UNIT-4

Multivariable Calucluss

Double Integrals

① Evaluate $\int_0^3 \int_1^2 xy\,(x+y)\,dx\,dy$

Sol:

$$\int_0^3 \int_1^2 xy\,(x+y)\,dx\,dy$$

$$= \int_{y=0}^{3} \int_{x=1}^{2} (x^2y + xy^2)\,dx\,dy$$

$$= \int_{y=0}^{3} \left(\frac{x^3}{3}y + \frac{x^2y^2}{2}\right)_1^2 dy$$

$$= \int_{y=0}^{3} \left[\left(\frac{8}{3}y + \frac{4}{2}\cdot y^2\right) - \left(\frac{y}{3} + \frac{y^2}{2}\right)\right] dy$$

$$= \int_{y=0}^{3} \left(\frac{8y}{3} + 2y^2 - \frac{y}{3} - \frac{y^2}{2}\right) dy$$

$$= \int_{y=0}^{3} \left(\frac{7}{3}y + \frac{3y^2}{2}\right) dy$$

$$= \left(\frac{7}{3}\cdot\frac{y^2}{2} + \cancel{3}\cdot\frac{y^3}{\cancel{3}}\right)_0^3$$

$$= \frac{7}{\cancel{6}_2}\cancel{9}^3 + 27$$

$$= \frac{21}{2} + 27$$

$$= \frac{21+54}{2} = \frac{75}{2}$$

Double Integrals in polar form.

(1) Evaluate $\int_0^{\pi} \int_0^{a\sin\theta} r \, dr \, d\theta$

Sol:- Given $\int_0^{\pi} \int_0^{a\sin\theta} r . dr . d\theta$

$$= \int_{\theta=0}^{\pi} \int_{r=0}^{a\sin\theta} . r . dr . d\theta$$

$$= \int_{\theta=0}^{\pi} \left(\frac{r^2}{2}\right)_0^{a\sin\theta} . d\theta$$

$$= \frac{1}{2} \int_{\theta=0}^{\pi} (a^2 . \sin^2\theta .) \, d\theta$$

$$\boxed{\begin{array}{l} \sin^2\theta = \frac{1}{2}(1-\cos 2\theta) \\ \cos^2\theta = \frac{1}{2}(1+\cos 2\theta) \end{array}}$$

$$= \frac{a^2}{2} \int_{\theta=0}^{\pi} \sin^2\theta \, d\theta$$

$$= \frac{a^2}{2} \int_{\theta=0}^{\pi} \frac{1}{2}(1-\cos 2\theta) \, d\theta$$

$$= \frac{a^2}{4} \left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi}$$

$$= \frac{a^2}{4} \left[\left(\pi - \frac{\sin 2\pi}{2}\right) - \left(0 - \frac{\sin 0}{2}\right)\right]$$

$$= \frac{a^2\pi}{4} = \frac{\pi a^2}{4} \,//$$

Change of order of integration :-

16) Change of order of integration. evaluate $\int_0^{4a}\int_{x^2/4a}^{2\sqrt{ax}} dy\, dx$.

Sol:- Given $\int_{x=0}^{4a}\int_{y=x^2/4a}^{2\sqrt{ax}} dy\, dx \longrightarrow (1)$

Here $x=0$ and $x=4a$

$y=\frac{x^2}{4a}$ and $y=2\sqrt{ax}$.

Take $x=0$ in y then $y=0$

Take $x=4a$ in y then $y=4a$

$\therefore$ y varies from 0 to $4a$.

we have $y=\frac{x^2}{4a} \Rightarrow x^2=4ay \Rightarrow x=2\sqrt{ay}$

$y=2\sqrt{ax} \Rightarrow y^2=4ax \Rightarrow x=\frac{y^2}{4a}$

$\therefore$ x varies from $\frac{y^2}{4a}$ to $2\sqrt{ay}$

$$\therefore \int_0^{4a}\int_{x^2/4a}^{2\sqrt{ax}} dy\, dx = \int_{y=0}^{4a}\int_{x=\frac{y^2}{4a}}^{2\sqrt{ay}} 1.\, dy\, dx$$

$$= \int_{y=0}^{4a} (x)_{\frac{y^2}{4a}}^{2\sqrt{ay}} .\, dy$$

$$= \int_{y=0}^{4a} \left(2\sqrt{ay}-\frac{y^2}{4a}\right) dy$$

$$= \int_0^{4a} \left(2\sqrt{a}\,.\, y^{\frac{1}{2}} - \frac{y^2}{4a}\right) dy$$

$$= \left(2\sqrt{a}\,.\, \frac{y^{\frac{3}{2}}}{\frac{3}{2}} - \frac{y^3}{12a}\right)_0^{4a}$$

$$= 2\sqrt{a}\,.\,\frac{2}{3}\,.\,(4a)^{\frac{3}{2}} - \frac{(4a)^3}{12\,.\,a}$$

$$(4)^{3/2} = (2^2)^{3/2} = 2^3$$

$$= \frac{4}{3}\sqrt{a}\,.\,(4)^{3/2}\,.\,(a)^{3/2} - \frac{\cancel{64}^{16}\,.\,a^{\cancel{3}\,2}}{\cancel{12}_{3}\,.\,a}$$

$$= \frac{4}{3}\sqrt{a}\,.\,8\,.\,a\sqrt{a} - \frac{16a^2}{3}$$

$$= \frac{32}{3}a^2 - \frac{16a^2}{3}$$

$$= \frac{16a^2}{3}$$

Triple integrals :-

① Evaluate $\int_0^1 dx \int_0^2 dy \int_1^2 x^2yz\, dz$

Sol:- Given $\int_0^1 dx \int_0^2 dy \int_1^2 x^2yz\, dz$

$$= \int_{x=0}^{1} dx \int_{y=0}^{2} dy\, (x^2y)\left(\frac{z^2}{2}\right)_1^2$$

$$= \int_{x=0}^{1} dx \int_{y=0}^{2} (x^2y).\, dy \left(\frac{2^2}{2} - \frac{1}{2}\right)$$

$$= \frac{3}{2}\int_{x=0}^{1} dx \int_0^2 (x^2)\, y\, dy$$

$$= \frac{3}{2}\int_0^1 x^2\, dx \left(\frac{y^2}{2}\right)_0^2$$

$$= \frac{3}{2}\left[\frac{x^3}{3}\right]_0^1 \left[\frac{4}{2} - 0\right]$$

$$= \frac{3}{2} \cdot \frac{1}{2}\left[\frac{1}{3} - 0\right]$$

$$= \frac{\cancel{3}}{2} \cdot \frac{1}{2} \cdot \frac{1}{\cancel{3}}$$

$$= \frac{1}{4}.$$

Volume as a Double Integrals :-

$$\text{Volume} = \iint z\,dx\,dy$$

pb) using double integration, find the volume of the tetrahedron bounded by the coordinate planes and the plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$.

Sol:- Given $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \longrightarrow (1)$

$$\text{Volume} = \iint z\,dx\,dy \longrightarrow (2)$$

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

$$\frac{z}{c} = 1 - \frac{x}{a} - \frac{y}{b} \Rightarrow z = c\left(1 - \frac{x}{a} - \frac{y}{b}\right)$$

On x-axis $y=0$, $z=0$, $\frac{x}{a} = 1 \Rightarrow x = a$

on xy plane, $z=0$, $\frac{x}{a} + \frac{y}{b} = 1$

$$\frac{y}{b} = 1 - \frac{x}{a} \Rightarrow y = b\left(1 - \frac{x}{a}\right)$$

$\therefore$ x varies from 0 to a

y varies from 0 to $b\left(1 - \frac{x}{a}\right)$

$$V = \iint z\,dx\,dy = \int_{x=0}^{a} \int_{y=0}^{b(1-x/a)} c\left(1 - \frac{x}{a} - \frac{y}{b}\right) dx\,dy$$

$$= c \int_{x=0}^{a} \left(y - \frac{x}{a}\cdot y - \frac{y^2}{2b}\right)_0^{b\left(1-\frac{x}{a}\right)} dx$$

$$= c \int_{x=0}^{a} \left[b - \frac{bx}{a} - \frac{x}{a}\left(b - \frac{bx}{a}\right) - \frac{1}{2b} b^2\left(1 - \frac{x}{a}\right)^2\right] dx$$

pb) Find the volume common to the Cylinders $x^2+y^2=a^2$ and $x^2+z^2=a^2$.

soln

Given $x^2+y^2=a^2$,

$x^2+z^2=a^2$

volume as double Integrals $= \iint z\,dx\,dy$ — (1)

$x^2+z^2=a^2 \Rightarrow z^2=a^2-x^2$

$\Rightarrow z=\pm\sqrt{a^2-x^2}$

$x^2=a^2$

$x=\pm\sqrt{a^2}$

$=\pm a$

on x-axis, $y=0$, $x^2=a^2$

$x=\pm a$

$\therefore$ x varies $-a$ to a

$x^2+y^2=a^2 \Rightarrow y^2=a^2-x^2 \Rightarrow y=\pm\sqrt{a^2-x^2}$

$\therefore$ y varies from $-\sqrt{a^2-x^2}$ to $\sqrt{a^2-x^2}$

$$\therefore \iint z\,dx\,dy = \int_{x=-a}^{a}\int_{y=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \sqrt{a^2-x^2}\,dy\,dx$$

$$=2\int_{x=0}^{a} 2\int_{y=0}^{\sqrt{a^2-x^2}} \sqrt{a^2-x^2}\cdot 1\,dy\,dx$$

$$=4\int_{x=0}^{a}\sqrt{a^2-x^2}\,(y)_0^{\sqrt{a^2-x^2}}\,dx$$

$$=4\int_{x=0}^{a}(\sqrt{a^2-x^2})(\sqrt{a^2-x^2})\,dx$$

$$=4\int_{x=0}^{a}(a^2-x^2)\,dx$$

$$= 4\left(a^2 \cdot x - \frac{x^3}{3}\right)_0^a$$

$$= 4\left(a^2 \cdot a - \frac{a^3}{3}\right)$$

$$= 4\left(a^3 - \frac{a^3}{3}\right)$$

$$= 4\left(\frac{3a^3 - a^3}{3}\right)$$

$$= \frac{4}{3} \cdot 2a^3$$

$$= \frac{8a^3}{3}.$$