}{y-z} = \frac{dy}{x-y} = \frac{dz}{z-x}\ . \qquad ..(2)$$

Using 1, 1, 1 as Lagrange's multipliers,

$$\text{Each ratio} = \frac{dx + dy + dz}{y - z + x - y + z - x} = \frac{dx + dy + dz}{0}$$

$$\therefore \qquad dx + dy + dz = 0$$

Integrating, $\qquad x + y + z = c_1 \qquad \ldots(3)$

Using x, z, y as Lagrange's Multipliers

$$\text{Each ratio} = \frac{xdx + zdy + ydz}{x(y-z) + z(x-y) + y(z-x)} = \frac{xdx + ydy + zdz}{0}$$

$$\therefore \qquad x\,dx + z\,dy + y\,dz = 0$$

$$\Rightarrow \qquad x\,dx + d\,(yz) = 0$$

Integrating

$$\frac{x^2}{2} + yz = c$$

$\Rightarrow$ $$x^2 + 2yz = c_2 \qquad \text{...(4)}$$

$\therefore$ From (3) and (4), the complete solution is

$$f(x + y + z, x^2 + 2yz) = 0$$

or $$x + y + z = f(x^2 + 2yz).$$

1. Solve $(x^3 + 3xy^2)\, p + (y^3 + 3x^2y)q = 2z(x^2+y^2)$ (OU 2017)

Sol. Given equation is $(x^3 + 3xy^2)\, p + (y^3 + 3x^2y)q = 2z(x^2+y^2)$ (1)

This equation in the form of $Pp + Qq = R$

where $P= x^3+3xy^2$, $Q = y^4+3x^2y$, $R= 2z\,(x^2+y^2)$

Lagrange's auxiliary equations are

$$\frac{dx}{x^3 + 3xy^2} = \frac{dy}{y^3 + 3x^2y} = \frac{dz}{2z(x^2 + y^2)} \qquad (2)$$

Using 1,1,0 as multipliers,

$$\text{Each ratio} = \frac{dx + dy}{x^3 + 3xy^2 + y^3 + 3x^2y} = \frac{d(x + y)}{(x + y)^3} \qquad (3)$$

Using 1,–1,0 as multipliers,

$$\text{Each ratio} = \frac{dx - dy}{x^3 + 3xy^2 - y^3 - 3x^2y} = \frac{d(x - y)}{(x - y)^3} \qquad (4)$$

From (3) and (4) $\dfrac{d(x+y)}{(x+y)^3} = \dfrac{d(x-y)}{(x-y)^3}$

$$\int (x + y)^{-3}\, d(x + y) = \int (x - y)^{-3}\, d(x - y)$$

$$\frac{(x + y)^{-2}}{-2} = \frac{(x - y)^{-2}}{-2} + c$$

$$\Rightarrow (x + y)^{-2} - (x-y)^{-2} = -2c$$

(or) $$(x + y)^{-2} - (x - y)^{-2} = c_1 \qquad (5)$$

again choosing $\dfrac{1}{x}, \dfrac{1}{y}, 0$ as multipliers

Each ratio = $\dfrac{\frac{1}{x}dx+\frac{1}{y}dy}{\frac{1}{x}(x^3+3xy^2)+\frac{1}{y}(y^3+3x^2y)}=\dfrac{\frac{1}{x}dx+\frac{1}{y}dy}{4(x^2+y^2)}$ (6)

from (1),(6), we obtain

$$\frac{\frac{1}{x}dx+\frac{1}{y}dy}{4(x^2+y^2)}=\frac{dz}{2z(x^2+y^2)}$$

$$\Rightarrow \frac{1}{x}dx+\frac{1}{y}dy=\frac{2}{z}dz$$

integrating on both sides

$$\int \frac{1}{x}dx+\frac{1}{y}dy=2\int\frac{1}{z}dz$$

$$\log x+\log y=2\log z+\log c_2$$

$$\log(xy)=\log(z^2c_2)$$

Taking anti log on both sides

$$\Rightarrow \frac{xy}{z^2}=c_2 \qquad (7)$$

from (5) and (7) the solution of (1) is

$$\phi\left((x+y)^{-2}-(x-y)^{-2},\frac{xy}{z^2}\right)=c_2$$

Problem 4: Solve $(x+2z)p + (4xz-y)q = 2x^2+y$ **(OU Dec 2017)**

Sol. Given equation is $(x+2z)p + (4xz-y)q = 2x^2+y$ (1)

This is in the form of $Pp + Qq = R$.

where $P = x + 2z$, $Q = 4xz - y$, $R = 2x^2 + y$

Lagrange's auxiliary equations are,

$$\frac{dx}{x+2z}=\frac{dy}{4xz-y}=\frac{dz}{2x^2+y} \qquad (2)$$

Using $y,x,-2z$ as multipliers,

Each ratio = $\dfrac{ydx + xdy - 2zdz}{y(x+2z) + x(4xz - y) - 2z(2x^2 + y)}$

$$= \frac{\text{ydx + xdy - 2zdz}}{0}$$

$$\Rightarrow d(yx) - 2zdz = 0$$

integrating on bothsides

$$yx - z^2 = c_1 \qquad (3)$$

again by the method of grouping, Using 2x,-1,-1 as multipliers,

Each ratio = $\dfrac{2xdx - dy - dz}{2x(x+2z) - (4xz - y) - (2x^2 + y)}$

$$= \frac{\text{2xdx - dy - dz}}{0}$$

$$\Rightarrow 2xdx - dy - dz = 0$$

integrating bothsides

$$x^2 - y - z = c_2 \quad (4)$$

$\therefore$ from (3) and (4), solution of (1) is given by

$$\phi(xy - z^2, x^2 - y - z) = 0$$

$$\text{(or) } xy - z^2 = f(x^2 - y - z)$$

Problem 5: Solve $y^2p - xyq = x(z - 2y)$ **(OU Dec 2013 & 2015)**

Sol. Given equation is $y^2p - xyq = x(z - 2y)$ (1)

This equation is in the form of Lagrange's equation

$$Pp + Qq = R.$$

where $P = y^2$, $Q = -xy$, $R = x(z - 2y)$

Langrange's auxiliary equations are

$$\frac{dx}{y^2} = \frac{dy}{-xy} = \frac{dz}{x(z - 2y)} \qquad (2)$$

Using 1st and 2nd fraction of (2)

$$\frac{dx}{y^2} = \frac{dy}{-xy} \Rightarrow xdx = -ydy$$

$$\Rightarrow xdx + ydy = 0,$$

Integrating both sides

$$\Rightarrow \frac{x^2}{2} + \frac{y^2}{2} = c$$

$$\Rightarrow x^2 + y^2 = c_1 \qquad (3)$$

where c_1 is an arbitrary constant

Now taking 2nd and 3rd fraction of (2)

$$\frac{dy}{-xy} = \frac{dz}{x(z-2y)}$$

$$\Rightarrow \frac{dz}{dy} = \frac{-z+2y}{y}$$

$$\Rightarrow \frac{dz}{dy} + \frac{z}{y} = 2 \qquad (4)$$

$\frac{dz}{dy} + \frac{z}{y} = 2$ which is a L.D.E in z

Comparing with $\frac{dz}{dy} + P(y)z = Q(y)$

where $P(y) = \frac{1}{y}, Q(y) = 2$

IF = $e^{\int p(y)dy} = e^{\int \frac{1}{y}dy} = e^{\log y} = y.$

Hence the general solution of (4) is

$$z.(\text{I.F}) = \int Q(y) IF dy + c_2$$

$$\Rightarrow zy = \int 2y dy + c_2$$

$$\Rightarrow zy = y_4^2 + c_2$$

$\therefore$ From (3) and (4), the complete solution is

$\phi(x^2+y^2, zy-y^2)=0$ (or) $x^2+y^2 = f(zy-y^2)$.

EXERCISE 4

Solve :

1. $x(y^2+z)p - y(x^2+z)q = (x^2-y^2)z$

2. $(x+y)zp + (x-y)zq = x^2+y^2$

3. $\left(\dfrac{y-z}{yz}\right)p + \left(\dfrac{z-x}{zx}\right)q = \left(\dfrac{x-y}{xy}\right)$

4. $(y^2+z^2)p - xy\,q + xz = 0$

5. $(3z-4y)p + (4x-2z)q = 2y-3x$

6. $(2z-y)p + (x+z)q + 2x + y = 0$

7. $\left(\dfrac{y^2z}{x}\right).p + xz\,q = y^2$

8. $p\tan x + q\tan y = \tan z$

9. $x(z^2-y^2)p + y(x^2-z^2)q = z(y^2-x^2)$

10. $x^2p + y^2q = (x+y)z$

ANSWER

1. $\phi\,(xyz, x^2+y^2-2z) = 0$

2. $\phi\,(x^2-y^2-z^2, z^2-2xy) = 0$

3. $\phi\,(x+y+z, xyz) = 0$

4. $\phi\left(\dfrac{y}{z}.x^2+y^2+z^2\right) = 0$

5. $\phi\,(x^2+y^2+z^2, 2x+3y+4z) = 0$

6. $\phi\,(x^2+y^2+z^2, x+2y-z) = 0$

7. $\phi\,(x^3-y^3, x^2-z^2) = 0$

8. $\phi\left(\dfrac{\sin x}{\sin y}, \dfrac{\sin y}{\sin z}\right) = 0$

9. $\phi\,(x^2+y^2+z^2, xyz) = 0$

10. $\phi\left(\dfrac{xy}{z}.\dfrac{1}{y}-\dfrac{1}{x}\right) = 0.$

4.5 NON-LINEAR PARTIAL DIFFERENTIAL EQUATION OF FIRST ORDER

A partial differential equation which involves first order partial derivatives p and q with degree higher than one and the products of p and q is called a non-linear partial differential equation of the first order.

The complete solution of such an equation involves only two arbitrary constants (*i.e.*, equal to the number of independent variables).

Here we discussed the solution of four standard types of first order non-linear partial differential equations.

4.5.1 STANDARD FORM: I

Equations of the form $f(p, q) = 0$, involving only p and q and not x, y, z explicitly.

Let $$Z = ax + by + c \quad ...(1)$$

be the solution of equation $f(p, q) = 0$.

Differentiating (1) partially w.r.t., x, we get

$$p = \frac{\partial z}{\partial x} = a$$

Similarly, $$q = \frac{\partial z}{\partial y} = b$$

$\therefore$ We get $$f(a, b) = 0$$

$\therefore$ $Z = ax + by + c$ will be a solution of

$f(p, q) = 0$ if $f(a, b) = 0$.

Now solving for b from $f(a, b) = 0$,

We get $$b = f(a)$$

Hence the complete integral of $f(p, q) = 0$ is

$$Z = ax + y\, f(a) + c \quad ...(2)$$

To obtain the singular integral, we have to eliminate a and c from (2) and equations

$\frac{\partial z}{\partial a} = 0, \ \frac{\partial z}{\partial c} = 0.$

$$\frac{\partial z}{\partial c} = 0 \Rightarrow 1 = 0 \text{ which is absurd.}$$

$\therefore$ There is no singular integral for the given equation.

To find General Solution,

Put $c = \phi(a)$ in (2), we get

$$Z = ax + y\, f(a) + \phi(a) \quad (3)$$

Differentiating (3) w.r.t., 'a', we get

$$x + y\ f'(a) + \phi'(a) = 0 \quad (4)$$

Eliminating 'a' from (3) and (4), we get the general solution.

SOLVED PROBLEMS

Problem 1. Solve $pq = p + q$

Solution. The given equation is

$$pq = p + q \tag{1}$$

This equation is in the form of

$$f(p, q) = 0 \tag{2}$$

The complete solution of (2) is given by

$$Z = ax + by + c$$

$$\Rightarrow \qquad \frac{\partial z}{\partial x} = p = a \text{ and } \frac{\partial z}{\partial y} = q = b.$$

Hence from (1), $ab = a + b$

$$\Rightarrow \qquad b(a-1) = a$$

$$\Rightarrow \qquad b = \frac{a}{a-1}$$

$\therefore$ The complete integral of (1) is,

$$Z = ax + \left(\frac{a}{a-1}\right) y + c \tag{3}$$

Differentiating (3) partially w.r.t., 'c'

we get 0 = 1 which is absurd.

Hence singular integral does not exist.

To find general integral, put $c = \phi(a)$ in (3), we get

$$Z = ax + \left(\frac{a}{a-1}\right) y + \phi(a) \tag{4}$$

Differentiating partially w.r.t., 'a' and equating to zero, we have

$$0 = x - \frac{y}{(a-1)^2} + \phi'(a) \tag{5}$$

Eliminating 'a' from (4) and (5), we get the general integral.

Problem 2: ***Find the complete integral of*** $\sqrt{p} + \sqrt{q}$ ***= 1.***

Solution. Given equation is

$$\sqrt{p} + \sqrt{q} = 1 \qquad (1)$$

The given equation is in the form of

$$f(p, q) = 0 \qquad (2)$$

The complete solution of (2) is given by

$$Z = ax + by + c \qquad (3)$$

$$\Rightarrow \qquad \frac{\partial z}{\partial x} = p = a, \ \frac{\partial z}{\partial y} = q = b$$

and hence from (1), $\sqrt{a} + \sqrt{b} = 1$

$$\Rightarrow \qquad \sqrt{b} = 1 - \sqrt{a}$$

$$\Rightarrow \qquad b = (1-\sqrt{a})^2$$

∴ From (3) the complete solution of (1) is

$$Z = ax + (1-\sqrt{a})^2\, y + c.$$

Problem 3: Find the complete integral of $p^2 + q^2 = mpq$

Sol. Given equation is $p^2 + q^2 = \text{mpq}$ (1)

can be written as $p^2 + q^2 - mpq = 0$ (2)

This is in the form of $f(p,q) = 0$ (3)

The complete solution of (3) is given by

$$z = ax + by + c$$

$$\Rightarrow \frac{\partial z}{\partial x} = p = a, \frac{\partial z}{\partial y} = q = b$$

Hence from (2) $a^2 + b^2 - mab = 0.$

$b^2 - mab + a^2 = 0$ is a quadratic equation in b

$$b = \frac{ma \pm \sqrt{m^2a^2 - 4a^2}}{2}$$

$$b = \frac{ma \pm a\sqrt{m^2 - 4}}{2}$$

Substitute *a, b* values in *z*.

$$\Rightarrow z = ax + a\left(\frac{m \pm \sqrt{m^2 - 4}}{2}\right)y + c$$

4.5.2 STANDARD FORM : II

Equations of the form Z = $px + qy + f(p, q)$, which are also known as CLAIRAUT'S TYPE. General form of the given equation is,

$$Z = px + qy + f(p, q) \quad \text{...(1)}$$

We can easily find that the solution of (1) is

$$Z = ax + by + f(a, b) \quad \text{...(2)}$$

where a and b are constants.

Since equation (2) contains two arbitrary constants, therefore it is complete integral. Thus the complete integral of a partial differential equation in CLAIRAUT'S form is obtained by simply putting $p = a$ and $q = b$ in the given equation.

The singular and general integral are obtained in the usual manner.

SOLVED PROBLEMS

Problem 1: Find the complete integral of *pq(px + qy - z)* = 1. (OU Dec 2015)

Sol. Given equation is pq(px + qy - z) = 1 (1)

$$\Rightarrow px + qy - z = \frac{1}{pq}$$

$$\Rightarrow z = px + qy - \frac{1}{pq} \quad (2)$$

This equation is in the form of Clairaut's form to obtain complete integral put $p = a$ and $q = b$ in (2)

$$\Rightarrow p^2x = a \text{ and } q\frac{y}{1+y^2} = a$$

$$\Rightarrow p = \sqrt{\frac{9}{x}} \text{ and } q = \frac{a(1+y^2)}{y}$$

Substitute p,q values in $dz = pdx + qdy$

$$\Rightarrow dz = \sqrt{\frac{a}{x}}dx + a\left(\frac{1+y^2}{y}\right)dy$$

$$\Rightarrow dz = \sqrt{a}x^{\frac{-1}{2}}dx + a\left(\frac{1}{y} + y\right)dy$$

integrating on bothsides

$$z = \frac{\sqrt{a}x^{\frac{1}{2}}}{\left(\frac{1}{2}\right)} + a\left(\log y + \frac{y^2}{2}\right) + b$$

(or) $\quad z = 2\sqrt{ax} + a\log y + \frac{ay^2}{2} + b$

where a and b are arbitrary constant.

Problem 2. Solve $z = px + qy + pq.$

Solution. Given differential equation is

$$z = px + qy + pq \qquad ...(1)$$

This equation is of the form Clairaut's type.

$\therefore$ The complete integral of (1) is given by

$$z = ax + by + ab \qquad ...(2)$$

where a and b are arbitrary constants.

Differentiating (2) partially w.r.t., a and b and equating to zero, we have

$$x + b = 0 \qquad ...(3)$$

and $\quad y + a = 0 \qquad ...(4)$

From (2), (3), (4) eliminate a and b we have $b = -x$ and $a = -y$ and putting in (2), we get

$$z = -xy - xy + xy = -xy.$$

$\therefore$ The singular integral of (1) is

$$z = -xy \qquad ...(5)$$

To get the general integral, put $b = f(a)$ where f is arbitrary in equation (2), we have

$$z = ax + y f(a) + a f(a) \qquad ...(6)$$

Differentiating (6) partially w.r.t., 'a' and equating to zero, we have

$$0 = x + y \ f'(a) + a \ f'(a) + f(a) \qquad ...(7)$$

By eliminating 'a' between the equations (6) and (7), we get the general integral.

Problem 5. Solve $z = px + qy + c\sqrt{(1+p^2+q^2)}$.

Solution. Given equation is

$$z = px + qy + c\sqrt{(1+p^2+q^2)} \qquad ...(1)$$

This equation is in the form of CLAIRAUT'S equation.

$\therefore$ The complete integral of (1) is given by

$$z = ax + by + c\sqrt{(1+a^2+b^2)} \qquad ...(2)$$

To find Singular Integral

Differentiating (2) partially w.r.t., a and b and equating to zero, we have

$$0 = x + \frac{ac}{\sqrt{(1+a^2+b^2)}} \qquad ...(3)$$

$$0 = y + \frac{bc}{\sqrt{(1+a^2+b^2)}} \qquad ...(4)$$

From (3) and (4),

$$x^2 + y^2 = \frac{a^2c^2 + b^2c^2}{1+a^2+b^2}$$

$$\Rightarrow \qquad c^2 - x^2 - y^2 = c^2 - \frac{a^2c^2+b^2c^2}{1+a^2+b^2} = \frac{c^2}{1+a^2+b^2}$$

$$\Rightarrow \qquad 1 + a^2 + b^2 = \frac{c^2}{c^2-x^2-y^2} \qquad ...(5)$$

From (3), we have

$$a = -\frac{x\sqrt{(1+a^2+b^2)}}{c} = -\frac{x}{\sqrt{(c^2-x^2-y^2)}}, - \qquad \text{[From (5)]}$$

Similarly from (4) and (5), we get

$$b = \frac{y}{\sqrt{(c^2-x^2-y^2)}}$$

Putting these values of a and b in (2), the singular solution is,

$$z = \frac{-x^2}{\sqrt{(c^2 - x^2 - y^2)}} - \frac{y^2}{\sqrt{(c^2 - x^2 - y^2)}} + \frac{c^2}{\sqrt{(c^2 - x^2 - y^2)}}$$

$$\Rightarrow \qquad z = \frac{c^2 - x^2 - y^2}{\sqrt{(c^2 - x^2 - y^2)}}$$

$$\Rightarrow \qquad z = \sqrt{(c^2 - x^2 - y^2)}$$

$$\Rightarrow \qquad z^2 = c^2 - x^2 - y^2$$

$$\Rightarrow \qquad x^2 + y^2 + z^2 = c^2$$

which is the singular solution of (1).

To find General Integral

Replace b by $f(a)$ in (2) and differentiate the resulting equation w.r.t., 'a'. Then eliminate 'a' between these two equations gives the desired general integral.

EXERCISE 5

Find the complete and singular integrals of the following :

1. $Z = px + qy + p^2 - q^2$

2. $Z = px + qy + \frac{p}{q} - p$

3. $Z = px + qy - p^2q$

4. $Z = px + qy - 2\sqrt{pq}$

5. $Z = px + qy + \sqrt{1 + p^2 + q^2}$

6. $Z = px + qy + \sqrt{\frac{pq}{p+q}}$

7. $Z = (p + q)(3 - px - qy) = 1$

8. $4xyz = pq + 2px^2y + 2qxy^2$

ANSWER

1. C.I. is $Z = ax + by + a^2 - b^2$

S.I. is $4Z = y^2 - x^2$.

2. C.I. is $Z = ax + by + \frac{a}{b} - a$

S.I. is $Zy = 1 - x$.

3. C.I. is $Z = ax + by - a^2b$,

S.I. is $Z = x^2y$

4. C.I. is $Z = ax + by - 2\sqrt{ab}$

S.I. is $xy = 1$.

5. C.I. is $Z = ax + by + \sqrt{1+a^2+b^2}$

S.I. is $x^2 + y^2 + z^2 = 1$.

6. C.I. is $Z = ax + by + \sqrt{\dfrac{ab}{a+qb}}$

7. C.I. is $Z = ax + by + \dfrac{1}{a+b}$

8. $Z = ax^2 + by^2 + ab$.

4.5.3 STANDARD FORM : III

Equations of the form $f(p, q, z) = 0$.

Given equation is of the form

$$f(p, q, z) = 0 \qquad \ldots(1)$$

in which independent variables x and y do not occur explicitly.

let
$$z = f(x + ay) \qquad \ldots(2)$$

be a trail solution of (1).

Put $u = x + ay$ so that $z = f(u)$.

$$\therefore \qquad p = \frac{dz}{du}\frac{\partial u}{\partial x} = \frac{dz}{du} \text{ and}$$

$$q = \frac{dz}{du}\frac{\partial u}{\partial y} = a\frac{dz}{du}$$

Substituting the values of p and q in equation (1), we get

$$f\left(z, \frac{dz}{du}, a\frac{dz}{du}\right) = 0$$

which is an ordinary differential equation of first order.

Solving for $\frac{dz}{du}$, we get

$$\frac{dz}{du} = g(a, z)$$

$$\Rightarrow \qquad \frac{dz}{g(a, z)} = du$$

integrating solution is given by

$$\phi(z, a) = u + b = x + ay + b$$

which is the complete integral of (1).

The singular and general solutions can be obtained by the usual methods.

Working rule to solve $f(p, q, z) = 0$.

1. Let $u = x + ay$ and substitute

$$p = \frac{dz}{du} \text{ and } q = a\frac{dz}{du} \text{ in the given equation.}$$

2. Solve the resulting ordinary differential equation in z and u.

3. Substitute $x + ay$ for u.

SOLVED PROBLEMS

Problem 1. Solve p(1 + q) = qz **(OU 2017)**

Sol. The equation is $p(1 + q) = qz$ (1)

This equation is in the form of $f(z,p,q) = 0$ (2)

Let $z = f(u)$ where $u = x + ay$ be the solution of (1)

So that $p = \frac{dz}{du}$ and $q = a\frac{dz}{du}$

Substituting these values in equation (1), we have

$$\frac{dz}{du}\left[1 + a\frac{dz}{du}\right] = a\frac{dz}{du}z$$

$$1 + a\frac{dz}{du} = az$$

$$a\frac{dz}{du} = (az - 1)$$

$$\frac{adz}{(az-1)} = du$$

variables are seperable, integrating on bothsides

$$\log(az - 1) = u + c$$

Replace $u = x + ay$

$$\Rightarrow \log(az - 1) = (x + ay) + c.$$

which is the complete solution.

Problem 2. Solve $z^2 = p^2 + q^2 + 1$.

Solution. The given equation is

$$z^2 = p^2 + q^2 + 1 \qquad \text{...(1)}$$

This equation is in the form

$$f(z, p, q) = 0 \qquad \text{...(2)}$$

So let $z = f(u)$ where $u = x + ay$ be a solution of equation (2),

So that $p = \frac{dz}{du}$ and $q = a\frac{dz}{du}$

Substituting these values of p and q in the given equation (1), we have

$$z^2 = 1 + \left(\frac{dz}{du}\right)^2 + a^2\left(\frac{dz}{du}\right)^2$$

$$\Rightarrow \qquad \sqrt{z^2 - 1} = \sqrt{1 + a^2}\frac{dz}{du}$$

$$\Rightarrow \qquad \frac{dz}{\sqrt{z^2 - 1}} = \frac{1}{\sqrt{1 + a^2}}du$$

Integrating, $\qquad \cosh^{-1} z = \frac{u}{\sqrt{1 + a^2}} + c$

$$\Rightarrow \qquad z = \cosh\left(\frac{x + ay}{\sqrt{1 + a^2}} + c\right)$$

where $u = x + ay$.

Problem 4. Solve $z^2(p^2 + q^2 + 1) = 1$.

Solution. Given $\quad z^2(p^2 + q^2 + 1) = 1 \qquad \text{...(1)}$

This equation is in the form of

$$f(z, p, q) = 0 \qquad \text{...(2)}$$

Let $z = f(u)$, where $u = x + ay \qquad \text{...(3)}$

be the solution of equation (1)

So that $\qquad p = \frac{dz}{du}$ and $q = a\frac{dz}{du}$.

Substituting these values in equation (1), we have

$$z^2\left[\left(\frac{dz}{du}\right)^2 + a^2\left(\frac{dz}{du}\right)^2 + 1\right] = 1$$

$$\Rightarrow \qquad \left(\frac{dz}{du}\right)^2 + a^2\left(\frac{dz}{du}\right)^2 + 1 = \frac{1}{z^2}$$

$$\Rightarrow \qquad (a^2 + 1)\left(\frac{dz}{du}\right)^2 = \frac{1}{z^2} - 1 = \frac{1 - z^2}{z^2}$$

$$\Rightarrow \qquad \frac{dz}{du} = \frac{\sqrt{1 - z^2}}{z\sqrt{a^2 + 1}}$$

$$\Rightarrow \qquad \frac{zdz}{\sqrt{1 - z^2}} = \frac{1}{\sqrt{a^2 + 1}}\,du$$

Integrating

$$-\frac{1}{2}(2\sqrt{1 - z^2}) = \frac{u}{\sqrt{a^2 + 1}} + c \qquad \left[\because \int \frac{f'(x)}{\sqrt{f(x)}}\,dx = 2\sqrt{f(x)}\right]$$

$$\Rightarrow \qquad -\sqrt{1 - z^2} = \frac{x + ay}{\sqrt{a^2 + 1}} + c$$

which is the required complete integral.

Problem 5. *Solve* $z = p^2 + q^2$.

Solution. Given equation is $z = p^2 + q^2$ ...(1)

This equation is in the form of

$$f(z, p, q) = 0 \qquad \text{...(2)}$$

Let the solution of (2) be

$$z = f(u) \text{ where } u = x + ay$$

So that $p = \dfrac{dz}{du}$ and $q = a\,\dfrac{dz}{du}$.

Substitute these values in equation (1), we have

$$z = \left(\frac{dz}{du}\right)^2 + a^2\left(\frac{dz}{du}\right)^2$$

$$\Rightarrow \qquad z = (1 + a^2)\left(\frac{dz}{du}\right)^2$$

$$\Rightarrow \qquad \frac{dz}{du} = \sqrt{\frac{z}{1+a^2}}$$

$$\Rightarrow \qquad \sqrt{1+a^2}\,\frac{dz}{\sqrt{z}} = du$$

Integrating,

$$\left(\sqrt{1+a^2}\right)\left(2\sqrt{2}\right) = u + c$$

$$(1 + a^2)\,4z = (u + c)^2$$

∴ The complete integral of (1) is,

$$4\,(1 + a^2)\,z = (x + ay + c)^2 \qquad \ldots(3)$$

To Find General Solution

Differentiating (3) partially w.r.t., a and b, we get

$$8az = 2y\,(x + ay + c) \qquad \ldots(4)$$

and

$$0 = 2\,(x + ay + c) \qquad \ldots(5)$$

Using (4) and (5), we get $8az = 0$

$z = 0$ is the singular integral of (1).

To find the General Solution :

Put $c = \text{f}\,(a)$ in equation (3), we have

$$4\,(1 + a^2)\,z = [x + ay + \text{f}\,(a)]^2 \qquad \ldots(6)$$

Differentiating (6) w.r.t. 'a', we get

$$8az\,(1 + a^2) = 2\,[x + ay + \text{f}\,(a)]\,[y + \text{f}\,¢\,(a)] \qquad \ldots(7)$$

Eliminating 'a' from (6) and (7)

we get the general solution of (1).

Problem 6. *Solve* $p^3 + q^3 = 27z.$

Solution. Given $p^3 + q^3 = 27z$ ...(1)

Let Z = $f\,(u)$, where $u = x + ay$ ...(2)

be the solution of equation (1),

so that $\dfrac{dz}{du} = p,\ q = a\,\dfrac{dz}{du}$

Substitute these values in (1), we have

$$\left(\frac{dz}{du}\right)^3 + a^3\left(\frac{dz}{du}\right)^3 = 27z$$

$$\Rightarrow \qquad (1 + a^3)\left(\frac{dz}{du}\right)^3 = 27z$$

$$\Rightarrow \qquad \frac{dz}{du} = \left(\frac{27z}{(1+a^3)}\right)^{\frac{1}{3}} = \frac{3z^{\frac{1}{3}}}{(1+a^3)^{\frac{1}{3}}}$$

$$\Rightarrow \qquad (1+a^3)^{\frac{1}{3}} \; z^{\frac{2}{3}} \; \frac{dz}{z^{\frac{1}{3}}} = 3\,du$$

Integrating on both sides

$$\frac{3}{2}\,(1+a^3)^{\frac{1}{3}} = 3u + 3c$$

where c is an arbitrary constant.

$$\therefore \qquad (1+a^3)^{\frac{1}{3}} \; z^{\frac{2}{3}} = 2\,(u + c)$$

$$\therefore \qquad (1 + a^3)\,z^2 = 8\,(u + c)^3$$

i.e., $$(1 + a^3)\,z^2 = 8\,(x + ay + c)^3 \qquad \ldots(3)$$

which is the required complete integral.

To Find Singular Integral

Now differentiating (3) partially w.r.t., 'a' and c, we get

$$3a^2z^2 = 24y\,(x + ay + c)^2 \qquad \ldots(4)$$

$$0 = 24\,(x + ay + c)^2 \qquad \ldots(5)$$

Using (5) in (4), we get

$$3a^2z^2 = 0$$

$\Rightarrow$ $z = 0$ is the singular integral.

To Find General Solution

Now put $c = \phi\,(a)$ in equation (3),

$$(1 + a^3)\,z^2 = 8\,[x + ay + \phi\,(a)]^3 \qquad \ldots(6)$$

Differentiating (6) w.r.t., 'a'

$$3a^2z^2 = 24\,[x + ay + \phi\ (a)]^2\,[y + f'\ (a)] \qquad \text{...(7)}$$

Eliminating 'a' from (6) and (7),
we get the general solution of equation (1).

EXERCISE

Find the complete integral of the following :

1. $4z = pq$

2. $z\,pq = p + q$

3. $p^2 z^2 + q^2 = p^2 q$

4. $p + q = \dfrac{z}{c}$

5. $p^3 + q^3 = 3pq\,z$

6. $p^2 z^2 + q^2 = 1$

7. $p\,(1 + q) = qz$

8. $p^3 = qz$

9. Solve $p^2 + pq = 4z$

10. Solve $p^3 + q^3 = 8z$.

ANSWER

1. $az = (x + ay + b)^2$

2. $x + ay + b = \dfrac{4az}{1-a}$ or $z = c$

3. $z = a \tan\,(x + ay + b)$

4. $x + ay + b = c\,(1 + a) \log z$

5. $\dfrac{1+a^3}{3a} \log az = (x + ay) + c$

6. f' $x + ay + b = \dfrac{z}{2}\sqrt{z^2 + a^2} + \dfrac{a^2}{2}\sinh^{-1}\left(\dfrac{z}{a}\right)$

7. $\log\,(az - 1) = x + ay + b$

8. $4z = a\,(x + ay - b)^2$

9. C.I. : $(1 + a)\,z = (x + ay + b)^2$
S.I. : $z = 0$.

10. C.I. : $3\,(1 + a^3)\,z^2 = 4\,(x + ay + b)^3$
S.I. : $z = 0$.

4.5.5 STANDARD FORM : IV

Equations of the form $f_1\,(x, p) = f_2\,(y, q)$.

Given equation is of the form

$$f_1\,(x, p) = f_2\,(y, q) \qquad \text{...(1)}$$

Let us assume each expression is equal to some constant 'a', so that

$$f_1\,(x, p) = a \text{ and } f_2\,(y, q) = a$$

Solving for p and q from these equations, we get

$$p = f_1\,(x, a) \text{ and } q = f_2\,(y, a)$$

Now we know that,

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

$\Rightarrow$ $$dz = p\ dx + q\ dy$$

$\Rightarrow$ $$dz = f_1\ (x,\ a)\ dx + f_2\ (y,\ a)\ dy$$

Integrating,

$$z = \int f_1(x,a)dx + \int f_2(y,a)dy + b$$

This is the complete integral of equation (1), we observe that

$\frac{\partial z}{\partial b}$ = 0 gives the absurd equation

1 = 0 and hence there is no singular integral.

The general integral is obtained in the usual manner.

Solved Problems

Problem 1. Find the complete integral of $p^2 - q^2 = x–y$

(OU July 2014, Dec 2015)

Sol. Given equation is $p^2 - q^2 = x - y$ (1)

This equation can be written as $p^2 - x = q^2 - y$

This is in the form $f_1(x,p) = f_2(y,q)$.

Let $p^2 - x = q^2 - y = a$

$\Rightarrow p^2 - x = a$ and $q^2 - y = a$.

$\Rightarrow p = \sqrt{a+x}$, $q = \sqrt{a-y}$

Put these values in dz = pdx + qdy

$\Rightarrow dz = \sqrt{a+x}dx + \sqrt{a-y}dy$

$\Rightarrow dz = (a+x)^{\frac{1}{2}}dx + (a-y)^{\frac{1}{2}}dy$

Integrating on bothsides

$$z = \frac{(a+x)^{\frac{3}{2}}}{\left(\frac{3}{2}\right)} + \frac{(a-y)^{\frac{3}{2}}}{\left(\frac{-3}{2}\right)} + c$$

$$\Rightarrow 3z = 2(a+x)^{\frac{3}{2}} - 2(a-y)^{\frac{3}{2}} + b \quad \left(\text{where } b = \frac{2}{3}c\right)$$

which is complete integral of (1).

Problem 2. Solve $\boldsymbol{p^2x(1 + y^2) = qy}$ **(OU Nov 2016)**

Sol. Given equation is $p^2x(1 + y^2) = qy$ .....(1)

This equation can be written as $p^2 x = q\frac{y}{1+y^2}$

and it is in the form of $f_1(x,p) = f_2(y,q)$

let $p^2 x = q\left(\frac{y}{1+y^2}\right) = a$

$$\Rightarrow p^2x = a \text{ and } q\frac{y}{1+y^2} = a$$

$$\Rightarrow p = \sqrt{\frac{9}{x}} \text{ and } q = \frac{a(1+y^2)}{y}$$

Substitute p,q values in $dz = pdx + qdy$

$$\Rightarrow dz = \sqrt{\frac{a}{x}}dx + a\left(\frac{1+y^2}{y}\right)dy$$

$$\Rightarrow dz = \sqrt{a}x^{\frac{-1}{2}}dx + a\left(\frac{1}{y} + y\right)dy$$

integrating on bothsides

$$z = \frac{\sqrt{a}x^{\frac{1}{2}}}{\left(\frac{1}{2}\right)} + a\left(\log y + \frac{y^2}{2}\right) + b$$

(or) $z = 2\sqrt{ax} + a\log y + \frac{ay^2}{2} + b$

where a and b are arbitrary constant.