UNIT - 2

Functions of single variables

Continuous function:- Let $f(x)$ be a function defined in $[a, b]$. A function $f(x)$ is said to be continuous at $x = c$ if

$$\lim_{x \to c} f(x) = f(c), \quad \text{(is finite)}.$$

Differentiable function:-

A function $f(x)$ is said to be diff'ble at $x = c$ if $\lim_{x \to c} \frac{f(x) - f(c)}{x - c}$ exist, and finite.

Note:- If $f(x)$ is continuous in $[a, b]$ then $f(x)$ is bounded and its attains bounds.

$f(c) = M =$ least upper bound.

$f(c) = m =$ greatest lower bound.

V.V.V. IMP

Rolles theorem:- If $f(x)$ is a function defined in $[a, b] \to R$ such that

(i) $f(x)$ is continuous in $[a, b]$

(ii) $f(x)$ is differentiable in (a, b)

(iii) If $f(a) = f(b)$ then there exists atleast one point $c \in (a, b)$ such that $f'(c) = 0$.

proof:- We have $f(x)$ is continuous, then $f(x)$ has bounds.

let $m =$ gretest Lower bound

$M =$ Least upper bound

$$\boxed{\begin{aligned} f(x) &= 1 \\ f'(x) &= 0 \end{aligned}}$$

<u>case (i)</u> :- If $M = m$, $f(x)$ is constant function, $f(x) = c$

$f'(x) = 0 \Rightarrow \underline{f'(c) = 0}$.

<u>case (ii)</u> :- If $M \neq m$. then let $f(c) = M$, $c \in (a,b)$

M is Least upper bound, $f(c+h) \leq f(c)$, h is small nbr.

$$f(c+h) - f(c) \leq 0$$

If $h > 0$, $\dfrac{f(c+h) - f(c)}{h} \leq 0. \longrightarrow (1)$

If $h < 0$, $\dfrac{f(c+h) - f(c)}{h} \geq 0 \longrightarrow (2)$

from (1) & (2) $\dfrac{f(c+h) - f(c)}{h} = 0 \Rightarrow \underline{f'(c) = 0}$

$\therefore$ Both cases $f'(c) = 0$.

Hence proved..

V.V.V.IMP

<u>Lagrange's Mean value theorem</u> :-

let $f(x)$ be a function ~~such that~~ defined, $[a,b] \to R \ni$

(i) $f(x)$ is continuous in $[a,b]$.

(ii) $f(x)$ is differentiable in (a,b),

Then there exist atleast one point $c \in (a,b)$ such that $f'(c) = \dfrac{f(b) - f(a)}{b-a}$.

<u>proof</u>:- Let $g(x)$ be a function defined in $[a,b] \to R$

$g(x)$ satisfies all the conditions of Rolles theorem

let $g(x) = f(x) + Ax \longrightarrow (1)$ A is constant

$g(a) = f(a) + Aa$

$g(b) = f(b) + Ab$.

Rolle's theorem

Taking 3^{rd} condition of

for $g(x)$, $g(a) = g(b)$

$$f(a) + Aa = f(b) + Ab.$$

$$Aa - Ab = f(b) - f(a)$$

$$A(a-b) = f(b) - f(a)$$

$$A = \frac{f(b) - f(a)}{a-b}$$

$$A = \frac{f(b) - f(a)}{-(b-a)}$$

$$A = -\left(\frac{f(b) - f(a)}{b-a}\right) \longrightarrow (2)$$

Taking 3^{rd} condition (2^{nd} part) of Rolles theorem,

for $g(x)$,

$$g'(c) = 0 \longrightarrow (3)$$

we have $g(x) = f(x) + A(x)$.

$$g'(x) = f'(x) + A$$

$$g'(c) = f'(c) + A \longrightarrow (4)$$

Sub (3) in (4), $f'(c) + A = 0$

$$f'(c) = -A \longrightarrow (5)$$

sub equ (2) in (5)

$$\therefore f'(c) = \frac{f(b) - f(a)}{b-a}$$

V.V.VVSMR

Cauchy's Mean Value theorem :-

If $f:[a,b] \to R$ and $g:[a,b] \to R$ such that

(i) $f(x), g(x)$ are continuous in $[a,b]$.

(ii) $f(x), g(x)$ are diff'ble in (a,b).

Then there exist atleast one point $c \in (a,b)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}.$$

Proof:- Let $h(x)$ be a function defined by

$$h(x) = f(x) + A\,g(x) \longrightarrow (1)$$

$h(x)$ satisfies all the conditions of Rolles theorem.

$$h(a) = f(a) + A\,g(a)$$

$$h(b) = f(b) + A\,g(b)$$

Taking 3rd condition of Rolles them for $h(x)$,

$$h(a) = h(b)$$

$$f(a) + A\,g(a) = f(b) + A\,g(b)$$

$$A\,g(a) - A\,g(b) = f(b) - f(a)$$

$$A[g(a)-g(b)] = f(b) - f(a)$$

$$A = \frac{f(b)-f(a)}{g(a)-g(b)}$$

$$A = \frac{f(b)-f(a)}{-[g(b)-g(a)]}$$

$$A = -\left[\frac{f(b)-f(a)}{g(b)-g(a)}\right] \longrightarrow (2)$$

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Taking 2^{nd} part & 3^{rd} condition of Rolle theorem for $g(x)$,

$h'(c) = 0 \longrightarrow (3)$

We have $h(x) = f(x) + A\,g(x)$

$h'(x) = f'(x) + A\,g'(x)$

$h'(c) = f'(c) + A\,g'(c) \longrightarrow (4)$

Sub (3) in (4)

$f'(c) + A\,g'(c) = 0$

$f'(c) = -A\,g'(c)$

$\dfrac{f'(c)}{g'(c)} = -A \longrightarrow (5)$

sub (2) in (5)

$\therefore \dfrac{f'(c)}{g'(c)} = \dfrac{f(b) - f(a)}{g(b) - g(a)}$

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(1) ~~Verif~~

Note :- 1) Every polynomial eqn is continuous and diff'ble

2) Every exponential function's are continuous and diff'ble.

3) Log function cont and diff [except $\log 0$]

4) Trigonometric functions also conti and diff'ble ($\tan\frac{\pi}{2}$, $\sec 0$, $\cot 0$.)

② Verify Rolle's theorem for $f(x) = 2x^3 + x^2 - 4x - 2$ in $(-\sqrt{2}, \sqrt{2})$.

Sol:- Given $f(x) = 2x^3 + x^2 - 4x - 2$ in $(-\sqrt{2}, \sqrt{2})$

Given function is a polynomial equ.

W.K.T Every polynomial equ is continuous

$\therefore f(x)$ is continuous in $[-\sqrt{2}, \sqrt{2}]$.

$$f'(x) = 6x^2 + 2x - 4$$

$\therefore f(x)$ is diff'ble in $(-\sqrt{2}, \sqrt{2})$.

$a = -\sqrt{2}$, $b = \sqrt{2}$

$$f(a) = f(-\sqrt{2}) = 2(-\sqrt{2})^3 + (-\sqrt{2})^2 - 4(-\sqrt{2}) - 2$$

$$= -2\cdot(2\sqrt{2}) + 2 + 4\sqrt{2} - 2$$

$$= -4\sqrt{2} + 4\sqrt{2}$$

$$= 0$$

$$f(b) = f(\sqrt{2}) = 2(\sqrt{2})^3 + (\sqrt{2})^2 - 4(\sqrt{2}) - 2$$

$$= 2(2\sqrt{2}) + \cancel{2} - 4\sqrt{2} - \cancel{2}$$

$$= 4\sqrt{2} - 4\sqrt{2}$$

$$= 0$$

$$\therefore f(-\sqrt{2}) = f(\sqrt{2})$$

Consider $f'(c) = 0$ $\quad c \in (a,b)$

$$6c^2 + 2c - 4 = 0$$

$ax^2 + bx + c = 0$

$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$2(3c^2 + c - 2) = 0$$

$$3c^2 + c - 2 = 0$$

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$c = 0.66, -1$

$\sqrt{2} = 1.41$

$\therefore \; C = 0.66, -1 \in (-\sqrt{2}, \sqrt{2})$

1.14

$\therefore$ All the conditions of Rolles theorem satisfied

$\therefore$ Rolle's theorem verified.

(Pb) If $a<b$, P.T $\frac{b-a}{1+b^2} < \tan^{-1}b - \tan^{-1}a < \frac{b-a}{1+a^2}$ and

: using Lagrange's mean value theorem.

deduce (i) $\frac{\pi}{4} + \frac{3}{25} < \tan^{-1}\frac{4}{3} < \frac{\pi}{4} + \frac{1}{6}$

(ii) $\frac{5\pi+4}{20} < \tan^{-1} 2 < \frac{\pi+2}{4}$.

sol:- Let $f(x) = \tan^{-1}x$ in $[a, b]$

$\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}$

$f'(x) = \frac{1}{1+x^2} \Rightarrow f'(c) = \frac{1}{1+c^2}$

W.K.T Lagrange's mean value theorem

is $f'(c) = \frac{f(b)-f(a)}{b-a}$

$\frac{1}{1+c^2} = \frac{\tan^{-1}(b) - \tan^{-1}(a)}{b-a} \longrightarrow (1)$

we have $c \in (a, b)$

$a < c < b$

$a^2 < c^2 < b^2$

$1+a^2 < 1+c^2 < 1+b^2$

$\frac{1}{1+a^2} > \frac{1}{1+c^2} > \frac{1}{1+b^2} \longrightarrow (2)$

sub (1) in (2)

$$\frac{1}{1+a^2} > \frac{\tan^{-1} b - \tan^{-1} a}{b-a} > \frac{1}{1+b^2}$$

$$\frac{(b-a)}{1+a^2} > \tan^{-1} b - \tan^{-1} a > \frac{b-a}{1+b^2}$$

$$\frac{b-a}{1+b^2} < \tan^{-1}(b) - \tan^{-1}(a) < \frac{b-a}{1+a^2} \longrightarrow (3)$$

(i) put $a = 1$, $b = \frac{4}{3}$ in (3)

$\tan\left(\frac{\pi}{4}\right) = 1$

$\tan^{-1}(1) = \frac{\pi}{4}$

$$\frac{\frac{4}{3}-1}{1+\left(\frac{4}{3}\right)^2} < \tan^{-1}\left(\frac{4}{3}\right) - \tan^{-1}(1) < \frac{\frac{4}{3}-1}{1+1}$$

$$\frac{\frac{1}{3}}{\frac{16+9}{9}} < \tan^{-1}\left(\frac{4}{3}\right) - \frac{\pi}{4} < \frac{\frac{1}{3}}{2}$$

$$\frac{3}{25} < \tan^{-1}\left(\frac{4}{3}\right) - \frac{\pi}{4} < \frac{1}{6}$$

[Addig $\frac{\pi}{4}$ on b.s]

$$\frac{\pi}{4} + \frac{3}{25} < \tan^{-1}\left(\frac{4}{3}\right) < \frac{\pi}{4} + \frac{1}{6}$$

(ii) Take $a = 1$ $b = 2$ in (3)

$$\frac{2-1}{1+4} < \tan^{-1}(2) - \tan^{-1}(1) < \frac{2-1}{1+1}$$

$$\frac{1}{5} < \tan^{-1}(2) - \frac{\pi}{4} < \frac{1}{2}$$

[Addig $\frac{\pi}{4}$ on b.s]

$$\frac{\pi}{4} + \frac{1}{5} < \tan^{-1}(2) < \frac{\pi}{4} + \frac{1}{2}$$

$$\frac{5\pi + 4}{20} < \tan^{-1}(2) < \frac{\pi + 2}{4}$$

Hence proved.

Pb) Verify Cauchy's mean value theorem for the function $f(x) = x^2$, $g(x) = x^3$ in $[1, 2]$.

Sol: Given $f(x) = x^2$, $g(x) = x^3$ in $[1, 2]$

$f(x)$ and $g(x)$ are polynomial equ's.

W.K.T Every polynomial equ's are Continue

$\therefore$ $f(x)$, and $g(x)$ are continuous in $[1, 2]$

$f(x) = x^2 \Rightarrow f'(x) = 2x \Rightarrow f'(c) = 2c$

$g(x) = x^3 \Rightarrow g'(x) = 3x^2 \Rightarrow g'(c) = 3c^2$

$\therefore$ $f(x)$ and $g(x)$ are diff'ble in $(1, 2)$.

consider $\dfrac{f'(c)}{g'(c)} = \dfrac{f(b) - f(a)}{g(b) - g(a)}$

$$\frac{2c}{3c^2} = \frac{(2)^2 - (1)^2}{(2)^3 - (1)^3}$$

$$\frac{2}{3c} = \frac{4-1}{8-1}$$

$$\frac{2}{3c} = \frac{3}{7}$$

$9c = 14 \Rightarrow c = \dfrac{14}{9} \Rightarrow c = 1.5 \in (1, 2)$

$\therefore$ All the conditions of Cauchy's mean value theorem satisfied.

$\therefore$ Cauchy's mean value theorem Verified.

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$\boxed{0! = 1}$

Taylor's theorem :-

$f(x) : [a, b] \to R$

$$f(b) = f(a) + \frac{(b-a)^1}{1!} f'(a) + \frac{(b-a)^2}{2!} f''(a) + \frac{(b-a)^3}{3!} f'''(a) + \cdots$$

is Taylor's them about the point $b = a$ (or) in powers of $(b-a)$.

Maclaurience series expansion :-

$f(x) : [0, x]$

Taking $a = 0$, $b = x$ in Taylors theorem

$$f(x) = f(0) + \frac{x^1}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{iv}(0) + \cdots$$

Maclaurience series expansion in powers of x about $x = 0$

① Find The Taylor's Series of $f(x) = \sin x$ about $x = \frac{\pi}{4}$.

Sol :- Given $f(x) = \sin x$, given $x = \frac{\pi}{4}$

using Taylor's theorem about the point $b = a$ is

$$f(b) = f(a) + \frac{(b-a)}{1!} f'(a) + \frac{(b-a)^2}{2!} f''(a) + \frac{(b-a)^3}{3!} f'''(a) + \cdots$$

put $a = \frac{\pi}{4}$, $b = x$ in above equ.

$$\therefore f(x) = f(\pi/4) + \frac{(x - \pi/4)}{1!} f'(\pi/4) + \frac{(x-\pi/4)^2}{2!} f''(\pi/4) + \frac{(x-\pi/4)^3}{3!} f'''(\pi/4) + \cdots \longrightarrow ①$$

$f(x) = \sin x \Rightarrow f(\pi/4) = \sin \pi/4 = \frac{1}{\sqrt{2}}$

$f^{1}(x) = \cos x \Rightarrow f^{1}(\pi/4) = \cos \pi/4 = \frac{1}{\sqrt{2}}$

$f^{11}(x) = -\sin x \Rightarrow f^{11}(\pi/4) = -\sin \pi/4 = -\frac{1}{\sqrt{2}}$

$f^{111}(\cancel{x}) = -\cos x \Rightarrow f^{111}(\pi/4) = -\cos(\pi/4) = -\frac{1}{\sqrt{2}}$

$f^{iv}(\pi/4) = \sin x \Rightarrow f^{iv}(\pi/4) = \sin(\pi/4) = \frac{1}{\sqrt{2}}$

Sub thu vale in 1

$$\therefore f(x) = \frac{1}{\sqrt{2}} + \frac{(x-\pi/4)^1}{1!}\left(\frac{1}{\sqrt{2}}\right) + \frac{(x-\pi/4)^2}{2!}\left(\frac{-1}{\sqrt{2}}\right) +$$

$$\frac{(x-\pi/4)^3}{3!}\left(\frac{-1}{\sqrt{2}}\right) + \frac{(x-\pi/4)^4}{4!}\left(\frac{1}{\sqrt{2}}\right) + \sim$$

$$f(x) = \frac{1}{\sqrt{2}}\left[1 + \frac{(x-\pi/4)^1}{1!} - \frac{(x-\pi/4)^2}{2!} - \frac{(x-\pi/4)^3}{3!} + \frac{(x-\pi/4)^4}{4!} + \cdots\right.$$

Curvature :- $\cancel{\delta}$ λ

Radius of Curvature :- $\rho = \frac{1}{\lambda}$ $|\rho| \Rightarrow$ —

Radius of Curvature in Cartesian form :-

1) Given eqn of the form $y = f(x)$ Then the radius of curvature $\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$

2) If $\frac{dy}{dx} = \infty$ Then $\cancel{\rho =}$ $\cancel{1+\left(\frac{dx}{dy}\right)^2}$

$$\rho = \frac{\left[1+\left(\frac{dx}{dy}\right)^2\right]^{3/2}}{\frac{d^2x}{dy^2}}$$

1) $$\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$$

2) $$\rho = \frac{\left[1+\left(\frac{dx}{dy}\right)^2\right]^{3/2}}{\frac{d^2x}{dy^2}} \quad \left[\text{if } \frac{dy}{dx} = \infty\right]$$

ρ = Always positive.

① Find the radius of curvature of the curve $\sqrt{x} + \sqrt{y} = \sqrt{a}$ at $(a/4, a/4)$.

Sol: Given $\sqrt{x} + \sqrt{y} = \sqrt{a} \longrightarrow ①$

Given point is $(a/4, a/4)$.

Clearly given equ is in cartesian form

∴ The radius of curvature in cartesian is

$$\rho = \frac{\left(1+\left(\frac{dy}{dx}\right)^2\right)^{3/2}}{\frac{d^2y}{dx^2}} \longrightarrow ②$$

diff equ ① w.r.to 'x'. $\frac{d}{dx}(\sqrt{x} + \sqrt{y}) = \frac{d}{dx}\sqrt{a}$

$$\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}} \cdot \frac{dy}{dx} = 0$$

$\boxed{\frac{d}{dx}(x) = 1}$

$1 \cdot x^{1-1}$

$1 \cdot x^0$

$$\frac{d}{dx} x^{\frac{1}{2}} = \frac{1}{2} x^{\frac{1}{2}-1} = \frac{1}{2} x^{-\frac{1}{2}} = \frac{1}{2}\frac{1}{x^{1/2}} = \frac{1}{2\sqrt{x}}$$

$$\frac{1}{2\sqrt{y}} \frac{dy}{dx} = -\frac{1}{2\sqrt{x}}$$

$$\frac{dy}{dx} = -\frac{\cancel{2}\sqrt{y}}{\cancel{2}\sqrt{x}}$$

$$\frac{dy}{dx} = -\frac{\sqrt{y}}{\sqrt{x}} \longrightarrow ③$$

$$\therefore \left(\frac{dy}{dx}\right)_{(a/4, a/4)} = -\left(\frac{\cancel{\sqrt{a/4}}}{\cancel{\sqrt{a/4}}}\right) = -1 \longrightarrow ④$$

$$\frac{d^2y}{dx^2} = (-)\frac{d}{dx}\left(\frac{\sqrt{y}}{\sqrt{x}}\right)$$

$\boxed{\frac{d}{dx}\frac{u}{v} = \frac{vu' - uv'}{v^2}}$

$$= (-1)\left[\frac{\sqrt{x} \cdot \frac{d}{dx}\sqrt{y} - \sqrt{y} \cdot \frac{d}{dx}\sqrt{x}}{(\sqrt{x})^2}\right]$$

$\frac{d}{dx}\sqrt{y} = \frac{1}{2\sqrt{y}}$

$$= (-1)\left[\frac{\sqrt{x}\,\frac{1}{2\sqrt{y}} \cdot \frac{dy}{dx} - \sqrt{y}\,\frac{1}{2\sqrt{x}}}{x}\right]$$

$\frac{d}{dx}(y) =$

$$\left(\frac{d^2y}{dx^2}\right)_{(a/4, a/4)} = (-1)\left[\frac{\sqrt{a/4}\,\frac{1}{2\sqrt{a/4}} \cdot (-1) - \sqrt{a/4} \cdot \frac{1}{2\sqrt{a/4}}}{\frac{a}{4}}\right]$$

$$=(-1)\left[\frac{-\frac{1}{2}-\frac{1}{2}}{\frac{a}{4}}\right]$$

$$=(-1)\left(\frac{-1}{\frac{a}{4}}\right)$$

$$=(-1)(-1)\ \frac{4}{a}$$

$$\left(\frac{d^2y}{dx^2}\right)_{(a/4,\ a/4)} = \frac{4}{a} \longrightarrow (5)$$

sub (4), (5) in (2)

$$\therefore\ \rho = \frac{\left[1+(-1)^2\right]^{3/2}}{\frac{4}{a}}$$

$$= (2)^{3/2} \cdot \frac{a}{4}$$

$4 = 2 \cdot \sqrt{2} \cdot \sqrt{2}$

$$= 2 \cdot \sqrt{2} \cdot \frac{a}{4}$$

$$\boxed{\rho = \frac{a}{\sqrt{2}}}$$

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Radius of Curvature in parametric form :

Given eqn is of the form $x = x(t)$, $y = y(t)$

$$\rho = \frac{\left[(\dot{x})^2 + (\dot{y})^2\right]^{3/2}}{\dot{x}\ddot{y} - \dot{y}\ddot{x}},$$

$$\dot{x} = \frac{dx}{dt}, \quad \dot{y} = \frac{dy}{dt}$$

$$\ddot{x} = \frac{d^2x}{dt^2}, \quad \ddot{y} = \frac{d^2y}{dt^2}$$

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Pb) Find the Curvature and radius of Curvature of the curve $x = a(t - \sin t)$
$y = a(1 - \cos t)$ at $t = \pi$.

Sol:- Given $x = a(t - \sin t)$, $y = a(1 - \cos t)$.
given point $t = \pi$.

The radius of curvature in parametric form $\dot{x} = \frac{dx}{dt}$, $\dot{y}$

$$\rho = \frac{\left[(\dot{x})^2 + (\dot{y})^2\right]^{3/2}}{\dot{x}\ddot{y} - \dot{y}\ddot{x}} \longrightarrow (1)$$

$$\dot{x} = \frac{dx}{dt} = \frac{d}{dt} a(t - \sin t) = a(1 - \cos t)$$

$$(\dot{x})_{t=\pi} = a(1 - \cos\pi) = a(1 - (-1))$$
$$= a(2) = 2a$$

$$\ddot{x} = \frac{d^2x}{dt^2} = \frac{d}{dt} a(1 - \cos t) = a(0 + \sin t) = a\sin t$$

$$(\ddot{x})_{t=\pi} = a \sin\pi = a(0) = 0$$

$$\dot{y} = \frac{dy}{dt} = \frac{d}{dt} a(1 - \cos t) = a \sin t$$

$$(\dot{y})_{t=\pi} = a \sin\pi = a(0) = 0$$

$$\ddot{y} = \frac{d^2y}{dt^2} = \frac{d}{dt}(a \sin t) = a \cos t$$

$$(\ddot{y})_{t=\pi} = a\cos\pi = a(-1) = -a$$

Sub these val in (1)

$$\rho = \frac{\left[(2a)^2 + (0)^2\right]^{3/2}}{2a(-a) - 0(0)} = $$

$$= \frac{\left[(2a)^{\cancel{2}}\right]^{3/\cancel{2}}}{-2a^2}$$

$$= \frac{(2a)^3}{-2a^2} = \frac{8a^3}{-2a^2}$$

$$= -4a$$

$$\therefore \underline{\underline{\rho = 4a}}$$

Q the

Radius of curvature in polar form :-

Given equ is of the form $f(r) = \theta$.
$r = f(\theta)$

$$\rho = \frac{(r^2 + r_1^2)^{3/2}}{r^2 + 2r_1^2 - r r_2}$$

$r_1 = \frac{dr}{d\theta}$

$r_2 = \frac{d^2 r}{d\theta^2}$

(1) Find the radius of curvature at any point of the cycloid $r = a(1-\cos\theta)$.

Sol: Given $r = a(1-\cos\theta)$

$$r_1 = \frac{dr}{d\theta} = a(0+\sin\theta) = a\sin\theta$$

$$r_2 = \frac{d^2r}{d\theta^2} = a.\cos\theta$$

The radius of curvature in polar form

$$\rho = \frac{(r^2+r_1^2)^{3/2}}{r^2+2r_1^2-rr_2} \longrightarrow (1)$$

$$r^2+r_1^2 = a^2(1-\cos\theta)^2 + a^2.\sin^2\theta$$

$$= a^2(1+\cos^2\theta - 2\cos\theta + \sin^2\theta)$$

$$= a^2(1+1-2\cos\theta)$$

$$= a^2(2-2\cos\theta)$$

$$= 2a^2(1-\cos\theta)$$

$$r^2+2r_1^2-rr_2 = a^2(1-\cos\theta)^2 + 2a^2\sin^2\theta - a(1-\cos\theta).a\cos\theta$$

$$= a^2\left[1+\cos^2\theta - 2\cos\theta + 2\sin^2\theta - \cos\theta + \cos^2\theta\right]$$

$$= a^2\left[1+2\sin^2\theta + 2\cos^2\theta - 3\cos\theta\right]$$

$$= a^2\left[1+2.\cancel{1}-3\cos\theta\right]$$

$$= a^2(3-3\cos\theta)$$

$$= 3a^2(1-\cos\theta)$$

$$\therefore \rho = \frac{\left[2a^2(1-\cos\theta)\right]^{3/2}}{3a^2(1-\cos\theta)} = \frac{2^{3/2}.(a^2)^{3/2}.(1-\cos\theta)^{3/2}}{3a^2\,(1-\cos\theta)}$$

$$= \frac{2\sqrt{2}.a^{3}.\cancel{(1-\cos\theta)}\,(1-\cos\theta)^{1/2}}{3\cancel{a^2}\,\cancel{(1-\cos\theta)}} = \frac{2\sqrt{2}}{3}a(1-\cos\theta)^{1/2}$$

$$= \frac{2\sqrt{2}\cdot a}{3}\sqrt{(1-\cos\theta)}$$

$\cos 2\theta = 1 - 2\sin^2\theta$

$2\sin^2\theta = 1 - \cos 2\theta$

$2\sin^2\theta/2 = 1 - \cos\theta$

$\sqrt{2}\cdot\sin\theta/2 = \sqrt{1-\cos\theta}$

$$= \frac{2\sqrt{2}\cdot a}{3}\cdot\sqrt{2}\cdot\sin\theta/2$$

$$\rho = \frac{4a}{3}\sin\theta/2$$

Envelope :-

A curve which touches each member of a given family of curves is called a envelope of that family.

The envelope of $f(x, y, \alpha) = 0$ which is a quadratic equation in α :-

$$A\alpha^2 + B\alpha + C = 0$$

The envelope is $B^2 - 4AC = 0$

(pb) Find the envelope of $y = mx + \frac{a}{m}$, where m is parameter.

Sol:- Given $y = mx + \frac{a}{m}$

$$y = \frac{xm^2 + a}{m}$$

$$ym = xm^2 + a$$

$$xm^2 - ym + a = 0$$ which is a quadratic equ in m.

$a = x$, $b = -y$, $c = a$

$\therefore$ Envelope is $b^2 - 4ac = 0$

$$(-y)^2 - 4xa = 0$$

$$y^2 - 4ax = 0 \Rightarrow y^2 = 4ax$$

(pb) Find the envelope of the family of curve $y = (x-p)^2$ where p is a parameter.

Sol:- Given $y = (x-p)^2$

$$y = x^2 + p^2 - 2xp.$$

$$x^2 + p^2 - 2xp - y = 0$$

$$p^2 - 2xp + (x^2 - y) = 0.$$ which is a quadratic in p,

Here $a = 1$, $b = -2x$, $c = (x^2 - y)$

Envelope is $b^2 - 4ac = 0$

$$(-2x)^2 - 4(1)(x^2 - y) = 0$$

$$\cancel{4x^2} - \cancel{4x^2} + 4y = 0$$

$$4y = 0$$

$y = 0$ is envelope.

① $y = mx + \sqrt{1+m^2}$, m is parameter.

② $\frac{x}{a} \cdot \cos\alpha + \frac{y}{b} \cdot \sin\alpha = 1$, α is par

Let $f(x, y, \alpha) = 0$ ...(1) α is a parameter

diff equ (1) w.r.to 'α' on b.s $\frac{\partial f}{\partial \alpha}$.

(Pb) Find the envelope of the family of Straight lines $x.\cos\alpha + y\sin\alpha = p$ where α is parameter.

Sol:- Given $x\cos\alpha + y\sin\alpha = P$ $\to$ (1)

diff equ (1) w.r.to 'α' on b.s

$-x\sin\alpha + y\cos\alpha = 0$ $\to$ (2)

$(1)^2 + (2)^2 \Rightarrow$

$x^2.\cos^2\alpha + y^2\sin^2\alpha + \cancel{2xy.\sin\alpha.\cos\alpha} = p^2$

$x^2.\sin^2\alpha + y^2\cos^2\alpha - \cancel{2xy\sin\alpha.\cos\alpha} = 0$

$x^2(\cos^2\alpha + \sin^2\alpha) + y^2(\sin^2\alpha + \cos^2\alpha) = p^2$

$\therefore x^2 + y^2 = p^2$ is envelope.

(Pb) Find the envelope of the family of straight lines $x\tan\alpha + y\sec\alpha = 5$, α is parameter.

Sol:- Given $x\tan\alpha + y\sec\alpha = 5$ $\to$ (1)

diff equ (1) w.r.to 'α' on b.s

$x\sec^2\alpha + y.\sec\alpha.\tan\alpha = 0$

$x\sec\alpha + y\tan\alpha = 0$ $\to$ (2)

$(1)^2 - (2)^2 \Rightarrow$ $x^2.\tan^2\alpha + y^2\sec^2\alpha + \cancel{2xy\tan\alpha.\sec\alpha} = 25$

$x^2\sec^2\alpha + y^2\tan^2\alpha + \cancel{2xy\tan\alpha.\sec\alpha} = 0$

$-x^2(\sec^2\alpha - \tan^2\alpha) + y^2(\sec^2\alpha - \tan^2\alpha) = 25$

$1 \qquad\qquad\qquad\qquad 1$

$y^2 - x^2 = 25$