UNIT-3

Multivariable Caluclus (Differentiation)

Limit function :- $\lim_{\substack{x \to a \\ y \to a}} f(x,y) = l$.

1) S.T $\lim_{(x,y) \to (0,0)} \dfrac{x^2 - x\sqrt{y}}{x^2+y}$ does not exist.

Sol:- Given $\lim_{(x,y) \to (0,0)} \dfrac{x^2 - x\sqrt{y}}{x^2+y}$

$$\lim_{\substack{x \to 0 \\ y \to 0}} \frac{x^2 - x\sqrt{y}}{x^2+y} = \lim_{y \to 0} \frac{0-0}{y}$$

$$= \lim_{y \to 0} (0)$$

$$= 0$$

$$\lim_{\substack{y \to 0 \\ x \to 0}} \frac{x^2 - x\sqrt{y}}{x^2+y} = \lim_{x \to 0} \frac{x^2}{x^2}$$

$$= \lim_{x \to 0} 1$$

$$= 1$$

$$\therefore \lim_{\substack{x \to 0 \\ y \to 0}} \frac{x^2 - x\sqrt{y}}{x^2+y} \neq \lim_{\substack{y \to 0 \\ x \to 0}} \frac{x^2 - x\sqrt{y}}{x^2+y}$$

$\therefore$ lim does not exist.

Continuous function:-

$$\lim_{\substack{x \to a \\ y \to b}} f(x,y) = f(a,b)$$

(Pb) Investigate the continuity of the function $f(x,y)$ at $(0,0)$ where $f(x,y) = \begin{cases} \frac{xy}{x^2+y^2}, & (x,y) \neq \\ 0, & (x,y) = \end{cases}$

Sol:- Given $f(x,y) = \begin{cases} \frac{xy}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$

$$\lim_{\substack{x \to 0 \\ y \to 0}} f(x,y) = \lim_{\substack{x \to 0 \\ y \to 0}} \frac{xy}{x^2+y^2}$$

$$= \lim_{y \to 0} \frac{0}{y^2}$$

$$= \lim_{y \to 0} 0$$

$$= 0 \longrightarrow (1)$$

$$\lim_{\substack{y \to 0 \\ x \to 0}} f(x,y) = \lim_{\substack{y \to 0 \\ x \to 0}} \frac{xy}{x^2+y^2}$$

$$= \lim_{x \to 0} \frac{0}{x^2}$$

$$= \lim_{x \to 0} 0$$

$$= 0 \longrightarrow (2)$$

Taking $y = mx$; $x \to 0$

$$\lim_{x \to 0} f(x,y) = \lim_{x \to 0} \frac{x \cdot mx}{x^2 + m^2 \cdot x^2}$$

$$= \lim_{x \to 0} \frac{\cancel{x^2} m}{\cancel{x^2}(1+m^2)}$$

$$= \frac{m}{(1+m^2)} \longrightarrow (3)$$

Equ (1), (2) and (3) are not unique.

$\therefore$ Given $f(x,y)$ is not continuous function.

Total derivative :- If u is a two variable (x, y) function, then

$$\boxed{du = \frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy}$$

① If $z = y + f(u)$, $u = \frac{x}{y}$, S.T $u\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 1$.

Sol:- Given $z = y + f(u) \rightarrow ①$ $\quad u = \frac{x}{y}$

P.d ① w.r.to 'x'

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}\left(y + f(u)\right)$$

$$= \frac{\partial}{\partial x}(y) + \frac{\partial}{\partial x} f(u)$$

$$= 0 + f'(u)\cdot\frac{\partial u}{\partial x}$$

$$= f'(u)\cdot\frac{\partial}{\partial x}\left(\frac{x}{y}\right)$$

$$= f'(u)\cdot\frac{1}{y}(1)$$

$$\frac{\partial z}{\partial x} = \frac{1}{y}f'(u) \rightarrow ②$$

P.d eqn ① w.r.to 'y'

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}\left[y + f(u)\right]$$

$$= \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial y} f(u)$$

$$= 1 + f'(u) \cdot \frac{\partial}{\partial y}(u)$$

$$= 1 + f'(u) \frac{\partial}{\partial y}\left(\frac{x}{y}\right)$$

$$\frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}$$

$$= 1 + f'(u) \cdot x \cdot \frac{\partial}{\partial y}(y^{-1})$$

$$= 1 + f'(u) \cdot x \cdot (-1) \cdot y^{-2}$$

$$= 1 + f'(u) \cdot x \cdot \frac{(-1)}{y^2}$$

$$\frac{\partial z}{\partial y} = 1 - \frac{x}{y^2} \cdot f'(u) \longrightarrow (3)$$

$$\therefore u \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = \frac{x}{y} \cdot \frac{1}{y} \cdot f'(u) + 1 - \frac{x}{y^2} \cdot f'(u)$$

$$= \cancel{\frac{x}{y^2} f'(u)} + 1 - \cancel{\frac{x}{y^2} f'(u)}$$

$$\therefore u \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 1$$

Jacobian :-

$$J\left(\frac{u,v}{x,y}\right) = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

(1) $J \cdot J' = 1$

$$\frac{\partial(u,v)}{\partial(x,y)} \cdot \frac{\partial(x,y)}{\partial(u,v)} = 1$$

(Pb) If $u = x^2 - 2y$, $v = x + y$, Find $\frac{\partial(u,v)}{\partial(x,y)}$

Sol:- Given $u = x^2 - 2y$, $v = x + y$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \rightarrow (1)$$

$\frac{\partial u}{\partial x} = 2x$, $\frac{\partial u}{\partial y} = -2$

$\frac{\partial v}{\partial x} = 1$, $\frac{\partial v}{\partial y} = 1$

$$\therefore \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 2x & -2 \\ 1 & 1 \end{vmatrix} = 2x + 2$$

Taylors theorem for two variables:-

$$f(x,y) = f(a,b) + \frac{1}{1!}\left[(x-a) f_x(a,b) + (y-b) f_y(a,b)\right] +$$

$$\frac{1}{2!}\left[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b) f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)\right] +$$

$$\frac{1}{3!}\left[(x-a)^3 f_{xxx}(a,b) + 3(x-a)^2(y-b) f_{xxy}(a,b) + 3(x-a)(y-b)^2 f_{xyy}(a,b) + (y-b)^3 f_{yyy}(a,b)\right] + \cdots$$

$f_x = \frac{\partial f}{\partial x}$
$f_{xx} = \frac{\partial^2 f}{\partial x^2}$

put $a = 0$, $b = 0$

$$f(x,y) = f(0,0) + \frac{1}{1!}\left[x f_x(0,0) + y f_y(0,0)\right] +$$
$$\frac{1}{2!}\left[x^2 f_{xx}(0,0) + 2xy f_{xy}(0,0) + y^2 f_{yy}(0,0)\right] +$$
$$\frac{1}{3!}\left[x^3 f_{xxx}(0,0) + 3x^2y f_{xxy}(0,0) + 3xy^2 f_{xyy}(0,0) + y^3 f_{yyy}(0,0)\right] + \cdots$$

$(x-1)$, $(y-2)$
$a=1$ $b=2$

① Expand $e^x \cdot \sin y$ in powers of x, y.

Sol:- Given $f(x,y) = e^x \cdot \sin y \longrightarrow ①$

By using Taylors series exp in powers of x, y

$$f(x,y) = f(0,0) + \frac{1}{1!}\left[x f_x(0,0) + y f_y(0,0)\right] +$$
$$\frac{1}{2!}\left[x^2 f_{xx}(0,0) + 2xy f_{xy}(0,0) + y^2 f_{yy}(0,0)\right] +$$
$$\frac{1}{3!}\left[x^3 f_{xxx}(0,0) + 3x^2y f_{xxy}(0,0) + 3xy^2 f_{xyy}(0,0) + y^3 f_{yyy}(0,0)\right]$$

$\downarrow$ ①

$f(x,y) = e^{x} \sin y \Rightarrow f(0,0) = e^{0} \cdot \sin 0 = 0$

$f_x = \frac{\partial f}{\partial x} = e^{x} \cdot \sin y \Rightarrow f_x(0,0) = e^{0} \cdot \sin 0 = 0$

$f_y = \frac{\partial f}{\partial y} = e^{x} \cdot \cos y \Rightarrow f_y(0,0) = e^{0} \cdot \cos 0 = 1 \cdot 1 = 1$

$f_{xx} = \frac{\partial^2 f}{\partial x^2} = e^{x} \cdot \sin y \Rightarrow f_{xx}(0,0) = e^{0} \cdot \sin 0 = 0$

$f_{xy} = \frac{\partial^2 f}{\partial x \partial y} = e^{x} \cos y \Rightarrow f_{xy}(0,0) = e^{0} \cdot \cos 0 = 1$

$f_{yy} = \frac{\partial^2 f}{\partial y^2} = -e^{x} \sin y \Rightarrow f_{yy}(0,0) = -e^{0} \cdot \sin 0 = 0$

$f_{xxx} = \frac{\partial^3 f}{\partial x^3} = e^{x} \cdot \sin y \Rightarrow f_{xxx}(0,0) = e^{0} \cdot \sin 0 = 0$

$f_{xxy} = \frac{\partial^3 f}{\partial x^2 \partial y} = e^{x} \cdot \cos y \Rightarrow f_{xxy}(0,0) = e^{0} \cdot \cos 0 = 1$

$f_{xyy} = \frac{\partial^3 f}{\partial x \partial y^2} = -e^{x} \sin y \Rightarrow f_{xyy}(0,0) = -e^{0} \sin 0 = 0$

$f_{yyy} = \frac{\partial^3 f}{\partial y^3} = -e^{x} \cdot \cos y \Rightarrow f_{yyy}(0,0) = -e^{0} \cdot \cos 0 = -1$

Sub these values in (1)

$\therefore f(x,y) = 0 + \frac{1}{1!}\left[x(0) + y(1)\right] + \frac{1}{2!}\left[x^2(0) + 2xy(1) + y^2(0)\right] +$

$\frac{1}{3!}\left[x^3(0) + 3x^2y(1) + 3xy^2(0) + y^3(-1)\right].$

$= y + \frac{2xy}{2} + \frac{3x^2y}{6} - \frac{y^3}{6}$

$\therefore f(x,y) = y + xy + \frac{x^2y}{2} - \frac{y^3}{6}$

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V.V.V.V IMP

maxima and minima :

Suppose $f(x,y)$ is given function

(1) To find $\frac{\partial f}{\partial x} = 0$, $\frac{\partial f}{\partial y} = 0$, solving these two equations we get (a_1, b_1), (a_2, b_2). are point

2) To find $l = \frac{\partial^2 f}{\partial x^2}$, $m = \frac{\partial^2 f}{\partial x \partial y}$. $n = \frac{\partial^2 f}{\partial y^2}$

$ln - m^2$

3) (i) If $ln - m^2 > 0$, $l > 0$, at the point (a, b_1) then minimum point and minimum value is $f(a_1, b_1)$.

(ii) If $ln - m^2 > 0$, and $l < 0$ at the point (a_1, b_1) then, this is maximum point. and maximum value is $f(a_1, b_1)$

(III) If $ln - m^2 \leq 0$ then there is no maxima and minima at the point (a_1, b_1)

(Pb) Find the maxima and minima of the fun $f(x,y) = x^3 + 3xy^2 - 3x^2 - 3y^2 + 4$.

Sol:- Given $f(x,y) = x^3 + 3xy^2 - 3x^2 - 3y^2 + 4 \rightarrow (1)$

p.d equ (1) w.r.to 'x' on bs

$$\frac{\partial f}{\partial x} = 3x^2 + 3y^2 - 6x \rightarrow (2)$$

p.d equ (1) w.r.to 'y' on b.s

$$\frac{\partial f}{\partial y} = 6xy - 6y \rightarrow (3)$$

$\frac{\partial f}{\partial x} = 0 \Rightarrow 3x^2 + 3y^2 - 6x = 0 \Rightarrow x^2 + y^2 - 2x = 0 \rightarrow (4)$

$\frac{\partial f}{\partial y} = 0 \Rightarrow 6xy - 6y = 0 \Rightarrow 6y(x-1) = 0 \rightarrow (5)$

$y = 0, \quad x = 1.$

Taking $y = 0$ in (4), $x^2 - 2x = 0$

$x(x-2) = 0$

$x = 0, \quad x = 2$

Takip $x = 1$ in (4) $1 + y^2 - 2 = 0$

$y^2 - 1 = 0 \Rightarrow y^2 = 1$

$\Rightarrow y = \pm 1$

$\therefore$ The points are $(1,1), (1,-1), (0,0), (2,0)$.

$$l = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$$

$$= \frac{\partial}{\partial x}(3x^2 + 3y^2 - 6x) = 6x - 6$$

$$m = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

$$= \frac{\partial}{\partial x}(6xy - 6y) = 6y$$

$$n = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}(6xy - 6y) = 6x - 6$$

$$ln - m^2 = (6x-6)(6x-6) - (6y)^2$$

$$ln - m^2 = (6x-6)^2 - 36y^2 \longrightarrow (6)$$

At the point $(1,1)$, $ln - m^2 = 0 - 36 = -36 < 0$

$\therefore$ There is no maxima and minima at $(1,1)$.

At the point $(1,-1)$, $ln - m^2 = 0 - 36 = -36 < 0$

$\therefore$ There is no maxima and minima at $(1,-1)$.

At the point $(0,0)$, $ln - m^2 = 36 > 0$

At the point $(0,0)$, $l = 6(0) - 6 = -6 < 0$

At the point $(0,0)$, $ln - m^2 > 0$ and $l < 0$.

$\therefore (0,0)$ is maximum point.

$\therefore$ The maximum value is $f(0,0) = 4$ [sub $x=0, y=0$ in (1)]

At the point $(2,0)$, $ln - m^2 = 36 > 0$.

At the point $(2,0)$, $l = 6(2) - 6 = 6 > 0$

$\therefore (2,0)$ is minimum point.

[$x=2, y=0$ in (1)]

$\therefore$ The minimum value is $f(2,0) = 8 + 0 - 12 + 4$

$= 0$

Lagrange's Method of Underteermine multipliers

$$F(x,y,z) = f(x,y,z) + \lambda\,\phi(x,y,z)$$

$\frac{\partial F}{\partial x} = 0 \Rightarrow \frac{\partial f}{\partial x} + \lambda \frac{\partial \phi}{\partial x} = 0$

$\frac{\partial F}{\partial y} = 0 \Rightarrow \frac{\partial f}{\partial y} + \lambda \frac{\partial \phi}{\partial y} = 0$

$\frac{\partial F}{\partial z} = 0 \Rightarrow \frac{\partial f}{\partial z} + \lambda \frac{\partial \phi}{\partial z} = 0$

Q) Find the maxima and minima for the function $x^2+y^2+z^2$, Subject to the condition $xyz = a^3$

Sol:- Let $f(x,y,z) = x^2+y^2+z^2 \rightarrow (1)$

$\phi(x,y,z) = xyz - a^3 \rightarrow (2)$

By the method of Lagranges term of Undetermine multipliers,

$$F(x,y,z) = f(x,y,z) + \lambda\,\phi(x,y,z)$$

$$F(x,y,z) = (x^2+y^2+z^2) + \lambda(xyz - a^3) \rightarrow (3)$$

P. d equ (3) w.r.to 'x' on b.s

$\frac{\partial F}{\partial x} = 0 \Rightarrow 2x + \lambda yz = 0$

$2x = -\lambda yz$

$\frac{2x}{yz} = -\lambda \rightarrow (4)$

P. d equ (3) w.r.to 'y' on b.s

$\frac{\partial F}{\partial y} = 0 \Rightarrow 2y + \lambda xz = 0$

$2y = -\lambda xz \Rightarrow \frac{2y}{xz} = -\lambda \rightarrow (5)$

P. d equ (3) w.r.to 'z' on b.s

$\frac{\partial F}{\partial z} = 0 \Rightarrow 2z + \lambda xy = 0 \Rightarrow 2z = -\lambda xy$

$\Rightarrow \frac{2z}{xy} = -\lambda \rightarrow (6)$

from (4), (5), (6)

$$\frac{2x}{yz} = \frac{2y}{xz} = \frac{2z}{xy} \longrightarrow (7)$$

Taking first and second Term $\rightarrow$ (7)

$$\frac{\cancel{2}x}{y\cancel{z}} = \frac{\cancel{2}y}{x\cancel{z}} \Rightarrow \frac{x}{y} = \frac{y}{x}$$

$$\Rightarrow x^2 = y^2 \Rightarrow x = y \rightarrow (8)$$

Taking second and Third term $\rightarrow$ (7)

$$\frac{\cancel{2}y}{\cancel{x}z} = \frac{\cancel{2}z}{\cancel{x}y} \Rightarrow \frac{y}{z} = \frac{z}{y}$$

$$\Rightarrow y^2 = z^2$$

$$\Rightarrow y = z \longrightarrow (9)$$

from (8), and (9), $x = y = z \longrightarrow (10)$

sub $y = x$, $z = x$ in (2)

$$xyz = a^3$$

$$x \cdot x \cdot x = a^3 \Rightarrow x^3 = a^3 \Rightarrow x = a.$$

Sub $x = a$ in (10)

$y = a$

$z = a$

Sub x, y, z in (1)

$$\therefore f(x,y,z) = a^2 + a^2 + a^2$$

$$= 3a^2$$ is minimum