UNIT 1

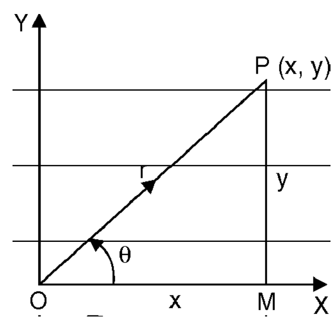
Solutions of Complex Variable

1.1 INTRODUCTION

A complex number z is an ordered pair (x, y) of real numbers and is written as

$$z = x + iy, \quad \text{where } i = \sqrt{-1}.$$

The real numbers x and y are called the real and imaginary parts of z. In the Argand's diagram, the complex number z is represented by the point $P(x, y)$. If (r, θ) are the polar coordinates of P, then $r = \sqrt{x^2 + y^2}$ is called the modulus of z and is denoted by $|z|$. Also $\theta = \tan^{-1} \dfrac{y}{x}$ is called the argument of z and is denoted by arg. z. Every non-zero complex number z can be expressed as



$$z = r(\cos\theta + i \sin\theta) = re^{i\theta}$$

If $z = x + iy$, then the complex number $x - iy$ is called the conjugate of the complex number z and is denoted by $\bar{z}$.

Clearly, $\quad |\bar{z}| = |z|, |z|^2 = z\bar{z},$

$$\text{Re}(z) = \frac{z + \bar{z}}{2}, \quad \text{Im}(z) = \frac{z - \bar{z}}{2i}.$$

1.2 DEFINITIONS

Let S be a non-empty set of complex numbers and δ be a positive real number.

1. Circle. $|z - a| = r$ represents a circle C with centre at the point a and radius r.

2. Open disk. The set of points which satisfies the equation $|z - z_0| < \delta$ defines an open disk of radius δ with centre at $z_0 = (x_0, y_0)$. This set consists of all points which lie inside circle C.

3. Closed disk. The set of points which satisfies the equation $|z - z_0| \le \delta$ defines a closed disk of radius δ with centre at $z_0 = (x_0, y_0)$. This set consists of all points which lie inside and on the boundary of circle C.

4. Annulus. The set of points which lie between two concentric circles $C_1 : |z - a| = r_1$ and $C_2 : |z - a| = r_2$ defines an open annulus *i.e.*, the set of points which satisfies the inequality $r_1 < |z - a| < r_2$.

The set of points which satisfies the inequality $r_1 \le |z - a| \le r_2$ defines a closed annulus.

It is to be noted that $r_1 \le |z - a| < r_2$ is neither open nor closed.

5. Neighbourhood. δ-Neighbourhood of a point z_0 is the set of all points z for which $|z - z_0| < \delta$ where δ is a positive constant. If we exclude the point z_0 from the open disk $|z - z_0| < \delta$ then it is called the deleted neighbourhood of the point z_0 and is written as $0 < |z - z_0| < \delta$.

6. Interior and exterior points. A point z is an interior point of S if all the points in some δ-neighbourhood of z are in S and an exterior point of S if they are outside S.

7. Boundary point. A point z is a boundary point of S if every δ-neighbourhood of z contains at least one point of S and at least one point not in S. For example, the points on the circle $|z - z_0| = r$ are the boundary points for the disk $|z - z_0| \leq r$.

8. Open and closed sets. A set S is open if every point of S is an interior point while a set S is closed if every boundary point of S belongs to S. *e.g.* $S = \{z : |z - z_0| < r\}$ is open set while $S = \{z : |z - z_0| \leq r\}$ is closed set.

9. Bounded set. An open set S is bounded if $\exists$ a positive real number M such that $|z| \leq M$ for all $z \in S$ otherwise unbounded.

For example : the set $S = \{z : |z - z_0| < r\}$ is a bounded set while the set $S = \{z : |z - z_0| > r\}$ is an unbounded set.

10. Connected set. An open set S is connected if any two points z_1 and z_2 belonging to S can be joined by a polygonal line which is totally contained in S.

11. Domain. An open connected set is called a domain denoted by D.

12. Region. A region is a domain together with all, some or none of its boundary points. Thus a domain is always a region but a region may or may not be a domain.

13. Finite complex plane. The complex plane without the point $z = \infty$ is called the finite complex plane.

14. Extended complex plane. The complex plane to which the point $z = \infty$ has been added is called the extended complex plane.

1.3 FUNCTION OF A COMPLEX VARIABLE

If x and y are real variables, then $z = x + iy$ is called a **complex variable.** If corresponding to each value of a complex variable $z(= x + iy)$ in a given region R, there correspond one or more values of another complex variable $w\ (= u + iv)$, then w is called a function of the complex variable z and is denoted by

$$w = f(z) = u + iv$$

For example, if $\quad w = z^2 \quad$ where $z = x + iy$ and $w = f(z) = u + iv$

then $\quad u + iv = (x + iy)^2 = (x^2 - y^2) + i(2xy)$

$\Rightarrow \quad u = x^2 - y^2 \quad$ and $\quad v = 2xy$

Thus u and v, the real and imaginary parts of w, are functions of the real variables x and y.

$$\therefore \quad w = f(z) = u(x, y) + iv(x, y)$$

If to each value of z, there corresponds one and only one value of w, then w is called a *single-valued function* of z. If to each value of z, there correspond more than one values of w, then w is called a *multi-valued function of* z. For example, $w = \sqrt{z}$ is a multi-valued function.

To represent $w = f(z)$ graphically, we take two Argand diagrams : one to represent the point z and the other to represent w. The former diagram is called the XOY-plane or the z-plane and the latter UOV-plane or the w-plane.

1.4 LIMIT OF f(z)

A function $f(z)$ tends to the limit l as z tends to z_0 *along any path,* if to each positive arbitrary number ε, however small, there corresponds a positive number δ, such that

$$|f(z) - l| < \varepsilon \quad \text{whenever} \quad 0 < |z - z_0| < \delta$$

and we write $\underset{z \quad z_0}{\text{Lt}} f(z) = l$, where l is finite

Note. In real variables, x ® x_0 implies that x approaches x_0 along the number line, either from left or from right. In complex variables, z ® z_0 implies that z approaches z_0 along any path, straight or curved, since the two points representing z and z_0 in a complex plane can be joined by an infinite number of curves.

Solved Problems

Example 1. Find the limit of $f(z) = z^2 + 4$ at $z = 3$.

Sol. $$\underset{z \to 3}{\text{Lt}} f(z) = \underset{z \to 3}{\text{Lt}} z^2 + 4$$

$$= (3^2) + 4 = 9 + 4 = 13$$

Example 2. Find limit of the function $f(z) = \dfrac{\bar{z}}{z}$ at $z = 0$

Sol. $g_{\text{ren}}\ f(z) = \dfrac{\bar{z}}{z}$

$$\Rightarrow f(z) = \frac{x - iy}{x + iy}$$

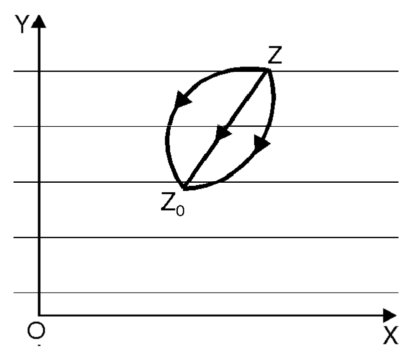
i) Suppose z ® 0 along x -axis. Then y=0, z=x and $z = x$ and $\bar{z} = x$

$$\underset{z \to 0}{\text{Lt}} \frac{\bar{z}}{z} = \underset{z \to 0}{\text{Lt}} \frac{x}{x} = 1$$

ii) Suppose z ® 0 along y -axis. Then x=0, z=iy and $\bar{z} = -iy$

$$\underset{z \to 0}{\text{Lt}} \frac{\bar{z}}{z} = \underset{iy \to 0}{\text{Lt}} \left(\frac{-iy}{iy} \right) = -1$$

$$\underset{z \to 0}{\text{Lt}} \frac{\bar{z}}{z} \text{ does not exist.}$$



Example 3 Find limit of $f(z) = \dfrac{z^2 + 3iz - 2}{z + i}$ at $z = -i$

Sol. Given $f(z) = \dfrac{z^2 + 3iz - 2}{z + i}$, we have $z = x+iy$.

$$z \to -i \Rightarrow x = 0, y = -1$$

(Now along $x \to 0$ and then $y \to -1$)

$$\underset{z\to -i}{\text{Lt}} = f(z) = \underset{\substack{x\to 0\\ y\to -1}}{\text{Lt}} \frac{(x+iy)^2 + 3i(3+iy) - 2}{(x+iy)+i}$$

$$= \underset{y\to -1}{\text{Lt}} \frac{(iy)^2 + 3i(iy) - 2}{iy+i}$$

$$= \underset{y\to -1}{\text{Lt}} \frac{-y^2 - 3y - 2}{i(y+i)}$$

$$= \underset{y\to -1}{\text{Lt}} \frac{-[y+1][y+2]}{(y+1)i}$$

$$= \underset{y\to -1}{\text{Lt}} \frac{-(y+2)}{i} = \frac{-1}{i} = i$$

also along $y \to -1$ and then $x \to 0$.

$$\underset{z\to -i}{\text{Lt}} f(z) = \underset{\substack{y\to -1\\ x\to 0}}{\text{Lt}} \frac{(x+iy)^2 + 3i(x+iy) - 2}{x+iy+i}$$

$$= \underset{x\to 0}{\text{Lt}} \frac{(x-i)^2 + 3i(x-i) - 2}{x-i+i} = \left(\frac{0}{0}\text{for n}\right)$$

$$= \underset{x\to 0}{\text{Lt}} \frac{2(x-i)+3i}{1} = -2i + 3i = i$$

$$\therefore \underset{z\to -i}{\text{Lt}} f(z) = i.$$

1.5 CONTINUITY OF f(z)

A single-valued function $f(z)$ is said to be continuous at a point $z = z_0$ if $f(z_0)$ exists, $\lim_{z\to z_0} f(z)$ exists and $\underset{z\to z_0}{\text{Lt}} f(z) = f(z_0)$.

A function $f(z)$ is said to be continuous in a region R of the z-plane if it is continuous at every point of the region. A function $f(z)$ which is not continuous at z_0 is said to be discontinuous at z_0.

If the function $f(z) = u + iv$ is continuous at $z_0 = x_0 + iy_0$ then the real functions u and v are also continuous at the point (x_0, y_0). Therefore, we can discuss the continuity of a complex valued function by studying the continuity of its real and imaginary parts.

If $f(z)$ and $g(z)$ are continuous at a point z_0 then the functions $f(z) \pm g(z)$, $f(z)\, g(z)$ and $\frac{f(z)}{g(z)}$, where $g(z_0) \neq 0$ are also continuous at z_0.

If $f(z)$ is continuous in a closed region S then it is bounded in S *i.e.*, $|f(z)| \leq M \quad \forall\, z \in S$.

Also, the function $f(z)$ is continuous at $z = \infty$ if the function $f\left(\frac{1}{z}\right)$ is continuous at x = 0

Solved Problems

Example 1. $f(z) = xy^3 + i(3x-2y)$ is continous for all z.

Sol. Given $f(z) = xy^3 + i(3x-2y)$, we have $f(z) = u + iv$ comparing on G.S. $u(x,y) = xy^3$, v e$x,y) = 3x-2y$.

Since $u(x,y)$ and $v(x,y)$ both are continuous

$\therefore$ $f(z)$ is also continuous every where.

Example 2. Verify the contintу of $\begin{cases} \dfrac{Z^2 - 2i}{Z^2 - 2z + 2} & Z \neq 1+i \\ 6 & Z = 1+i \end{cases}$

Sol. $f(z) = \dfrac{z^2 - 2i}{z^2 + 2z + 2}$

$$\text{Now } \operatorname*{Lt}_{z\to 1+i} \frac{z^2 - 2i}{(z^2 + 2z + 2)} = \operatorname*{Lt}_{z\to 1+i} \frac{(z+1+i)(z-1-i)}{(z-1+i)(z-1-i)}$$

$$= \operatorname*{Lt}_{z\to 1+i} \frac{(z+1-i)}{(z-1+i)}$$

$$= \frac{(1+i)+1-i}{(1+i)-1+i} = \frac{2+2\text{i}}{2\text{i}} = 1 - i$$

but $f(1+i) \neq 1 - i$ $\therefore$ $f(z)$ is not coninous at $1+i$

Example 3. Verify $f(z) = \dfrac{\bar{z}}{z}$ is continuous at $z = 0$

Sol. Limit $f(z) = \dfrac{\bar{z}}{z}$ is does not exist at $z = 0$

$\therefore$ $f(z)$ is not continous at $z = 0$.

Example 4. $f(z) = \bar{z}$ is continous at z_0

Sol. Given $f(z) = \bar{z}$

Now $|f(z) - f(z_0)| = |\bar{z} - \bar{z}_0|$

For given $\in > 0$ choose $\in > \delta$, we get

$$|f(z) f(z_0)| < \in \quad \text{for } |z - z_0| < \in = \delta.$$

i.e., whenever $|z - z_0| < \delta$ there exist $|f(z) - f(z_0)| < \in$

$f(z)$ is continuous at $z = z_0$

Example 5 Discuss the continuty of $f(z) = \dfrac{z^2 + 4}{z - 2i}$ at $z = 2i$

Sol. By defination we have to prove for $\in > 0$ there exists a d>0

Such that $|f(z) - f(zi)| < \in$ for all $|z\text{-}zi| < \delta$

Now $\underset{z \to 2i}{\text{Lt}} f(z) = \underset{z \to 2i}{\text{Lt}} \frac{z^2+4}{z-2i} = \underset{z \to 2i}{\text{Lt}} \frac{(z-2i)(z+2i)}{(z-2i)} = \underset{z \to 2i}{\text{Lt}} (z+2i) \Rightarrow f(2i) = 4i$

Let $|f(z) - f(2i)| \in$

$$\Rightarrow \left|\frac{z^2+4}{z-2i} - 4i\right| = \left|\frac{(z+2i)(z-2i)}{(z-2i)} - 4i = |z-2i|\right|$$

Choose $\in = \delta \Rightarrow |z-2i| < \delta$ for $|f(z)\text{-}f(2i)| < \in$

$\therefore f(z)$ is continous at $z = 2i$

1.6. DERIVATIVE OF f(z)

Let $w = f(z)$ be a single-valued function of the variable $z\ (= x + iy)$, then the derivative or differential co-efficient of $w = f(z)$ is defined as

$$\frac{dw}{dz} \quad f \ (z) \quad \underset{z \quad 0}{\text{Lt}} \frac{f(z \quad z) \quad f(z)}{z}$$

provided the limit exists, independent of the manner in which dz ® 0.

Solved Problems

Example 1. Find derivative of $f(z) = z^2$ by using defination of derivative.

Sol. Then $f(z) = z^2$.

$$f'(z) = \underset{\delta z \to 0}{\text{Lt}} \frac{f(z+\delta z) - f(z)}{\delta z}$$

$$= \underset{\delta z \to 0}{\text{Lt}} \frac{(z+\delta z)^2 - z^2}{\delta z}$$

$$= \underset{\delta z \to 0}{\text{Lt}} \frac{z^2 + 2z(\delta z) + (\delta z)^2 - z^2}{\delta z}$$

$$= \underset{\delta z \to 0}{\text{Lt}} 2z + \delta z = 2z.$$

Example 2. If $f(z)$ is differentiable at z_0 then show that $f(z)$ is continous at z_0.

Sol. To show $f(z)$ is continous at z_0, we need to prove

$\underset{z \to z_0}{\text{Lt}} f(z) = f(z_0)$ (or) $\underset{z \to z_0}{\text{Lt}} f(z) - f(z_0) = 0$

Let $f(z)$ is differentiable at z_0

Now conside $\underset{z \to z_0}{\text{Lt}} f(z) = f(z_0)$

$$= \operatorname*{Lt}_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \times (z - z_0)$$
$$= f'(z_0) \operatorname*{Lt}_{z \to z_0} (z - z_0)$$
$$= f'(z_0) - 0 = 0$$

$$\therefore \operatorname*{Lt}_{z \to z_0} f(z) - f(z_0) = 0$$
$$\operatorname*{Lt}_{z \to z_0} f(z) = f(z_0)$$

Hence proved.

1.7. ANALYTIC FUNCTION AT A POINT

A function $f(z)$ is said to be **analytic** at a point z_0 if it is one-valued and differentiable not only at z_0 but at every point of some neighbourhood of z_0. *e.g.* e^x (cos y + i sin y).

1.7.1. Analytical Function

A function $f(z)$ is said to be analytic in a certain domain D if it is analytic at every point of D.

1.8. ENTIRE FUNCTION

A function $f(z)$ which is analytic at every point of the finite complex plane is called an entire function.

Since the derivative of a polynomial exists at every point, a polynomial of any degree is an entire function. Rational functions are also entire functions.

1.9. NECESSARY AND SUFFICIENT CONDITIONS FOR f(z) TO BE ANALYTIC

The necessary and sufficient conditions for the function

$$w = f(z) = u(x, y) + iv(x, y)$$

to be analytic in a region R, are

(*i*) $\frac{u}{x}, \frac{u}{y}, \frac{v}{x}, \frac{v}{y}$ *are continuous functions of x and y in the region R.*

(*ii*) $\frac{u}{x} \quad \frac{v}{y}, \frac{u}{y} \quad \frac{v}{x}$.

The conditions in (*ii*) are known as **Cauchy-Riemann equations** or briefly **C-R equations.**

Proof. (*a*) **Necessary Condition.** Let $w = f(z) = u(x, y) + iv(x, y)$ be analytic in a region R, then $\frac{dw}{dz}$ = f ¢(z) exists uniquely at every point of that region.

Let dx and dy be the increments in x and y respectively. Let du, dv and dz be the corresponding increments in u, v and z respectively. Then,

$$f ¢(z) = \operatorname*{Lt}_{z \quad 0} \frac{f(z \quad z) \quad f(z)}{z} \quad \operatorname*{Lt}_{z \quad 0} \frac{(u \quad u) \quad i(v \quad v) \quad (u \quad iv)}{z}$$

$$= \underset{\delta z \to 0}{\text{Lt}} \left(\frac{\delta u}{\delta z} + i \frac{\delta v}{\delta z} \right) \qquad ...(1)$$

Since the function $w = f(z)$ is analytic in the region R, the limit (1) must exist independent of the manner in which $\delta z \to 0$, *i.e.*, along whichever path δx and $\delta y \to 0$.

First, let $\delta z \to 0$ along a line parallel to x-axis so that $\delta y = 0$ and $\delta z = \delta x$.

[since $z = x + iy$, $z + \delta z = (x + \delta x) + i(y + \delta y)$ and $\delta z = \delta x + i\delta y$]

$\therefore$ From (1),
$$f'(z) = \underset{\delta x \to 0}{\text{Lt}} \left(\frac{\delta u}{\delta x} + i \frac{\delta v}{\delta x} \right) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \qquad ...(2)$$

Now, let $\delta z \to 0$ along a line parallel to y-axis so that $\delta x = 0$ and $\delta z = i\,\delta y$.

$\therefore$ From (1),
$$f'(z) = \underset{\delta y \to 0}{\text{Lt}} \left(\frac{\delta u}{i\,\delta y} + i \frac{\delta v}{i\,\delta y} \right) = \frac{1}{i} \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$
$$= \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y} \qquad ...(3) \qquad \Big|\ \because \frac{1}{i} = -i$$

From (2) and (3), we have $\dfrac{\partial u}{\partial x} + i \dfrac{\partial v}{\partial x} = \dfrac{\partial v}{\partial y} - i \dfrac{\partial u}{\partial y}$

Equating the real and imaginary parts, $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ and $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$.

Hence the necessary condition for f(z) to be analytic is that the C-R equations must be satisfied.

(*b*) **Sufficient Condition.** Let $f(z) = u + iv$ be a single-valued function possessing partial derivatives $\dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}, \dfrac{\partial v}{\partial x}, \dfrac{\partial v}{\partial y}$ at each point of a region R and satisfying C-R equations.

i.e., $$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

We shall show that $f(z)$ is analytic, *i.e.*, $f'(z)$ exists at every point of the region R.

By Taylor's theorem for functions of two variables, we have, on omitting second and higher degree terms of δx and δy.

$$f(z + \delta z) = u(x + \delta x, y + \delta y) + iv(x + \delta x, y + \delta y)$$
$$= \left[u(x, y) + \left(\frac{\partial u}{\partial x} \delta x + \frac{\partial u}{\partial y} \delta y \right) \right] + i \left[v(x, y) + \left(\frac{\partial v}{\partial x} \delta x + \frac{\partial v}{\partial y} \delta y \right) \right]$$
$$= [u(x, y) + iv(x, y)] + \left(\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \right) \delta x + \left(\frac{\partial u}{\partial y} + i \frac{\partial v}{\partial y} \right) \delta y$$
$$= f(z) + \left(\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \right) \delta x + \left(\frac{\partial u}{\partial y} + i \frac{\partial v}{\partial y} \right) \delta y$$

or
$$f(z + \delta z) - f(z) = \left(\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \right) \delta x + \left(\frac{\partial u}{\partial y} + i \frac{\partial v}{\partial y} \right) \delta y$$
$$= \left(\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \right) \delta x + \left(-\frac{\partial v}{\partial x} + i \frac{\partial u}{\partial x} \right) \delta y \qquad \text{| Using C-R equations}$$
$$= \left(\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \right) \delta x + \left(\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \right) i\delta y \qquad |\ \because -1 = i^2$$

$$= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)(dx + i\,dy) = \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) dz \qquad | \because \; dx + i\,dy = dz$$

$$\Rightarrow \qquad \frac{f(z+\delta z) - f(z)}{\delta z} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$\therefore \qquad f'(z) = \underset{\delta z \to 0}{\text{Lt}} \frac{f(z+\delta z) - f(z)}{\delta z} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

Thus $f'(z)$ exists, because $\frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}$ exist.

Hence $f(z)$ is analytic.

Note 1. The real and imaginary parts of an analytic function are called **conjugate functions.** Thus, if $f(z) = u(x, y) + iv(x, y)$ is an analytic function, then $u(x, y)$ and $v(x, y)$ are conjugate functions. The relation between two conjugate functions is given by C-R equations.

Note 2. When a function $f(z)$ is known to be analytic, it can be differentiated in the ordinary way as if z is a real variable.

Thus, $f(z) = z^2 \quad \Rightarrow \quad f'(z) = 2z$

$f(z) = \sin z \quad \Rightarrow \quad f'(z) = \cos z$ etc.

1.10. CAUCHY-RIEMANN EQUATIONS IN POLAR COORDINATES

Let (r, θ) be the polar coordinates of the point whose cartesian coordinates are (x, y), then

$$x = r\cos\theta, \; y = r\sin\theta,$$

$$z = x + iy = r(\cos\theta + i\sin\theta) = re^{i\theta}$$

$$\therefore \qquad u + iv = f(z) = f(re^{i\theta}) \qquad \text{...(1)}$$

Differentiating (1) partially w.r.t. r, we have

$$\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r} = f'(re^{i\theta}) \,.\, e^{i\theta} \qquad \text{...(2)}$$

Differentiating (1) partially w.r.t. θ, we have

$$\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta} = f'(re^{i\theta}) \,.\, ire^{i\theta} = ir\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right) \qquad | \text{ Using (2)}$$

$$= -r\frac{\partial v}{\partial r} + ir\frac{\partial u}{\partial r}$$

Equating real and imaginary parts, we get

$$\frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r} \quad \text{and} \quad \frac{\partial v}{\partial \theta} = r\frac{\partial u}{\partial r}$$

or $$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$ which is the polar form of C-R equations.

Soved Problems

Example 1. *Find the values of c_1 and c_2 such that the function*

$$f(z) = x^2 + c_1 y^2 - 2xy + i(c_2 x^2 - y^2 + 2xy)$$

is analytic. Also find $f'(z)$.

Sol. Here $f(z) = (x^2 + c_1 y^2 - 2xy) + i(c_2 x^2 - y^2 + 2xy)$ ...(1)

Comparing (1) with $f(z) = u(x, y) + iv(x, y)$, we get

$$u(x, y) = x^2 + c_1 y^2 - 2xy \qquad \text{...(2)}$$

and $$v(x, y) = c_2x^2 - y^2 + 2xy \qquad \text{...(3)}$$

For the function $f(z)$ to be analytic, it should satisfy Cauchy-Riemann equations.

Now, from (2), $\dfrac{\partial u}{\partial x} = 2x - 2y$ and $\dfrac{\partial u}{\partial y} = 2c_1y - 2x$

Also, from (3), $\dfrac{\partial v}{\partial x} = 2c_2x + 2y$ and $\dfrac{\partial v}{\partial y} = -2y + 2x$

Cauchy-Riemann equations are

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$\Rightarrow$ $2x - 2y = -2y + 2x$ which is true.

and $$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$\Rightarrow$ $$2c_1y - 2x = -2c_2x - 2y \qquad \text{...(4)}$$

Comparing the coefficients of x and y in equation (4), we get

$$2c_1 = -2 \quad \Rightarrow \quad c_1 = -1$$

and $$-2 = -2c_2 \quad \Rightarrow \quad c_2 = 1$$

Hence $c_1 = -1$ and $c_2 = 1$

Now, $$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = 2x - 2y + i(2c_2x + 2y)$$

$$= 2x - 2y + i(2x + 2y) \qquad | \because \ c_2 = 1$$

$$= 2(x + iy) + 2i(x + iy)$$

$$= 2z + 2iz = 2(1 + i)z.$$

Example 2. *Find p such that the function $f(z)$ expressed in polar coordinates as $f(z) = r^2 \cos 2\theta + ir^2 \sin p\theta$ is analytic.*

Sol. Let $f(z) = u + iv$, then $u = r^2 \cos 2\theta$, $v = r^2 \sin p\theta$

$$\frac{\partial u}{\partial r} = 2r \cos 2\theta, \ \frac{\partial v}{\partial r} = 2r \sin p\theta$$

$$\frac{\partial u}{\partial \theta} = -2r^2 \sin 2\theta, \ \frac{\partial v}{\partial \theta} = pr^2 \cos p\theta$$

For $f(z)$ to be analytic, $\dfrac{\partial u}{\partial r} = \dfrac{1}{r}\dfrac{\partial v}{\partial \theta}$ and $\dfrac{\partial v}{\partial r} = -\dfrac{1}{r}\dfrac{\partial u}{\partial \theta}$

$\therefore$ $2r \cos 2\theta = pr \cos p\theta$ and $2r \sin p\theta = 2r \sin 2\theta$

Both these equations are satisfied if $p = 2$.

Example 3. (*i*) *Prove that the function sinh z is analytic and find its derivative.*

(*ii*) *Show that $f(z) = \log z$ is analytic everywhere in the complex plane except at the origin and that its derivative is $\left(\dfrac{1}{z}\right)$.*

Sol. (*i*) Here $$f(z) = u + iv = \sinh z = \sinh (x + iy) = \sinh x \cos y + i \cosh x \sin y$$

$\therefore$ $u = \sinh x \cos y$ and $v = \cosh x \sin y$

$$\frac{\partial u}{\partial x} = \cosh x \cos y, \ \frac{\partial u}{\partial y} = -\sinh x \sin y$$

$$\frac{\partial v}{\partial x} = \sinh x \sin y,\ \frac{\partial v}{\partial y} = \cosh x \cos y$$

$$\therefore \qquad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Thus C-R equations are satisfied.

Since sinh x, cosh x, sin y and cos y are continuous functions, $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ are also continuous functions satisfying C-R equations.

Hence $f(z)$ is analytic everywhere.

Now $$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$= \cosh x \cos y + i \sinh x \sin y = \cosh (x + iy) = \cosh z.$$

(ii) Here $f(z) = u + iv = \log z = \log (x + iy)$

Let $x = r \cos \theta$ and $y = r \sin \theta$ so that

$$x + iy = r (\cos \theta + i \sin \theta) = re^{i\theta}$$

$$\log (x + iy) = \log (r e^{i\theta}) = \log r + i\theta$$

$$= \frac{1}{2} \log (x^2 + y^2) + i \tan^{-1} \left(\frac{y}{x}\right)$$

Separating real and imaginary parts, we get

$$u = \frac{1}{2} \log (x^2 + y^2) \quad \text{and} \quad v = \tan^{-1} \left(\frac{y}{x}\right)$$

Now, $$\frac{\partial u}{\partial x} = \frac{x}{x^2 + y^2},\ \frac{\partial u}{\partial y} = \frac{y}{x^2 + y^2}$$

and $$\frac{\partial v}{\partial x} = \frac{-y}{x^2 + y^2},\ \frac{\partial v}{\partial y} = \frac{x}{x^2 + y^2}$$

We observe that the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

are satisfied except when $x^2 + y^2 = 0$ *i.e.*, when $x = 0, y = 0$

Hence the function $f(z) = \log z$ is analytic everywhere in the complex plane except at the origin.

Also, $$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{x - iy}{x^2 + y^2}$$

$$= \frac{x - iy}{(x + iy)(x - iy)} = \frac{1}{x + iy} = \frac{1}{z}$$

Example 4. *Show that the function $e^x (\cos y + i \sin y)$ is holomorphic and find its derivative.*

Sol. $$f(z) = e^x \cos y + i\, e^x \sin y = u + iv$$

Here, $$u = e^x \cos y, \quad v = e^x \sin y$$

$$\frac{\partial u}{\partial x} = e^x \cos y \qquad \frac{\partial v}{\partial x} = e^x \sin y$$

$$\frac{u}{y} = -e^x \sin y \qquad \frac{v}{y} = e^x \cos y$$

Since, $\quad \frac{u}{x} \quad \frac{v}{y}$ and $\frac{u}{y} \quad \frac{v}{x}$

hence, C-R equations are satisfied. Also first order partial derivatives of u and v are continuous everywhere. Therefore $f(z)$ is analytic.

Now, $\quad f\,¢(z) = \frac{u}{x} \quad i\,\frac{v}{x} = e^x \cos y + i\, e^x \sin y$

$$= e^x (\cos y + i \sin y) = e^x \,.\, e^{iy} = e^{x+iy} = e^z$$

Example 5. *If n is real, show that r^n (cos nq + i sin nq) is analytic except possibly when r = 0 and that its derivative is*

$$nr^{n-1} [\cos (n-1)\, q + i \sin (n-1)\, q].$$

Sol. Let $\quad w = f(z) = u + iv = r^n (\cos nq + i \sin nq)$

Here, $\quad u = r^n \cos nq, \qquad v = r^n \sin nq$

then, $\quad \frac{u}{r} = nr^{n-1} \cos nq \qquad \frac{v}{r} = nr^{n-1} \sin nq$

$$\frac{u}{} = -nr^n \sin nq \qquad \frac{v}{} = nr^n \cos nq$$

Thus, we see that, $\quad \frac{u}{r} \quad \frac{1}{r}\frac{v}{} \quad$ and $\quad \frac{v}{r} \quad \frac{1}{r}\frac{u}{}$

\ Cauchy-Riemann equations are satisfied. Also first order partial derivatives of u and v are continuous everywhere.

Hence $f(z)$ is analytic if $f\,¢(z)$ or $\frac{dw}{dz}$ exists for all finite values of z.

We have, $\quad \frac{dw}{dz} = (\cos q - i \sin q)\, \frac{w}{r}$

$$= (\cos q - i \sin q) \,.\, nr^{n-1} (\cos nq + i \sin nq)$$

$$= nr^{n-1} [\cos (n-1)\, q + i \sin (n-1)\, q]$$

This exists for all finite values of r including zero, except when $r = 0$ and n £ 1.

Example 6. *Show that if f(z) is analytic and*

(i) Re f(z) = constant

(ii) Im f(z) = constant then f(z) is a constant. (Anna 2007, 2009)

Sol. Since the function $f(z) = u(x, y) + iv(x, y)$ is analytic, it satisfies the Cauchy-Riemann equations

$$\frac{u}{x} \quad \frac{v}{y} \quad \text{and} \quad \frac{u}{y} \quad \frac{v}{x}$$

(i) $\quad$ Re $f(z)$ = constant, therefore $u(x, y) = c_1$

\ $\quad \frac{u}{x} = 0 = \frac{u}{y}.$

Using C-R equations, $\frac{\partial v}{\partial x} = 0 = \frac{\partial v}{\partial y}$

Hence $v(x, y) = c_2$ = a real constant

Therefore $f(z) = u(x, y) + iv(x, y) = c_1 + ic_2$ = a complex constant.

(*ii*) Im $f(z)$ = constant. Therefore $v(x, y) = c_3$

$$\therefore \quad \frac{\partial v}{\partial x} = 0 = \frac{\partial v}{\partial y}$$

Using C-R equations, $\frac{\partial u}{\partial y} = 0 = \frac{\partial u}{\partial x}$

Hence $u(x, y) = c_4$ = a real constant.

Therefore $f(z) = u(x, y) + iv(x, y) = c_4 + ic_3$ = a complex constant.

Example 7. *Given that* $u(x, y) = x^2 - y^2$ and $v(x, y) = -\left(\frac{y}{x^2+y^2}\right)$.

Prove that both u and v are harmonic functions but u + iv is not an analytic function of z.

Sol.

$$u = x^2 - y^2$$

$$\frac{\partial u}{\partial x} = 2x \quad \Rightarrow \quad \frac{\partial^2 u}{\partial x^2} = 2$$

$$\frac{\partial u}{\partial y} = -2y \quad \Rightarrow \quad \frac{\partial^2 u}{\partial y^2} = -2$$

Since $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ Hence $u(x, y)$ is harmonic.

Also,

$$v = -\frac{y}{x^2+y^2}$$

$$\frac{\partial v}{\partial x} = \frac{2xy}{(x^2+y^2)^2} \quad \Rightarrow \quad \frac{\partial^2 v}{\partial x^2} = \frac{2y^3-6x^2y}{(x^2+y^2)^3}$$

$$\frac{\partial v}{\partial y} = \frac{y^2-x^2}{(x^2+y^2)^2} \quad \Rightarrow \quad \frac{\partial^2 v}{\partial y^2} = \frac{6x^2y-2y^3}{(x^2+y^2)^3}$$

Since $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$. Hence $v(x, y)$ is also harmonic.

But, $\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}$ and $\frac{\partial v}{\partial x} \neq -\frac{\partial u}{\partial y}$

Therefore $u + iv$ is not an analytic function of z.

Example 8. *If* φ *and* ψ *are functions of x and y satisfying Laplace's equation, show that s + it is analytic, where*

$$s = \frac{\partial \phi}{\partial y} - \frac{\partial \psi}{\partial x} \quad and \quad t = \frac{\partial \phi}{\partial x} + \frac{\partial \psi}{\partial y}.$$

Sol. Since ϕ and ψ are functions of x and y satisfying Laplace's equations,

$$\therefore \qquad \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \qquad ...(1)$$

and
$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0. \qquad ...(2)$$

For the function $s + it$ to be analytic,

$$\frac{\partial s}{\partial x} = \frac{\partial t}{\partial y} \qquad ...(3)$$

and
$$\frac{\partial s}{\partial y} = -\frac{\partial t}{\partial x} \qquad ...(4)$$

must satisfy.

Now,
$$\frac{\partial s}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial y} - \frac{\partial \psi}{\partial x}\right) = \frac{\partial^2 \phi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial x^2} \qquad ...(5)$$

$$\frac{\partial t}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial x} + \frac{\partial \psi}{\partial y}\right) = \frac{\partial^2 \phi}{\partial y \partial x} + \frac{\partial^2 \psi}{\partial y^2} \qquad ...(6)$$

$$\frac{\partial s}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial y} - \frac{\partial \psi}{\partial x}\right) = \frac{\partial^2 \phi}{\partial y^2} - \frac{\partial^2 \psi}{\partial y \partial x} \qquad ...(7)$$

and
$$\frac{\partial t}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial x} + \frac{\partial \psi}{\partial y}\right) = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \psi}{\partial x \partial y}. \qquad ...(8)$$

From (3), (5) and (6), we have

$$\frac{\partial^2 \phi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial x^2} = \frac{\partial^2 \phi}{\partial y \partial x} + \frac{\partial^2 \psi}{\partial y^2} \qquad \Rightarrow \qquad \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$$

which is true by (2).

Again from (4), (7) and (8), we have,

$$\frac{\partial^2 \phi}{\partial y^2} - \frac{\partial^2 \psi}{\partial y \partial x} = -\frac{\partial^2 \phi}{\partial x^2} - \frac{\partial^2 \psi}{\partial x \partial y} \qquad \Rightarrow \qquad \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

which is also true by (1).

Hence the function $s + it$ is analytic.

Example 9. *Verify if* $f(z) = \dfrac{xy^2(x + iy)}{x^2 + y^4}$, $z \neq 0$; $f(0) = 0$ *is analytic or not?*

Sol.
$$u + iv = \frac{xy^2(x + iy)}{x^2 + y^4} \; ; z \neq 0$$

$$\therefore \qquad u = \frac{x^2 y^2}{x^2 + y^4}, \; v = \frac{xy^3}{x^2 + y^4}$$

At the origin,
$$\frac{\partial u}{\partial x} = \lim_{x \to 0} \frac{u(x, 0) - u(0, 0)}{x} = \lim_{x \to 0} \frac{0 - 0}{x} = 0$$

$$\frac{\partial u}{\partial y} = \lim_{y \to 0} \frac{u(0, y) - u(0, 0)}{y} = \lim_{y \to 0} \frac{0 - 0}{y} = 0$$

$$\frac{\partial v}{\partial x} = \lim_{x \to 0} \frac{v(x, 0) - v(0, 0)}{x} = \lim_{x \to 0} \frac{0 - 0}{x} = 0$$

$$\frac{\partial v}{\partial y} = \lim_{y \to 0} \frac{v(0, y) - v(0, 0)}{y} = \lim_{y \to 0} \frac{0 - 0}{y} = 0$$

Hence Cauchy-Riemann equations are satisfied at the origin.

But $$f'(0) = \lim_{z \to 0} \frac{f(z) - f(0)}{z} = \lim_{\substack{x \to 0 \\ y \to 0}} \left\{ \frac{xy^2 (x + iy)}{x^2 + y^4} - 0 \right\} \cdot \frac{1}{x + iy} = \lim_{\substack{x \to 0 \\ y \to 0}} \frac{xy^2}{x^2 + y^4}$$

Let $z \to 0$ along the real axis $y = 0$, then

$$f'(0) = 0$$

Again let $z \to 0$ along the curve $x = y^2$, then

$$f'(0) = \lim_{x \to 0} \frac{x^2}{x^2 + x^2} = \frac{1}{2}$$

which shows that $f'(0)$ does not exist. Hence $f(z)$ is not analytic at origin although Cauchy-Riemann equations are satisfied there.

Example 10. *Show that the function defined by* $f(z) = \sqrt{|xy|}$ *is not regular at the origin, although Cauchy-Riemann equations are satisfied*.

Sol. Let $f(z) = u(x, y) + iv(x, y) = \sqrt{|xy|}$ then $u(x, y) = \sqrt{|xy|}$, $v(x, y) = 0$

At the origin (0, 0), we have

$$\frac{\partial u}{\partial x} = \lim_{x \to 0} \frac{u(x, 0) - u(0, 0)}{x} = \lim_{x \to 0} \frac{0 - 0}{x} = 0$$

$$\frac{\partial u}{\partial y} = \lim_{y \to 0} \frac{u(0, y) - u(0, 0)}{y} = \lim_{y \to 0} \frac{0 - 0}{y} = 0$$

$$\frac{\partial v}{\partial x} = \lim_{x \to 0} \frac{v(x, 0) - v(0, 0)}{x} = \lim_{x \to 0} \frac{0 - 0}{x} = 0$$

$$\frac{\partial v}{\partial y} = \lim_{y \to 0} \frac{v(0, y) - v(0, 0)}{y} = \lim_{y \to 0} \frac{0 - 0}{y} = 0$$

Clearly, $$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Hence C-R equations are satisfied at the origin.

Now $$f'(0) = \lim_{z \to 0} \frac{f(z) - f(0)}{z} = \lim_{z \to 0} \frac{\sqrt{|xy|} - 0}{x + iy}$$

If $z \to 0$ along the line $y = mx$, we get

$$f'(0) = \lim_{x \to 0} \frac{\sqrt{|mx^2|}}{x(1 + im)} = \lim_{x \to 0} \frac{\sqrt{|m|}}{1 + im}$$

Now this limit is not unique since it depends on m. Therefore, $f'(0)$ does not exist.

Hence the function $f(z)$ is not regular at the origin.

Example 11. *Prove that the function f(z) defined by*

$$f(z) = \frac{x^3(1 + i) - y^3(1 - i)}{x^2 + y^2}, \; z \neq 0 \text{ and } f(0) = 0$$

is continuous and the Cauchy-Riemann equations are satisfied at the origin, yet $f'(0)$ does not exist.

Sol. Here, $f(z) = \dfrac{(x^3 - y^3) + i(x^3 + y^3)}{x^2 + y^2}, \; z \neq 0$

Let $f(z) = u + iv = \dfrac{x^3 - y^3}{x^2 + y^2} + i\,\dfrac{x^3 + y^3}{x^2 + y^2}$,

then $u = \dfrac{x^3 - y^3}{x^2 + y^2}, \; v = \dfrac{x^3 + y^3}{x^2 + y^2}$

Since $z \neq 0 \;\Rightarrow\; x \neq 0, y \neq 0$

$\therefore$ u and v are rational functions of x and y with non-zero denominators. Thus, u, v and hence $f(z)$ *are continuous functions when $z \neq 0$*. To test them for continuity at $z = 0$, on changing u, v to polar co-ordinates by putting $x = r\cos\theta$, $y = r\sin\theta$, we get

$$u = r(\cos^3\theta - \sin^3\theta) \text{ and } v = r(\cos^3\theta + \sin^3\theta)$$

When $z \to 0$, $r \to 0$

$$\therefore \quad \lim_{z \to 0} u = \lim_{r \to 0} r(\cos^3\theta - \sin^3\theta) = 0$$

Similarly, $\lim_{z \to 0} v = 0$

$$\therefore \quad \lim_{r \to 0} f(z) = 0 = f(0)$$

$\Rightarrow$ **$f(z)$ is continuous at $z = 0$**.

Hence $f(z)$ is continuous for all values of z.

At the origin (0, 0), we have

$$\frac{\partial u}{\partial x} = \lim_{x \to 0} \frac{u(x, 0) - u(0, 0)}{x} = \lim_{x \to 0} \frac{x - 0}{x} = 1$$

$$\frac{\partial u}{\partial y} = \lim_{y \to 0} \frac{u(0, y) - u(0, 0)}{y} = \lim_{y \to 0} \frac{-y - 0}{y} = -1$$

$$\frac{\partial v}{\partial x} = \lim_{x \to 0} \frac{v(x, 0) - v(0, 0)}{x} = \lim_{x \to 0} \frac{x - 0}{x} = 1$$

$$\frac{\partial v}{\partial y} = \lim_{y \to 0} \frac{v(0, y) - v(0, 0)}{y} = \lim_{y \to 0} \frac{y - 0}{y} = 1$$

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Hence C-R equations are satisfied at the origin.

Now $f'(0) = \lim_{z \to 0} \dfrac{f(z) - f(0)}{z} = \lim_{z \to 0} \dfrac{(x^3 - y^3) + i(x^3 + y^3) - 0}{(x^2 + y^2)(x + iy)}$

Let $z \to 0$ along the line $y = x$, then

$$f'(0) = \lim_{x \to 0} \frac{0 + 2ix^3}{2x^3(1 + i)} = \frac{i}{1 + i} = \frac{i(1 - i)}{2} = \frac{1 + i}{2} \qquad \text{...(1)}$$

Also, let $z \to 0$ along the x-axis (*i.e.*, $y = 0$), then

$$f'(0) = \lim_{x \to 0} \frac{x^3 + ix^3}{x^3} = 1 + i \qquad \text{...(2)}$$

Since the limits (1) and (2) are different, $f'(0)$ does not exist.

Example 12. *(i) Examine the nature of the function*

$$f(z) = \frac{x^2 y^5 (x + iy)}{x^4 + y^{10}} \; ; z \neq 0$$
$$f(0) = 0$$

in the region including the origin.

(*ii*) *If* $f(z) = \begin{cases} \dfrac{x^3 y(y - ix)}{x^6 + y^2}, & z \neq 0 \\ 0, & z = 0 \end{cases}$ *prove that* $\dfrac{f(z) - f(0)}{z} \to 0$ *as* $z \to 0$ *along any radius vector but not as* $z \to 0$ *in any manner and also that* $f(z)$ *is not analytic at* $z = 0$.

Sol. (*i*) Here, $$u + iv = \frac{x^2 y^5 (x + iy)}{x^4 + y^{10}} \; ; z \neq 0$$

$$\therefore \quad u = \frac{x^3 y^5}{x^4 + y^{10}}, \; v = \frac{x^2 y^6}{x^4 + y^{10}}$$

At the origin,
$$\frac{\partial u}{\partial x} = \lim_{x \to 0} \frac{u(x, 0) - u(0, 0)}{x} = \lim_{x \to 0} \frac{0 - 0}{x} = 0$$
$$\frac{\partial u}{\partial y} = \lim_{y \to 0} \frac{u(0, y) - u(0, 0)}{y} = \lim_{y \to 0} \frac{0 - 0}{y} = 0$$

Similarly, $\dfrac{\partial v}{\partial x} = 0 = \dfrac{\partial v}{\partial y}$

Hence Cauchy-Riemann eqns. are satisfied at the origin.

But
$$f'(0) = \lim_{z \to 0} \frac{f(z) - f(0)}{z} = \lim_{\substack{x \to 0 \\ y \to 0}} \left\{ \frac{x^2 y^5 (x + iy)}{x^4 + y^{10}} - 0 \right\} \cdot \frac{1}{x + iy}$$
$$= \lim_{\substack{x \to 0 \\ y \to 0}} \frac{x^2 y^5}{x^4 + y^{10}}$$

Let $z \to 0$ along the radius vector $y = mx$, then
$$f'(0) = \lim_{x \to 0} \frac{m^5 x^7}{x^4 + m^{10} x^{10}} = \lim_{x \to 0} \frac{m^5 x^3}{1 + m^{10} x^6} = 0$$

Again let $z \to 0$ along the curve $y^5 = x^2$
$$f'(0) = \lim_{x \to 0} \frac{x^4}{x^4 + x^4} = \frac{1}{2}$$

which shows that $f'(0)$ does not exist. Hence $f(z)$ is not analytic at origin although Cauchy-Riemann equations are satisfied there.

(*ii*)
$$\frac{f(z) - f(0)}{z} = \left\{ \frac{x^3 y(y - ix)}{x^6 + y^2} - 0 \right\} \cdot \frac{1}{x + iy}$$
$$= \frac{-ix^3 y(x + iy)}{(x^6 + y^2)} \cdot \frac{1}{x + iy} = -i \frac{x^3 y}{x^6 + y^2}$$

Let $z \to 0$ along radius vector $y = mx$ then,

$$\lim_{z \to 0} \frac{f(z) - f(0)}{z} = \lim_{x \to 0} \frac{ix^3(mx)}{x^6 + m^2x^2} = \lim_{x \to 0} \frac{imx^2}{x^4 + m^2} = 0$$

Hence $\frac{f(z) - f(0)}{z} \to 0$ as $z \to 0$ along any radius vector.

Now let $z \to 0$ along a curve $y = x^3$ then,

$$\lim_{z \to 0} \frac{f(z) - f(0)}{z} = \lim_{x \to 0} \frac{ix^3 . x^3}{x^6 + x^6} = \frac{i}{2}$$

Hence $\frac{f(z) - f(0)}{z}$ does not tend to zero as $z \to 0$ along the curve $y = x^3$.

We observe that $f'(0)$ does not exist hence $f(z)$ is not analytic at $z = 0$.

Example 13. *Show that the following functions are harmonic and find their harmonic conjugate functions.*

(*i*) $u = \frac{1}{2} \log(x^2 + y^2)$ (*ii*) $v = \sinh x \cos y$.

(*iii*) $u = e^x \cos y$. (Tirunelveli 2010)

Sol. (*i*)

$$u = \frac{1}{2} \log (x^2 + y^2) \quad ...(1)$$

$$\frac{\partial u}{\partial x} = \frac{1}{2} . \frac{1}{x^2 + y^2} . 2x = \frac{x}{x^2 + y^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{(x^2 + y^2) . 1 - x . 2x}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2} \quad ...(2)$$

Also,

$$\frac{\partial u}{\partial y} = \frac{1}{2} . \frac{1}{x^2 + y^2} . 2y = \frac{y}{x^2 + y^2}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{(x^2 + y^2) . 1 - y . 2y}{(x^2 + y^2)^2} = \frac{x^2 - y^2}{(x^2 + y^2)^2} \quad ...(3)$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0. \quad \text{[From (2) and (3)]}$$

Since u satisfies Laplace's equation hence u is a harmonic function.

Let

$$dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy$$

$$= \left(-\frac{\partial u}{\partial y}\right) dx + \left(\frac{\partial u}{\partial x}\right) dy \quad \text{[Using C-R equations]}$$

$$= \left(\frac{-y}{x^2 + y^2}\right) dx + \left(\frac{x}{x^2 + y^2}\right) dy$$

$$= \frac{x\,dy - y\,dx}{(x^2 + y^2)} = d\left[\tan^{-1}\left(\frac{y}{x}\right)\right]$$

Integration yields, $v = \tan^{-1}\left(\dfrac{y}{x}\right) + c$ | c is a constant

which is the required harmonic conjugate function of u.

(*ii*) $v = \sinh x \cos y$ ...(1)

$$\frac{\partial v}{\partial x} = \cosh x \cos y \quad \Rightarrow \quad \frac{\partial^2 v}{\partial x^2} = \sinh x \cos y \qquad ...(2)$$

$$\frac{\partial v}{\partial y} = -\sinh x \sin y \quad \Rightarrow \quad \frac{\partial^2 v}{\partial y^2} = -\sinh x \cos y \qquad ...(3)$$

Since, $\dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2} = 0$

Hence v is harmonic.

Now, $du = \dfrac{\partial u}{\partial x}dx + \dfrac{\partial u}{\partial y}dy = \dfrac{\partial v}{\partial y}dx - \dfrac{\partial v}{\partial x}dy$

$= -\sinh x \sin y\, dx - \cosh x \cos y\, dy$

$= -[\sinh x \sin y\, dx + \cosh x \cos y\, dy]$

$= -d(\cosh x \sin y)$.

Integration yields, $u = -\cosh x \sin y + c$ | c is a constant

which is the required harmonic conjugate function of v.

(*iii*) $u = e^x \cos y$

$$\frac{\partial u}{\partial x} = e^x \cos y \quad \Rightarrow \quad \frac{\partial^2 u}{\partial x^2} = e^x \cos y$$

$$\frac{\partial u}{\partial y} = -e^x \sin y \quad \Rightarrow \quad \frac{\partial^2 u}{\partial y^2} = -e^x \cos y$$

Since $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0 \quad \therefore \quad u$ is harmonic.

Let $v = v(x, y)$

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$$

$$= \left(-\frac{\partial u}{\partial y}\right)dx + \left(\frac{\partial u}{\partial x}\right)dy$$

$= e^x \sin y\, dx + e^x \cos y\, dy$

$= d(e^x \sin y)$

Integration yields, $v = e^x \sin y + c$.

Example 14. *Determine the analytic function $w = u + iv$ if*

(*i*) $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$. (*ii*) $u = \dfrac{x}{x^2 + y^2}$ (Tirunelveli 2010)

Sol. (*i*) $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$ ...(1)

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2 + 6x = \phi_1(x, y) \qquad |\text{say}$$

$$\therefore \quad \phi_1(z, 0) = 3z^2 + 6z. \qquad ...(2)$$

Again, $$\frac{\partial u}{\partial y} = -6xy - 6y = \phi_2(x, y) \qquad |\text{say}$$

$$\therefore \quad \phi_2(z, 0) = 0$$

By Milne's Thomson method,

$$f(z) = \int [\phi_1(z, 0) - i\,\phi_2(z, 0)]\,dz + c$$

$$= \int (3z^2 + 6z)\,dz + c = z^3 + 3z^2 + c. \qquad |\ c \text{ is a constant}$$

Hence, $$w = z^3 + 3z^2 + c$$

(*ii*) $$u = \frac{x}{x^2 + y^2}$$

$$\frac{\partial u}{\partial x} = \frac{(x^2 + y^2).1 - x.2x}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2} = \phi_1(x, y) \qquad |\text{ say}$$

$$\therefore \quad \phi_1(z, 0) = -\frac{1}{z^2}$$

Again, $$\frac{\partial u}{\partial y} = \frac{-2xy}{(x^2 + y^2)^2} = \phi_2(x, y) \qquad |\text{ say}$$

$$\therefore \quad \phi_2(z, 0) = 0$$

By Milne-Thomson method,

$$f(z) = \int [\phi_1(z, 0) - i\,\phi_2(z, 0)]\,dz + c = \frac{1}{z} + c \text{ where } c \text{ is a constant.}$$

Example 15. *(i) In a two-dimensional fluid flow, the stream function is* $\psi = -\frac{y}{x^2 + y^2}$*, find the velocity potential* ϕ.

(*ii*) *An electrostatic field in the xy-plane is given by the potential function* $\phi = 3x^2y - y^3$*, find the stream function.*

Sol. (*i*) $$\psi = -\frac{y}{x^2 + y^2} \qquad ...(1)$$

$$\frac{\partial \psi}{\partial x} = \frac{2xy}{(x^2 + y^2)^2}, \qquad \frac{\partial \psi}{\partial y} = \frac{y^2 - x^2}{(x^2 + y^2)^2}$$

We know that, $$d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy = \frac{\partial \psi}{\partial y}dx - \frac{\partial \psi}{\partial x}dy$$

$$= \frac{(y^2 - x^2)}{(x^2 + y^2)^2}dx - \frac{2xy}{(x^2 + y^2)^2}dy$$

$$= \frac{(x^2 + y^2)dx - 2x^2\,dx - 2xy\,dy}{(x^2 + y^2)^2}$$

$$= \frac{(x^2 + y^2)\,d(x) - x(2x\,dx + 2y\,dy)}{(x^2 + y^2)^2}$$

$$= \frac{(x^2 + y^2)\,d(x) - x\,d(x^2 + y^2)}{(x^2 + y^2)^2} = d\left(\frac{x}{x^2 + y^2}\right).$$

Integration yields, $\phi = \dfrac{x}{x^2 + y^2} + c$ where c is a constant.

(*ii*) Let $\psi(x, y)$ be a stream function.

$$d\psi = \frac{\partial \psi}{\partial x}dx + \frac{\partial \psi}{\partial y}dy = \left(-\frac{\partial \phi}{\partial y}\right)dx + \left(\frac{\partial \phi}{\partial x}\right)dy$$

$$= \{-(3x^2 - 3y^2)\}\,dx + 6xy\,dy$$

$$= -3x^2\,dx + (3y^2\,dx + 6xy\,dy)$$

$$= -d(x^3) + 3d(xy^2)$$

Integrating, we get

$$\psi = -x^3 + 3xy^2 + c \qquad |c \text{ is a constant}$$

Example 16. (*i*) *If $u = e^x(x\cos y - y\sin y)$ is a harmonic function, find an analytic function $f(z) = u + iv$ such that $f(1) = e$.* (Anna 2011, 2009)

(*ii*) *Determine an analytic function f(z) in terms of z whose real part is $e^{-x}(x\sin y - y\cos y)$.*

Sol. (*i*) We have, $u = e^x(x\cos y - y\sin y)$

$$\frac{\partial u}{\partial x} = e^x(x\cos y - y\sin y) + e^x\cos y = \phi_1(x, y) \qquad |\text{say}$$

$$\frac{\partial u}{\partial y} = e^x[-x\sin y - y\cos y - \sin y] = \phi_2(x, y) \qquad |\text{say}$$

$\therefore$ $\phi_1(z, 0) = e^z z + e^z = (z + 1)e^z$

$\phi_2(z, 0) = 0$

By Milne's Thomson method,

$$f(z) = \int\{\phi_1(z, 0) - i\phi_2(z, 0)\}\,dz + c \qquad |\, c \text{ is a constant}$$

$$= \int(z + 1)\,e^z\,dz + c = (z - 1)\,e^z + e^z + c = ze^z + c \qquad \ldots(1)$$

$f(1) = e + c$ |From (1)

$e = e + c$ $|f(1) = e$ (given)

$\Rightarrow$ $c = 0$

$\therefore$ From (1), $f(z) = ze^z$.

(*ii*) $u = e^{-x}(x\sin y - y\cos y)$

$$\frac{\partial u}{\partial x} = e^{-x}\sin y - e^{-x}(x\sin y - y\cos y) = \phi_1(x, y) \qquad |\text{ say}$$

$$\frac{\partial u}{\partial y} = e^{-x}(x\cos y - \cos y + y\sin y) = \phi_2(x, y) \qquad |\text{ say}$$

$\therefore$ $\phi_1(z, 0) = 0$

$\phi_2(z, 0) = e^{-z}(z - 1)$

By Milne's Thomson method,

$$f(z) = \int [\phi_1(z, 0) - i\,\phi_2(z, 0)]\, dz + c$$

$$= -i \int e^{-z}(z - 1)\, dz + c$$

$$= -i \left[(z - 1)(-e^{-z}) - \int (-e^{-z})\, dz\right] + c$$

$$= -i\,[(1 - z)\, e^{-z} - e^{-z}] + c$$

$\Rightarrow$ $f(z) = ize^{-z} + c$ | where c is a constant

Example 17. (*i*) *Determine the analytic function whose real part is*

$$e^{2x}\,(x \cos 2y - y \sin 2y).$$

(*ii*) *Find an analytic function whose imaginary part is* $e^{-x}(x \cos y + y \sin y)$.

Sol. (*i*) Let $f(z) = u + iv$ be the required analytic function.

Here, $u = e^{2x}\,(x \cos 2y - y \sin 2y)$

$\therefore$ $$\frac{\partial u}{\partial x} = e^{2x}\,(2x \cos 2y - 2y \sin 2y + \cos 2y) = \phi_1\,(x, y)$$ | say

and $$\frac{\partial u}{\partial y} = -e^{2x}\,(2x \sin 2y + \sin 2y + 2y \cos 2y) = \phi_2\,(x, y)$$ | say

Now, $\phi_1\,(z, 0) = e^{2z}\,(2z + 1)$

$\phi_2\,(z, 0) = -e^{2z}\,(0) = 0$

By Milne's Thomson method,

$$f(z) = \int \{\phi_1(z, 0) - i\,\phi_2(z, 0)\}\, dz + c = \int e^{2z}(2z + 1)\, dz + c$$

$$= (2z + 1)\,\frac{e^{2z}}{2} - \int 2\,.\,\frac{e^{2z}}{2}\, dz + c$$

$$= (2z + 1)\,\frac{e^{2z}}{2} - \frac{1}{2}\, e^{2z} + c$$

$$= ze^{2z} + c$$

where c is an arbitrary constant.

(*ii*) Let $f(z) = u + iv$ be the required analytic function.

Here $v = e^{-x}(x \cos y + y \sin y)$

$$\frac{\partial v}{\partial y} = e^{-x}\,(-x \sin y + y \cos y + \sin y) = \psi_1(x, y)$$ | say

$$\frac{\partial v}{\partial x} = e^{-x} \cos y - e^{-x}\,(x \cos y + y \sin y) = \psi_2\,(x, y)$$ | say

$\therefore$ $\psi_1\,(z, 0) = 0$

$\psi_2(z, 0) = e^{-z} - e^{-z}\,(z) = (1 - z)\, e^{-z}$

By Milne's Thomson method,

$$f(z) = \int [\psi_1(z, 0) + i\,\psi_2(z, 0)]\, dz + c$$

$$= i \int (1 - z)\, e^{-z}\, dz + c$$

$$= i \left[(1 - z)(-e^{-z}) - \int (-1)(-e^{-z})\, dz\right] + c$$

$$= i\,[(z - 1)\, e^{-z} + e^{-z}] + c$$

$\Rightarrow \qquad f(z) = ize^{-z} + c$

Example 18. *Show that the function $u = e^{-2xy} \sin (x^2 - y^2)$ is harmonic. Find the conjugate function v and express u + iv as an analytic function of z.*

Sol. Here, $\quad u = e^{-2xy} \sin (x^2 - y^2)$

$$\therefore \quad \frac{\partial u}{\partial x} = -2y\, e^{-2xy} \sin (x^2 - y^2) + 2x e^{-2xy} \cos (x^2 - y^2)$$

$$\frac{\partial^2 u}{\partial x^2} = 4y^2 e^{-2xy} \sin (x^2 - y^2) - 4xy\, e^{-2xy} \cos (x^2 - y^2) + 2e^{-2xy} \cos (x^2 - y^2) - 4xy\, e^{-2xy} \cos (x^2 - y^2) - 4x^2 e^{-2xy} \sin (x^2 - y^2) \qquad ...(1)$$

$$\frac{\partial u}{\partial y} = -2x\, e^{-2xy} \sin (x^2 - y^2) - 2y\, e^{-2xy} \cos (x^2 - y^2)$$

$$\frac{\partial^2 u}{\partial y^2} = 4x^2 e^{-2xy} \sin (x^2 - y^2) + 4xy\, e^{-2xy} \cos (x^2 - y^2) - 2e^{-2xy} \cos (x^2 - y^2) + 4xy\, e^{-2xy} \cos (x^2 - y^2) - 4y^2 e^{-2xy} \sin (x^2 - y^2) \qquad ...(2)$$

Adding (1) and (2), we get

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \text{which proves that } u \text{ is harmonic.}$$

Now, $\quad \phi_1(z, 0) = 2z \cos z^2, \quad \phi_2(z, 0) = -2z \sin z^2$

By Milne's Thomson method,

$$f(z) = \int [\phi_1(z, 0) - i\,\phi_2(z, 0)]\, dz + c$$

$$= 2\int (z \cos z^2 + iz \sin z^2)\, dz + c$$

$$= 2\int z e^{iz^2}\, dz + c \qquad \text{Put } iz^2 = t$$

$$= \frac{1}{i}\int e^t\, dt + c = -i e^{iz^2} + c \qquad \therefore \quad 2z\, dz = \frac{dt}{i}$$

Since, $\quad u + iv = -i e^{iz^2} + c = -i e^{i(x+iy)^2} + c$

$$= -i e^{i(x^2 - y^2 + 2ixy)} + c = -ie^{-2xy} \cdot e^{i(x^2 - y^2)} + c$$

$$= -ie^{-2xy} [\cos (x^2 - y^2) + i \sin (x^2 - y^2)] + c$$

$$= e^{-2xy} \sin (x^2 - y^2) + i[-e^{-2xy} \cos (x^2 - y^2)] + c$$

$$\therefore \quad v = -e^{-2xy} \cos (x^2 - y^2) + b \qquad |\text{if } c = a + ib \text{ is complex constant}$$

Example 19. Construct the analytic function $f(z)=u+iv$ if $u(x,y)=y^3-3x^2y$

Sol. Given $u=y^3-3x^2y$

$$\frac{\partial u}{\partial x} = -6xy, \quad \frac{\partial u}{\partial y} = 3y^2 - 3x^2$$

Now $Q_1(z,0) = 0 \quad Q_2(z, 0) = -3z^2$

By Milne's Thomson Method

$$f(z) = \int \left(\phi_1(z,0) - i\phi_2(z,0)\right)dz + c$$
$$= \int 0 - i(-3z^2)dz + c$$
$$= \frac{i3z^3}{3} + c$$
$$f(z) = z^3 i$$
$$f(z) = u + iv = i(x+iy)^3$$
$$= i(x + iy^3 + 3x^2yi - 3xy^2)$$
$$= (y^3 - 3x^2y) + i(x^3 - 3x^2y)$$
$$\therefore\ v(x,y) = x^3 - 3x^2y$$

Solved Example

1. Evaluate $\int_c (\bar{z})^2 dz$ where c is the straight line path joining O(0,0) to A(2,1).

Sol. $f(z) = (\bar{z})^2$
$$= (x-iy)^2 = (x^2 - y^2) - 2ixy$$

Now along the straight line OA The equation OA is

$$(y-0) = \frac{1-0}{2-0}(x-0)$$
$$y = \frac{x}{2} \Rightarrow x = 2y$$
$$dx = 2dy$$
$$\Rightarrow dz = (dx + idy) = (2+i)dy$$

also y varies from 0 to 1

$$\Rightarrow \int_c (z)^2 dz = \int_c \left[(x^2 - y^2) - 2ixy\right](dx + idy)$$
$$= \int_0^1 \left[(2y)^2 - y^2 - 2i(2y)y\right](2+i)dy$$
$$= \int_0^1 \left(4y^2 - y^2 - 4y^2 i\right)(2+i)dy$$

$$= \int_0^1 \left(3y^2 - 4y^2 i\right)(2+i)dy$$

$$= \int_0^1 (3-4i)(2+i)y^2 dy$$

$$= \int_0^1 (3-4i)(2+i)\left(\frac{y^3}{3}\right)_0^1 = \frac{10}{3} - \frac{5i}{3}$$

1. Expand $f(z) = \dfrac{1}{z^2 - 7z + 6}$ in the regions

(i) |z| < 1 (ii) 1 < |z| < 6 (iii) |z| > 6

$$f(z) = \frac{1}{z^2 - 7z + 6} = \frac{1}{(z-1)(z-6)}$$

Sol. $= \dfrac{1}{5}\left[\dfrac{1}{z-6} - \dfrac{1}{z-1}\right]$

(i) **|z| < 1**

|z| < 1 $\Rightarrow$ **|z| < 6** $\Rightarrow \left|\dfrac{z}{6}\right|$ **< 1**

$$(z) = \frac{1}{5}\left[\frac{-1}{6}\left[1 - \frac{2}{6}\right]^{-1} + (1-z)^{-1}\right]$$

$$= \frac{1}{5}\left[\frac{-1}{6}\sum\left(\frac{z}{6}\right)^n + \sum z^n\right]$$

(ii) 1 < |z| < 6

$\Rightarrow \left|\dfrac{1}{z}\right| < 1$ **and** $\left|\dfrac{z}{6}\right| < 1$

it is a Laurent's series within the assuming 1< |z| <6.

(iii) |z| > 6

$$|z| > 6 \Rightarrow \left|\frac{6}{z}\right| < 1$$

also $|z| > 6 \Rightarrow |z| > 1 \Rightarrow \left(\frac{1}{z}\right) < 1$

$$\therefore f(z) = \frac{1}{5}\left\{\frac{1}{z}\left[\frac{1}{(1-\frac{6}{z})}\right] - \frac{1}{z(1-\frac{1}{z})}\right\}$$

$$= \frac{1}{5}\left\{\frac{1}{z}\left(1-\frac{6}{z}\right)^{-1} - \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}\right\}$$

$$= \frac{1}{5}\left\{\frac{1}{z}\sum\left(\frac{6}{z}\right)^{n} - \sum\left(\frac{1}{z}\right)^{n+1}\right\}$$

it is a Laurent's series within the assuming 1< |z| <6.

(iii) |z| > 6

$$|z| > 6 \Rightarrow \left|\frac{6}{z}\right| < 1$$

also $|z| > 6 \Rightarrow |z| > 1 \Rightarrow \left(\frac{1}{z}\right) < 1$

$$\therefore f(z) = \frac{1}{5}\left\{\frac{1}{z}\left[\frac{1}{(1-\frac{6}{z})}\right] - \frac{1}{z(1-\frac{1}{z})}\right\}$$

$$= \frac{1}{5}\left\{\frac{1}{z}\left(1-\frac{6}{z}\right)^{-1} - \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}\right\}$$

$$= \frac{1}{5}\left\{\frac{1}{z}\sum\left(\frac{6}{z}\right)^{n} - \sum\left(\frac{1}{z}\right)^{n+1}\right\}$$

Q. 1. *Determine a, b, c, d such that*

$$f(z) = (x^2 + axy + by^2) + i(cx^2 + dxy + y^2)$$

is analytic.

Sol. Here $u = x^2 + axy + by^2$, $v = cx^2 + dxy + y^2$, given $f(z)$ is analytic.

Therefore C.R. equations must be satisfied.

Now $$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$\Rightarrow$ $$2x + ay = dx + 2y$$

$\Rightarrow$ $$(2 - d)x + (a - 2)y = 0 \qquad ...(1)$$

Again, $$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$\Rightarrow$ $$ax + 2by = -2cx - dy$$

$\Rightarrow$ $$(a + 2c)x + (2b + d)y = 0 \qquad ...(2)$$

Solving (1) and (2) for a, b, c, d, we get

$$2 - d = 0,\ a - 2 = 0$$ | On equating the co-efficient of x, y in (1)

$\Rightarrow$ $$d = 2,\ a = 2$$

Similarly from (2),

$$a + 2c = 0 \ \Rightarrow\ c = -1,\ 2b + d = 0 \ \Rightarrow\ b = -1.$$

Q. 2. *Determine p such that the function* $f(z) = \frac{1}{2}\log(x^2 + y^2) + i\tan^{-1}\frac{px}{y}$ *be an analytic function.*

Sol. Take $x = r\cos\theta$, $y = r\sin\theta$. Then

$$f(z) = \frac{1}{2}\log r^2 + i\tan^{-1}(p\cot\theta) = u + iv, \text{ say,}$$

Here $$u = \frac{1}{2}\log r^2 = \log r \quad \text{and} \quad v = \tan^{-1}(p\cot\theta).$$

Now given $f(z)$ is analytic therefore it must satisfy C.R. equations.

Here $$\frac{\partial u}{\partial r} = \frac{1}{r},\ \frac{\partial u}{\partial \theta} = 0$$

$$\frac{\partial v}{\partial r} = 0,\ \frac{\partial v}{\partial \theta} = \frac{1}{1 + p^2\cot^2\theta}(-p\,\text{cosec}^2\theta)$$

Now $$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$$ | From C.R. equations

$\Rightarrow$ $$\frac{1}{r} = \frac{1}{r}\frac{(-p\,\text{cosec}^2\theta)}{1 + p^2\cot^2\theta} \ \Rightarrow\ 1 + p^2\cot^2\theta = -p\,\text{cosec}^2\theta$$

$\Rightarrow$ $-1 = p(p\cot^2\theta + \text{cosec}^2\theta)$. This equation is true if $p = -1$.

Q. 3. *Evaluate* $\displaystyle\int \frac{dx}{(x^2 + a^2)(x^2 + b^2)}$.

Sol. Consider $f(z) = \dfrac{1}{(z^2 + a^2)(z^2 + b^2)}$.

The poles are given by $z = \pm ai, \pm bi$. Only $z = ai, bi$ lie in the upper half of the plane.

We now find the residues of $f(z)$ at $z = ai, bi$

Now residue of $f(z)$ at $z\ (= ai)$ = Res. $(f(z), ai)$

$$= \underset{z \to ai}{\text{Lt}} \, (z - ai) f(z)$$

$$= \underset{z \to ai}{\text{Lt}} \, (z - ai) \frac{1}{(z + ai)(z - ai)(z + bi)(z - bi)}$$

$$= \underset{z \to ai}{\text{Lt}} \, \frac{1}{(z + ai)(z + bi)(z - bi)}$$

$$= \frac{1}{2ai(ai + bi)(ai - bi)}$$

$$= \frac{1}{2ai(-a^2 + b^2)} \cdot \frac{i}{i}$$

$$= \frac{i}{2a(a^2 - b^2)}$$

Similarly residue of $f(z)$ at $z(= bi) = \dfrac{i}{2b(b^2 - a^2)}$

Therefore by Cauchy Residue Theorem,

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2 + a^2)(x^2 + b^2)} = 2\pi i \text{ [sum of residues in the upper half of the plane]}$$

$$= 2\pi i \left[\frac{i}{2(a^2 - b^2)} \left(\frac{1}{a} - \frac{1}{b} \right) \right]$$

$$= \frac{\pi}{a^2 - b^2} \cdot \frac{b - a}{ab} = \frac{\pi}{(a + b)ab}.$$

Q. 4. *Evaluate* $\displaystyle\int_{-\infty}^{\infty} \frac{x^2 - x + 2}{x^4 + 10x^2 + 9} \, dx.$

Sol. Here $f(z) = \dfrac{z^2 - z + 2}{z^4 + 10z^2 + 9}$.

The poles are given by $z^4 + 10z^2 + 9 = 0$

$\Rightarrow \quad z^4 + 9z^2 + z^2 + 9 = 0$

$\Rightarrow \quad (z^2 + 9)(z^2 + 1) = 0 \quad \Rightarrow \quad z = \pm 3i, z = \pm i$

Only $z = 3i, i$ lie in the upper half of the plane.

We now find the residues of $f(z)$ at $z = 3i, i$

Now residue of $f(z)$ at $z(= 3i) = \text{Res.}(f(z), 3i)$

$$= \underset{z \to 3i}{\text{Lt}} \, (z - 3i) f(z)$$

$$= \underset{z \to 3i}{\text{Lt}} \, (z - 3i) \cdot \frac{z^2 - z + 2}{(z + 3i)(z - 3i)(z^2 + 1)}$$

$$= \underset{z \to 3i}{\text{Lt}} \, \frac{z^2 - z + 2}{(z + 3i)(z^2 + 1)}$$

$$= \frac{-9 - 3i + 2}{(3i + 3i)(-9 + 1)}$$

$$= \frac{-7-3i}{6i(-8)} = \frac{7 \quad 3i}{48i}$$

Also residue of $f(z)$ *at* $z(=i)$ = Res.$(f(z), i)$

$$= \underset{z \quad i}{\text{Lt}} (z-i) f(z)$$

$$= \underset{z \quad i}{\text{Lt}} (z-i) \frac{z^2 - z \quad 2}{(z^2 \quad 9)(z \quad i)(z-i)}$$

$$= \underset{z \quad i}{\text{Lt}} \frac{z^2 - z \quad 2}{(z^2 \quad 9)(z \quad i)} = \frac{-1-i \quad 2}{(-1 \quad 9)(i \quad i)}$$

$$= \frac{1-i}{8(2i)} = \frac{1-i}{16i}$$

Hence by using Cauchy Residue Theorem,

$$\int \frac{x^2 \quad x \quad 2}{x^4 \quad 10x^2 \quad 9} dx = 2\text{p}i \text{ [sum of residues in the upper half of the plane]}$$

$$= 2\text{p}i \left[\frac{7 \quad 3i}{48i} \quad \frac{1 \quad i}{16i}\right] \quad \frac{(7 \quad 3i \quad 3 \quad 3i)}{24} = \frac{5}{12}.$$

Q. 5. *Evaluate* $\int_0 \frac{x^2}{(x^2 \quad 9)(x^2 \quad 4)^2} dx.$

Sol. Consider $\int \frac{x^2}{(x^2 \quad 9)(x^2 \quad 4)^2} dx \quad 2\int_0 \frac{x^2}{(x^2 \quad 9)(x^2 \quad 4)^2} dx$ ...(1)

$$\left| \int_a^a f(x)\, dx \quad 2\int_0^a f(x)\, dx \text{ if } f(x) \text{ is even}\right.$$

Here $f(z) = \dfrac{z^2}{(z^2+9)(z^2+4)^2}.$

The poles are given by $z = \pm 3i, \pm 2i$ and $\pm 2i$.

Out of these $3i$, $2i$ lie in the upper half of the plane.

$z = 3i$ is a simple pole whereas $z = 2i$ is a double pole.

We now find the residues of $f(z)$ at those poles

Now Res. $(f(z), 3i) = \underset{z \quad 3i}{\text{Lt}} (z-3i) \cdot f(z) = \underset{z \to 3i}{\text{Lt}} \dfrac{(z-3i)z^2}{(z+3i)(z-3i)(z^2+4)^2}$

$$= \underset{z \quad 3i}{\text{Lt}} \frac{z^2}{(z \quad 3i)(z^2 \quad 4)^2}$$

$$= \frac{9}{6i \cdot 25} \quad \frac{9i}{150} \quad \frac{3i}{50}$$

Also Res.$(f(z), 2i) = \underset{z \quad 2i}{\text{Lt}} \dfrac{1}{1!} \dfrac{d}{dz} [(z-2i)^2 \cdot f(z)]$

$$= \underset{z \to 2i}{\text{Lt}} \frac{d}{dz}\left[(z-2i)^2 \cdot \frac{z^2}{(z^2+9)(z+2i)^2(z-2i)^2}\right]$$

$$= \underset{z \to 2i}{\text{Lt}} \frac{d}{dz}\left[\frac{z^2}{(z^2+9)(z+2i)^2}\right]$$

$$= \underset{z \to 2i}{\text{Lt}} \left[\frac{(z^2+9)(z+2i)^2 \,.\, 2z - z^2(2(z^2+9)(z+2i) + 2z(z+2i)^2)}{[(z^2+9)(z+2i)^2]^2}\right]$$

$$= \frac{(-4+9)(4i)^2(4i) - 4(2(-4+9)(4i) + 4i(4i)^2)}{[(-4+9)(4i)^2]^2}$$

$$= \frac{-320i - 4(20i - 64i)}{(-80)^2}$$

$$= \frac{-320i - 176i}{6400} = \frac{-496i}{6400} = \frac{-3i}{200}$$

Hence by using Cauchy Residue Theorem, we have from (1)

$$\int_0^\infty \frac{x^2}{(x^2+9)(x^2+4)^2}\,dx = \frac{1}{2} \,.\, 2\pi i \text{ (Sum of residues in the upper half plane)}$$

$$= \pi i\left(\frac{3i}{50} - \frac{13i}{200}\right) = \frac{\pi}{200}.$$

Q. 6 *Evaluate* $\displaystyle\int_0^\infty \frac{x \sin x}{x^2+a^2}\,dx.$

Sol. Consider $\displaystyle\int_{-\infty}^{\infty} \frac{x \sin x}{x^2+a^2}\,dx = 2\int_0^\infty \frac{x \sin x}{x^2+a^2}\,dx$

$$\left|\; \int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx, \text{ If } f(x) \text{ is even}\right.$$

$$\Rightarrow \int_0^\infty \frac{x \sin x}{x^2+a^2}\,dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{x \sin x}{x^2+a^2}\,dx = \text{I.P. } \frac{1}{2}\int_{-\infty}^{\infty} \frac{xe^{ix}}{x^2+a^2}\,dx \quad \left| e^{i\theta} = \cos\theta + i\sin\theta\right.$$

Here $f(z) = \dfrac{ze^{iz}}{z^2+a^2}$.

The poles are given by $z = \pm ai$. But $z = ai$ lies in the upper half of the plane.
Further Res. $(f(z), ai)$

$$= \underset{z \to ai}{\text{Lt}} (z-ai)\frac{ze^{iz}}{(z-ai)(z+ai)} = \frac{aie^{-a}}{2ai} = \frac{e^{-a}}{2}$$

Hence by Cauchy Residue Theorem, we have

$$\int_0^\infty \frac{x \sin x}{x^2+a^2}\,dx = \frac{1}{2}\text{ I.P. } 2\pi i \,.\left[\frac{e^{-a}}{2}\right] = \frac{\pi}{2}e^{-a}.$$

Q. 7. *Evaluate* $\displaystyle\int_{-\infty}^{\infty} \frac{\cos x}{(x^2+a^2)(x^2+b^2)}\,dx,\ a > b > 0.$

Sol. $\int \frac{\cos x}{(x^2 + a^2)(x^2 + b^2)} dx$

$$= \text{Real part} \int \frac{e^{iz}}{(z^2 + a^2)(z^2 + b^2)} dz \qquad ...(1)$$

Here $f(z) = \frac{e^{iz}}{(z^2 + a^2)(z^2 + b^2)}$.

The poles are given by $z = \pm ai$, $z = \pm bi$.

Only $z = ai$, bi lie in the upper half of the plane.

Further Res. $(f(z), ai) = \underset{z \to ai}{\text{Lt}} (z - ai) . \frac{e^{iz}}{(z - ai)(z + ai)(z^2 + b^2)} = \frac{e^{-a}}{2ai(b^2 - a^2)}$

Similarly Res. $(f(z), bi) = \underset{z \to bi}{\text{Lt}} (z - bi) \frac{e^{iz}}{(z^2 + a^2)(z + bi)(z - bi)} = \frac{e^{-b}}{2bi(a^2 - b^2)}$

Therefore by using Cauchy Residue Theorem, we have, from (1)

$$\int \frac{\cos x}{(x^2 + a^2)(x^2 + b^2)} dx = \text{R.P. } 2\pi i \text{ [Sum of residues in upper half of the plane]}$$

$$= \text{R.P. } \frac{2\pi i}{2i} \left[\frac{e^{-a}}{a(b^2 - a^2)} + \frac{e^{-b}}{2b(a^2 - b^2)} \right]$$

$$= \frac{\pi}{a^2 - b^2} \left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a} \right).$$

Q. 8. *Evaluate* $\int_0^{\infty} \frac{\sin mx}{x(x^2 + a^2)} dx$, *m > 0, a > 0.*

Sol. Here $\int \frac{\sin mx}{x(x^2 + a^2)} dx$

$$= \frac{1}{2} \int_0^{\infty} \frac{\sin mx}{x(x^2 + a^2)} dx$$

$$= \text{I.P. } \frac{1}{2} \int_0^{\infty} \frac{e^{imx}}{x(x^2 + a^2)} dx \qquad ...(1)$$

Here $f(z) = \frac{\sin mz}{z(z^2 + a^2)}$. The poles are given by $z = 0$, $z = \pm ai$. The pole $z = 0$ lies on the real axis.

Therefore we choose the contour C to be a large semi-circle $| z | = \text{R}$ and a small circle of radius r. Then the only pole within C is $z = ai$.

Now Res. $(f(z), ai) = \underset{z \to ai}{\text{Lt}} (z - ai) . f(z) = \underset{z \to ai}{\text{Lt}} (z - ai) . \frac{e^{imz}}{z(z^2 + a^2)}$

$$= \underset{z \to ai}{\text{Lt}} (z - ai) \frac{c^{imz}}{z(z - ai)(z + ai)} = \underset{z \to ai}{\text{Lt}} \frac{c^{imz}}{z(z + ai)} = \frac{e^{-am}}{-2a^2}$$

Therefore by using Cauchy Residue theorem,

$$\oint_C f(z)\, dz = 2\pi i \text{ (sum of residues in the upper half plane)}$$

$$= 2\pi i \frac{e^{-am}}{-2a^2} = \frac{-\pi i}{a^2} e^{-am}$$

$$\Rightarrow \int_r^R f(z)\, dz + \int_{C_R} f(z)\, dz + \int_{-R}^{-r} f(z)\, dz + \int_{C_r} f(z)\, dz = -\frac{\pi i}{a^2} e^{-am}$$

$$\Rightarrow I_1 + I_2 + I_3 + I_4 = -\frac{\pi i}{a^2} e^{-am}, \text{ say} \qquad \ldots(*)$$

Consider $\quad I_2 = \int_{C_R} f(z)\, dz = \int_{C_R} \frac{e^{izm}}{z(z - ai)(z + ai)}\, dz$

$$= \frac{1}{Ri} \int_0^{\pi} \frac{e^{i(R\cos\theta + i\sin\theta)m}}{Re^{i\theta}\,(Re^{i\theta} - ai)(Re^{i\theta} + ai)} \frac{d\theta}{e^{i\theta}} \to 0, \text{ as } R \to \infty$$

| Put $x = R\cos\theta$, $y = R\sin\theta$, $0 \le \theta \le \pi$, for the upper half $z = Re^{i\theta}$, $dz = Ri\, e^{i\theta}\, d\theta$

Further $\quad I_4 = \int_{C_r} f(z)\, dz$

$$= \int_{C_r} \frac{e^{imz}}{z(z^2 + a^2)}\, dz \qquad \text{Put } z = r\cos\theta + i\sin\theta = re^{i\theta},\ dz = ire^{i\theta}\, d\theta$$

$$\Rightarrow \quad |I_4| = \left| \int_{C_r} \frac{e^{imz}}{z(z^2 + a^2)}\, dz \right| \le \frac{1}{a^2} \left| \int_{C_r} \frac{e^{imz}}{z}\, dz \right|$$

$$= \frac{1}{a^2} \int_{C_r} \frac{1}{z}\, dz \quad | \; |e^{imz}| = 1$$

$$= \frac{1}{a^2} \int_{\pi}^{0} \frac{ire^{i\theta}\, d\theta}{re^{i\theta}} = -\frac{\pi i}{a^2}$$

Therefore from (*), we get

$$\int_r^R f(z)\, dz + \int_{-R}^{-r} f(z)\, dz - \frac{\pi i}{a^2} = -\frac{\pi i}{a^2} e^{-am}$$

Taking $r \to 0$, $R \to \infty$, we get

$$\Rightarrow \quad \int_0^{\infty} f(z)\, dz + \int_{-\infty}^{0} f(z)\, dz = -\frac{\pi i}{a^2} e^{-am} + \frac{\pi i}{a^2} = \frac{\pi i}{a^2} (1 - e^{-am}).$$

Therefore from (1),

$$\int_0^{\infty} \frac{\sin mx}{x(x^2+a^2)}\,dx = \frac{1}{2}\,.\,\text{I.P.}\int_{-\infty}^{\infty} \frac{e^{imx}}{x(x^2+a^2)}\,dx$$

$$= \frac{1}{2}\,\text{I.P.}\int_C f(z)\,dz = \frac{1}{2}\,\text{I.P.}\,\frac{\pi i}{a^2}(1-e^{-am}) = \frac{\pi}{2a^2}(1-e^{-am}).$$

Q. 9. *Evaluate* $\int_0^{2\pi} \frac{1}{(5-3\cos\theta)^2}\,d\theta$.

Sol. Consider $\int_0^{2\pi} \frac{1}{a+b\cos\theta}\,d\theta = \frac{2\pi}{\sqrt{a^2-b^2}}$ (as in Q. 2)

Differentiating w.r.t. a, by Leibnitz's rule for differentiation under the integral sign, we get

$$\frac{d}{da}\int_0^{2\pi} \frac{1}{(a+b\cos\theta)}\,d\theta = \frac{d}{da}\,2\pi(a^2-b^2)^{-1/2}$$

$$\Rightarrow \quad \int_0^{2\pi} \frac{d}{da}\frac{1}{(a+b\cos\theta)}\,d\theta = 2\pi\left(-\frac{1}{2}\right)(a^2-b^2)^{-\frac{3}{2}}(2a)$$

$$\Rightarrow \quad \int_0^{2\pi} \frac{-1}{(a+b\cos\theta)^2}\,d\theta = \frac{-2\pi a}{(a^2-b^2)^{3/2}}$$

$$\Rightarrow \quad \int_0^{2\pi} \frac{1}{(a+b\cos\theta)^2}\,d\theta = \frac{2\pi a}{(a^2-b^2)^{3/2}}$$

Take $a = 5$, $b = -3$,

$$\int_0^{2\pi} \frac{1}{(5-3\cos\theta)^2}\,d\theta = \frac{10\pi}{16^{3/2}} = \frac{5\pi}{32}.$$

Q. 10. *Evaluate* $\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)(x^2+4)}\,dx$.

Sol. Here $f(z) = \frac{z^2}{(z^2+1)(z^2+4)}$.

The poles are given by $z = \pm i$, $z = \pm 2i$.

Out of these $z = i$, $2i$ lie in the upper half of the plane.

$$\text{Res.}\,(f(z), i) = \underset{z\to i}{\text{Lt}}\,(z-i)\,.\,f(z) = \underset{z\to i}{\text{Lt}}\,(z-i)\,.\,\frac{z^2}{(z+i)(z-i)(z^2+4)}$$

$$= \underset{z\to i}{\text{Lt}}\,\frac{z^2}{(z+i)(z^2+4)}$$

$$= \frac{-1}{2i(-1 \quad 4)} = \frac{-1}{6i}$$

Similarly Res. $(f(z), 2i) = \underset{z \quad 2i}{\text{Lt}} (z - 2i) \,.\, f(z)$

$$= \underset{z \quad 2i}{\text{Lt}} (z \quad 2i) \,.\, \frac{z^2}{(z^2 \quad 1)(z \quad 2i)(z \quad 2i)}$$

$$= \underset{z \to 2i}{\text{Lt}} \frac{z^2}{(z^2 + 1)(z + 2i)} = \frac{4}{12i^3} \quad \frac{1}{3i}.$$

Therefore, by using Cauchy Residue Theorem, we have

$$\int \frac{x^2}{(x^2 \quad 1)(x^2 \quad 4)} dx = 2\text{p}i \text{ (sum of residues in the upper half of the plane)}$$

$$= 2\text{p}i \left(\frac{-1}{6i} + \frac{1}{3i} \right) = 2 \quad i \frac{(\quad 1 \quad 2)}{6i} \quad \frac{2 \quad i}{6i} \qquad / 3.$$

Q. 11. *Evaluate* $\int_0 \frac{\cos x}{x} dx.$

Sol. Consider the integral $\int_C \frac{e^{iz}}{z} dz \quad \int_C f(z)\, dz,$

where C consists of (*i*) The real axis from r to R

(*ii*) The upper half of the circle $| z | = \text{R}$, say C_R

(*iii*) The real axis $-\text{R}$ to $-r$

(*iv*) The upper half of the circle $| z | = r$, say C_r and $\text{R} > r$.

Now the singularities of $f(z)$ is $z = 0$. As

$z = 0$ lies outside C \ By Cauchy Theorem,

$$\int_C f(z)\, dz = 0$$

$$\text{Þ} \qquad \int_r^R f(z)\, dz \quad \int_{C_R} f(z)\, dz \quad \int_R^r f(z)\, dz \quad \int_{C_r} f(z)\, dz = 0$$

$$I_1 + I_2 + I_3 + I_4 = 0, \text{ say} \qquad ...(*)$$

Consider $\quad I_2 = \int_{C_R} f(z)\, dz \quad \int_{C_R} \frac{e^{iz}}{z} dz \qquad$ | Put $z = \text{R}e^{i\text{q}}$, $dz = \text{R}ie^{i\text{q}}\, d\text{q}$, 0 £ q £ p

$$= i \int_0 \frac{e^{i\text{R}(\cos \quad i \sin \ )} \text{R}e^{i}}{\text{R}e^{i}} d \qquad i \int_0 e^{\ \text{R} \sin} \,.\, e^{i\text{R} \cos} \ d$$

$$\Rightarrow \left|\int_{C_R} f(z)\,dz\right| \le \left| i\int_0^{\pi} e^{-iR\sin\theta}\, e^{iR\cos\theta}\, d\theta\right| \le \int_0^{\pi} |e^{iR\cos\theta}|\, e^{-R\sin\theta}\, d\theta$$

$$= \int_0^{\pi} e^{-R\sin\theta}\, d\theta \qquad \because |e^{iR\cos\theta}| = 1$$

$$= 2\int_0^{\pi/2} e^{-R\sin\theta}\, d\theta \le 2\int_0^{\pi/2} e^{-R.\frac{2\theta}{\pi}}\, d\theta$$

$$\left[\text{For } 0 \le \theta \le \frac{\pi}{2},\ \frac{\sin\theta}{\theta} \ge \frac{2}{\pi} \text{ or } \sin\theta \ge \frac{2\theta}{\pi}\right]$$

$$= 2\left|\frac{e^{-\frac{2R\theta}{\pi}}}{-\frac{2R}{\pi}}\right|_0^{\pi/2} = \frac{\pi}{R}(1 - e^{-R}) \to 0, \text{ as } R \to \infty$$

$\Rightarrow$ $I_2 = 0.$

Now consider $I_4 = \int_{C_r} f(z)\,dz = \int_{C_r} \frac{e^{imz}}{z}\,dz$

$$= \int_{C_r} \frac{1 + e^{imz} - 1}{z}\,dz = \int_{C_r} \frac{1}{z}\,dz + \int_{C_r} \frac{e^{imz} - 1}{z}\,dz$$

Take $z = re^{i\theta}$ $\Rightarrow$ $dz = rie^{i\theta}\,d\theta$, $\pi \le \theta \le 0$.

Therefore $\int_{C_r} \frac{1}{z}\,dz = \int_{\pi}^{0} \frac{rie^{i\theta}}{re^{i\theta}}\,d\theta = -\pi i$

and $\left|\int_{C_r} \frac{e^{imz} - 1}{z}\,dz\right| \le \int_{C_r} \left|\frac{e^{imz} - 1}{z}\right| |dz|$

$$= \int_{C_r} \left|\frac{e^{i(r\cos\theta + i\sin\theta)} - 1}{re^{i\theta}}\right| |dz| \le \int_{C_r} \left|\frac{e^{-r\sin\theta}\, e^{ir\cos\theta} - 1}{i}\right| d\theta \to 0, \text{ as } r \to \infty$$

Therefore $I_4 = -\pi i$. Hence from (*), we have

$$I_1 + I_3 - \pi i = 0$$

$$\Rightarrow \int_r^R f(z)\,dz + \int_{-R}^{-r} f(z)\,dz = \pi i \Rightarrow \int_0^{\infty} f(z)\,dz + \int_{-\infty}^{0} f(z)\,dz = \pi i \qquad \left| \begin{matrix} r \to 0 \\ R \to \infty \end{matrix} \right.$$

$$= \int_{-\infty}^{\infty} f(z)\,dz = \pi i.$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{e^{iz}}{z}\,dz = \pi i.$$

Equating real part,

$$\int_{-\infty}^{\infty} \frac{\cos z}{z}\,dz = 0 \Rightarrow 2\int_0^{\infty} \frac{\cos x}{x}\,dx = 0 \Rightarrow \int_0^{\infty} \frac{\cos x}{x}\,dx = 0.$$

Q. 12. *Evaluate* $\int \frac{\sin x}{x^2 + 4x + 5} dx$.

Sol. Consider $f(z) = \frac{e^{iz}}{z^2 + 4z + 5}$.

The poles are given by $z^2 + 4z + 5 = 0$

$\Rightarrow \quad z = -2 - i, -2 + i.$

Only $z = -2 + i$ lies in the upper half of the plane.

$$\text{Res. } (f(z), -2 + i) = \underset{z \to -2+i}{\text{Lt}} (z + 2 - i) \frac{e^{iz}}{(z + 2 - i)(z + 2 + i)}$$

$$= \frac{e^{i(-2+i)}}{-2 + i + 2 + i} = \frac{e^{-2i} e^{-1}}{2i}$$

Therefore by Cauchy Residue Theorem,

$$\int \frac{e^{iz}}{z^2 + 4z + 5} dz = 2\pi i \left(\frac{e^{-2i} e^{-1}}{2i} \right) = \frac{\pi}{e} (\cos 2 - i \sin 2)$$

Equating imaginary part,

$$\int \frac{\sin x}{x^2 + 4x + 5} dx = -\frac{\pi \sin 2}{e}.$$

Section C

SOME MORE IMPORTANT PROBLEMS

Q. 1. *Solve* $\int_0 \frac{1 - \cos x}{x^2} dx$ (P.T.U. B. Tech., May 2005)

Sol. Try yourself as in Q. 19. **Ans.** $\frac{\pi}{2}$

Q. 2. *Apply calculus of residues to prove that* $\int_0 \frac{dx}{x^4 + a^4} = \frac{\pi\sqrt{2}}{4a^3}$; *a* > 0.

(P.T.U. B. Tech., Dec. 2006)

Sol. Consider the integral $\int_C f(z)\, dz$ where $f(z) = \frac{1}{z^4 + a^4}$.

The poles of $f(z)$ are given by

$$z^4 + a^4 = 0 \quad \Rightarrow \quad z^4 = a^4 e^{\pi i} = a^4 e^{2n\pi i + \pi i}$$

or $$z = a e^{(2n+1)\pi i/4};\ n = 0, 1, 2, 3.$$

Since there is no pole on the real axis, therefore, we may take the closed contour C consisting of the upper half C_R of a large semi-cicle $|z| = R$ and the real axis from – R to R.

∴ By Cauchy's residue theorem, we have

$$\int_C f(z)\, dz = 2\pi i \sum R$$

$\Rightarrow \quad \int_{-R}^{R} f(x)\, dx + \int_{C_R} f(z)\, dz = 2\pi i \sum R$

or $$\int_{-R}^{R} \frac{1}{x^4 + a^4}\, dx + \int_{C_R} \frac{1}{z^4 + a^4}\, dz = 2\pi i \sum R \qquad \text{...(1)}$$

| where $\sum R$ = sum of residue of $f(z)$ at poles within C.

The poles $z = ae^{\frac{i\pi}{4}}$ and $z = ae^{3\pi i/4}$ are the only two poles which the lie within the contour C.

Let α denote any one of these poles, then

$$\alpha^4 + a^4 = 0$$

$\Rightarrow \qquad \alpha^4 = -a^4.$

Residue of $f(z)$ at $z = \alpha$ is

$$= \left[\frac{1}{\frac{d}{dz}(z^4 + a^4)}\right]_{z=\alpha} = \frac{1}{4\alpha^3} = \frac{\alpha}{4\alpha^4} = -\frac{\alpha}{4a^4}$$

$\therefore$ Residue at $z = ae^{\pi i/4}$ is $= -\dfrac{1}{4a^3} e^{\pi i/4}$

and residue at $z = ae^{3\pi i/4}$ is $= -\dfrac{1}{4a^3} e^{3\pi i/4}$

$\therefore$ Sum of residues $= -\dfrac{1}{4a^3} e^{i\pi/4} - \dfrac{1}{4a^3} e^{i3\pi/4} = -\dfrac{1}{4a^3}\left[e^{i\frac{\pi}{4}} + e^{i\frac{3\pi}{4}}\right]$

$$= -\frac{1}{4a^3}\left[\cos\frac{\pi}{4} + i\sin\pi/4 + \cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right]$$

$$= -\frac{1}{4a^3}\left[\cos\frac{\pi}{4} + i\sin\frac{\pi}{4} + \cos\left(\pi - \frac{\pi}{4}\right) + i\sin\left(\pi - \frac{\pi}{4}\right)\right]$$

$$= -\frac{1}{4a^3}\left[\cos\frac{\pi}{4} + i\sin\frac{\pi}{4} - \cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right]$$

$$= -\frac{i}{2a^3}\sin\frac{\pi}{4} = -\frac{i}{2a^3}\frac{1}{\sqrt{2}}$$

∴ From (1), $\int_{-R}^{R}\frac{dx}{x^4+a^4}+\int_{C_R}\frac{dz}{z^4+a^4}=2\pi i\left(\frac{-i}{2\sqrt{2}\,a^3}\right)=\frac{\pi\sqrt{2}}{2a^3}$ ...(2)

Now, $\left|\int_{C_R}\frac{1}{z^4+a^4}\,dz\right|\le\int_{C_R}\frac{|dz|}{|z^4+a^4|}\le\int_{C_R}\frac{|dz|}{|z^4|-|a^4|}$

$$=\int_0^{\pi}\frac{R\,d\theta}{R^4-a^4}\qquad \because\ |z|=R \text{ on } C_R$$

$$=\frac{\pi R}{R^4-a^4}\to 0 \text{ as } R\to\infty.$$

Hence when $R\to\infty$, relation (2) becomes

$$\int_{-\infty}^{\infty}\frac{dx}{x^4+a^4}=\frac{\pi\sqrt{2}}{2a^3}\quad\text{or}\quad\int_0^{\infty}\frac{dx}{x^4+a^4}=\frac{\pi\sqrt{2}}{4a^3}.$$

Q. 3. *Show by the method of residues that :*

$$\int_0^{\pi}\frac{d\theta}{17-8\cos\theta}=\frac{\pi}{15}.$$ (P.T.U. B. Tech., Dec. 2003)

Sol. $\int_0^{2\pi}\frac{d\theta}{17-8\cos\theta}=2\int_0^{\pi}\frac{1}{17-8\cos\theta}\,d\theta$

Since $\int_0^{2a}f(x)\,dx=2\int_0^{a}f(x)\,dx$, if $f(2a-x)=f(x)$

Here $\cos(2\pi-\theta)=\cos\theta$

⇒ $\int_0^{\pi}\frac{1}{17-8\cos\theta}\,d\theta=\frac{1}{2}\int_0^{2\pi}\frac{1}{17-8\cos\theta}\,d\theta$ ...(1)

Putting $z=e^{i\theta}$, so that

$$\cos\theta=\frac{1}{2}\left(z+\frac{1}{z}\right)\quad\text{and}\quad dz=ie^{i\theta}\,d\theta$$

⇒ $$d\theta=\frac{1}{iz}\,dz$$

∴ $\int_0^{2\pi}\frac{1}{17-8\cos\theta}\,d\theta=\oint_C\frac{1}{17-8\cdot\frac{1}{2}\left(z+\frac{1}{z}\right)}\frac{dz}{iz}$, where C : $|z|=1$

$$=\oint_C\frac{1}{17z-4(z^2+1)}\frac{dz}{i}$$

$$=-\frac{1}{i}\oint_C\frac{1}{4z^2-17z+4}\,dz$$ ...(2)

The poles are given by $4z^2-17z+4=0$

$$\Rightarrow \qquad z = \frac{17 \pm \sqrt{289 - 64}}{8} = \frac{17 \pm \sqrt{225}}{8} = \frac{17 \pm 15}{8} = 4, \frac{1}{4}$$

Out of these, $z = 1/4$ lies in $|z| = 1$

$\therefore$ Residue of $f(z)$ at $z = \frac{1}{4}$ $= \underset{z \to \frac{1}{4}}{\text{Lt}} \left(z - \frac{1}{4}\right) . f(z)$

$$= \frac{1}{i} \underset{z \to \frac{1}{4}}{\text{Lt}} \left(z - \frac{1}{4}\right) . \frac{1}{4\left(z - \frac{1}{4}\right)(z - 4)}$$

$$= \frac{1}{4i} \underset{z \to \frac{1}{4}}{\text{Lt}} \frac{1}{z - 4} = \frac{1}{4i} . \frac{1}{\frac{1}{4} - 4} = \frac{1}{15i}$$

Hence from (2), by using Cauchy Residue theorem,

$$\int_0^{2\pi} \frac{1}{17 - 8\cos\theta}\, d\theta = 2\pi i . \frac{1}{15\, i} = \frac{2\pi}{15}$$

$\therefore$ From (1), we have

$$\int_0^{\pi} \frac{1}{17 - 8\cos\theta}\, d\theta = \frac{1}{2} . \frac{2\pi}{15} = \frac{\pi}{15}$$

PROBLEMS FOR PRACTICE

1. Evaluate $\displaystyle\int_0^{2\pi} \frac{d\theta}{2 + \cos\theta}$.

2. Use Residue Calculus to evaluate the integral $\displaystyle\int_0^{2\pi} \frac{d\theta}{5 + 4\sin\theta}$. (*P.T.U.B.Tech Dec. 2006*)

3. Evaluate $\displaystyle\int_0^{2\pi} \frac{\cos\theta}{3 + \sin\theta}\, d\theta$

4. Using Residue theorem evaluate the integral $\displaystyle\int_0^{\pi} \frac{1 + 2\cos\theta}{5 + 4\cos\theta}\, d\theta$.

5. Evaluate $\displaystyle\int_0^{2\pi} \frac{\sin 2\theta\, d\theta}{1 - 2p\cos\theta + p^2}$, $0 < p < 1$

6. Evaluate $\displaystyle\int_0^{2\pi} \frac{\sin^2\theta}{5 + 4\cos\theta}\, d\theta$

7. Evaluate by Residue theorem $\displaystyle\int_0^{2\pi} \frac{\sin^2\theta - 2\cos\theta}{2 + \cos\theta}\, d\theta$

8. Show by method of residues that $\displaystyle\int_0^{\pi} \frac{a\, d\theta}{a^2 + \sin^2\theta} = \frac{\pi}{\sqrt{1 + a^2}}$.

9. Evaluate $\displaystyle\int_0^{2\pi} \frac{d\theta}{1 - 2a\sin\theta + a^2}$, $0 < a < 1$. (*P.T.U.B.Tech. Dec. 2001*)

10. Prove that $\int_0 \frac{\sin^2}{a \quad b\cos} d \quad = \frac{2}{b^2}(a - \sqrt{a^2 \quad b^2})$, where $0 < b < a$.

11. Evaluate $\int_0 \frac{d}{a \quad b\cos}$, where $a > |b|$.

12. Evaluate $\int \frac{x^2}{(x^2 \quad a^2)(x^2 \quad b^2)} dx$. (*P.T.U.B.Tech. Dec. 2003*)

13. Evaluate $\int_0 \frac{dx}{x^4 \quad 1}$. (*P.T.U.B.Tech. Dec. 2006*)

14. Evaluate $\int \frac{x^2}{(1 \quad x^2)^3} dx$.

15. Evaluate $\int \frac{x^2\, dx}{x^4 \quad 5x^2 \quad 4}$.

16. Evaluate $\int_0 \frac{dx}{x^6 \quad 1}$.

17. Evaluate $\int_0 \frac{\cos ax}{x^2 \quad 1} dx (a\ ^{3}\ 0)$

18. Evaluate $\int_0 \frac{\cos 3x}{(x^2 \quad 1)(x^2 \quad 4)} dx$.

19. Evaluate $\int_0 \frac{\cos 2x}{(x^2 \quad 9)^2 (x^2 \quad 16)} dx$

20. Using Calculus of Residue evaluate the integral given by $\int_0 \frac{\cos ax}{(x^2 \quad b^2)^2} dx$; $a >$, $b > 0$

21. Evaluate $\int_0 \frac{x \sin ax}{x^4 \quad a^4} dx$; $a > 0$.

22. Evaluate $\int_0 \frac{\sin mx}{x} dx$, $m > 0$.

23. Show that if $a\ ^{3}\ b\ ^{3}\ 0$, then $\int_0 \frac{\cos 2ax \quad \cos 2bx}{x^2} dx = \text{p}(b - a)$.

24. Evaluate $\int_0 \frac{\sin x}{x(x^2 \quad a^2)} dx$; $a > 0$.

25. By contour integration, show that $\int_0 \frac{1 \quad \cos x}{x^2} dx = \frac{}{2}$. (*P.T.U. May 2005*)

Answers

1. $\frac{2}{\sqrt{3}}$

2. $\frac{2}{3}$

3. 0

4. 0

5. 0

6. $\frac{}{4}$

7. $\frac{2}{\sqrt{3}}$

9. $\frac{2}{1 \quad a^2}$

11. $\frac{}{\sqrt{a^2 \quad b^2}}$

12. $\frac{}{a \quad b}$

13. $\frac{}{2\sqrt{2}}$

14. $\frac{}{8}$

15. $\frac{}{3}$

16. $\frac{}{3}$

17. $\frac{e^{-a}}{2}$

18. $\frac{}{12}(2e^{-3} - e^{-6})$

19. $\frac{}{196}\left(\frac{e^{-8}}{2} \quad \frac{31\, e^{-6}}{27}\right)$

20. $\frac{(ab \quad 1)\, e^{-ab}}{4b^3}$

21. $\frac{}{4a^2} e^{\frac{-a^2}{\sqrt{2}}} \sin \frac{a^2}{\sqrt{2}}$ **22.** $\frac{}{2}$ **24.** $\frac{}{2a^2}(1 - e^{-a})$

PRoblEM

An analytic function with constant modulus is constant.

Proof. Let $f(z) = u + iv$ be an analytic function with constant modulus. Then,

$$|f(z)| = |u + iv| = \text{constant}$$

$$\Rightarrow \quad \sqrt{u^2 + v^2} = \text{constant} = c \text{ (say)}$$

Squaring both sides, we get

$$u^2 + v^2 = c^2 \qquad ...(1)$$

Differentiating eqn. (1) partially w.r.t. x, we get

$$2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x} = 0$$

$$\Rightarrow \quad u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x} = 0 \qquad ...(2)$$

Again, differentiating eqn. (1) partially w.r.t. y, we get

$$2u\frac{\partial u}{\partial y} + 2v\frac{\partial v}{\partial y} = 0$$

$$\Rightarrow \quad u\frac{\partial u}{\partial y} + v\frac{\partial v}{\partial y} = 0$$

$$\Rightarrow \quad u\left(-\frac{\partial v}{\partial x}\right) + v\left(\frac{\partial u}{\partial x}\right) = 0 \qquad ...(3) \qquad \left| \because \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \text{ and } \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}\right.$$

Squaring and adding eqns. (2) and (3), we get

$$(u^2 + v^2)\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right] = 0$$

$$\Rightarrow \quad \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 = 0 \qquad |\because u^2 + v^2 = c^2 \neq 0$$

$$\Rightarrow \quad |f'(z)|^2 = 0 \qquad \left| \because f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right.$$

$$\Rightarrow \quad |f'(z)| = 0$$

$\Rightarrow$ $f(z)$ is constant.

SOLVED EXAMPLES

Example 1. *If* φ *and* ψ *are functions of x and y satisfying Laplace's equation, show that s + it is analytic, where*

$$s = \frac{\partial \phi}{\partial y} - \frac{\partial \psi}{\partial x} \quad \text{and} \quad t = \frac{\partial \phi}{\partial x} + \frac{\partial \psi}{\partial y}.$$

Sol. Since φ and ψ are functions of x and y satisfying Laplace's equations,

$$\therefore \qquad \frac{\partial^2}{\partial x^2} \quad \frac{\partial^2}{\partial y^2} = 0 \qquad \text{...(1)}$$

and $$\frac{\partial^2}{\partial x^2} \quad \frac{\partial^2}{\partial y^2} = 0. \qquad \text{...(2)}$$

For the function $s + it$ to be analytic,

$$\frac{\partial s}{\partial x} \quad \frac{\partial t}{\partial y} \qquad \text{...(3)}$$

and $$\frac{\partial s}{\partial y} \quad \frac{\partial t}{\partial x} \qquad \text{...(4)}$$

must satisfy.

Now, $$\frac{\partial s}{\partial x} \quad \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y} \quad \frac{\partial}{\partial x}\right) = \frac{\partial^2}{\partial x \partial y} \quad \frac{\partial^2}{\partial x^2} \qquad \text{...(5)}$$

$$\frac{\partial t}{\partial y} \quad \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x} \quad \frac{\partial}{\partial y}\right) = \frac{\partial^2}{\partial y \partial x} \quad \frac{\partial^2}{\partial y^2} \qquad \text{...(6)}$$

$$\frac{\partial s}{\partial y} \quad \frac{\partial}{\partial y}\left(\frac{\partial}{\partial y} \quad \frac{\partial}{\partial x}\right) \quad \frac{\partial^2}{\partial y^2} \quad \frac{\partial^2}{\partial y \partial x} \qquad \text{...(7)}$$

and $$\frac{\partial t}{\partial x} \quad \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x} \quad \frac{\partial}{\partial y}\right) = \frac{\partial^2}{\partial x^2} \quad \frac{\partial^2}{\partial x \partial y}. \qquad \text{...(8)}$$

From (3), (5) and (6), we have

$$\frac{\partial^2}{\partial x \partial y} \quad \frac{\partial^2}{\partial x^2} \quad \frac{\partial^2}{\partial y \partial x} \quad \frac{\partial^2}{\partial y^2} \qquad \Rightarrow \quad \frac{\partial^2}{\partial x^2} \quad \frac{\partial^2}{\partial y^2} = 0$$

which is true by (2).

Again from (4), (7) and (8), we have,

$$\frac{\partial^2}{\partial y^2} \quad \frac{\partial^2}{\partial y \partial x} \quad \frac{\partial^2}{\partial x^2} \quad \frac{\partial^2}{\partial x \partial y} \qquad \Rightarrow \quad \frac{\partial^2}{\partial x^2} \quad \frac{\partial^2}{\partial y^2} = 0$$

which is also true by (1).

Hence the function $s + it$ is analytic.

Example 2. *Verify if* $f(z) = \dfrac{xy^2(x + iy)}{x^2 + y^4}$, $z \neq 0$; $f(0) = 0$ *is analytic or not?*

Sol. $$u + iv = \frac{xy^2(x + iy)}{x^2 + y^4} \ ; z \neq 0$$

$$\therefore \qquad u = \frac{x^2 y^2}{x^2 + y^4}, \quad v = \frac{xy^3}{x^2 + y^4}$$

At the origin, $$\frac{\partial u}{\partial x} = \lim_{x \to 0} \frac{u(x, 0) - u(0, 0)}{x} = \lim_{x \to 0} \frac{0 - 0}{x} = 0$$

$$\frac{\partial u}{\partial y} = \lim_{y \to 0} \frac{u(0, y) - u(0, 0)}{y} = \lim_{y \to 0} \frac{0 - 0}{y} = 0$$

$$\frac{\partial v}{\partial x} = \lim_{x \to 0} \frac{v(x, 0) - v(0, 0)}{x} = \lim_{x \to 0} \frac{0 - 0}{x} = 0$$

$$\frac{\partial v}{\partial y} = \lim_{y \to 0} \frac{v(0, y) - v(0, 0)}{y} = \lim_{y \to 0} \frac{0 - 0}{y} = 0$$

Hence Cauchy-Riemann equations are satisfied at the origin.

But $$f'(0) = \lim_{z \to 0} \frac{f(z) - f(0)}{z} = \lim_{\substack{x \to 0 \\ y \to 0}} \left\{ \frac{xy^2 (x + iy)}{x^2 + y^4} - 0 \right\} \cdot \frac{1}{x + iy} = \lim_{\substack{x \to 0 \\ y \to 0}} \frac{xy^2}{x^2 + y^4}$$

Let $z \to 0$ along the real axis $y = 0$, then

$$f'(0) = 0$$

Again let $z \to 0$ along the curve $x = y^2$, then

$$f'(0) = \lim_{x \to 0} \frac{x^2}{x^2 + x^2} = \frac{1}{2}$$

which shows that $f'(0)$ does not exist. Hence $f(z)$ is not analytic at origin although Cauchy-Riemann equations are satisfied there.

Example 3. *Show that the function defined by* $f(z) = \sqrt{|xy|}$ *is not regular at the origin, although Cauchy-Riemann equations are satisfied*.

Sol. Let $f(z) = u(x, y) + iv(x, y) = \sqrt{|xy|}$ then $u(x, y) = \sqrt{|xy|}$, $v(x, y) = 0$

At the origin (0, 0), we have

$$\frac{\partial u}{\partial x} = \lim_{x \to 0} \frac{u(x, 0) - u(0, 0)}{x} = \lim_{x \to 0} \frac{0 - 0}{x} = 0$$

$$\frac{\partial u}{\partial y} = \lim_{y \to 0} \frac{u(0, y) - u(0, 0)}{y} = \lim_{y \to 0} \frac{0 - 0}{y} = 0$$

$$\frac{\partial v}{\partial x} = \lim_{x \to 0} \frac{v(x, 0) - v(0, 0)}{x} = \lim_{x \to 0} \frac{0 - 0}{x} = 0$$

$$\frac{\partial v}{\partial y} = \lim_{y \to 0} \frac{v(0, y) - v(0, 0)}{y} = \lim_{y \to 0} \frac{0 - 0}{y} = 0$$

Clearly, $$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Hence C-R equations are satisfied at the origin.

Now $$f'(0) = \lim_{z \to 0} \frac{f(z) - f(0)}{z} = \lim_{z \to 0} \frac{\sqrt{|xy|} - 0}{x + iy}$$

If $z \to 0$ along the line $y = mx$, we get

$$f'(0) = \lim_{x \to 0} \frac{\sqrt{|mx^2|}}{x(1 + im)} = \lim_{x \to 0} \frac{\sqrt{|m|}}{1 + im}$$

Now this limit is not unique since it depends on m. Therefore, $f'(0)$ does not exist.

Hence the function $f(z)$ is not regular at the origin.

Example 4. *Prove that the function f(z) defined by*

$$f(z) = \frac{x^3(1+i) - y^3(1-i)}{x^2+y^2},\ z \neq 0 \text{ and } f(0) = 0$$

is continuous and the Cauchy-Riemann equations are satisfied at the origin, yet $f'(0)$ does not exist.

Sol. Here, $f(z) = \dfrac{(x^3 - y^3) + i(x^3 + y^3)}{x^2 + y^2},\ z \neq 0$

Let $f(z) = u + iv = \dfrac{x^3 - y^3}{x^2 + y^2} + i\,\dfrac{x^3 + y^3}{x^2 + y^2}$,

then $u = \dfrac{x^3 - y^3}{x^2 + y^2},\ v = \dfrac{x^3 + y^3}{x^2 + y^2}$

Since $z \neq 0 \Rightarrow x \neq 0, y \neq 0$

$\therefore$ u and v are rational functions of x and y with non-zero denominators. Thus, u, v and hence $f(z)$ *are continuous functions when $z \neq 0$*. To test them for continuity at $z = 0$, on changing u, v to polar co-ordinates by putting $x = r\cos\theta$, $y = r\sin\theta$, we get

$$u = r(\cos^3\theta - \sin^3\theta) \text{ and } v = r(\cos^3\theta + \sin^3\theta)$$

When $z \to 0$, $r \to 0$

$$\therefore \quad \lim_{z \to 0} u = \lim_{r \to 0} r(\cos^3\theta - \sin^3\theta) = 0$$

Similarly, $\lim_{z \to 0} v = 0$

$$\therefore \quad \lim_{r \to 0} f(z) = 0 = f(0)$$

$\Rightarrow$ **f(z) is continuous at z = 0**.

Hence $f(z)$ is continuous for all values of z.

At the origin (0, 0), we have

$$\frac{\partial u}{\partial x} = \lim_{x \to 0} \frac{u(x, 0) - u(0, 0)}{x} = \lim_{x \to 0} \frac{x - 0}{x} = 1$$

$$\frac{\partial u}{\partial y} = \lim_{y \to 0} \frac{u(0, y) - u(0, 0)}{y} = \lim_{y \to 0} \frac{-y - 0}{y} = -1$$

$$\frac{\partial v}{\partial x} = \lim_{x \to 0} \frac{v(x, 0) - v(0, 0)}{x} = \lim_{x \to 0} \frac{x - 0}{x} = 1$$

$$\frac{\partial v}{\partial y} = \lim_{y \to 0} \frac{v(0, y) - v(0, 0)}{y} = \lim_{y \to 0} \frac{y - 0}{y} = 1$$

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Hence C-R equations are satisfied at the origin.

Now $f'(0) = \lim_{z \to 0} \dfrac{f(z) - f(0)}{z} = \lim_{z \to 0} \dfrac{[(x^3 - y^3) + i(x^3 + y^3)] - 0}{(x^2 + y^2)(x + iy)}$

Let $z \to 0$ along the line $y = x$, then

$$f'(0) = \lim_{x \to 0} \frac{0 + 2ix^3}{2x^3(1+i)} = \frac{i}{1+i} = \frac{i(1-i)}{2} = \frac{1+i}{2} \qquad \text{...(1)}$$

Also, let $z \to 0$ along the x-axis (*i.e.*, $y = 0$), then

$$f'(0) = \lim_{x \to 0} \frac{x^3 + ix^3}{x^3} = 1 + i \qquad \text{...(2)}$$

Since the limits (1) and (2) are different, $f'(0)$ does not exist.

Example 5. *(i) Examine the nature of the function*

$$f(z) = \frac{x^2 y^5 (x + iy)}{x^4 + y^{10}} \; ; \; z \neq 0$$

$$f(0) = 0$$

in the region including the origin.

(*ii*) *If* $f(z) = \begin{cases} \dfrac{x^3 y(y - ix)}{x^6 + y^2}, & z \neq 0 \\ 0, & z = 0 \end{cases}$ *prove that* $\dfrac{f(z) - f(0)}{z} \to 0$ *as* $z \to 0$ *along any radius vector but not as* $z \to 0$ *in any manner and also that* $f(z)$ *is not analytic at* $z = 0$.

Sol. (*i*) Here, $u + iv = \dfrac{x^2 y^5 (x + iy)}{x^4 + y^{10}}$; $z \neq 0$

$$\therefore \quad u = \frac{x^3 y^5}{x^4 + y^{10}}, \; v = \frac{x^2 y^6}{x^4 + y^{10}}$$

At the origin,

$$\frac{\partial u}{\partial x} = \lim_{x \to 0} \frac{u(x, 0) - u(0, 0)}{x} = \lim_{x \to 0} \frac{0 - 0}{x} = 0$$

$$\frac{\partial u}{\partial y} = \lim_{y \to 0} \frac{u(0, y) - u(0, 0)}{y} = \lim_{y \to 0} \frac{0 - 0}{y} = 0$$

Similarly, $\dfrac{\partial v}{\partial x} = 0 = \dfrac{\partial v}{\partial y}$

Hence Cauchy-Riemann eqns. are satisfied at the origin.

But

$$f'(0) = \lim_{z \to 0} \frac{f(z) - f(0)}{z} = \lim_{\substack{x \to 0 \\ y \to 0}} \left\{ \frac{x^2 y^5 (x + iy)}{x^4 + y^{10}} - 0 \right\} \cdot \frac{1}{x + iy}$$

$$= \lim_{\substack{x \to 0 \\ y \to 0}} \frac{x^2 y^5}{x^4 + y^{10}}$$

Let $z \to 0$ along the radius vector $y = mx$, then

$$f'(0) = \lim_{x \to 0} \frac{m^5 x^7}{x^4 + m^{10} x^{10}} = \lim_{x \to 0} \frac{m^5 x^3}{1 + m^{10} x^6} = 0$$

Again let $z \to 0$ along the curve $y^5 = x^2$

$$f'(0) = \lim_{x \to 0} \frac{x^4}{x^4 + x^4} = \frac{1}{2}$$

which shows that $f'(0)$ does not exist. Hence $f(z)$ is not analytic at origin although Cauchy-Riemann equations are satisfied there.

(*ii*) $$\frac{f(z)-f(0)}{z} = \left[\frac{x^3 y(y-ix)}{x^6+y^2} - 0\right] \cdot \frac{1}{x+iy}$$

$$= \frac{-ix^3 y(x+iy)}{(x^6+y^2)} \cdot \frac{1}{x+iy} = -i\,\frac{x^3 y}{x^6+y^2}$$

Let $z \to 0$ along radius vector $y = mx$ then,

$$\lim_{z\to 0} \frac{f(z)-f(0)}{z} = \lim_{x\to 0} \frac{-ix^3(mx)}{x^6+m^2x^2} = \lim_{x\to 0} \frac{-imx^2}{x^4+m^2} = 0$$

Hence $\frac{f(z)-f(0)}{z} \to 0$ as $z \to 0$ along any radius vector.

Now let $z \to 0$ along a curve $y = x^3$ then,

$$\lim_{z\to 0} \frac{f(z)-f(0)}{z} = \lim_{x\to 0} \frac{-ix^3 . x^3}{x^6+x^6} = -\frac{i}{2}$$

Hence $\frac{f(z)-f(0)}{z}$ does not tend to zero as $z \to 0$ along the curve $y = x^3$.

We observe that $f'(0)$ does not exist hence $f(z)$ is not analytic at $z = 0$.

Example 6. *Show that the following functions are harmonic and find their harmonic conjugate functions.*

(*i*) $u = \frac{1}{2}\log(x^2+y^2)$ (*ii*) $v = \sinh x \cos y$.

(*iii*) $u = e^x \cos y$. (Tirunelveli 2010)

Sol. (*i*) $$u = \frac{1}{2}\log(x^2+y^2) \quad ...(1)$$

$$\frac{\partial u}{\partial x} = \frac{1}{2} . \frac{1}{x^2+y^2} . 2x = \frac{x}{x^2+y^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{(x^2+y^2) . 1 - x . 2x}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2} \quad ...(2)$$

Also, $$\frac{\partial u}{\partial y} = \frac{1}{2} . \frac{1}{x^2+y^2} . 2y = \frac{y}{x^2+y^2}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{(x^2+y^2) . 1 - y . 2y}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2} \quad ...(3)$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0. \quad \text{[From (2) and (3)]}$$

Since u satisfies Laplace's equation hence u is a harmonic function.

Let $$dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy$$

$$= \left(-\frac{\partial u}{\partial y}\right) dx + \left(\frac{\partial u}{\partial x}\right) dy \qquad \text{[Using C-R equations]}$$

$$= \left(\frac{-y}{x^2 + y^2}\right) dx + \left(\frac{x}{x^2 + y^2}\right) dy$$

$$= \frac{x\,dy - y\,dx}{(x^2 + y^2)} = d\left[\tan^{-1}\left(\frac{y}{x}\right)\right]$$

Integration yields, $v = \tan^{-1}\left(\frac{y}{x}\right) + c$ | c is a constant

which is the required harmonic conjugate function of u.

(*ii*) $v = \sinh x \cos y$ ...(1)

$$\frac{\partial v}{\partial x} = \cosh x \cos y \quad \Rightarrow \quad \frac{\partial^2 v}{\partial x^2} = \sinh x \cos y \qquad \text{...(2)}$$

$$\frac{\partial v}{\partial y} = -\sinh x \sin y \quad \Rightarrow \quad \frac{\partial^2 v}{\partial y^2} = -\sinh x \cos y \qquad \text{...(3)}$$

Since, $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$

Hence v is harmonic.

Now, $du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy = \frac{\partial v}{\partial y} dx - \frac{\partial v}{\partial x} dy$

$$= -\sinh x \sin y\, dx - \cosh x \cos y\, dy$$
$$= -[\sinh x \sin y\, dx + \cosh x \cos y\, dy]$$
$$= -d(\cosh x \sin y).$$

Integration yields, $u = -\cosh x \sin y + c$ | c is a constant

which is the required harmonic conjugate function of v.

(*iii*) $u = e^x \cos y$

$$\frac{\partial u}{\partial x} = e^x \cos y \quad \Rightarrow \quad \frac{\partial^2 u}{\partial x^2} = e^x \cos y$$

$$\frac{\partial u}{\partial y} = -e^x \sin y \quad \Rightarrow \quad \frac{\partial^2 u}{\partial y^2} = -e^x \cos y$$

Since $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \therefore \quad u$ is harmonic.

Let $v = v(x, y)$

$$dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy$$

$$= \left(-\frac{\partial u}{\partial y}\right) dx + \left(\frac{\partial u}{\partial x}\right) dy$$

$$= e^x \sin y\, dx + e^x \cos y\, dy$$
$$= d(e^x \sin y)$$

Integration yields, $v = e^x \sin y + c.$

1.11. HARMONIC FUNCTION

A function of x, y which possesses continuous partial derivatives of the first and second orders and satisfies Laplace's equation is called a Harmonic function.

1.11.1 THEOREM

If f(z) = u + iv is an analytic function then u and v are both harmonic functions.

Proof. Let $f(z) = u + iv$ be analytic in some region of the z-plane, then u and v satisfy C-R equations.

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \text{...(1)}$$

and
$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \qquad \text{...(2)}$$

Differentiating (1) partially w.r.t. x and (2) w.r.t. y, we get

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} \qquad \text{...(3)}$$

and
$$\frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \partial x} \qquad \text{...(4)}$$

Assuming $\frac{\partial^2 v}{\partial x \partial y} = \frac{\partial^2 v}{\partial y \partial x}$ and adding (3) and (4), we get

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \text{...(5)}$$

Now, differentiating (1) partially w.r.t. y and (2) w.r.t. x, we get

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 v}{\partial y^2} \qquad \text{...(6)}$$

and
$$\frac{\partial^2 u}{\partial x \partial y} = -\frac{\partial^2 v}{\partial x^2} \qquad \text{...(7)}$$

Assuming $\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y}$ and subtracting (7) from (6), we get

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \qquad \text{...(8)}$$

Equations (5) and (8) show that the real and imaginary parts u and v of an analytic function satisfy the Laplace's equation.

Hence u and v are harmonic functions.

Note. Here u and v are called conjugate harmonic functions.

Example 4. *Show that the function $e^x (\cos y + i \sin y)$ is holomorphic and find its derivative.*

Sol. $f(z) = e^x \cos y + i\, e^x \sin y = u + iv$

Here, $u = e^x \cos y, \quad v = e^x \sin y$

$$\frac{\partial u}{\partial x} = e^x \cos y \qquad \frac{\partial v}{\partial x} = e^x \sin y$$

$$\frac{\partial u}{\partial y} = -e^x \sin y \qquad \frac{\partial v}{\partial y} = e^x \cos y$$

Since, $\quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

hence, C-R equations are satisfied. Also first order partial derivatives of *u* and *v* are continuous everywhere. Therefore *f*(*z*) is analytic.

Now, $\quad f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = e^x \cos y + i\, e^x \sin y$

$$= e^x(\cos y + i \sin y) = e^x \cdot e^{iy} = e^{x+iy} = e^z$$

Example 5. *Given that* $u(x, y) = x^2 - y^2$ and $v(x, y) = -\left(\frac{y}{x^2 + y^2}\right)$.

Prove that both u and v are harmonic functions but u + iv is not an analytic function of z.

Sol. $\quad u = x^2 - y^2$

$$\frac{\partial u}{\partial x} = 2x \quad \Rightarrow \quad \frac{\partial^2 u}{\partial x^2} = 2$$

$$\frac{\partial u}{\partial y} = -2y \quad \Rightarrow \quad \frac{\partial^2 u}{\partial y^2} = -2$$

Since $\quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad$ Hence *u*(*x*, *y*) is harmonic.

Also, $\quad v = \frac{y}{x^2 + y^2}$

$$\frac{\partial v}{\partial x} = \frac{2xy}{(x^2 + y^2)^2} \quad \Rightarrow \quad \frac{\partial^2 v}{\partial x^2} = \frac{2y^3 - 6x^2 y}{(x^2 + y^2)^3}$$

$$\frac{\partial v}{\partial y} = \frac{y^2 - x^2}{(x^2 + y^2)^2} \quad \Rightarrow \quad \frac{\partial^2 v}{\partial y^2} = \frac{6x^2 y - 2y^3}{(x^2 + y^2)^3}$$

Since $\quad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$. Hence *v*(*x*, *y*) is also harmonic.

But, $\quad \frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}$ and $\frac{\partial v}{\partial x} \neq -\frac{\partial u}{\partial y}$

Therefore *u* + *iv* is not an analytic function of *z*.

Example 8. *If* ϕ *and* ψ *are functions of x and y satisfying Laplace's equation, show that s + it is analytic, where*

$$s = \frac{\partial \phi}{\partial y} - \frac{\partial \psi}{\partial x} \quad \text{and} \quad t = \frac{\partial \phi}{\partial x} + \frac{\partial \psi}{\partial y}.$$

Sol. Since ϕ and ψ are functions of x and y satisfying Laplace's equations,

$$\therefore \qquad \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \qquad \text{...(1)}$$

and

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0. \qquad \text{...(2)}$$

For the function $s + it$ to be analytic,

$$\frac{\partial s}{\partial x} = \frac{\partial t}{\partial y} \qquad \text{...(3)}$$

and

$$\frac{\partial s}{\partial y} = -\frac{\partial t}{\partial x} \qquad \text{...(4)}$$

must satisfy.

Now,

$$\frac{\partial s}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial y} - \frac{\partial \psi}{\partial x}\right) = \frac{\partial^2 \phi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial x^2} \qquad \text{...(5)}$$

$$\frac{\partial t}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial x} + \frac{\partial \psi}{\partial y}\right) = \frac{\partial^2 \phi}{\partial y \partial x} + \frac{\partial^2 \psi}{\partial y^2} \qquad \text{...(6)}$$

$$\frac{\partial s}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial y} - \frac{\partial \psi}{\partial x}\right) = \frac{\partial^2 \phi}{\partial y^2} - \frac{\partial^2 \psi}{\partial y \partial x} \qquad \text{...(7)}$$

and

$$\frac{\partial t}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial x} + \frac{\partial \psi}{\partial y}\right) = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \psi}{\partial x \partial y}. \qquad \text{...(8)}$$

From (3), (5) and (6), we have

$$\frac{\partial^2 \phi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial x^2} = \frac{\partial^2 \phi}{\partial y \partial x} + \frac{\partial^2 \psi}{\partial y^2} \quad \Rightarrow \quad \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$$

which is true by (2).

Again from (4), (7) and (8), we have,

$$\frac{\partial^2 \phi}{\partial y^2} - \frac{\partial^2 \psi}{\partial y \partial x} = -\frac{\partial^2 \phi}{\partial x^2} - \frac{\partial^2 \psi}{\partial x \partial y} \quad \Rightarrow \quad \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

which is also true by (1).

Hence the function $s + it$ is analytic.

1.12. DETERMINATION OF CONJUGATE FUNCTION

If $f(z) = u + iv$ is an analytic function where both $u(x, y)$ and $v(x, y)$ are conjugate functions, then we determine the other function v when one of these say u is given as follows:

$$\because \qquad v = v(x, y)$$

$$\therefore \qquad dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy$$

$$\Rightarrow \qquad dv = -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy \qquad \text{...(1)} \qquad \text{| By C-R eqns.}$$

$$M = -\frac{\partial u}{\partial y}, \quad N = \frac{\partial u}{\partial x}$$

$$\therefore \quad \frac{\partial M}{\partial y} = -\frac{\partial^2 u}{\partial y^2} \text{ and } \frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x^2}$$

Now, $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ gives

$$-\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial x^2}$$

or $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

which is true as u being a harmonic function satisfies Laplace's equation.

$\therefore$ dv is exact.

$\therefore$ dv can be integrated to get v.

However, if we are to construct $f(z) = u + iv$ when only u is given, we first of all find v by above procedure and then write $f(z) = u + iv$.

Similarly, if we are to determine u and only v is given then we use $du = \frac{\partial v}{\partial y} dx - \frac{\partial v}{\partial x} dy$ and integrate it to find u. Consequently $f(z) = u + iv$ can also be determined.

Example 1. *Show that the following functions are harmonic and find their harmonic conjugate functions.*

(*i*) $u = \frac{1}{2} \log(x^2 + y^2)$ (*ii*) $v = \sinh x \cos y$.

(*iii*) $u = e^x \cos y$. (Tirunelveli 2010)

Sol. (*i*) $$u = \frac{1}{2} \log (x^2 + y^2) \quad ...(1)$$

$$\frac{\partial u}{\partial x} = \frac{1}{2} \cdot \frac{1}{x^2 + y^2} \cdot 2x = \frac{x}{x^2 + y^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{(x^2 + y^2) \cdot 1 - x \cdot 2x}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2} \quad ...(2)$$

Also, $$\frac{\partial u}{\partial y} = \frac{1}{2} \cdot \frac{1}{x^2 + y^2} \cdot 2y = \frac{y}{x^2 + y^2}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{(x^2 + y^2) \cdot 1 - y \cdot 2y}{(x^2 + y^2)^2} = \frac{x^2 - y^2}{(x^2 + y^2)^2} \quad ...(3)$$

$$\therefore \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0. \quad \text{[From (2) and (3)]}$$

Since u satisfies Laplace's equation hence u is a harmonic function.

Let $$dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy$$

$$= \left(-\frac{\partial u}{\partial y}\right) dx + \left(\frac{\partial u}{\partial x}\right) dy$$ [Using C-R equations]

$$= \left(-\frac{y}{x^2+y^2}\right) dx + \left(\frac{x}{x^2+y^2}\right) dy$$

$$= \frac{x\,dy - y\,dx}{(x^2+y^2)} = d\left[\tan^{-1}\left(\frac{y}{x}\right)\right]$$

Integration yields, $v = \tan^{-1}\left(\frac{y}{x}\right) + c$ | c is a constant

which is the required harmonic conjugate function of u.

(*ii*) $$v = \sinh x \cos y \qquad ...(1)$$

$$\frac{\partial v}{\partial x} = \cosh x \cos y \Rightarrow \frac{\partial^2 v}{\partial x^2} = \sinh x \cos y \qquad ...(2)$$

$$\frac{\partial v}{\partial y} = -\sinh x \sin y \Rightarrow \frac{\partial^2 v}{\partial y^2} = -\sinh x \cos y \qquad ...(3)$$

Since, $$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

Hence v is harmonic.

Now, $$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy = \frac{\partial v}{\partial y} dx - \frac{\partial v}{\partial x} dy$$

$$= -\sinh x \sin y\, dx - \cosh x \cos y\, dy$$

$$= -[\sinh x \sin y\, dx + \cosh x \cos y\, dy]$$

$$= -d(\cosh x \sin y).$$

Integration yields, $u = -\cosh x \sin y + c$ | c is a constant

which is the required harmonic conjugate function of v.

(*iii*) $$u = e^x \cos y$$

$$\frac{\partial u}{\partial x} = e^x \cos y \Rightarrow \frac{\partial^2 u}{\partial x^2} = e^x \cos y$$

$$\frac{\partial u}{\partial y} = -e^x \sin y \Rightarrow \frac{\partial^2 u}{\partial y^2} = -e^x \cos y$$

Since $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \therefore \quad u \text{ is harmonic.}$$

Let $$v = v(x, y)$$

$$dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy$$

$$= \left(-\frac{\partial u}{\partial y}\right) dx + \left(\frac{\partial u}{\partial x}\right) dy$$

$$= e^x \sin y \, dx + e^x \cos y \, dy$$

$$= d(e^x \sin y)$$

Integration yields, $\quad v = e^x \sin y + c.$

Example 15. *(i) In a two-dimensional fluid flow, the stream function is* $\psi = -\dfrac{y}{x^2 + y^2}$, *find the velocity potential* ϕ.

(*ii*) *An electrostatic field in the xy-plane is given by the potential function* $\phi = 3x^2y - y^3$, *find the stream function.*

Sol. (*i*) $$\psi = -\frac{y}{x^2 + y^2} \qquad \text{...(1)}$$

$$\frac{\partial \psi}{\partial x} = \frac{2xy}{(x^2 + y^2)^2}, \qquad \frac{\partial \psi}{\partial y} = \frac{y^2 - x^2}{(x^2 + y^2)^2}$$

We know that, $$d\phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy = \frac{\partial \psi}{\partial y} dx - \frac{\partial \psi}{\partial x} dy$$

$$= \frac{(y^2 - x^2)}{(x^2 + y^2)^2} dx - \frac{2xy}{(x^2 + y^2)^2} dy$$

$$= \frac{(x^2 + y^2)dx - 2x^2 \, dx - 2xy \, dy}{(x^2 + y^2)^2}$$

$$= \frac{(x^2 + y^2) \, d(x) - x(2x \, dx + 2y \, dy)}{(x^2 + y^2)^2}$$

$$= \frac{(x^2 + y^2) \, d(x) - x d(x^2 + y^2)}{(x^2 + y^2)^2} = d\left(\frac{x}{x^2 + y^2}\right).$$

Integration yields, $\quad \phi = \dfrac{x}{x^2 + y^2} + c \quad$ where c is a constant.

(*ii*) Let $\psi(x, y)$ be a stream function.

$$d\psi = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy = \left(-\frac{\partial \phi}{\partial y}\right) dx + \left(\frac{\partial \phi}{\partial x}\right) dy$$

$$= \{-(3x^2 - 3y^2)\} \, dx + 6xy \, dy$$

$$= -3x^2 \, dx + (3y^2 \, dx + 6xy \, dy)$$

$$= -d(x^3) + 3d(xy^2)$$

Integrating, we get

$$\psi = -x^3 + 3xy^2 + c \qquad |c \text{ is a constant}$$

1.17. MILNE'S THOMSON METHOD

With the help of this method, we can directly construct $f(z)$ in terms of z without first finding out v when u is given or u when v is given.

$$z = x + iy$$
$$\bar{z} = x - iy$$

$$\Rightarrow \quad x = \frac{1}{2}(z + \bar{z}) \text{ and } y = \frac{1}{2i}(z - \bar{z})$$

$$\therefore \quad f(z) = u(x, y) + iv(x, y)$$
$$= u\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right) + iv\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right) \qquad ...(1)$$

Relation (1) is an identity in z and $\bar{z}$. Putting $\bar{z} = z$, we get

$$f(z) = u(z, 0) + iv(z, 0) \qquad ...(2)$$

Now, $\quad f(z) = u + iv$

$$\Rightarrow \quad f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} \qquad | \text{ By C-R eqns.}$$
$$= \phi_1(x, y) - i\,\phi_2(x, y)$$
$$\left| \text{ where } \phi_1(x, y) = \frac{\partial u}{\partial x} \text{ and } \phi_2(x, y) = \frac{\partial u}{\partial y} \right.$$

Now, $\quad f'(z) = \phi_1(z, 0) - i\,\phi_2(z, 0) \qquad$ | Replacing x by z and y by 0

Integrating, we get $\quad f(z) = \int \{\phi_1(z, 0) - i\,\phi_2(z, 0)\}\, dz + c \qquad$ | c is an arbitrary constant.

Hence the function is constructed directly in terms of z.

Similarly if $v(x, y)$ is given, then

$$f(z) = \int [\psi_1(z, 0) + i\psi_2(z, 0)]\, dz + c \qquad \left| \ \psi_1(x, y) = \frac{\partial v}{\partial y} \text{ and } \psi_2(x, y) = \frac{\partial v}{\partial x} \right.$$

Milne's Thomson method can easily be grasped by going through the steps involved in following various cases.

Case I. When only real part **u(x, y)** is given.

To construct analytic function $f(z)$ directly in terms of z when only real part u is given, we use following steps:

1. Find $\dfrac{\partial u}{\partial x}$
2. Write it as equal to $\phi_1(x, y)$
3. Find $\dfrac{\partial u}{\partial y}$
4. Write it as equal to $\phi_2(x, y)$
5. Find $\phi_1(z, 0)$ by replacing x by z and y by 0 in $\phi_1(x, y)$.
6. Find $\phi_2(z, 0)$ by replacing x by z and y by 0 in $\phi_2(x, y)$.
7. $f(z)$ is obtained by the formula

$$f(z) = \int \{\phi_1(z, 0) - i\,\phi_2(z, 0)\}\, dz + c$$

directly in terms of z.

Case II. When only imaginary part **v(x, y)** is given.

To construct analytic function $f(z)$ directly in terms of z when only imaginary part v is given, we use following steps:

1. Find $\frac{v}{y}$
2. Write it as equal to $y_1(x, y)$
3. Find $\frac{v}{x}$
4. Write it as equal to $y_2(x, y)$
5. Find $y_1(z, 0)$ by replacing x by z and y by 0 in $y_1(x, y)$
6. Find $y_2(z, 0)$ by replacing x by z and y by 0 in $y_2(x, y)$
7. $f(z)$ is obtained by the formula

$$f(z) = \int \{ \quad _1(z, 0) \quad i \quad _2(z, 0)\} \, dz \quad c$$

directly in terms of z.

Case III. When **u – v** is given.

To construct analytic function $f(z)$ directly in terms of z when $u - v$ is given, we follow the following steps:

1. $f(z) = u + iv$ ...(1)
2. $i\, f(z) = iu - v$ ...(2)
3. Add (1) and (2) to get

$$(1 + i)\, f(z) = (u - v) + i(u + v)$$

or, $\qquad F(z) = U + iV$

where $\qquad F(z) = (1 + i)\, f(z), \;\; U = u - v, \; V = u + v$

4. Since $u - v$ is given hence $U(x, y)$ is given
5. Find $\frac{U}{x}$
6. Write it as equal to $f_1(x, y)$
7. Find $\frac{U}{y}$
8. Write it as equal to $f_2(x, y)$
9. Find $f_1(z, 0)$
10. Find $f_2(z, 0)$
11. $F(z)$ is obtained by the formula

$$F(z) = \int \{ \quad _1(z, 0) - i \quad _2(z, 0)\} \, dz \quad c$$

12. $f(z)$ is determined by $f(z) = \frac{F(z)}{1 \quad i}$ directly in terms of z.

Case IV. When **u + v** is given.

To construct analytic function $f(z)$ directly in terms of z when $u + v$ is given, we follow the following steps:

1. $f(z) = u + iv$ ...(1)
2. $i\,f(z) = iu - v$ ...(2)
3. Add (1) and (2) to get

$$(1 + i)\,f(z) = (u - v) + i(u + v)$$

$\Rightarrow$ $$F(z) = U + iV$$

where, $$F(z) = (1 + i)\,f(z),\ U = u - v,\ V = u + v$$

4. Since $u + v$ is given hence $V(x, y)$ is given
5. Find $\dfrac{\partial V}{\partial y}$
6. Write it as equal to $\psi_1(x, y)$
7. Find $\dfrac{\partial V}{\partial x}$
8. Write it as equal to $\psi_2(x, y)$
9. Find $\psi_1(z, 0)$
10. Find $\psi_2(z, 0)$
11. F(z) is obtained by the formula

$$F(z) = \int \{\psi_1(z, 0) + i\,\psi_2(z, 0)\}\,dz + c$$

12. $f(z)$ is determined by $f(z) = \dfrac{F(z)}{1 + i}$ directly in terms of z.

Solved Example

Example 14. *Determine the analytic function $w = u + iv$ if*

(*i*) $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$. (*ii*) $u = \dfrac{x}{x^2 + y^2}$ (Tirunelveli 2010)

Sol. (*i*) $$u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1 \quad ...(1)$$

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2 + 6x = \phi_1(x, y) \quad |\text{say}$$

$\therefore$ $$\phi_1(z, 0) = 3z^2 + 6z. \quad ...(2)$$

Again, $$\frac{\partial u}{\partial y} = -6xy - 6y = \phi_2(x, y) \quad |\text{say}$$

$\therefore$ $$\phi_2(z, 0) = 0$$

By Milne's Thomson method,

$$f(z) = \int [\phi_1(z, 0) - i\,\phi_2(z, 0)]\,dz + c$$

$$= \int (3z^2 + 6z)\,dz + c = z^3 + 3z^2 + c. \quad |\ c \text{ is a constant}$$

Hence, $w = z^3 + 3z^2 + c$

(*ii*) $$u = \frac{x}{x^2 + y^2}$$

$$\frac{\partial u}{\partial x} = \frac{(x^2 + y^2) \,.\, 1 - x \,.\, 2x}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2} = \phi_1(x, y) \qquad | \text{ say}$$

\ $f_1(z, 0) = -\frac{1}{z^2}$

Again, $$\frac{\partial u}{\partial y} = \frac{-2xy}{(x^2 + y^2)^2} = \phi_2(x, y) \qquad | \text{ say}$$

\ $f_2(z, 0) = 0$

By Milne-Thomson method,

$$f(z) = \int [\phi_1(z, 0) - i\, \phi_2(z, 0)]\, dz + c = \frac{1}{z} + c \text{ where } c \text{ is a constant.}$$

Example 17. (*i*) *Determine the analytic function whose real part is*

$$e^{2x} (x \cos 2y - y \sin 2y).$$

(*ii*) *Find an analytic function whose imaginary part is* $e^{-x}(x \cos y + y \sin y)$.

Sol. (*i*) Let $f(z) = u + iv$ be the required analytic function.

Here, $u = e^{2x} (x \cos 2y - y \sin 2y)$

\ $$\frac{u}{x} = e^{2x} (2x \cos 2y - 2y \sin 2y + \cos 2y) = f_1 (x, y) \qquad | \text{ say}$$

and $$\frac{u}{y} = -e^{2x} (2x \sin 2y + \sin 2y + 2y \cos 2y) = f_2 (x, y) \qquad | \text{ say}$$

Now, $f_1 (z, 0) = e^{2z} (2z + 1)$

$f_2 (z, 0) = -e^{2z} (0) = 0$

By Milne's Thomson method,

$$f(z) = \int \{ {}_1(z, 0) \quad i \quad {}_2(z, 0)\} dz \quad c = \int e^{2z} (2z \quad 1)\, dz \quad c$$

$$= (2z + 1) \frac{e^{2z}}{2} \quad \int 2 \,.\, \frac{e^{2z}}{2}\, dz \quad c$$

$$= (2z + 1) \frac{e^{2z}}{2} \quad \frac{1}{2} e^{2z} + c$$

$$= ze^{2z} + c$$

where c is an arbitrary constant.

(*ii*) Let $f(z) = u + iv$ be the required analytic function.

Here $v = e^{-x}(x \cos y + y \sin y)$

$$\frac{v}{y} = e^{-x} (-x \sin y + y \cos y + \sin y) = y_1(x, y) \qquad | \text{ say}$$

$$\frac{v}{x} = e^{-x} \cos y - e^{-x} (x \cos y + y \sin y) = y_2 (x, y) \qquad | \text{ say}$$

\ $y_1 (z, 0) = 0$

$y_2(z, 0) = e^{-z} - e^{-z} (z) = (1 - z)\, e^{-z}$

By Milne's Thomson method,

$$f(z) = \int [\phi_1(z, 0) - i\phi_2(z, 0)]\, dz + c$$

$$= i \int (1 - z)\, e^{-z}\, dz + c$$

$$= i \left[(1 - z)(-e^{-z}) - \int (-1)(-e^{-z})\, dz \right] + c$$

$$= i\,[(z - 1)\, e^{-z} + e^{-z}] + c$$

$$\Rightarrow \qquad f(z) = ize^{-z} + c$$

Example 18. *Show that the function $u = e^{-2xy} \sin (x^2 - y^2)$ is harmonic. Find the conjugate function v and express u + iv as an analytic function of z.*

Sol. Here, $u = e^{-2xy} \sin (x^2 - y^2)$

$$\therefore \qquad \frac{\partial u}{\partial x} = -2y\, e^{-2xy} \sin (x^2 - y^2) + 2xe^{-2xy} \cos (x^2 - y^2)$$

$$\frac{\partial^2 u}{\partial x^2} = 4y^2 e^{-2xy} \sin (x^2 - y^2) - 4xy\, e^{-2xy} \cos (x^2 - y^2) + 2e^{-2xy} \cos (x^2 - y^2) - 4xy\, e^{-2xy} \cos (x^2 - y^2) - 4x^2 e^{-2xy} \sin (x^2 - y^2) \qquad \ldots(1)$$

$$\frac{\partial u}{\partial y} = -2x\, e^{-2xy} \sin (x^2 - y^2) - 2y\, e^{-2xy} \cos (x^2 - y^2)$$

$$\frac{\partial^2 u}{\partial y^2} = 4x^2 e^{-2xy} \sin (x^2 - y^2) + 4xy\, e^{-2xy} \cos (x^2 - y^2) - 2e^{-2xy} \cos (x^2 - y^2) + 4xy\, e^{-2xy} \cos (x^2 - y^2) - 4y^2 e^{-2xy} \sin (x^2 - y^2) \qquad \ldots(2)$$

Adding (1) and (2), we get

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \text{which proves that } u \text{ is harmonic.}$$

Now, $\phi_1(z, 0) = 2z \cos z^2, \quad \phi_2(z, 0) = -2z \sin z^2$

By Milne's Thomson method,

$$f(z) = \int [\phi_1(z, 0) - i\phi_2(z, 0)]\, dz + c$$

$$= 2 \int (z \cos z^2 + iz \sin z^2)\, dz + c$$

$$= 2 \int ze^{iz^2}\, dz + c \qquad \text{Put } iz^2 = t$$

$$= \frac{1}{i} \int e^t\, dt + c = -i e^{iz^2} + c \qquad \therefore \quad 2z\, dz = \frac{dt}{i}$$

Since, $u + iv = -i e^{iz^2} + c = -i e^{i(x + iy)^2} + c$

$$= -i e^{i(x^2 - y^2 + 2ixy)} + c = -ie^{-2xy} \cdot e^{i(x^2 - y^2)} + c$$

$$= -ie^{-2xy} [\cos (x^2 - y^2) + i \sin (x^2 - y^2)] + c$$

$$= e^{-2xy} \sin (x^2 - y^2) + i[-e^{-2xy} \cos (x^2 - y^2)] + c$$

$$\therefore \qquad v = -e^{-2xy} \cos (x^2 - y^2) + b \qquad |\text{if } c = a + ib \text{ is complex constant}$$

Example 19. *If* $u - v = (x - y)\ (x^2 + 4xy + y^2)$ *and* $f(z) = u + iv$ *is an analytic function of* $z = x + iy$, *find* $f(z)$ *in terms of z.*

Sol. Here, $f(z) = u + iv$

$\therefore \quad if(z) = iu - v$

Adding, $(1 + i) f(z) = (u - v) + i(u + v)$

$\Rightarrow \quad F(z) = U + iV$

where, $F(z) = (1 + i) f(z)$, $U = u - v$ and $V = u + v$.

Now, $U = u - v = (x - y)(x^2 + 4xy + y^2)$

$$\Rightarrow \quad \frac{\partial U}{\partial x} = x^2 + 4xy + y^2 + (x - y)(2x + 4y) = 3x^2 + 6xy - 3y^2 = \phi_1(x, y) \qquad | \text{ say}$$

and
$$\frac{\partial U}{\partial y} = -(x^2 + 4xy + y^2) + (x - y)(4x + 2y) = 3x^2 - 6xy - 3y^2 = \phi_2(x, y) \qquad | \text{ say}$$

Now, $\phi_1(z, 0) = 3z^2$

$\phi_2(z, 0) = 3z^2$

By Milne's Thomson method,

$$F(z) = \int [\phi_1(z, 0) - i\,\phi_2(z, 0)]\, dz + c = \int [3z^2 - i(3z^2)]\, dz + c$$

$$F(z) = (1 - i)\, z^3 + c$$

$$\Rightarrow \quad (1 + i) f(z) = (1 - i)\, z^3 + c$$

or
$$f(z) = \left(\frac{1 - i}{1 + i}\right) z^3 + \frac{c}{1 + i} = \left(\frac{-2i}{2}\right) z^3 + c_1 \qquad \left(\text{where } c_1 = \frac{c}{1 + i}\right)$$

or
$$f(z) = -iz^3 + c_1.$$

Example 20. *If* $u + v = \dfrac{2 \sin 2x}{e^{2y} + e^{-2y} - 2 \cos 2x}$ *and* $f(z) = u + iv$ *is an analytic function of* $z = x + iy$, *find* $f(z)$ *in terms of z.*

Sol. Let $f(z) = u + iv$ ...(1)

Multiplying both sides by i

$$i\, f(z) = iu - v \qquad ...(2)$$

Adding (1) and (2), we get

$$(1 + i) f(z) = (u - v) + i(u + v) \qquad ...(3)$$

$$\Rightarrow \quad F(z) = U + iV \qquad ...(4)$$

where
$$F(z) = (1 + i) f(z) \qquad ...(5)$$

$$U = u - v \quad \text{and} \quad V = u + v \qquad ...(6)$$

It means that we have been given

$$V = \frac{2 \sin 2x}{e^{2y} + e^{-2y} - 2 \cos 2x} \qquad ...(7)$$

or $$V = \frac{\sin 2x}{\cosh 2y - \cos 2x} \qquad \Big|\ \because\ e^{2y} + e^{-2y} = 2\cosh 2y$$

Now, $$\frac{\partial V}{\partial y} = \frac{-2\sin 2x \sinh 2y}{(\cosh 2y - \cos 2x)^2} = \psi_1(x, y) \qquad |\text{ say}$$

and $$\frac{\partial V}{\partial x} = \frac{2\cos 2x\,(\cosh 2y - \cos 2x) - 2\sin^2 2x}{(\cosh 2y - \cos 2x)^2}$$

$$= \frac{2\cos 2x \cosh 2y - 2}{(\cosh 2y - \cos 2x)^2} = \psi_2(x, y) \qquad |\text{ say}$$

$\therefore$ $$\psi_1(z, 0) = 0$$

$$\psi_2(z, 0) = \frac{2(\cos 2z - 1)}{(1 - \cos 2z)^2} = \frac{-2}{1 - \cos 2z} = \frac{-2}{1 - 1 + 2\sin^2 z} = -\operatorname{cosec}^2 z$$

By Milne's Thomson method, we have

$$F(z) = \int \{\psi_1(z, 0) + i\,\psi_2(z, 0)\}\, dz + c$$

$$= \int -i \operatorname{cosec}^2 z\, dz + c = i \cot z + c$$

Replacing F(z) by (1 + i) $f(z)$, from eqn. (5), we get

$$(1 + i)\, f(z) = i \cot z + c$$

$\Rightarrow$ $$f(z) = \frac{i}{1+i}\cot z + \frac{c}{1+i}$$

$\therefore$ $$f(z) = \tfrac{1}{2}(1 + i)\cot z + c_1 \qquad \text{where } c_1 = \frac{c}{1+i}.$$

Example 21. *If f(z) = u + iv is an analytic function of z and* $u - v = \dfrac{\cos x + \sin x - e^{-y}}{2\cos x - 2\cosh y}$, *prove that* $f(z) = \dfrac{1}{2}\left(1 - \cot \dfrac{z}{2}\right)$ *when* $f\left(\dfrac{\pi}{2}\right) = 0.$

Sol. Let $$f(z) = u + iv \qquad \text{...(1)}$$

$\therefore$ $$i\, f(z) = iu - v$$

Add, $$(1 + i)\, f(z) = (u - v) + i(u + v) \qquad \text{...(2)}$$

$\Rightarrow$ $$F(z) = U + iV \qquad \text{...(3)}$$

where $u - v = U$, $u + v = V$ and $(1 + i)\, f(z) = F(z)$.

We have, $$u - v = \frac{\cos x + \sin x - e^{-y}}{2\cos x - 2\cosh y}$$

or $$U = \frac{\cos x + \sin x - \cosh y + \sinh y}{2\cos x - 2\cosh y} \qquad [\because\ e^{-y} = \cosh y - \sinh y]$$

$$= \frac{1}{2} + \frac{\sin x + \sinh y}{2(\cos x - \cosh y)} \qquad \text{...(4)}$$

Diff. (4) w.r.t. x partially, we get

$$\frac{\partial U}{\partial x} = \frac{1}{2}\left[\frac{(\cos x - \cosh y)\cos x - (\sin x - \sinh y)(-\sin x)}{(\cos x - \cosh y)^2}\right]$$

$$\phi_1(x, y) = \frac{1}{2}\left[\frac{1 - \cosh y \cos x - \sinh y \sin x}{(\cos x - \cosh y)^2}\right]$$

$$\phi_1(z, 0) = \frac{1}{2}\left[\frac{1 - \cos z}{(\cos z - 1)^2}\right] = \frac{1}{2(1 - \cos z)}. \qquad ...(5)$$

Diff. (4) partially w.r.t. y, we get

$$\frac{\partial U}{\partial y} = \frac{1}{2}\left[\frac{(\cos x - \cosh y)\,.\,(-\cosh y) - (\sin x - \sinh y)(-\sinh y)}{(\cos x - \cosh y)^2}\right]$$

$$\phi_2(x, y) = \frac{1}{2}\left[\frac{\cos x \cosh y - \sin x \sinh y - 1}{(\cos x - \cosh y)^2}\right]$$

$$\therefore \quad \phi_2(z, 0) = \frac{1}{2}\left[\frac{\cos z - 1}{(\cos z - 1)^2}\right] = \frac{1}{2}\,.\left(-\frac{1}{1 - \cos z}\right). \qquad ...(6)$$

By Milne's Thomson Method,

$$F(z) = \int [\phi_1(z, 0) - i\,\phi_2(z, 0)]\, dz + c$$

$$= \int \left[\frac{1}{2}\,.\,\frac{1}{(1 - \cos z)} + \frac{i}{2}\,.\,\frac{1}{1 - \cos z}\right] dz + c$$

$$= \frac{1 + i}{2} \int \frac{1}{2 \sin^2 z/2}\, dz + c = \frac{1 + i}{4} \int \text{cosec}^2 (z/2)\, dz + c$$

$$= -\left(\frac{1 + i}{2}\right) \cot \frac{z}{2} + c$$

or $$(1 + i)\, f(z) = -\left(\frac{1 + i}{2}\right) \cot \frac{z}{2} + c$$

$$\Rightarrow \quad f(z) = -\frac{1}{2} \cot \frac{z}{2} + \frac{c}{1 + i} \qquad ...(7)$$

$$f\left(\frac{\pi}{2}\right) = -\frac{1}{2} \cot \frac{\pi}{4} + \frac{c}{1 + i} \qquad \text{[From (7)]}$$

$$0 = -\frac{1}{2} + \frac{c}{1 + i} \quad \Rightarrow \quad \frac{c}{1 + i} = \frac{1}{2} \qquad ...(8)$$

$\therefore$ From (7), $f(z) = -\dfrac{1}{2} \cot \dfrac{z}{2} + \dfrac{1}{2} = \dfrac{1}{2}\left(1 - \cot \dfrac{z}{2}\right).$ [Using (8)]

Example 22. (*i*) *If f(z) is a regular function of z, prove that*

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) |f(z)|^2 = 4\, |f'(z)|^2.$$

(Coimbatore 2010, Anna 2006, 2009, 2010)

(*ii*) *If f(z) is a harmonic function of z, show that*

$$\left\{\frac{\partial}{\partial x}|f(z)|\right\}^2 + \left\{\frac{\partial}{\partial y}|f(z)|\right\}^2 = |f'(z)|^2.$$

Sol. (*i*) Let $f(z) = u + iv$ so that $|f(z)| = \sqrt{u^2 + v^2}$

or $$|f(z)|^2 = u^2 + v^2 = \phi(x, y) \text{ (say)}$$

∴ $$\frac{\partial \phi}{\partial x} = 2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x}$$

$$\frac{\partial^2 \phi}{\partial x^2} = 2\left[u\frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial u}{\partial x}\right)^2 + v\frac{\partial^2 v}{\partial x^2} + \left(\frac{\partial v}{\partial x}\right)^2\right]$$

Similarly, $$\frac{\partial^2 \phi}{\partial y^2} = 2\left[u\frac{\partial^2 u}{\partial y^2} + \left(\frac{\partial u}{\partial y}\right)^2 + v\frac{\partial^2 v}{\partial y^2} + \left(\frac{\partial v}{\partial y}\right)^2\right]$$

Adding, we get

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 2\left[u\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + v\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right] \quad ...(1)$$

Since $f(z) = u + iv$ is a regular function of z, u and v satisfy C-R equations and Laplace's equation.

∴ $$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \text{and} \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}$$

∴ From (1), we get

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 2\left[0 + \left(\frac{\partial u}{\partial x}\right)^2 + \left(-\frac{\partial v}{\partial x}\right)^2 + 0 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial x}\right)^2\right]$$

$$= 4\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right] \quad ...(2)$$

Now, $$f(z) = u + iv$$

∴ $$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \quad \text{and} \quad |f'(z)|^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2$$

From (2), we get

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\phi = 4|f'(z)|^2 \quad \text{or} \quad \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4|f'(z)|^2.$$

(*ii*) We have, $$f(z) = u + iv \quad ...(1)$$

∴ $$|f(z)| = \sqrt{u^2 + v^2} \quad ...(2)$$

Partially differentiating eqn. (2) w.r.t. x and y, we get

$$\frac{\partial}{\partial x}|f(z)| = \frac{1}{2}(u^2 + v^2)^{-1/2}\left(2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x}\right) = \frac{u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x}}{|f(z)|} \qquad \text{...(3)}$$

Similarly, $$\frac{\partial}{\partial y}|f(z)| = \frac{u\frac{\partial u}{\partial y} + v\frac{\partial v}{\partial y}}{|f(z)|} \qquad \text{...(4)}$$

Squaring and adding (3) and (4), we get

$$\left\{\frac{\partial}{\partial x}|f(z)|\right\}^2 + \left\{\frac{\partial}{\partial y}|f(z)|\right\}^2 = \frac{\left(u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x}\right)^2 + \left(u\frac{\partial u}{\partial y} + v\frac{\partial v}{\partial y}\right)^2}{|f(z)|^2}$$

$$= \frac{\left(u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x}\right)^2 + \left(-u\frac{\partial v}{\partial x} + v\frac{\partial u}{\partial x}\right)^2}{|f(z)|^2} \qquad \text{| Using C-R eqns.}$$

$$= \frac{(u^2 + v^2)\left\{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right\}}{|f(z)|^2}$$

$$= \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 \qquad |\because\ |f(z)|^2 = u^2 + v^2$$

$$= |f'(z)|^2 \qquad \because\ f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

Example 23. (*i*) *Show that a harmonic function satisfies the formal differential equation*

$$\frac{\partial^2 u}{\partial z\,\partial \bar{z}} = 0$$

(ii) If w = f(z) is a regular function of z, prove that

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\log|f'(z)| = 0. \qquad \text{(Anna 2009)}$$

Further, if $|f'(z)|$ *is the product of a function of x and function of y, show that* $f'(z) = \exp.(\alpha z^2 + \beta z + \gamma)$ *where* α *is a real and* β, γ *are complex constants.*

Sol. (*i*) We have $x + iy = z$ and $x - iy = \bar{z}$

so that $$x = \frac{1}{2}(z + \bar{z}), \quad y = -\frac{i}{2}(z - \bar{z})$$

$\therefore$
$$\frac{\partial x}{\partial z} = \frac{1}{2}, \qquad \frac{\partial y}{\partial z} = -\frac{i}{2}$$
$$\frac{\partial x}{\partial \bar{z}} = \frac{1}{2}, \qquad \frac{\partial y}{\partial \bar{z}} = \frac{i}{2}$$

Now, $$\frac{\partial}{\partial z} = \frac{\partial}{\partial x}\cdot\frac{\partial x}{\partial z} + \frac{\partial}{\partial y}\cdot\frac{\partial y}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$$

and $$\frac{\partial}{\partial \bar{z}} = \frac{\partial}{\partial x}\cdot\frac{\partial x}{\partial \bar{z}} + \frac{\partial}{\partial y}\cdot\frac{\partial y}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

Hence, $$\frac{\partial^2}{\partial z\,\partial \bar{z}} = \frac{1}{4}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$$

or, $$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\frac{\partial^2}{\partial z\,\partial \bar{z}}$$

A harmonic function u satisfies the eqn.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{which implies that } 4\frac{\partial^2 u}{\partial z\,\partial \bar{z}} = 0 \quad \text{or} \quad \frac{\partial^2 u}{\partial z\,\partial \bar{z}} = 0.$$

(*ii*) $$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\log|f'(z)|$$

$$= 4\frac{\partial^2}{\partial z\,\partial \bar{z}}\left\{\frac{1}{2}\log|f'(z)|^2\right\} = 2\frac{\partial^2}{\partial z\,\partial \bar{z}}[\log\{f'(z)\,\bar{f}'(\bar{z})\}]$$

$$= 2\frac{\partial^2}{\partial z\,\partial \bar{z}}[\log f'(z) + \log \bar{f}'(\bar{z})]$$

$$= 2\frac{\partial}{\partial z}\left[\frac{\bar{f}''(\bar{z})}{\bar{f}'(\bar{z})}\right] = 0. \qquad \text{| Since } \bar{f}'(\bar{z}) \text{ and } \bar{f}''(\bar{z}) \text{ are independent of } z$$

Further, let $|f'(z)| = \phi(x)\,\psi(y)$

where $\phi(x)$ is a function of x only and $\psi(y)$ is a function of y only. Here $\phi(x)$ and $\psi(y)$ are either both positive or negative.

Now, $$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\log|f'(z)| = 0$$

$$\Rightarrow \quad \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\{\log\phi(x) + \log\psi(y)\} = 0$$

$$\Rightarrow \quad \frac{d^2}{dx^2}\{\log\phi(x)\} + \frac{d^2}{dy^2}\{\log\psi(y)\} = 0$$

$$\Rightarrow \quad \frac{d^2}{dx^2}\{\log\phi(x)\} = -\frac{d^2}{dy^2}\{\log\psi(y)\} = c \text{ (a constant)} \qquad \text{| say}$$

$$\therefore \quad \frac{d^2}{dx^2}\{\log\phi(x)\} = c \quad \text{and} \quad \frac{d^2}{dy^2}\{\log\psi(y)\} = -c$$

$$\Rightarrow \quad \log\phi(x) = \frac{1}{2}cx^2 + dx + e \quad \text{and} \quad \log\psi(y) = -\frac{1}{2}cy^2 + d'y + e'$$

$$\Rightarrow \quad \phi(x) = \exp.\left(\frac{1}{2}cx^2 + dx + e\right) \quad \text{and} \quad \psi(y) = \exp.\left(-\frac{1}{2}cy^2 + d'y + e'\right)$$

where d, e, d' and e' are real constants.

$$\therefore \quad |f'(z)| = \phi(x)\,\psi(y) = \exp.\left[\frac{c}{2}(x^2 - y^2) + dx + d'y + e + e'\right] \qquad ...(1)$$

Similarly,

$$|\exp.(\alpha z^2 + \beta z + \gamma)| = |\exp.\ a(x + iy)^2 + (a + ib)(x + iy) + (c + id)|$$

$$= \exp.[a(x^2 - y^2) + ax - by + c] \qquad ...(2) \qquad \left|\ \because\ |e^{A+iB}| = e^A\right.$$

where $\beta = a + ib$, $\gamma = c + id$

Expression (2) is of the same form as (1).

Hence we can write $f'(z) = \exp.(\alpha z^2 + \beta z + \gamma)$.

EXERCISE

1. (*i*) Determine a, b, c, d so that the function $f(z) = (x^2 + axy + by^2) + i(cx^2 + dxy + y^2)$ is analytic.

(*ii*) Find the constants a, b, c such that the function $f(z)$ where

$f(z) = -x^2 + xy + y^2 + i(ax^2 + bxy + cy^2)$ is analytic. Express $f(z)$ in terms of z.

(*iii*) Find the value of the constants a and b such that the following function $f(z)$ is analytic.

$f(z) = \cos x(\cosh y + a \sinh y) + i \sin x(\cosh y + b \sinh y)$

(*iv*) Determine p such that the function $f(z) = \frac{1}{2}\log(x^2 + y^2) + i\tan^{-1}\frac{px}{y}$ is an analytic function.

(*v*) For what values of a, b and c, the function $f(z) = x - 2ay + i(bx - cy)$ is analytic ?

2. Discuss the analyticity of the following functions:

(*i*) $\sin z$ (*ii*) $\cosh z$ (*iii*) $\frac{1}{z}$ (*iv*) z^3.

3. (*i*) If $f(z) = (x - y)^2 + 2i(x + y)$, show that C-R equations are satisfied along the curve $x - y = 1$.

(*ii*) Show that the function $f(z) = (x^3 - 3xy^2) + i(3x^2y - y^3)$ satisfies Cauchy-Riemann equations.

(*Coimbatore 2010*)

(*iii*) Find the analytic region of $f(z) = (x - y)^2 + 2i(x + y)$.

(*iv*) Check whether $w = \bar{z}$ is analytic everywhere ?

(*v*) Determine whether the function $2xy + i(x^2 - y^2)$ is analytic or not ?

(*vi*) If $w = f(z)$ is analytic, prove that $\frac{dw}{dz} = \frac{\partial w}{\partial x} = -i\frac{\partial w}{\partial y}$ where $z = x + iy$ and prove that $\frac{\partial^2 w}{\partial z\,\partial \bar{z}} = 0$.

(*vii*) Find where the following function ceases to be analytic: $f(z) = \frac{z^2 - 4}{z^2 + 1}$.

(*viii*) Verify if the function $e^{-2x}\cos 2y$ can be the real/imaginary part of an analytic function.

4. Show that the polar form of Cauchy-Riemann equations are $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$, $\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$. Deduce that $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$.

(*Anna 2009*)

5. Show that if $f(z)$ is differentiable at a point z, then

$$|f'(z)|^2 = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$$

6. (*i*) Show that an analytic function $f(z)$, whose derivative is identically zero, is constant.

(*ii*) It is given that a function $f(z)$ and its conjugate $\overline{f(z)}$ are both analytic. Determine the function $f(z)$.

7. (*i*) Show that the function $f(z)$ defined by $f(z) = \dfrac{x^3y^5(x+iy)}{x^6+y^{10}}$, $z \neq 0$, $f(0) = 0$, is not analytic at the origin even though it satisfies Cauchy-Riemann equations at the origin.

(*ii*) Show that for the function

$$f(z) = \begin{cases} \dfrac{(\bar{z})^2}{z}, & z \neq 0 \\ 0, & z = 0 \end{cases}$$

the Cauchy-Riemann equations are satisfied at the origin. Does $f'(0)$ exist?

(*iii*) Show that for the function

$$f(z) = \begin{cases} \dfrac{2xy\,(x+iy)}{x^2+y^2}, & z \neq 0 \\ 0, & z = 0 \end{cases}$$

the C-R equations are satisfied at origin but derivative of $f(z)$ does not exist at origin.

8. (*i*) If u is a harmonic function then show that $w = u^2$ is not a harmonic function unless u is a constant.

(*ii*) If $f(z)$ is an analytic function, show that $|f(z)|$ is not a harmonic function.

(*iii*) Show that the function $y + e^x \cos y$ is harmonic.

Also find the analytic function $f(z) = u(x, y) + iv(x, y)$ whose real part is $y + e^x \cos y$.

(*iv*) Show that $v = \log(x^2 + y^2)$ is harmonic. Find a function u such that $u + iv$ is analytic.

(*Anna 2009*)

(*v*) Show that the function $u = 2x - x^3 + 3xy^2$ is harmonic.

(*vi*) Show that $u = 3x^2y - y^3$ is a harmonic function.

9. (*i*) Show that the function $u(x, y) = 2x + y^3 - 3x^2y$ is harmonic. Find its conjugate harmonic function $v(x, y)$ and the corresponding analytic function $f(z)$.

(*ii*) Show that the function $u(x, y) = 3x^2y + 2x^2 - y^3 - 2y^2$ is harmonic. Find the conjugate harmonic function v and express $u + iv$ as an analytic function of z. (*Coimbatore 2010*)

(*iii*) Show that the function $v(x, y) = e^x \sin y$ is harmonic. Find its conjugate harmonic function $u(x, y)$ and the corresponding analytic function $f(z)$.

(*iv*) Show that $v = x^3y - xy^3 + x + y$ is harmonic and also find the analytic function $w = u + iv$ interms of z. (*Anna 2010*)

10. (*i*) Show that the function $u(r, \theta) = r^2 \cos 2\theta$ is harmonic. Find its conjugate harmonic function and the corresponding analytic function $f(z)$.

(*ii*) Determine constant '*b*' such that $u = e^{bx} \cos 5y$ is harmonic.

(*iii*) Prove that $u = x^2 - y^2 - 2xy - 2x + 3y$ is harmonic. Find a function v such that $f(z) = u + iv$ is analytic. Also express $f(z)$ in terms of z.

11. Determine the analytic function $f(z)$ in terms of z whose real part is

(*i*) $\frac{1}{2} \log (x^2 + y^2)$ (*ii*) $\cos x \cosh y$

(*iii*) $e^x \cos y$ (*iv*) $\frac{\sin 2x}{\cosh 2y \quad \cos 2x}$

(*v*) $\frac{\sin 2x}{\cosh 2y \quad \cos 2x}$ (*vi*) $e^{2x} \sin 2y$.

12. Find the regular function $f(z)$ in terms of z whose imaginary part is

(*i*) $\frac{x \quad y}{x^2 \quad y^2}$ (*ii*) $\cos x \cosh y$ (*iii*) $\sinh x \cos y$

(*iv*) $6xy - 5x + 3$ (*v*) $\frac{x}{x^2 \quad y^2} + \cosh x \cos y$.

13. (*i*) Show that $v = e^{2x}(y \cos 2y + x \sin 2y)$ is harmonic and find the corresponding analytic function $f(z) = u + iv$.

(*ii*) Construct the analytic function $f(z) = u + iv$ given that $2u + 3v = e^x(\cos y - \sin y)$.

(*iii*) Show that the function $u = x^3 + x^2 - 3xy^2 + 2xy - y^2$ is harmonic and find the corresponding analytic function $f(z) = u + iv$.

14. (*i*) An electrostatic field in the xy-plane is given by the potential function $\phi = x^2 - y^2$, find the stream function.

(*ii*) If the potential function is $\log (x^2 + y^2)$, find the flux function and the complex potential function.

15. (*i*) In a two dimensional fluid flow, the stream function is $\psi = \tan^{-1}\left(\frac{y}{x}\right)$, find the velocity potential ϕ.

(*ii*) If $w = \phi + i\psi$ represents the complex potential for an electric field and $\psi = x^2 - y^2 \quad \frac{x}{x^2 \quad y^2}$, determine the function ϕ.

16. If $f(z)$ is an analytic function of z, prove that

$$\left(\frac{\ ^2}{x^2} \quad \frac{\ ^2}{y^2}\right) |R\, f(z)|^2 = 2\, |f'(z)|^2.$$

17. Find an analytic function $f(z) = u(r, \theta) + iv(r, \theta)$ such that $v(r, \theta) = r^2 \cos 2\theta - r \cos \theta + 2$.

18. If $f(z) = u + iv$ is an analytic function, find $f(z)$ in terms of z if

(*i*) $u - v = e^x (\cos y - \sin y)$ (*Anna 2012*) (*ii*) $u + v = \frac{x}{x^2 \quad y^2}$, when $f(1) = 1$

(*iii*) $u - v = \frac{e^y \quad \cos x \quad \sin x}{\cosh y \quad \cos x}$ when $f\left(\frac{\ }{2}\right) \quad \frac{3 \quad i}{2}$

(*iv*) $u - v = \frac{\sin 2x}{\cosh 2y - \cos 2x}$. (*Anna 2009*)

19. (*i*) If $f(z) = u + iv$ is an analytic function of $z = x + iy$ and $u + v = (x + y)(2 - 4xy + x^2 + y^2)$ then construct $f(z)$ interms of z.

(*ii*) If $f(z) = u + iv$ is an analytic function of $z = x + iy$ and $u - v = e^{-x}[(x - y)\sin y - (x + y)\cos y]$ then construct $f(z)$ interms of z.

20. If $f = u + iv$ is analytic show that $g = -v + iu$ and $h = v - iu$ are also analytic. Also show that u and $-v$ are conjugate harmonic. (*Anna 2009*)

21. Show that the function

(*i*) $f(z) = \dfrac{z}{z+1}$ is analytic at $z = \infty$. (*ii*) $f(z) = z$ is not analytic at $z = \infty$.

22. If $f(z) = u(x, y) + iv(x, y)$ where $x = \dfrac{z \quad \bar{z}}{2}$, $y = \dfrac{z \quad \bar{z}}{2i}$ is continuous as a function of two variables z and $\bar{z}$ then show that $\dfrac{f}{\bar{z}} = 0$ is equivalent to the Cauchy-Riemann equations.

$$\left[\textbf{Hint: } \frac{\partial f}{\partial \bar{z}} = \left(\frac{\partial u}{\partial x}\frac{\partial x}{\partial \bar{z}} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \bar{z}}\right) + i\left(\frac{\partial v}{\partial x}\frac{\partial x}{\partial \bar{z}} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial \bar{z}}\right)\right]$$

23. If $f(z) = u + iv$ is a regular function of z in a domain D. Prove that the following relations hold in D:

(*i*) $\nabla^2 [\arg f(z)] = 0$ *i.e.*, $\arg f(z)$ is harmonic in D.

(*ii*) $\nabla^2 |\text{Im } f(z)|^2 = 2|f'(z)|^2$.

24. If $f(z) = u + iv$ is a regular function of z in a domain D, show that the following relation holds in D $\nabla^2[f(z)]^p = p^2 |f(z)|^{p-2} |f'(z)|^2$.

25. If $f(z) = u + iv$ is a regular function of z in a domain D, prove that the following relation holds in D.

$\nabla^2 \log |f(z)| = 0$ if $f(z) f'(z) \neq 0$ in D Or $\log |f(z)|$ is harmonic in D.

Answers

1. (*i*) $a = 2, b = -1, c = -1, d = 2$ (*ii*) $a = \frac{1}{2}, b = -2, c = \frac{1}{2}$; $f(z) = \frac{1}{2}(2 + i)z^2$

(*iii*) $a = -1, b = -1$ (*iv*) $p = -1$ (*v*) $2a = b, c = 1$

3. (*iii*) $x - y = 1$ (*iv*) nowhere analytic (*v*) No

(*vii*) $z = \pm i$ (*viii*) yes

6. (*ii*) constant function **7.** (*ii*) No.

8. (*iii*) $e^z - iz + c$ (*iv*) $u = 2\tan^{-1}\left(\frac{x}{y}\right) \quad c$

9. (*i*) $v = 2y - 3xy^2 + x^3 + c$; $f(z) = 2z + iz^3 + ic$ (*ii*) $v = 3xy^2 + 4xy - x^3 + c$, $f(z) = -iz^3 + 2z^2 + ic$

(*iii*) $u = e^x \cos y + c$; $f(z) = e^z + c$ (*iv*) $w = \dfrac{z^4}{4} + (1 + i)z + c$

10. (*i*) $v = r^2 \sin 2\theta + c$; $f(z) = z^2 + ic$ (*ii*) $b = \pm 5$

(*iii*) $v = x^2 - y^2 + 2xy - 2y - 3x$, $f(z) = (1 + i)z^2 - (2 + 3i)z$

11. (*i*) $\log z + c$ (*ii*) $\cos z + c$ (*iii*) $e^z + c$

(*iv*) $\cot z + c$ (*v*) $\tan z + c$ (*vi*) $-ie^{2z} + c$

12. (*i*) $\frac{1 \quad i}{z} \quad c$ (*ii*) $i \cos z + c$ (*iii*) $i \sinh z + c$

(*iv*) $3z^2 - 5iz + c$ (*v*) $\frac{i}{z} + i \cosh z$

13. (*i*) $ze^{2z} + c$ (*ii*) $\left(\frac{1 \quad 5i}{13}\right) e^{z} \quad \frac{c}{2i \quad 3}$ (*iii*) $z^3 + z^2(1 - i) + c$

14. (*i*) $y = 2xy + c$ (*ii*) $2 \tan^{-1}\left(\frac{y}{x}\right)$, $2 \log z + c$

15. (*i*) $\frac{1}{2} \log (x^2 + y^2)$ (*ii*) $-2xy + \frac{y}{x^2 \quad y^2} \quad c$ **17.** $i(z^2 - z + 2) + c$

18. (*i*) $e^z + c$ (*ii*) $\frac{1}{1 \quad i}\left(\frac{i}{z} \quad 1\right)$ (*iii*) $\cot \frac{z}{2} \quad \frac{1}{2}(1 - i)$

(*iv*) $f(z) = \frac{\cot z}{1 \quad i} \quad c_1$. **19.** (*i*) $2z + iz^3 + c$ (*ii*) $ize^{-z} + c$