UNIT-5

Vector Caluclus

gradient :-

$\nabla = \text{grad} = \bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}$

$\nabla f = \text{grad } f = \bar{i}\frac{\partial f}{\partial x} + \bar{j}\frac{\partial f}{\partial y} + \bar{k}\frac{\partial f}{\partial z}$

unit normal vector :-

The unit normal vector to f is $\frac{\nabla f}{|\nabla f|}$.

Directional derivative :-

The Directional derivative of the function f in the direction vector $\bar{a}$ is $\frac{\bar{a}\cdot\nabla f}{|\bar{a}|}$.

Angle between two surfaces :- (n_1 and n_2)

The angle between normals to the two surfaces is $\cos\theta = \frac{(\bar{n}_1)\cdot(\bar{n}_2)}{|n_1|\,|n_2|}$.

$\bar{i}\cdot\bar{i} = 1$

$\bar{j}\cdot\bar{j} = 1$

$\bar{k}\cdot\bar{k} = 1$

$i\cdot j = 0$

$j\cdot i = 0$

$j\cdot k = 0$

$k\cdot i = 0$

$i\cdot k = 0$

(1) Find grad ϕ where $\phi = 3x^2y - y^3z^2$ at $(1, -2, -1)$.

Sol:- Given $\phi = 3x^2y - y^3z^2$, given $(1, -2, -1)$.

$$\text{grad}\,\phi = i\frac{\partial \phi}{\partial x} + j\frac{\partial \phi}{\partial y} + \bar{k}\frac{\partial \phi}{\partial z}$$

$$= i\frac{\partial}{\partial x}(3x^2y - y^3z^2) + j\frac{\partial}{\partial y}(3x^2y - y^3z^2) + \bar{k}\frac{\partial}{\partial z}(3x^2y - y^3z^2)$$

$$\text{grad}\,\phi = \bar{i}(6xy) + \bar{j}(3x^2 - 3y^2z^2) + \bar{k}(-2y^3z)$$

$$\therefore (\text{grad}\,\phi)_{(1,-2,-1)} = \bar{i}\; 6(1)(-2) + \bar{j}\,(3 - 3.(4)(1)) + \bar{k}\,(-2(-8)(-1))$$

$$\therefore (\text{grad}\,\phi)_{1,-2,-1} = -12\bar{i} - 9\bar{j} - 16\bar{k}$$

(Pb) Find grad ϕ where $\phi = x^2 + yz$ at $(1,2,3)$

Sol:- Given $\phi = x^2 + yz$, given $(1,2,3)$:

$$\text{grad}\,\phi = \bar{i}\frac{d\phi}{dx} + \bar{j}\frac{d\phi}{dy} + \bar{k}\frac{d\phi}{dz}$$

$$= \bar{i}\frac{d}{dx}(x^2+yz) + \bar{j}\frac{d}{dy}(x^2+yz) + \bar{k}\frac{d}{dz}(x^2+yz)$$

$$\text{grad}\,\phi = \bar{i}(2x) + \bar{j}(z) + \bar{k}(y)$$

$$\therefore (\text{grad}\,\phi)_{(1,2,3)} = \bar{i}\,2(1) + \bar{j}(3) + \bar{k}(2)$$

$$\therefore (\text{grad}\,\phi)_{(1,2,3)} = 2\bar{i} + 3\bar{j} + 2\bar{k}$$

(Pb) let $f = x^2y + 2xz - 4$, given $(2, -2, 3)$

$$\nabla f = \bar{i}\frac{\partial f}{\partial x} + \bar{j}\frac{\partial f}{\partial y} + \bar{k}\frac{\partial f}{\partial z}$$

$$\nabla f = \bar{i}\frac{\partial}{\partial x}(x^2y + 2xz - 4) + \bar{j}\frac{\partial}{\partial y}(x^2y + 2xz - 4) + \bar{k}\frac{\partial}{\partial z}(x^2y + 2xz -$$

$$\nabla f = \bar{i}(2xy + 2z) + \bar{j}(x^2) + \bar{k}(2x)$$

$$\nabla f_{(2,-2,3)} = \bar{i}(2(-4) + 2(3)) + \bar{j}(4) + \bar{k}(2(2))$$

$$\nabla f_{(2,-2,3)} = \bar{i}(-2) + 4\bar{j} + 4\bar{k}$$

$$\nabla f = -2\bar{i} + 4\bar{j} + 4\bar{k}$$

$$|\nabla f| = \sqrt{(-2)^2 + (4)^2 + (4)^2}$$

$$= \sqrt{4 + 16 + 16}$$

$$= \sqrt{36}$$

$$= \cancel{\sqrt{9 \times 4}}$$

$$= 6$$

The unit normal vector of f is $\frac{\nabla f}{|\nabla f|}$

$$= \frac{-2\bar{i}+4\bar{j}+4\bar{k}}{6}$$

$$= \frac{\cancel{2}(-\bar{i}+2\bar{j}+2\bar{k})}{\cancel{6}\,3}$$

$$= \frac{-\bar{i}+2\bar{j}+2\bar{k}}{3}$$

P6) Find the directional derivative of $f(x,y) = x^2+y^2$ at $(1,1)$ in the direction of $2\hat{i}-4\hat{j}$.

Sol:- Given $f(x,y) = x^2+y^2$ given $(1,1)$

let $\bar{a} = 2\hat{i}-4\bar{j} \Rightarrow |\bar{a}| = \sqrt{2^2+(-4)^2} = \sqrt{20}$

The direction derivative of f in the direction vector $\bar{a} = \frac{\bar{a}\cdot\nabla f}{|\bar{a}|}$

$$\nabla f = \hat{i}\frac{\partial f}{\partial x} + \bar{j}\frac{\partial f}{\partial y} + \bar{k}\frac{\partial f}{\partial z}$$

$$= \hat{i}\frac{\partial}{\partial x}(x^2+y^2) + \bar{j}\frac{\partial}{\partial y}(x^2+y^2) + \bar{k}\frac{\partial}{\partial z}(x^2+y^2)$$

$$= \hat{i}(2x) + \bar{j}(2y) + 0$$

$$\therefore (\nabla f)_{(1,1)} = 2\hat{i}+2\bar{j}$$

$(2)(-4)$

$$\bar{a}\cdot\nabla f = (2\hat{i}-4\bar{j})\cdot(2\hat{i}+2\bar{j})$$

$$= 4-8 = -4$$

$\therefore$ The directional derivative $\Rightarrow \frac{-4}{\sqrt{20}}$

$$\Rightarrow \frac{-4}{2\sqrt{5}} \Rightarrow \frac{-2}{\sqrt{5}}$$

P6) Find the directional derivative of xyz^2+xz at $(1,1,1)$ in a direction of the normal to the surface $3xy^2+y=z$ at $(0,1,1)$.

Sol:- let $f = xyz^2 + xz$, given point $(1,1,1)$

let $\phi = 3xy^2 + y - z$, $(0,1,1)$

$$\nabla\phi = \bar{i}\frac{\partial}{\partial x}\phi + \bar{j}\frac{\partial}{\partial y}\phi + \bar{k}\frac{\partial}{\partial z}\phi$$

$$= \bar{i}\frac{\partial}{\partial x}(3xy^2+y-z) + \bar{j}\frac{\partial}{\partial y}(3xy^2+y-z) + \bar{k}\frac{\partial}{\partial z}(3xy^2+y-z)$$

$$= \bar{i}(3y^2) + \bar{j}(6xy+1) + \bar{k}(-1)$$

$$(\nabla\phi)_{(0,1,1)} = 3\bar{i} + \bar{j} - \bar{k}$$

let $\bar{a} = 3\bar{i} + \bar{j} - \bar{k}$

$$|\bar{a}| = \sqrt{(3)^2+(1)^2+(-1)^2} = \sqrt{11}$$

The directional derivative of f in the directional vector $\bar{a}$ $= \dfrac{\bar{a}\cdot\nabla f}{|\bar{a}|}$

$$\nabla f = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)(xyz^2+xz)$$

$$= \bar{i}(yz^2+z) + \bar{j}(xz^2) + \bar{k}(2xyz+x)$$

$$(\nabla f)_{(1,1,1)} = 2\bar{i} + \bar{j} + 3\bar{k}$$

$\bar{a} \cdot \nabla f = (3i+j-k)\cdot(2\bar{i}+\bar{j}+3\bar{k})$

$= 6+1-3 = 4$

$\therefore$ The directional derivative $\Rightarrow \frac{4}{\sqrt{11}}$

p6) Find the directional derivative of the function $f = x^2-y^2+2z^2$ at the point $P(1,2,3)$ in the direction of the line $\overrightarrow{PQ}$ where Q is the point $(5,0,4)$.

sol:- Given $f = x^2-y^2+2z^2$, given $P(1,2,3)$

and $Q = (5,0,4)$

$\overrightarrow{PQ} = OQ - OP$

$= (5i+0\bar{j}+4\bar{k}) - (i+2\bar{j}+3\bar{k})$

$\bar{a} = \overrightarrow{PQ} = 4\bar{i}-2\bar{j}+\bar{k}$

$|\bar{a}| = \sqrt{16+4+1} = \sqrt{21}$

The directional derivative of f in the direction of the vector $\bar{a} = \frac{\bar{a}\cdot\nabla f}{|\bar{a}|}$

$\nabla f = \left(i\frac{\partial}{\partial x}+\bar{j}\frac{\partial}{\partial y}+\bar{k}\frac{\partial}{\partial z}\right)(x^2-y^2+2z^2)$

$= i(2x) + \bar{j}(-2y) + \bar{k}(4z)$

$\therefore (\nabla f)_{(1,2,3)} = 2i-4\bar{j}+12\bar{k}$

$\bar{a}\cdot\nabla f = (4i-2\bar{j}+\bar{k})\cdot(2i-4j+12\bar{k}) = 8+8+12 = 28$

$\therefore$ The directional derivative $= \frac{28}{\sqrt{21}}$

Divergence of a vector point function :-

$\vec{F} = f_1\vec{i} + f_2\vec{j} + f_3\vec{k}$

$\vec{i}\cdot\vec{i} = 1$
$\vec{j}\cdot\vec{j} = 1$
$\vec{k}\cdot\vec{k} = 1$

$$\nabla\cdot F = \operatorname{div}\bar{F} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

Solenoidal vector :-

If $\operatorname{div}\bar{F} = 0$, then $\bar{F}$ is said to be solenoidal vector.

Curl of a vector point function :-

$\nabla = \bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}$

If $\bar{F} = f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$,

$$\nabla\times\bar{F} = \operatorname{curl}\bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

Irrotational vector :- (conservative field) :-

If $\operatorname{curl}\bar{F} = 0$, then $\bar{F}$ is said to Irrotational vector.

(1) If $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$, then S.T (i) $\operatorname{div}\bar{r} = 3$
(ii) $\operatorname{curl}\bar{r} = 0$

sol:- Given $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

$r_1 = x, \quad r_2 = y, \quad r_3 = z$

$$\operatorname{div}\bar{r} = \frac{\partial}{\partial x}(r_1) + \frac{\partial}{\partial y}(r_2) + \frac{\partial}{\partial z}(r_3)$$

$$= \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z)$$

$$= 1 + 1 + 1$$

$$= 3$$

$r_1 = x, \quad r_2 = y, \quad r_3 = z$

$$\text{curl}\,\bar{r} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ r_1 & r_2 & r_3 \end{vmatrix}$$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix}$$

$$= i\left(\frac{\partial}{\partial y}(z) - \frac{\partial}{\partial z}(y)\right) - \bar{j}\left(\frac{\partial}{\partial x}(z) - \frac{\partial}{\partial z}(x)\right) + \bar{k}\left(\frac{\partial}{\partial x}(y) - \frac{\partial}{\partial y}\right.$$

$$= i\,\bar{0} + \bar{j}(0) + \bar{k}(0)$$

$\therefore \text{Curl}\,\bar{r} = 0.$

(3) Find the divergence and curl of the vector $\vec{V} = (xyz)\bar{i} + (3x^2y)\bar{j} + (xz^2 - y^2z)\bar{k}$, at the point $(2, -1, 1)$

Sol:- Given $\vec{V} = (xyz)\bar{i} + (3x^2y)\bar{j} + (xz^2 - y^2z)\bar{k}$

Given point $(2, -1, 1)$

$$\text{div}\,\vec{V} = \frac{\partial}{\partial x}(V_1) + \frac{\partial}{\partial y}(V_2) + \frac{\partial}{\partial z}(V_3)$$

$$V_1 = xyz \,;\; V_2 = 3x^2y \,;\; V_3 = xz^2 - y^2z$$

$$\text{div}\,\vec{V} = \frac{\partial}{\partial x}(xyz) + \frac{\partial}{\partial y}(3x^2y) + \frac{\partial}{\partial z}(xz^2 - y^2z)$$

$$\text{div}\,\vec{V} = yz + 3x^2 + 2xz - y^2$$

$$\text{div}\,\vec{V}_{(2,-1,1)} = -1 + 12 + 4 - 1$$

$$\text{div}\,\vec{V}_{(2,-1,1)} = 14$$

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$v_1 = xyz, \; v_2 = 3x^2y, \; v_3 = xz^2 - y^2z$

$$\text{curl } \vec{v} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xyz & 3x^2y & xz^2 - y^2z \end{vmatrix}$$

$$= i\left[\frac{\partial}{\partial y}(xz^2 - y^2z) - \frac{\partial}{\partial z}(3x^2y)\right] - j\left[\frac{\partial}{\partial x}(xz^2 - y^2z) - \frac{\partial}{\partial z}(xyz)\right]$$

$$+ k\left[\frac{\partial}{\partial x}(3x^2y) - \frac{\partial}{\partial y}(xyz)\right]$$

$$= i[-2yz - 0] - j[z^2 - xy] + k[6xy - xz]$$

$$\text{curl } \vec{v}(2,-1,1) = i(-2(-1)(1)) - j(1+2) + k[-12-2]$$

$$\text{curl } \vec{v}_{(2,-1,1)} = 2\bar{i} - 3\bar{j} - 14\bar{k}$$

If $\bar{A}$ is irrotational Vector then there exist a Scalar potential f such that

$$\boxed{\bar{A} = \text{grad } \bar{f}}$$

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}$$

Total deri

Pb S.T $\bar{V} = 12x\bar{i} - 15y^2\bar{j} + \bar{k}$ is irrotational and find the scalar function $f(x, y, z)$ suchthat $\vec{V} = \text{grad} f$.

Sol:- Given $\vec{V} = 12x\bar{i} - 15y^2\bar{j} + \bar{k}$

$V_1 = 12x, \quad V_2 = -15y^2, \quad V_3 = 1$

$$\text{curl}\,\vec{V} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix}$$

$$= \begin{vmatrix} i & j & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 12x & -15y^2 & 1 \end{vmatrix}$$

$$= i\,[0-0] - \bar{j}\,(0-0) + \bar{k}\,(0-0)$$

$\therefore \text{curl}\,\vec{V} = 0$

$\therefore \vec{V}$ is an irrotational vector.

Let F is the scalar potential function

$$\vec{V} = \text{grad}\, f$$

$$i\frac{\partial f}{\partial x} + \bar{j}\frac{\partial f}{\partial y} + \bar{k}\frac{\partial f}{\partial z} = 12x\bar{i} - 15y^2\bar{j} + \bar{k}$$

$$\frac{\partial f}{\partial x} = 12x$$

$$\frac{\partial f}{\partial y} = -15y^2$$

$$\frac{\partial f}{\partial z} = 1$$

$\therefore$ we have $df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$

$$df = 12x\,dx - 15y^2dy + 1\,dz$$

Integrating on b.s

$$\int 1\,df = \int 12x\,dx - 15\int y^2dy + \int 1\,dz$$

$$f = 12\frac{x^2}{2} + 15\frac{y^3}{3} + z$$

$\therefore f(x, y, z) = 6x^2 - 5y^3 + z$

PB) A vector field is given by $\bar{A} = (x^2 + xy^2)\bar{i} + (y^2 + x^2y)\bar{j}$. Shows that the field is irrotational and find the scalar potential.

Sol:- Given $\bar{A} = (x^2 + xy^2)\bar{i} + (y^2 + x^2y)\bar{j}$

$A_1 = x^2 + xy^2, \quad A_2 = y^2 + x^2y, \quad A_3 = 0$

$$\text{curl } \bar{A} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix}$$

$$= \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2+xy^2 & y^2+x^2y & 0 \end{vmatrix}$$

$$= \bar{i}(0-0) - \bar{j}(0-0) + \bar{k}(2xy - 2xy)$$

$$= 0$$

$\therefore \text{curl } \vec{A} = 0$

$\therefore \vec{A}$ is an irrotational vector.

If f is a scalar potential function Then

$$\vec{A} = \text{grad } f$$

$$\bar{i}\frac{\partial f}{\partial x} + \bar{j}\frac{\partial f}{\partial y} + \bar{k}\frac{\partial f}{\partial z} = (x^2 + xy^2)\bar{i} + (y^2 + x^2y)\bar{j} + 0\bar{k}$$

$$\frac{\partial f}{\partial x} = x^2 + xy^2$$

$$\frac{\partial f}{\partial y} = y^2 + x^2y$$

$$\frac{\partial f}{\partial z} = 0.$$

we have $\quad df = \frac{\partial f}{\partial x} \cdot dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} \cdot dz$

$$df = (x^2 + xy^2)\, dx + (y^2 + x^2 y)\, dy + 0\, dz$$

Integrating on b.s

$$\int 1\, df = \int (x^2 + xy^2)\, dx + \int (y^2 + x^2 y)\, dy$$

$$\int 1\, dx = x$$

$$\int x^0\, dx = \frac{x^{0+1}}{0+1} = \frac{x^1}{1} = x$$

$$f = \frac{x^3}{3} + \frac{x^2 y^2}{2} + \frac{y^3}{3} + \frac{x^2 \cdot y^2}{2}$$

$$\therefore \; f(x, y, z) = \frac{x^3}{3} + \frac{y^3}{3} + x^2 y^2.$$

Pb. S.T $\operatorname{div}(r^n \bar{r}) = (n+3)r^n$ (or)

P.T $r^n \bar{r}$ is Solenoidal if $n = -3$

Sol:- Let $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

$$|r| = \sqrt{x^2+y^2+z^2}$$

$$r^2 = |r|^2 = x^2+y^2+z^2 \rightarrow (1)$$

p.d equ (1) w.r.to 'x'

$$2r.\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$$

p.d equ (1) w.r.to 'y'

$$2r\frac{\partial r}{\partial y} = 2y \Rightarrow \frac{\partial r}{\partial y} = \frac{y}{r}$$

p.d equ (1) w.r. to 'z' on b.s

$$2r.\frac{\partial r}{\partial z} = 2z \Rightarrow \frac{\partial r}{\partial z} = \frac{z}{r}$$

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mentar.

$r^n \bar{r} = r^n (x\bar{i} + y\bar{j} + z\bar{k})$

$$r^n \bar{r} = \underbrace{r^n x}_{r_1} \bar{i} + \underbrace{r^n y}_{r_2} \bar{j} + \underbrace{r^n z}_{r_3} \bar{k}$$

$$\operatorname{div}(r^n \bar{r}) = \frac{\partial}{\partial x}(r^n x) + \frac{\partial}{\partial y}(r^n y) + \frac{\partial}{\partial z}(r^n z)$$

$$= \underline{r^n}(1) + x \cdot n r^{n-1} \cdot \frac{\partial r}{\partial x} + \underline{r^n(1)} + y \cdot n \cdot r^{n-1} \cdot \frac{\partial r}{\partial y} +$$

$$\underline{r^n(1)} + z \cdot n \cdot r^{n-1} \cdot \frac{\partial r}{\partial z}$$

$$= 3r^n + x \cdot n r^{n-1} \cdot \frac{x}{r} + y \cdot n r^{n-1} \cdot \frac{y}{r} + z \cdot n r^{n-1} \cdot \frac{z}{r}$$

$$= 3r^n + x^2 \cdot \underline{n \cdot r^{n-2}} + y^2 \underline{n\, r^{n-2}} + z^2 \underline{n\, r^{n-2}}$$

$$= 3r^n + n r^{n-2} \cdot (x^2 + y^2 + z^2)$$

$$= 3r^n + n r^{n-2} \cdot r^2$$

$$= 3r^n + n r^n$$

$$\operatorname{div}(r^n \bar{r}) = (3+n) r^n$$

put $n = -3$ in above equation,

$\operatorname{div}(r^n \bar{r}) = 0$.

$\therefore r^n \bar{r}$ is solenoidal if $n = -3$.

(P3) Find curl $(r^n \bar{r})$.

Sol:- let $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

$|\bar{r}| = \sqrt{x^2+y^2+z^2}$

$r^2 = x^2+y^2+z^2$

$2r \cdot \frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}$

$r^n \bar{r} \Rightarrow r^n(xi + y\bar{j} + z\bar{k})$

$\Rightarrow r^n x\,\bar{i} + r^n y\,\bar{j} + r^n z\,\bar{k}$

$$\text{curl}(r^n\bar{r}) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ r^n x & r^n y & r^n z \end{vmatrix}$$

$$= i\left[z n \cdot r^{n-1} \cdot \frac{\partial r}{\partial y} - y \cdot n \cdot r^{n-1} \cdot \frac{\partial r}{\partial z}\right] -$$

$$\bar{j}\left[z \cdot n r^{n-1} \frac{\partial r}{\partial x} - x \cdot n \cdot r^{n-1} \frac{\partial r}{\partial z}\right] +$$

$$\bar{k}\left[y \cdot n \cdot r^{n-1} \frac{\partial r}{\partial x} - x \cdot n \cdot r^{n-1} \cdot \frac{\partial r}{\partial y}\right]$$

$$= i\left[z \cdot n r^{n-1} \cdot \frac{y}{r} - y \cdot n r^{n-1} \cdot \frac{z}{r}\right] -$$

$$\bar{j}\left[z \cdot n r^{n-1} \cdot \frac{x}{r} - x n r^{n-1} \frac{z}{r}\right] +$$

$$\bar{k}\left[y\, n \cdot r^{n-1} \frac{x}{r} - x \cdot n\, r^{n-1} \frac{y}{r}\right]$$

$$= i\left[n y z r^{n-2} - n y z r^{n-2}\right] - \bar{j}\left[x z \cdot n r^{n-2} - x z n r^{n-2}\right] + \bar{k}\left[x y n r^{n-2} - x y n r^{n-2}\right]$$

$\therefore \text{curl}(r^n \bar{r}) = 0$

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Work done by a Force :-

Work done by a force $\int_A^B \bar{F}\cdot d\vec{r}$

① If $\bar{F} = 3xy\bar{i} - y^2\bar{j}$, evaluate $\int_C \bar{F}\cdot d\vec{r}$, where c is the arc of the parabola $y = 2x^2$ from (0,0) to (1,2)

Sol:- Given $\bar{F} = 3xy\bar{i} - y^2\bar{j}$.

Given $y = 2x^2$. given (0,0) to (1,2)

let $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

$d\bar{r} = dx\,\bar{i} + dy\,\bar{j} + dz\,\bar{k}$

$\bar{F}\cdot d\bar{r} = (3xy\bar{i} - y^2\bar{j})\cdot(dx\,\bar{i} + dy\,\bar{j} + dz\,\bar{k})$

$\bar{F}\cdot dr = 3xy\,dx - y^2dy$

Given $y = 2x^2 \Rightarrow dy = 4x\,dx$

$\therefore x$ varies from 0 to 1.

$$\therefore \int_C \bar{F}\cdot d\bar{r} = \int_{x=0}^{1} 3xy\,dx - y^2\,dy$$

$$= \int_{x=0}^{1} 3x(2x^2)\,dx - (2x^2)^2\cdot 4x\,dx$$

$$= \int_{x=0}^{1} 6x^3\,dx - 16x^5\,dx$$

$$= \left(6\frac{x^4}{4} - 16\frac{x^6}{6}\right)_0^1$$

$$= \frac{3}{2} - \frac{8}{3}$$

$$= \frac{9-16}{6} = \frac{-7}{6}$$

(16) Find the work done by the force $\bar{F} = (3x^2 - 6yz)\bar{i} + (2y + 3xz)\bar{j} + (1 - 4xyz^2)\bar{k}$ in moving particle from the point $(0,0,0)$ to the $(1,1,1)$ along the curve $C: x = t,\ y = t^2,\ z = t^3$.

Sol:- Given $\bar{F} = (3x^2 - 6yz)\bar{i} + (2y + 3xz)\bar{j} + (1 - 4xyz^2)\bar{k}$

let $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

$d\bar{r} = dx\,\bar{i} + dy\,\bar{j} + dz\,\bar{k}$

$\bar{F} \cdot d\bar{r} = (3x^2 - 6yz)\,dx + (2y + 3xz)\,dy + (1 - 4xyz^2)\,dz$

$x = t \Rightarrow dx = dt$

$y = t^2 \Rightarrow dy = 2t\,dt$

$z = t^3 \Rightarrow dz = 3t^2\,dt$

At the point $(0,0,0)$, $t = 0$

At the point $(1,1,1)$, $t = 1$

$\therefore$ t varies from 0 to 1.

The work done by a force $\int_C \bar{F} \cdot d\bar{r}$

$$= \int_{t=0}^{1} (3x^2 - 6yz)\,dx + (2y + 3xz)\,dy + (1 - 4xyz^2)\,dz$$

$$= \int_{t=0}^{1} (3t^2 - 6t^5)\,dt + (2t^2 + 3t^4)\,2t\,dt + (1 - 4t^9)\,3t^2\,dt$$

$$= \int_{t=0}^{1} \left(3t^2 - \cancel{6t^5} + 4t^3 + \cancel{6t^5} + 3t^2 - 12t^{11}\right) dt$$

$$= \int_{t=0}^{1} (6t^2 + 4t^3 - 12t^4)\, dt$$

$$= \left(\cancel{6}^{2} \frac{t^3}{\cancel{3}} + \cancel{4} \frac{t^4}{\cancel{4}} - \cancel{12} \frac{t^{12}}{\cancel{12}} \right)_0^1$$

$$= 2(1)^3 + 1(1)^4 - 1(1)^{12}$$

$$= 2 + 1 - 1 - 0$$

$$\therefore \int_c \bar{F} \cdot d\bar{r} = 2$$

V.V.V.IMP

Gauss divergence theorem :-

If $\bar{F}$ is a vector point function having continuous first order partial derivatives in the region V bounded by a closed surface S

Then $\iiint_V \operatorname{div} \bar{F}\, dv = \iint_S \bar{F} \cdot \bar{n}\, ds$, where $\bar{n}$ is the outward drawn unit normal vector to the surface S.

(Pb) Verify divergence theorem for $\bar{F} = x^2 \bar{i} + z\bar{j} + yz\bar{k}$ taken over the cube bounded by $x=0$, $x=1$, $y=0$, $y=1$, $z=0$ and $z=1$.

Sol:- Given $\bar{F} = x^2 \bar{i} + z\bar{j} + yz\bar{k}$

given $x=0$, $x=1$, $y=0$, $y=1$, $z=0$ and $z=1$

using Gauss divergence theorem, we have

$$\iiint_V \operatorname{div} \bar{F}\, dv = \iint_S \bar{F} \cdot \bar{n}\, ds \longrightarrow (1)$$

$f_1 = x^2$, $f_2 = z$, $f_3 = yz$

$$\operatorname{div} \bar{F} = \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(z) + \frac{\partial}{\partial z}(yz)$$

$$= 2x + 0 + y$$

$$\operatorname{div} \bar{F} = 2x + y$$

$$\therefore \iiint_V \operatorname{div}\bar{F}\, dv = \int_{x=0}^{1}\int_{y=0}^{1}\int_{z=0}^{1} (2x+y)\, dx\, dy\, dz$$

$$= \int_{x=0}^{1}\int_{y=0}^{1} (\cancel{2xz+yz})(2x+y)\int 1\, dz \cdot dx\, dy$$

$$= \int_{x=0}^{1}\int_{y=0}^{1} (2x+y)\,(z)_0^1\, dx\, dy \qquad (2xz+yz)_0^1$$

$$= \int_{x=0}^{1}\int_{y=0}^{1} (2x+y)\cdot(1-0)\cdot dy\, dx$$

$$= \int_{x=0}^{1} \left(2xy + \frac{y^2}{2}\right)_{y=0}^{1} dx$$

$$= \int_{x=0}^{1} \left[\left(2x + \frac{1}{2}\right) - (0-0)\right] dx$$

$$= \int_{x=0}^{1} \left(2x + \frac{1}{2}\right) dx$$

$$= \left(\cancel{2}\frac{x^2}{\cancel{2}} + \frac{1}{2}x\right)_0^1$$

$$\int_V \operatorname{div}\bar{F}\, dv = 1 + \frac{1}{2} = \frac{3}{2} \longrightarrow (2)$$

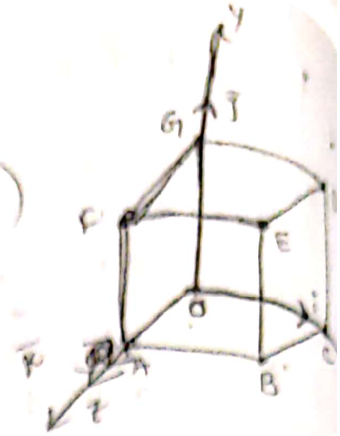
consider

$$\iint_S \bar{F}\cdot\bar{n}\, ds = \iint_{OABC} \bar{F}\cdot\bar{n}\, ds + \iint_{OCDG} \bar{F}\cdot\bar{n}\, ds + \iint_{BCDE} \bar{F}\cdot\bar{n}\, ds +$$

$$\iint_{DEFG} \bar{F}\cdot\bar{n}\, ds + \iint_{OAFG} \bar{F}\cdot\bar{n}\, ds + \iint_{ABEF} \bar{F}\cdot\bar{n}\, ds \longrightarrow (3)$$

$F = x^2 i + z j + yz k$

Case (i) :- On the Face OABC



$i \cdot i = 1$
$j \cdot j = 1$
$k \cdot k = 1$

$\bar{n} = -\bar{j}$

$\bar{F} \cdot \bar{n} = 0$

$\therefore \iint_{OABC} \bar{F} \cdot \bar{n}\, dS = 0 \longrightarrow (4)$

Case (ii) :- on the Face OCDG

$\bar{n} = -\bar{k}$

$\therefore \bar{F} \cdot \bar{n} = 0$

$\therefore \iint_{OCDG} \bar{F} \cdot \bar{n}\, dS = 0 \longrightarrow (5)$

Case (iii) :- On the Face BCDE

$\bar{n} = \bar{i}$, $x = 1$

$\bar{F} \cdot \bar{n} = (x^2 \bar{i} + z \bar{j} + yz \bar{k}) \cdot (\bar{i})$

$= x^2$

$\bar{F} \cdot \bar{n} = 1$

$dS = dy\, dz$

$y = 0$ to 1 , $z = 0$ to 1

$\therefore \iint_{BCDE} \bar{F} \cdot \bar{n}\, dS = \int_{y=0}^{1} \int_{z=0}^{1} 1 \, dy\, dz$

$= \int_{y=0}^{1} (z)_0^1 \, dy$

$= \int_{y=0}^{1} (1-0)\, dy$

$$= (y)_0^1$$

$$= 1-0$$

$$\iint_{BCDE} \bar{F}\cdot\bar{n}\, ds = 1 \longrightarrow (6)$$

Case (iv) :- on the face DEFG

$$\bar{n} = \bar{j}, \quad y = 1$$

$$\bar{F}\cdot\bar{n} = (x^2\bar{i} + z\bar{j} + yz\bar{k})\cdot(\bar{j})$$

$$\bar{F}\cdot\bar{n} = z$$

$$ds = dx\, dz$$

$x = 0$ and $x = 1$, $\therefore z = 0$ to 1

$$\therefore \iint_{DEFG} \bar{F}\cdot\bar{n}\, ds = \int_{x=0}^{1}\int_{z=0}^{1} z\, dz\, dx$$

$$= \int_{x=0}^{1} \left(\frac{z^2}{2}\right)_0^1 dx$$

$$= \int_{x=0}^{1} \left(\frac{1}{2} - 0\right) dx$$

$$= \frac{1}{2}\int_0^1 1\, dx$$

$$= \frac{1}{2}(x)_0^1$$

$$= \frac{1}{2}(1-0)$$

$$\iint_{DEFG} \bar{F}\cdot\bar{n}\, ds = \frac{1}{2} \longrightarrow (7)$$

Case (v) :- on the face OAFG

$$n = -\bar{i}$$

$$\bar{F}\cdot\bar{n} = 0$$

$$\therefore \iint_{OAFG} \bar{F}\cdot\bar{n}\, dS = 0 \longrightarrow (8)$$

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Case (vi): On the Face ABEF

$\bar{n} = \bar{k}$, $z=1$

$\bar{F}\cdot\bar{n} = (x^2 i + zj + yz k)\cdot(\bar{k})$

$\bar{F}\cdot\bar{n} = yz \Rightarrow \bar{F}\cdot\bar{n} = y$

$ds = dx\,dy$

$x=0,\quad x=1,\qquad y=0,\ y=1$

$$\therefore \iint_{ABEF} \bar{F}\cdot\bar{n}\,ds = \int_{x=0}^{1}\int_{y=0}^{1} y\,.\,dx\,dy$$

$$= \int_{x=0}^{1} \left(\frac{y^2}{2}\right)_0^1 . dx$$

$$= \int_{x=0}^{1} \left(\frac{1}{2}\right) dx$$

$$= \frac{1}{2}(x)_0^1$$

$$= \frac{1}{2}(1-0)$$

$$= \frac{1}{2} \longrightarrow (8)$$

Sub (3), (4), (5), (6), (7), (8) in (2)

$$\therefore \iint_S \bar{F}\cdot\bar{n}\,ds = 0+0+1+\frac{1}{2}+0+\frac{1}{2}$$

$$= 2 \longrightarrow (9)$$

$$\iiint_V \operatorname{div}\bar{F}\,dv = \frac{3}{2}$$

$$\iint_S \bar{F}\cdot\bar{n}\,ds = 2$$

$$\therefore \iiint_V \operatorname{div}\bar{F}\,dv \neq \iint_S \bar{F}\cdot\bar{n}\,ds$$

$\therefore$ Gauss divergence theorem not verified.

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Green's theorem :-

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

$\cancel{M}$ $\quad M$

$\frac{\partial}{\partial x}$ $\quad \frac{\partial}{\partial y}$

(P5) using Green's theorem, $\int_C (2xy - x^2)\,dx + (x^2+y^2)\,dy$ where c is a closed curve of the region bounded by $y = x^2$ and $y^2 = x$

Sol:- Given $\int_C (2xy - x^2)\,dx + (x^2+y^2)\,dy$

Given $y = x^2$ and $y^2 = x$.

By using Green's theorem

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy \rightarrow (1)$$

$M = 2xy - x^2$

$N = x^2 + y^2$

$\frac{\partial M}{\partial y} = 2x$

$\frac{\partial N}{\partial x} = 2x$

- $y = x^2$ and $y^2 = x \Rightarrow y = \sqrt{x}$

$\therefore$ y varies from x^2 to $\sqrt{x}$,

Given $x = y^2$

$x = (x^2)^2 \Rightarrow x = x^4$

$x^4 - x = 0 \Rightarrow x(x^3 - 1) = 0$

$x = 0$ and $x = 1$.

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$\therefore \oint_C M\,dx + N\,dy = \iint \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

$\therefore \oint_C (2xy - x^2)\,dx + (x^2+y^2)\,dy = \int_{x=0}^{1} \int_{y=x^2}^{\sqrt{x}} (2x - 2x)\,dx\,dy$

$= \int \int 0 \,dx\,dy$

$= 0.$

p6) Using green's theorem, $\int_C (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy$ where c is the region bounded by $x=0,\ y=0,\ x+y=1$.

Sol:- Given $\int_C (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy$

Given $x=0, \quad y=0, \quad x+y=1$

on x-axis, $y=0, \quad x=1$

$x+y=1 \Rightarrow y = 1-x$

$\therefore x$ varies from 0 to 1

y varies from 0 to $1-x$

$M = 3x^2 - 8y^2, \quad N = 4y - 6xy$

$\frac{\partial M}{\partial y} = -16y, \quad \frac{\partial N}{\partial x} = -6y$

By using Green's theorem,

$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

$= \int_{x=0}^{1} \int_{y=0}^{1-x} (-6y + 16y)\,dx\,dy$

$= \int_{x=0}^{1} \int_{y=0}^{1-x} 10y\,dy\,dx$

$= 10 \int_{x=0}^{1} \left(\frac{y^2}{2}\right)_0^{1-x} dx$

$$= 5 \int_{x=0}^{1} (1-x)^2 \cdot dx$$

$$= 5 \int_{x=0}^{1} (1+x^2-2x)\, dx$$

$$= 5 \cdot \left[x + \frac{x^3}{3} - \cancel{2}\frac{x^2}{\cancel{2}} \right]_0^1$$

$$= 5 \left[\cancel{1} + \frac{1}{3} - \cancel{1} \right]$$

$$= \frac{5}{3}$$

Pb) Verify Green's theorem in the plane for $\oint_C (x^2 - xy^3)\, dx + (y^2 - 2xy)\, dy$ where c is a square with vertices $(0,0), (2,0), (2,2), (0,2)$.

Sol:- Given $\int_C (x^2 - xy^3)\, dx + (y^2 - 2xy)\, dy \rightarrow (1)$

Given $(0,0), (2,0), (2,2)$ and $(0,2)$.

using Green them, we have

$$\oint_C M\, dx + N\, dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\, dy \rightarrow (2)$$

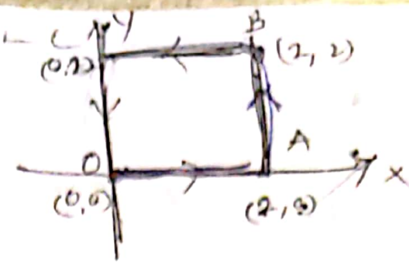
$$M = x^2 - xy^3, \quad N = y^2 - 2xy$$

$$\frac{\partial M}{\partial y} = -3xy^2, \quad \frac{\partial N}{\partial x} = -2y$$

x varia from 0 to 2

y varia from 0 to 2

consider $\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy =$



$$\int_{x=0}^{2} \int_{y=0}^{2} (-2y + 3xy^2)\, dx\,dy$$

$$= \int_{x=0}^{2} \left(-2\frac{y^2}{2} + 3x\frac{y^3}{3}\right)_0^2 dx$$

$$= \int_{x=0}^{2} (-4 + 8x)\, dx$$

$$= \left(-4x + 8\frac{x^2}{2}\right)_0^2$$

$$= -4(2) + 4(4)$$

$$= 16 - 8$$

$$= 8$$

$$\therefore \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = 8 \longrightarrow (3)$$

Consider $\int_C M\,dx + N\,dy = \int_{OA} M\,dx + N\,dy + \int_{AB} M\,dx + N\,dy +$

$\int_{BC} M\,dx + N\,dy + \int_{CO} M\,dx + N\,dy \longrightarrow (4)$

Case (i) :- Along the path OA

The points are (0,0) and (2,0)

Here $y = 0 \Rightarrow dy = 0$

x varies from 0 to 2

$$\int_{OA} M\,dx + N\,dy = \int_{x=0}^{2} x^2\,dx = \left(\frac{x^3}{3}\right)_0^2 = \frac{8}{3} \rightarrow (5)$$

Case (ii) :- Along the curve AB.

The point are $(2, 0)$ and $(2, 2)$

$$x = 2 \Rightarrow dx = 0$$

y varies from 0 to 2.

$$\therefore \int_{AB} M\,dx + N\,dy = \int_{y=0}^{2} (y^2 - 4y)\,dy$$

$$= \left[\frac{y^3}{3} - 4\frac{y^2}{2} \right]_0^2$$

$$= \frac{8}{3} - 8 \rightarrow (6)$$

Case (iii) :- Along the curve BC.

The points are $(2, 2)$, $(0, 2)$

Here $y = 2 \Rightarrow dy = 0$

x varies from 2 to 0

$$\therefore \int_{BC} M\,dx + N\,dy = \int_{x=2}^{0} (x^2 - 8x)\,dx$$

$$= \left(\frac{x^3}{3} - 8\frac{x^2}{2} \right)_2^0$$

$$= 0 - \left[\frac{8}{3} - 16 \right]$$

$$= 16 - \frac{8}{3} \rightarrow (7)$$

Case (iv) :- Along the curve CO

The points $(0, 2)$ and $(0, 0)$

$$x = 0 \Rightarrow dx = 0$$

y varies from 2 to 0.

$$\int_{C_0} M\,dx + N\,dy = \int_{y=2}^{0} y^2\,dy$$

$$= \left(\frac{y^3}{3}\right)_2^0$$

$$= 0 - \left(\frac{8}{3}\right)$$

$$= -\frac{8}{3} \longrightarrow (8)$$

sub (5), (6), (7) and (8) in (4)

$$\therefore \int_C M\,dx + N\,dy = \cancel{\frac{8}{3}} + \cancel{\frac{8}{3}} - 8 + 16 - \cancel{\frac{8}{3}} - \cancel{\frac{8}{3}}$$

$$= 16 - 8$$

$$\int_C M\,dx + N\,dy = 8 \longrightarrow (9)$$

From (3)

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = 8$$

$$\therefore \int_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

$\therefore$ Green's thm verified.

30/11/19

* **Stoke's Theorem**:- Let 's' be a open $\overline{S}$ surface bounded by closed curve 'c', If $\bar{F}$ is differentiable vector point function then

$$\oint_c \bar{F} \cdot d\bar{r} = \int_S \text{curl}\, \bar{F} \cdot \bar{n} \cdot ds$$ when 'c' is travels in the positive direction

where, $\bar{n}$ is unit outward normal at any point on the surface.

1. verify stoke's Theorem for $\bar{F} = (x^2 - y^2)\bar{i} + 2xy\bar{j}$ over the box bounded by the planes $x=0, x=a, y=0, y=b$

Sol:- Given $\bar{F} = (x^2-y^2)\bar{i} + 2xy\bar{j}$

Given points are $x=0, x=a, y=0, y=b$

By using stoke's Theorem we have

$$\oint_c \bar{F} \cdot d\bar{r} = \int_S \text{curl}\, \bar{F} \cdot \bar{n}\, ds \longrightarrow (1)$$

Here $f_1 = x^2 - y^2$, $f_2 = 2xy$, $f_3 = 0$

$$\text{curl}\, \bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix} = \begin{vmatrix} \overset{+}{\bar{i}} & \overset{-}{\bar{j}} & \overset{+}{\bar{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2-y^2 & 2xy & 0 \end{vmatrix}$$

$$= \bar{i}\left[\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}(2xy)\right] - \bar{j}\left[\frac{\partial}{\partial x}(0) - \frac{\partial}{\partial z}(x^2-y^2)\right] + \bar{k}\left[\frac{\partial}{\partial x}(\ldots\right.$$

$$\left. - \frac{\partial}{\partial y}(x^2-y^2)\right]$$

$$= \bar{i}(0) - \bar{j}(0) + \bar{k}(2y-(-2y))$$

$$= \bar{k}(4y)$$

$$= 4y\bar{k}$$

$\text{curl}\,\bar{F} = 4y\bar{k}$

Consider $\int_S \text{curl}\,\bar{F}\cdot\bar{n}\,ds = \iint_S (4y\bar{k})\cdot\bar{k}(dx\,dy)$

$$= \int_{x=0}^{a}\int_{y=0}^{b}(4y\,dx\,dy)$$

$$= \int_{x=0}^{a}\left(4\cdot\frac{y^2}{2}\right)_0^b dx$$

$$= \int_{x=0}^{a} 2(b^2-0)\,dx$$

$$= 2b^2\int_{x=0}^{a} 1\,dx$$

$$= 2b^2(a-0)$$

$$= 2b^2a$$

$$\therefore \int_S \text{curl}\,\bar{F}\cdot\bar{n}\,ds = 2ab^2 \rightarrow (2)$$

$F = (x^2-y^2)\bar{i} + 2xy\bar{j}$

$x=0, x=a, y=0, y=b$

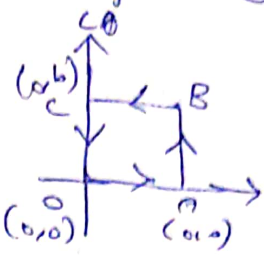
$\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

$d\bar{r} = dx\bar{i} + dy\bar{j} + dz\bar{k}$

r to ay

$\bar{F}\cdot d\bar{r} = (x^2-y^2)\,dx + 2xy\,dy$

Consider $\oint_C \bar{F}\cdot dr = \int_{OA} \bar{F}\cdot dr + \int_{AB} \bar{F}\cdot dr + \int_{BC} \bar{F}\cdot d\bar{r} + \int \bar{F}\cdot dr \rightarrow (3)$



Case i):- Along the path OA

The points are $(0,0),(a,0)$

Here $y=0 \Rightarrow dy=0$

x varies from 0 to a

$$\int_{OA} \bar{F}\cdot d\bar{r} = \int_{x=0}^{a} x^2\,dx = \left(\frac{x^3}{3}\right)_0^a = \frac{a^3}{3} - 0 = \frac{a^3}{3} \rightarrow (4)$$

Case ii):- Along the path AB

The ~~path~~ points $(a,0)$ to (a,b)

Here $x=a \Rightarrow dx=0$

y varies from 0 to b

$$\int_{AB} \bar{F}\cdot dr = \int_{y=0}^{b} 2ay\,dy = 2a\int y\,dy$$

$$= 2a\left(\frac{y^2}{2}\right)_0^b$$

$$= a(b^2-0)$$

$$= ab^2 \rightarrow (5)$$

Case iii) Along the path BC

The points are (a,b) to $(0,b)$

Here $y=b \Rightarrow dy=0$

x varies from a to 0

$$\int_{BC} \bar{F}\cdot d\bar{r} = \int_{x=a} (x^2-b^2)\,dx$$

$$= \left(\frac{x^3}{3} - b^2x\right)_a^0 = (0-0) - \left(\frac{a^3}{3} - b^2a\right)$$

$$= -\frac{a^3}{3} + ab^2 \rightarrow (6)$$

Case iv) :- Along the path co

The point $(0,b)$, $(0,0)$

Here $x=0 \Rightarrow dx=0$

y varies from b to c.

$$\int_{co} \bar{F}\cdot d\bar{r} = \int_{y=b}^{0} 0\, dy = 0 \longrightarrow (7)$$

sub (4), (5), (6), (7) in (3)

$$\oint_C \bar{F}\cdot d\bar{r} = \cancel{\frac{a^3}{3}} + ab^2 - \cancel{\frac{a^3}{3}} + ab^2 + 0$$

$$= 2ab^2 \longrightarrow (8)$$

$\therefore$ LHS = RHS.

$\therefore$ stokes theorem verified.