Chapter 5

Introduction:

Classification of PDEs is an important concept because the general theory and methods of solution usually apply only to a given class of equations.

In addition to the distinction between linear and nonlinear PDEs, it is important for the computational scientist to know that there are different classes of PDEs. Just as different solution techniques are called for in the linear versus the nonlinear case, different numerical methods are required for the different classes of PDEs, whether the PDE is linear or nonlinear. The need for this specialization in numerical approach is rooted in the physics from which the different classes of PDEs arise.

By analogy with conic sections (ellipse, parabola and hyperbola) partial differential equations have been classified as elliptic, parabolic and hyperbolic.

Just as an ellipse is a smooth, rounded object, solutions to elliptic equations tend to be quite smooth. Elliptic equations generally arise from a physical problem that involves a diffusion process that has reached equilibrium, a steady state temperature distribution, for example. The hyperbola is the disconnected conic section. By analogy, hyperbolic equations are able to support solutions with discontinuities, for example a shock wave. Hyperbolic PDEs usually arise in connection with mechanical oscillators, such as a vibrating string, or in convection driven transport problems.

Mathematically, parabolic PDEs serve as a transition from the hyperbolic PDEs to the elliptic PDEs. Physically, parabolic PDEs tend to arise in time dependent diffusion problems, such as the transient flow of heat in accordance with Fourier's law of heat conduction.

5.1.1 CLASSIFICATIOC WITH TWO INDEPENDENT VARIABLES

Consider the following general second order linear PDE in two independent variables:

$$A\frac{\partial^2 u}{\partial x^2}+B\frac{\partial^2 u}{\partial x\partial y}+C\frac{\partial^2 u}{\partial y^2}+D\frac{\partial u}{\partial x}+E\frac{\partial u}{\partial y}+Fu+G=0, \qquad (1)$$

where A, B, C, D, E F and G are functions of the independent variables **x** and **y.** The equation (1) may be written in the form

$$Au_{xx}+Bu_{xy}+Cu_{yy}+f(x,y,u_x,u_y,u)=0 \qquad (2)$$

where

$$u_x=\frac{\partial u}{\partial x},\ u_y=\frac{\partial u}{\partial y},\ u_{xx}=\frac{\partial^2 u}{\partial x^2},\ u_{xy}=\frac{\partial^2 u}{\partial x\partial y},\ \frac{\partial^2 u}{\partial y^2}$$

Assume that *A*,*B* and *C* are continuous functions of *x* and *y* possessiong continuous partial derivative of as high order as necessary.

5.1.2 LINEAR PDE WITH CONSTANT COEFFICIENTS

Let us first consider the following general linear second order PDE in two independent variables x and y with constants coefficients:

$$Au_{xx}+Bu_{xy}+Cu_{yy}+Du_x+Eu_y+Fu+G=0 \qquad (3)$$

where the coefficients A, B, C, D, E, F and G are constants. The nature of the equation (3) is determined by the principal part containing highest partial derivatives i.e.,

$$Lu\equiv Au_{xx}+Bu_{xy}+Cu_{yy} \qquad (4)$$

For classification, we attach a symbol to (5) as $P(x, y) = Ax^2 + Bxy + Cy^2$ (as if we have replaced x by $\frac{\partial}{\partial x}$ and y by $\frac{\partial}{\partial y}$). Now depending on the sign of the discriminant $(B^2 - 4AC)$, the classifiacation of (4) is done as follows:

Eq. (3) is hyperbolic ...(5)

$B^2 - 4AC = 0 \Rightarrow$ Eq. (3) is parabolic ...(6)

$B^2 - 2AC < 0 \Rightarrow$ Eq. (3) is elliptic ...(7)

5.1.3 Linear PDE with variable coefficients : The above classification of (3) is still valid if the coefficients A, B, C, D, E and F depend on x, y**.** In this case, the conditions (5), (6) and (7) should be satisfied at each point (x, y) in the region where we want to describe its nature e.d., for elliptic we need to verify

$$B^2(x, y) - 4A(x, y)C(x, y) < 0$$

for each (x, y) in the region of interest. Thus we classify linear PDE with variable coefficients as follows:

$B^2(x, y) - 4A(x, y)C(x, y) > 0$ at $(x, y) \Rightarrow$ Eq. (3) is hyperbolic at (x, y) ...(8)

$B^2(x, y) - 4A(x, y)C(x, y) = 0$ at $(x, y) \Rightarrow$ Eq. (3) parabolic at (x, y) ...(9)

$B^2(x, y) - 4A(x, y)C(x, y) < 0$ at $(x, y) \Rightarrow$ Eq. (3) elliptic at (x, y) ...(10)

Note: Eq. (3) is hyperbolic, parabolic or elliptic depends only on the coeeficients of the second derivatives. It has nothing to do with the first-derivative terms, the term in u, or the nonhomogeneous term.

Solved Problems

Problem 1: Classify the partial differential equation

$$\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x \partial y} + 4\frac{\partial^2 u}{\partial y^2} = 0$$

Solution : This equation is in the form of of

$$AU_{xx} + BU_{xy} + CU_{yy} = 0$$

where **A =** 1, B = 2**, C =** 4

Now $B^2 - 4AC = (2)^2 - 4(1)(4)$

$= 4 - 16$

$= -12 < 0$

So the given equation is an elliptic.

Problem 2 : Classify the partial differential equation

$$3U_{xx} + 4U_{xy} + 6U_{yy} - 2U_x + U_y - 4 = 0$$

Solution : This equation is in the form of of

$$AU_{xx} + BU_{xy} + CU_{yy} = 0$$

where **A =** 3, B = 4**, C =** 6

Now $$B^2-4AC=(4)^2-4(3)(6)$$
$$=16-72$$
$$=-56<0$$
So the given equation is an elliptic.

Problem 3: Classify $24U_{xx}-3U_{xy}-U_{yy}=0$

Solution : This equation is in the form of of

$$AU_{xx}+BU_{xy}+CU_{yy}=0$$

where **A =** 2, B = -3**, C =** -1

Now $$B^2-4AC=(-3)^2-4(2)(-1)$$
$$=9+8$$
$$=17>0$$
So the given equation is Hyperbolic.

Problem 4: Classify $U_{xx}+5U_{xy}+3U_{yy}=0$

Solution : This equation is in the form of of

$$AU_{xx}+BU_{xy}+CU_{yy}=0$$

where **A =** 1, B = 5**, C =** 3

Now $$B^2-4AC=(5)^2-4(1)(3)$$
$$=25-12$$
$$=13>0$$
So the given equation is Hyperbolic.

Problem 5: Classify $f_{xx}+2f_{xy}+f_{yy}=0$

Solution : This equation is in the form of of

$$Af_{xx}+Bf_{xy}+Cf_{yy}=0$$

where **A =** 1, B = 2**, C =** 3

Now $$B^2-4AC=(2)^2-4(1)(1)$$
$$=4-4=0$$
So the given equation is parabolic.

Problem 6: Classify $xU_{xy}+yU_{yy}=0$

Solution : This equation is in the form of of

$$A(x,y)U_{xx}+B(x,y)U_{xy}+C(x,y)U_{yy}=0$$

where **A =** 0, B = x**, C =** y

Now $$B^2-4AC=x^2-4(0)(y)$$
$$=x^2$$

Here $B^2-4AC>0\ \forall\ x\neq 0$ so the the given equation is Hyperbolic at $x\neq 0$**.**

and $B^2-4AC=0$ at x = 0 so the given equation is parabolic at x = 0.

Problem 7: Classify $xU_{xy}+U_{yy}=0$

Solution : This equation is in the form of of

$$A(x,y)U_{xx} + B(x,y)U_{xy} + C(x,y)U_{yy} = 0$$

where **A =** x, B = 0**,** C =1

Now $B^2 - 4AC = 0 - 4(x)(1)$

$$= -4x$$

If $x > 0 \Rightarrow B^2 - 4AC < 0$ so the equation is elliptic

If $x = 0 \Rightarrow B^2 - 4AC = 0$ so the equation is parabolic

If $x < 0 \Rightarrow B^2 - 4AC > 0$ so the equation is Hyperbolic.

Problem 8: **Classify** $U_{xy} - xU_{yy} = \frac{1}{2x} U_x$ **(x > 0)**

Solution : The given equation can be re-write as

$$2xU_{xx} - 2x^2(x,y)U_{xy} + U_x = 0 \qquad \textbf{(x > 0)}$$

This in the form of

$$A(x,y)U_{xx} + B(x,y)U_{xy} + C(x,y)U_{yy} + f(x,y,U_x,U_y,u) = 0$$

where A = 2x. B = $-2x^2$, c = 0

Now $B^2 - 4AC = (-2x^2)^2 - 4(2x)(0)$

$$= 4x^4$$

Since x > 0 so $B^2 - 4AC = 4x^4 > 0$

Hence given equation is hyperbolic.

Problem 9: **Classify** $x^2U_{xx} + 4xyU_{xy} + (x^2 + 4y^2)U_{yy} = \sin(x+y)$

Solution : The given equation can be written as

$$x^2U_{xx} + 4xyU_{xy} + (x^2 + 4y^2)U_{yy} - \sin(x+y) = 0$$

This in the form of

$$A(x,y)U_{xx} + B(x,y)U_{xy} + C(x,y)U_{yy} + f(x,y,U_x,U_y,U) = 0$$

where A = x^2 . B = 4x y, c = $x^2 + 4y^2$

Now $B^2 - 4AC = (4xy)^2 - 4x^2(x^2 + 4y^2)$

$$= 16x^2y^2 - 4x^4 - 16x^2y^2$$

$$= -4x^4$$

If $x \neq 0$ $B^2 - 4AC < 0$ so the given equation is elliptic at $x \neq 0$

If $x = 0 \Rightarrow B^2 - 4AC = 0$ so the given equation is parabolic at x = 0.

Problem 10: **Classify the PDE**

$$x^2U_{xx} + 2xyU_{xy} + y^2U_{yy} = 0$$

Solution : The equation is in the form of

$$AU_{xx} + BU_{xy} + CU_{yy} = 0$$

where A = x^2 . B = 2x y, c = y^2

Now $B^2 - 4AC = (2xy)^2 - 4(x^2)(y^2)$

$$= 4x^2y^2 - 4x^2y^2 = 0$$

So the equation is parabolic.

Problem 11: Classify the partial differential equation

$$5U_{xx} - 3U_{xy} + (\cos x)U_x + e^y U_y + u = 0$$

Solution : This equation is in the form of

$$AU_{xx} + BU_{xy} + CU_{yy} + f(x, y, U_x, U_y, u) = 0$$

where A = 5. B = 0, c = -3

Now $B^2 - 4AC = 0 - 4(5)(-3)$

$$= 60 > 0$$

So the given equation is Hyperbolic.

Problem 12: Classify $\sin^2 xU_{xx} + \sin 2xyU_{xy} + \cos^2 xU_{yy} = x$

Solution : The given equation can be re-write as

$$\sin^2 xU_{xx} + \sin 2xyU_{xy} + \cos^2 xU_{yy} - x = 0$$

This in the form of

$$AU_{xx} + BU_{xy} + CU_{yy} + f(x, y, U_x, U_y, u) = 0$$

where A = $\sin^2 x$, B = $\sin 2x$, c = $\cos^2 x$

Now $B^2 - 4AC = (\sin 2x)^2 - 4(\sin^2 x)(\cos^2 x)$

$$= (\sin 2x)^2 - (2\sin x \cos x)^2 \qquad [\because \sin 2\theta = 2\sin\theta\cos\theta]$$

$$= (\sin 2x)^2 - (\sin 2x)^2 = 0$$

So the given equation is parabolic.

Problem 13: Classify $e^x U_{xx} + e^y U_{xy} - 4 = 0$

Solution : This equation is in the form of

$$\sin^2 xU_{xx} + \sin 2xyU_{xy} + \cos^2 xU_{yy} - x = 0$$

This in the form of

$$A(x,y)U_{xx} + B(x,y)U_{xy} + C(x,y)U_{yy} + f(x,y,U_x,U_y,U) = 0$$

where A = e^x, B = 0, c = e^y

Now $B^2 - 4AC = 0 - 4e^x.e^y$

$$= -4e^{x+y}$$

Since e^{x+y} gives non-zero positive values of x and y in R

$$\therefore \quad B^2 - 4AC = -4e^{x+y} < 0$$

So the given equation is elliptic.

Exercise-1

1. Classify the following partial differential equation

(i) $\frac{\partial^2 x}{\partial x^2}+2\frac{\partial^2 x}{\partial x\partial y}+\frac{\partial^2 x}{\partial y^2}=0$ (ii) $\frac{\partial^2 x}{\partial x^2}=5\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}=0$

(iii) $\frac{\partial^2 x}{\partial x^2}+3\frac{\partial^2 x}{\partial x\partial y}+\frac{\partial^2 x}{\partial y^2}=0$. (iv) $\frac{\partial^2 x}{\partial x^2}+\frac{\partial^2 x}{\partial y^2}=\frac{\partial u}{\partial x}$

(v) $U_{xx}-2U_{xy}+U_{xy}+3U_x-4U_y=3x-2y$ (vi) $U_{xx}+4U_{xy}+(x^2+4y^2)U_{yy}=\sin(x+y)$

2. Show that the equation:

$z_{xx}+2xz_{xy}+(1-y^3)z_{yy}=0$ is elliptic for all values of x, y in the region $x^2+y^2<1$, parabolic on the boundary and hyperbolic outside the region.

Answers

I. (i) Parabolic (ii) Parabolic (iii) Hyperbolic

(iv) Elliptic (v) Parabolic

(vi) Elliptic outside the region of the ellipse $\frac{x^2}{4}+\frac{y^2}{1}=1$

Parabolic on the region of the ellipse $\frac{x^2}{4}+\frac{y^2}{1}=1$

Hyperbolic inside the region of the ellipse $\frac{x^2}{4}+\frac{y^2}{1}=1$

5.2 Method of Seperation of Variables partial differential solutions to many (but not all).

Seperation of variables is one of the oldest technique for solving initial boundary value problems (IBVP) and applies to problems, where,

- PDE is linear and homogeneous (not necessarily constant coefficients)
- and Boundary conditions are linear and homogeneous

it is based on the fact that, if *f(x)* and *g(t)* are functions of idenpendent variables *x, t* respectively and if

f(x) = g(t)

then there must be a constant l for which *f(x)*=λ and *g(t)*= λ

The proof is straight forward, in that

$$\frac{\partial}{\partial x}f(x)=\frac{\partial}{\partial x}g(t)=0 \Rightarrow f'(x)=0\Rightarrow f(x)$$

$$\frac{\partial}{\partial x}f(t)=\frac{\partial}{\partial x}f(x)=0 \Rightarrow g'(x)=0\Rightarrow g(x)$$

In separation of variables, we first assume that the solution is of the separated from

$$u(x,t) = X(x)T(t)$$

We then substitute the separated from into the equation, and it possible move the '*x*'-terms to one side and '*t*'-terms to the otherside.

If not possible hen this method will not work and correspondingly, we say that the partial differential equation is not possible.

Once separated, the two sides of the equation most be constant, thus requiring the solutions to PDE, the product *X(x)*, *T(t)* is the separated solution of the partial differential equation.

SOLVED PROBLEMS

Problem 1. By the method of seperation of variables.

$$4\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 3u \quad \text{and} \quad u(0, y) = e^{-5y}$$ **(OU Dec 2011) (June 2012)**

Solution. Given equation $4\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 3u$ ...(1)

Here, U is a function of x and y.

Let us assume its solution as U = X Y ...(2)

$$4\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 3u \quad \text{and} \quad u(0, y) = e^{-5y}$$

Substitute these values in (1)

$$4X'Y + XY' = 3XY$$

$$\frac{4X'}{X} + \frac{Y'}{Y} = 3$$

$$\frac{4X'}{X} = \frac{3 - Y'}{Y}$$

$$\frac{4X'}{X} = \frac{3Y - Y'}{Y} \qquad \text{...(3)}$$

Since x and y are independent variables

$\therefore$ equation (3) will hold if each side of (3) is constant say 'K'.

$$\frac{4X'}{X} = \frac{3 - Y'}{Y} = K$$

(i) $\frac{4X'}{X} = k$

$4X' - KX = 0$

$(4D - K)X = 0$

A.E. is 4m - k = 0

4 m = k

$m = \frac{k}{4}$

$\therefore X = c_1 e^{\frac{k}{4}x}$

(ii) $\frac{3 - Y'}{Y} = k$

$3 - Y' = KY \Rightarrow Y'3Y + KY = 0$

$Y' - (3 - K)Y = 0$

[D - (3 - k)]Y = 0

A.E. is m - (3 - k) =0

m = 3 - k

$Y = c_2 e^{(3-k)y}$

Substitute X and Y values in (2)

$$U = c_1 e^{\frac{k}{4}x} . c_2 e^{(3-k)y}$$

$$U(x, y) = Ae^{\left[e^{\frac{k}{4}x + (3-k)y}\right]}$$ ...(4) where $A = c_1\, c_2$

and given $4(0, y) = e^{-5y}$

Put x = 0 in (4)

$\Rightarrow e^{-5y} = Ae^{(3-k)y}$

Comparing on both sides

A = 1 and -5 = 3 - k.

$\Rightarrow$ k = 3 + 5

$\Rightarrow$ k = 8

Substitute A and k values in (4)

$$U(x, y) = Ae^{2x - 5y}$$

Problem 2. Solve by seperation variables method for $U_x = U_y$ (OU Dec 2011)

Solution. Given equation $U_x = U_y$ ...(1)

Here u is a function of x and y

Let us assume its solution as U = X Y ...(2)

$$U_x = X'Y \text{ and } U_y = X\,Y'$$

divide with X Y

$$\frac{X'}{X} = \frac{Y'}{Y} \quad ...(3)$$

Since x and y are independent variables.

$\therefore$ equation (3) will hold if each side of (3) is constant say ‘K’.

$$\frac{X'}{X} = \frac{Y'}{Y} = K$$

(i) $\frac{X'}{X} = K$

$X' = K\,X$

$X' - K\,X = 0$

$(D - K)X = 0$

A.E. is m - k = 0

m = k

$X = c_1\, e^{kx}$

(ii) $\frac{Y'}{Y} = K$

$Y' = K\,Y$

$Y' - K\,Y = 0$

$(D - K)Y = 0$

A.E. is m - k = 0

m = k

$Y = c_2\, e^{ky}$

Substitute X and Y in equation (2)
which gives required solution

$$U \quad c_1 e^{Kx}.c_2 e^{Ky}$$

$$U(x, y) \quad c_1 c_2 e^{Kx \quad Ky}$$

$$U(x, y) \quad A e^{K(x \quad y)} \qquad \text{where } A \quad c_1 c_2$$

Problem 3. Solve $3\frac{u}{t} \quad 2\frac{u}{x} \quad u$ with $u(t, 0) \quad 6e^{\ t}$ (OU July 2014)

Solution. Given equation is $3\frac{u}{t} \quad 2\frac{u}{x} \quad u$ ...(1)

with $u(t, 0) \quad 6e^{\ t}$

i.e., U is a function of (t, x)

Let u = T X is a solution ...(2)

$$u_t \quad T\,X$$
$$u_x \quad TX$$ ...(3)

Substitute in equation (1)

$$3T\,X \quad 2TX \quad XT$$

$$\frac{3T}{T} \quad \frac{2X}{X} \quad 1$$

$$\frac{3T}{T} \quad 1 \quad \frac{2X}{X} \quad \frac{3T}{T} \quad \frac{X \quad 2X}{X}$$ ...(4)

Since T and X are independent variables

(4) will hold if each side of (4) is equal to constant say ‘K’

$$\frac{3T}{T} \quad \frac{X \quad 2X}{X} \quad K$$

(i) $\frac{3T}{T} \quad K \quad 3T \quad K\,T$

$$3T \quad K\,T \quad 0$$

$$(3D \quad K)T \quad 0$$

A.E. is 3m - k = 0

3m = K

m = $\frac{K}{3}$

$T \quad c_1 e^{\frac{K}{3}t}$ ...(5)

(ii) $\frac{X \quad 2X}{X} \quad K$

$X \quad 2X \quad K\,X$

$2X \quad X \quad KX \quad 0$

$2X \quad KX \quad X \quad 0 \quad (2D \quad (K \quad 1)X \quad 0$

$D \quad \frac{(K \quad 1)}{2} \quad X \quad 0$

A.E. is $m \quad \frac{(K \quad 1)}{2} \quad 0$

$m \quad \frac{1 \quad K}{2}$

$X \quad c_2 e^{\frac{(1 \quad K)x}{2}}$ ...(6)

Substitute equation (5) and (6) in (2)
which gives the required solution

$u \quad c_1 e^{\frac{K}{3}t} . c_2 e^{\frac{(1 \quad K)x}{2}}$

$u(t,x) \quad c_1\, c_2\, e^{\frac{K}{3}t \quad \frac{(1 \quad K)x}{2}}$

$u(t,x) \quad Ae^{\frac{K}{3}t \quad \frac{(1 \quad K)x}{2}}$ ...(7) where A = $c_1\, c_2$

and given $u(t,0) \quad 6e^{\ t}$;
Put x = 0 in (7)

$6e^{\ t} \quad Ae^{\frac{K}{3}t}$

Comparing on both sides

A = 6, $\frac{K}{3} \quad 1 \quad K \quad 3$

Substitute A & K values in (7)

$u(t,x) \quad 6e^{\frac{3t}{3}t \quad \frac{1 \quad 3}{2} \quad x}$

$u(t,x) \quad 6e^{\ t \quad 2x}$

which is the required solution.

Problem 4. Solve $3\frac{\partial u}{\partial x}+2\frac{\partial u}{\partial y}=0$ **where** $u(x,0)=4e^{-x}$ **(OU Dec 2014)**

Solution. Given equation $3\frac{\partial u}{\partial x}+2\frac{\partial u}{\partial y}=0$ **...(1)**

with $u(x,0)=4e^{-x}$

i.e., u is a function of (x, y)

Let u = (x, y) = X Y is a solution **...(2)**

$$\left.\begin{array}{l} u_x = X'Y \\ u_y = X\,Y' \end{array}\right\} \quad ...(3)$$

Substitute in equation (1)

$$3X'Y+2X\,Y'=0$$

divide bothsides with X Y

$$\frac{3X'Y+2X\,Y'}{X\,Y}=\frac{0}{X\,Y}$$

$$\frac{3X'}{X}+\frac{2Y'}{Y}=0 \Rightarrow \frac{3X'}{X}=\frac{-2Y'}{Y} \quad ...(4)$$

Since x and y are indepent variables

$\therefore$ (4) will hold if each side of (4) is equal to constant say 'K'

$$\frac{3X'}{X}=\frac{-2Y'}{Y}=K$$

(i) $\frac{3X'}{X}=K$

$3X'=K\,X$

$3X'-K\,X=0$

$(3D-K)X=0$

A.E. is 3m - k = 0

3m = k

m = $\frac{k}{3}$

$\Rightarrow$ $X=c_1 e^{\frac{kx}{3}}$ ...(5)

(ii) $\frac{-2Y'}{Y}=K$

$2Y'=K\,Y$

$2Y'+K\,Y=0$

$(2D+K)Y=0$

A.E. is 2m + k = 0

2m = -K

$m=\frac{-K}{2}$

$Y=c_2\,e^{\frac{-ky}{2}}$ ...(6)

Substitute X and Y values in (2)

$$U = c_1 e^{\frac{kx}{3}} . c_2 e^{\frac{-Ky}{2}}$$

$$U(x, y) = c_1 c_2 e^{K\left(\frac{x}{3} - \frac{y}{2}\right)}$$

$\Rightarrow$ $$u(x, y) = Ae^{K\left(\frac{x}{3} - \frac{y}{2}\right)} \quad ..(7) \quad \text{where } A = c_1 c_2$$

and given u(x, 0) = $4e^{-x}$;

Put y = 0 in (7)

$$4e^{-x} = Ae^{K\left(\frac{kx}{3}\right)}$$

$$A = 4, \quad \frac{k}{3} = -1 \Rightarrow k = -3$$

Substitute A and K values in (7)

$$u(x, y) = 4e^{-x\left(\frac{x}{3} - \frac{y}{2}\right)}$$

$$u(x, y) = 4e^{-x + \frac{3y}{2}}$$

(or) $$u(x, y) = 4e^{\frac{3y - 2x}{2}}$$ which is required solution.

Problem 5. **Solve** $\dfrac{\partial u}{\partial x} = 4 \dfrac{\partial u}{\partial y}$ **where** $u(0, y) = 8e^{-3y}$ **(OU Dec 2013)**

Solution. Given equation $\dfrac{\partial u}{\partial x} = 4 \dfrac{\partial u}{\partial y}$ **...(1)**

with $u(0, y) = 8e^{-3y}$

i.e., u is a function of (x, y)

Let u = (x, y) = X Y is a solution **...(2)**

$$\left.\begin{aligned} U_x &= X'Y \\ U_y &= XY' \end{aligned}\right\} \quad ...(3)$$

Substitute in equation (1)

$$X'Y = 4XY'$$

divide with X Y on both sides

$$\frac{X'Y}{XY} = \frac{4XY'}{XY}$$

$$\frac{X'}{X} = \frac{4Y'}{Y} \qquad ...(4)$$

Since x and y are indepent variables

$\therefore$ (4) will hold if each side of (4) is equal to constant say 'K'

$$\frac{X'}{X} = \frac{4Y'}{Y} = K$$

(i)

$$\frac{X'}{X} = K$$

$$X' = K\,X$$

$$X' - K\,X = 0$$

$$(D-K)X = 0$$

A.E. is m - k = 0

m = k

$$X = c_1 e^{Kx} \qquad ...(5)$$

(ii)

$$\frac{4Y'}{Y} = K$$

$$4Y' = K\,Y$$

$$4Y' - K\,Y = 0$$

$$(4D-K)Y = 0$$

A.E. is 4m - k = 0

4m = K

$$m = \frac{K}{4}$$

$$\Rightarrow Y = c_2\, e^{\frac{ky}{4}} \qquad ...(6)$$

Substitute X and Y values in (2)

$$u = c_1\, e^{Kx} \,.\, c_2\, e^{\frac{Ky}{4}}$$

$$u = c_1\, c_2 e^{K\left(x+\frac{y}{4}\right)}$$

$\Rightarrow$ $$u(x,y) = Ae^{K\left(\frac{4x+y}{2}\right)} \qquad ..(7)$$ where A = $c_1 c_2$

and also given u (0, y) = $8e^{-3y}$;

Put x = 0 in (7)

$$8\,e^{-3y} = A\,e^{\left(\frac{ky}{4}\right)}$$

Comparing on both sides

A = 8, $-3 = \frac{k}{4} \Rightarrow k = -12$

Substitute A and K values in (7)

$$u\,(x,y) = 8\,e^{-12\left(\frac{4x+y}{4}\right)}$$

$$u(x,y) = 8e^{-3(4x+y)}$$

Problem 6. Solve $3xU_x - 4yU_y = 0$. **(OU Dec-2017)**

Solution. Given equation is $3xU_x - 4yU_y = 0$ ...(1)

Here U is a function x and y.

Let us assume its solution as U = X Y ...(2)

$$U_x = X'Y \text{ and } U_y = X\,Y'$$

Substitute these values in (1)

$$3\,x\,X'Y + 4\,y\,X\,Y' = 0$$

$$\frac{3\,x\,X'}{X} + \frac{4\,yY'}{Y} = 0$$

$$\frac{3\,x\,X'}{X} = \frac{-4\,yY'}{Y} = 0 \quad \text{...(3)}$$

Since x and y are independent variables.

$\therefore$ (3) will hold if each side of (3) is equal to constant say 'K'

$$\frac{3\,x\,X'}{X} = \frac{-4\,y\,Y'}{Y} = K$$

$$\frac{3\,x\,X'}{X} = K \text{ and } \frac{-4\,yY'}{Y} = K$$

$$\frac{3\,X'}{X} = \frac{K}{x} \qquad \frac{4\,Y'}{Y} = \frac{-K}{y}$$

(i) $\dfrac{X'}{X} = K$

integrating $3\int \dfrac{X'}{X} = k\int \dfrac{1}{x}$

3log X = K log x + log c_1

$$\log X^3 = \log(x^K c_1)$$

$$X^3 = c_1 x^K$$

$$X = c_1^{1/3} x^{\frac{K}{3}} \quad \text{...(4)}$$

(ii) $\dfrac{-4yY'}{Y} = K$

$4\int \dfrac{Y'}{Y} = -k\int \dfrac{1}{y}$

$$4\log Y = -K\log y + \log c_2$$

$$Y^4 = \log\left(\frac{c_2}{y^K}\right)$$

$$Y^4 = c_2 y^{-K}$$

$$Y = c_2^{1/4} y^{\frac{-K}{4}} \quad \text{...(5)}$$

Substitute (4) and (5) in (1)

$$u(x, y) = \left(c_1^{1/3} x^{\frac{K}{3}} \right) . \left(c_2^{1/4} y^{\frac{-K}{4}} \right)$$

$\Rightarrow$ $$u = A x^{\frac{K}{3}} . y^{\frac{-K}{4}}$$

where $$A = c_1^{1/3},\ c_2^{1/4}$$

Exercise

1. Solve the following equations by the method of separation of variables

(*i*) $3\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} = 0$, where $u\ (x,\ 0) = 4e^{-x}$ (*ii*) $y^3 \frac{\partial z}{\partial x} + x^2 \frac{\partial z}{\partial y} = 0$

(*iii*) $\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial y} = 0$ (*iv*) $\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} = 0$, where $u(x,\ 0) = 2e^{3x}$

(*v*) $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0$, where $u(x,\ 0) = e^x + 3e^{2x}$

2. Find the solution of the wave equation $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$, corresponding to the triangular initial deflection

$$f\ (x) = \begin{cases} \frac{2k\ x}{l}, & \text{when } 0 < x < \frac{l}{2} \\ \frac{2k\ (l - x)}{l}, & \text{when } \frac{l}{2}\ 0 < x < l \end{cases}$$ and initial velocity zero.

3. A tightly stretch string with fixed and points $x = 0$ and $x = l$ is initially in a position given by $y = y_0 \sin^3 \left(\frac{\pi x}{l} \right)$. If it is released from rest from this position, find the displacement $y(x,\ t)$.

4. Solve $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$ with conditions given : $\frac{\partial u}{\partial x}(0, t) = 0, \frac{\partial u}{\partial x}(20, t) = 0$ for all t and $u(x,\ 0)$ = $\cos\left(\frac{\pi x}{20}\right) + 3\cos\left(\frac{3\pi x}{20}\right)$ for $0 \le x \le 20$.

5. A bar 40 cm long, with insulated sides, has its ends kept at 40° and 0° until steady state conditions prevail. The two ends are then suddenly insulated and kept so. Find the temperature distribution.

Answers

1. (*i*) $u(x,\ y) = 4e^{-\frac{1}{2}(2x - 3y)}$ (*ii*) $z(x, y) = ce^{\lambda\left(\frac{x^3}{3} - \frac{y^4}{4}\right)}$ (*iii*) $u(x,\ y) = e^{2ky}(c_1 e^{\sqrt{k}x} + c_2 e^{-\sqrt{k}x})$

(*iv*) $u(x,\ y) = 2\ e^{3(x\ +\ y)}$ (*v*) $u(x,\ y) = e^{x\ +\ y} + 3e^{2\ (x\ +\ y)}$

2. $u(x,\ t) = \frac{8k}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \cos\left(\frac{n\pi ct}{l}\right) \sin\left(\frac{n\pi x}{l}\right)$

3. $y(x,\ t) = \frac{y_0}{4}\left[3\sin\left(\frac{\pi x}{l}\right)\cos\left(\frac{\pi ct}{l}\right) - \sin\left(\frac{3\pi x}{l}\right)\cos\left(\frac{3\pi ct}{l}\right)\right]$

4. $u(x,\ t) = \cos\left(\frac{\pi x}{20}\right) e^{-\frac{\pi^2 c^2 t}{400}} + 3\cos\left(\frac{3\pi x}{20}\right) e^{-\frac{9\pi^2 c^2 t}{400}}$

5. $u(x,\ t) = 20 + \dfrac{160}{\pi^2}\displaystyle\sum_{n=1}^{\infty} \dfrac{1}{(2n-1)^2}\cos\left(\dfrac{(2n-1)\pi x}{40}\right) e^{-\frac{(2n-1)^2\pi^2 c^2 t}{1600}}$

5.3 SOLUTION OF ONE DIMENSIONAL WAVE EQUATION $\dfrac{\partial^2 y}{\partial t^2} = c^2 \dfrac{\partial^2 y}{\partial x^2}$

Consider a uniform elastic string of length *l* stretched tightly between two points O and A, and displaced slightly from its equilibrium position OA. Taking the end O as the origin, OA as the *x*-axis and a perpendicular line through O as the *y*-axis, we shall find the displacement *y* as a function of the distance *x* and the time *t*.

We shall obtain the equation of motion for the string under the following assumptions:

(*i*) The motion takes place entirely in the *xy*-plane and each particle of the string moves perpendicular to the equilibrium position OA of the string.

(*ii*) The string is perfectly flexible and does not offer resistance to bending.

(*iii*) The tension in the string is so large that the forces due to weight of the string can be neglected.

(*iv*) The displacement *y* and the slope $\dfrac{\partial y}{\partial x}$ are small, so that their higher powers can be neglected.

Let *m* be the mass per unit length of the string. Consider the motion of an element PQ of length δs. Since the string does not offer resistance to bending (by assumption), the tensions T_1 and T_2 at P and Q respectively are tangential to the curve.

Since there is no motion in the horizontal direction, we have

$$T_1 \cos\alpha = T_2 \cos\beta = T \text{ (constant)} \quad \text{...(1)}$$

Mass of element PQ is $m\delta s$. By Newton's second law of motion, the equation of motion in the vertical direction is

$$m\delta s \frac{\partial^2 y}{\partial t^2} = T_2 \sin\beta - T_1 \sin\alpha$$

or $$\frac{m\,\delta s}{T}\frac{\partial^2 y}{\partial t^2} = \frac{T_2 \sin\beta}{T_2\cos\beta} - \frac{T_1\sin\alpha}{T_1\cos\alpha} \quad \text{[By using (1)]}$$

or $$\frac{\partial^2 y}{\partial t^2} = \frac{T}{m\,\delta s}(\tan\beta - \tan\alpha)$$

or $$\frac{\partial^2 y}{\partial t^2} = \frac{T}{m\,\delta x}\left[\left(\frac{\partial y}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial y}{\partial x}\right)_x\right]$$

[Since $\delta s = \delta x$ to a first approximation, and $\tan\alpha$ and $\tan\beta$ are the slopes of the curve of the string at x and $x + \delta x$]

or $$\frac{\partial^2 y}{\partial t^2} = \frac{T}{m}\left[\frac{\left(\frac{\partial y}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial y}{\partial x}\right)_x}{\delta x}\right] = \frac{T}{m}\frac{\partial^2 y}{\partial x^2}, \text{ as } \delta x \to 0$$

or $$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2}, \quad \text{where } c^2 = \frac{T}{m}$$

This is the partial differential equation giving the transverse vibrations of the string. It is also called the *one dimensional wave equation*.

The *boundary conditions*, which the equation $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$ has to satisfy are

$\left.\begin{array}{l}(i)\ y = 0, \text{ when } x = 0 \\ (ii)\ y = 0, \text{ when } x = l\end{array}\right\}$. These should be satisfied for every value of *t*.

If the string is made to vibrate by pulling it into a curve $y = f(x)$ and then releasing it, the *initial conditions* are:

$(i)\ y = f(x)$, when $t = 0$ $\qquad$ $(ii)\ \frac{\partial y}{\partial t} = 0$, when $t = 0$.

Now, Consider the wave equation

$$\frac{\partial^2 y}{\partial t^2} = c^2 \cdot \frac{\partial^2 y}{\partial x^2} \qquad \text{...(1)}$$

Let $\qquad y = \text{XT}$, be a solution of (1) $\qquad$ ...(2)

where X is a function of *x* only and T is a function of *t* only.

Then, $\qquad \frac{\partial^2 y}{\partial t^2} = \text{XT}''$ $\quad$ and $\quad \frac{\partial^2 y}{\partial x^2} = \text{X}''\text{T}$

Substituting in (1), we have $\qquad \text{XT}'' = c^2\text{X}''\text{T}$

Separating the variables, we get $\qquad \frac{\text{X}''}{\text{X}} = \frac{1}{c^2} \cdot \frac{\text{T}''}{\text{T}}$ $\qquad$ ...(3)

Now, the LHS of (3) is a function of *x* only and the RHS is a function of *t* only. Since *x* and *t* are independent variables, this equation can hold only when both sides reduce to a constant, say *k*. Then equation (3) leads to the ordinary linear differential equations

$$\text{X}'' - k\text{X} = 0 \quad \text{and} \quad \text{T}'' - kc^2\text{T} = 0 \qquad \text{...(4)}$$

Solving equations (4), we get

(*i*) *When k is positive* and = p^2, say

$$\text{X} = c_1 e^{px} + c_2 e^{-px},\ \text{T} = c_3 e^{cpt} + c_4 e^{-cpt}$$

(*ii*) *When k is negative* and = $-p^2$, say

$$\text{X} = c_1 \cos px + c_2 \sin px$$
$$\text{T} = c_3 \cos cpt + c_4 \sin cpt$$

(*iii*) *When k = 0*

$$\text{X} = c_1 x + c_2 \qquad \text{T} = c_3 t + c_4$$

Thus, the various possible solutions of the wave equation (1) are :

$$y = (c_1 e^{px} + c_2 e^{-px})(c_3 e^{cpt} + c_4 e^{-cpt})$$
$$y = (c_1 \cos px + c_2 \sin px)(c_3 \cos cpt + c_4 \sin cpt)$$
$$y = (c_1 x + c_2)(c_3 t + c_4)$$

Of these three solutions, we have to choose that solution which is consistent with the physical nature of the problem. Since we are dealing with a problem on vibrations, *y* must be a periodic function of *x* and *t*. Therefore, the solution must involve trigonometric terms.

Accordingly $\quad y = (c_1 \cos px + c_2 \sin px)(c_3 \cos cpt + c_4 \sin cpt)$ $\qquad$ ...(5)

is the only suitable solution of the wave equation and it corresponds to $k = -p^2$.

Now, applying boundary conditions that

$y = 0$, $\quad$ when $\quad x = 0$

and $\quad y = 0$, $\quad$ when $\quad x = l$, we get

$$0 = c_1(c_3 \cos cpt + c_4 \sin cpt) \qquad \text{...(6)}$$

and $$0 = (c_1 \cos pl + c_2 \sin pl)(c_3 \cos cpt + c_4 \sin cpt) \qquad \text{...(7)}$$

From (6), we have $c_1 = 0$ and equation (7) reduces to

$$c_2 \sin pl(c_3 \cos cpt + c_4 \sin cpt) = 0$$

which is satisfied when $\sin pl = 0$ or $pl = n\pi$ or $p = \frac{n\pi}{l}$, where $n = 1, 2, 3, ...$

∴ A solution of the wave equation satisfying the boundary conditions is

$$y = c_2\left(c_3 \cos \frac{n\pi ct}{l} + c_4 \sin \frac{n\pi ct}{l}\right) \sin \frac{n\pi x}{l}$$

$$= \left(a_n \cos \frac{n\pi ct}{l} + b_n \sin \frac{n\pi ct}{l}\right) \sin \frac{n\pi x}{l}$$

on replacing $c_2 c_3$ by a_n and $c_2 c_4$ by b_n.

Adding up the solutions for different values of n, we get

$$y = \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi ct}{l} + b_n \sin \frac{n\pi ct}{l}\right) \sin \frac{n\pi x}{l} \quad ...(8)$$

is also a solution.

Now, applying the initial conditions

$$y = f(x) \quad \text{and} \quad \frac{\partial y}{\partial t} = 0, \quad \text{when } t = 0, \text{ we have}$$

$$f(x) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{l} \quad ...(9)$$

and

$$0 = \sum_{n=1}^{\infty} \frac{n\pi c}{l} b_n \sin \frac{n\pi x}{l} \quad ...(10)$$

Since equation (9) represents Fourier series for $f(x)$, we have

$$a_n = \frac{2}{l}\int_0^l f(x) \sin \frac{n\pi x}{l}\, dx \quad ...(11)$$

From (10), $b_n = 0$, for all n

Hence (8) reduces to $y = \sum_{n=1}^{\infty} a_n \cos \frac{n\pi ct}{l} \sin \frac{n\pi x}{l}$ ...(12)

where a_n is given by (11) when $f(x)$ *i.e.*, $y(x, 0)$ is known.

SOLVED PROBLEMS

Example 1. *A string is stretched and fastened to two points l apart. Motion is started by displacing the string in the form $y = A \sin \frac{\pi x}{l}$ from which it is released at time t = 0. Show that the displacement of any point at a distance x from one end at time t is given by*

$y(x, t) = A \sin \frac{\pi x}{l} \cos \frac{\pi ct}{l}$. (U.K.T.U., 2011, 2012; M.D.U., 2012)

Sol. The equation of the string is

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \quad ...(1)$$

Since, the string is stretched between two fixed points (0, 0) and (l, 0) hence the displacement of the string at these points will be zero

∴ $y(0, t) = 0$ ...(2)

and $y(l, t) = 0$ ...(3)

Since, the string is released from rest hence its initial velocity will be zero

∴ $\frac{\partial y}{\partial t} = 0$ at $t = 0$ ...(4)

Since, the string is displaced from its initial position at time $t = 0$ hence the initial displacement is

$$y(x, 0) = A \sin \frac{\pi x}{l} \quad ...(5)$$

Conditions (2), (3), (4) and (5) are the boundary conditions.

Let us now proceed to solve equation (1),

Let $y = XT.$ ...(6)

where X is a function of x only and T is a function of t only.

$$\frac{y}{t} \quad \frac{}{t}(XT) \quad X\frac{dT}{dt}$$

$$\frac{{}^2 y}{t^2} \quad \frac{}{t}\left(X\frac{dT}{dt}\right) \quad X\frac{d^2T}{dt^2}.$$

Similarly, $\frac{{}^2 y}{x^2} \quad T\frac{d^2X}{dx^2}.$

Substituting the above in equation (1), we get

$$X\frac{d^2T}{dt^2} \quad c^2T\frac{d^2X}{dx^2} \quad \Rightarrow \quad XT'' = c^2TX''$$

Case I. $\frac{1}{c^2}\frac{T}{T} \quad \frac{X}{X} = -p^2$ (say)

(i) $\frac{1}{c^2}\frac{T}{T} = -p^2$

$$\frac{d^2T}{dt^2} + c^2p^2T = 0.$$

Auxiliary equation is $m^2 + c^2p^2 = 0$

$$m^2 = c^2p^2i^2$$

$$m = \pm cpi$$

$\therefore$ C.F. $= c_1 \cos cpt + c_2 \sin cpt$

P.I. $= 0$

$\therefore$ T = C.F. + P.I. $= c_1 \cos cpt + c_2 \sin cpt$ ...(7)

(ii) $\frac{X}{X} \quad p^2 \quad \Rightarrow \quad \frac{d^2X}{dx^2} + p^2X = 0.$

Auxiliary equation is $m^2 + p^2 = 0$

$$m = \pm pi$$

C.F. $= c_3 \cos px + c_4 \sin px$

P.I. $= 0$

$\therefore$ $X = c_3 \cos px + c_4 \sin px$ ...(8)

Hence, $y(x, t) = (c_1 \cos cpt + c_2 \sin cpt)(c_3 \cos px + c_4 \sin px)$ ...(9)

Case II. $\frac{1}{c^2}\frac{T}{T} \quad \frac{X}{X} = p^2$ (say)

(i) $\frac{1}{c^2}\frac{T}{T} = p^2 \quad \Rightarrow \quad \frac{d^2T}{dt^2} - p^2c^2T = 0$

Auxiliary equation is $m^2 - p^2c^2 = 0 \quad \Rightarrow \quad m = \pm pc$

$\therefore$ C.F. $= c_5\, e^{pct} + c_6\, e^{-pct}$

P.I. $= 0$

$\therefore$ $T = c_5\, e^{pct} + c_6\, e^{-pct}.$

$(ii) \qquad \frac{X}{X} \quad p^2 \quad \Rightarrow \quad \frac{d^2X}{dx^2} - p^2X = 0$

Auxiliary equation is

$$m^2 - p^2 = 0 \quad \Rightarrow \quad m = \pm p$$

$\therefore$ C.F. $= c_7 e^{px} + c_8 e^{-px}$

P.I. = 0

$\therefore$ $X = c_7 e^{px} + c_8 e^{-px}$

Hence, $y(x, t) = (c_5 e^{pct} + c_6 e^{-pct})(c_7 e^{px} + c_8 e^{-px})$ ...(10)

Case III. $\frac{1}{c^2} \frac{T}{T} \quad \frac{X}{X} = 0$ (say)

$(i) \qquad \frac{1}{c^2} \frac{T}{T} \quad 0 \quad \Rightarrow \quad T'' = 0 \quad \text{or} \quad \frac{d^2T}{dt^2} = 0$

Auxiliary equation is

$$m^2 = 0 \quad \Rightarrow \quad m = 0, 0$$

$\therefore$ C.F. $= c_9 + c_{10} t$

P.I. = 0

$\therefore$ $T = c_9 + c_{10} t$

$(ii) \qquad \frac{X}{X} \quad 0 \quad \Rightarrow \quad X'' = 0 \quad \text{or} \quad \frac{d^2X}{dx^2} \quad 0$

Auxiliary equation is

$$m^2 = 0 \quad \Rightarrow \quad m = 0, 0$$

$\therefore$ C.F. $= c_{11} + c_{12}x$

P.I. = 0

$\therefore$ $X = c_{11} + c_{12}x$

Hence, $y(x, t) = (c_9 + c_{10} t)(c_{11} + c_{12}x)$ ...(11)

Out of these three above solutions (9), (10) and (11), we have to choose the solution which is consistent with the physical nature of the problem. Since, we are dealing with a problem on vibrations, the solution must contain periodic functions. Hence the solution which contains trigonometric terms must be the required solution.

Hence solution (9) is the general solution of one dimensional wave equation given by equation (1).

Now, $y(x, t) = (c_1 \cos cpt + c_2 \sin cpt)(c_3 \cos px + c_4 \sin px)$

Applying the boundary condition,

$$y(0, t) = 0 = (c_1 \cos cpt + c_2 \sin cpt)\, c_3$$

$\Rightarrow$ $c_3 = 0.$

$\therefore$ From (9), $y(x, t) = (c_1 \cos cpt + c_2 \sin cpt)\, c_4 \sin px$ ...(12)

Again, $y(l, t) = 0 = (c_1 \cos cpt + c_2 \sin cpt)\, c_4 \sin pl$

$\Rightarrow$ $\sin pl = 0 = \sin n\pi \ (n \in I)$

$\therefore$ $p = \frac{n}{l}.$

Hence from (12), $y(x, t) = \left(c_1 \cos \frac{n \ ct}{l} \quad c_2 \sin \frac{n \ ct}{l}\right) c_4 \sin \frac{n \ x}{l}$ ...(13)

$$\frac{y}{t} \quad \frac{n \ c}{l}\left(\quad c_1 \sin \frac{n \ ct}{l} \quad c_2 \cos \frac{n \ ct}{l}\right) c_4 \sin \frac{n \ x}{l}$$

At $t = 0$,

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0 = \frac{n\pi c}{l}\left(c_2 c_4 \sin\frac{n\pi x}{l}\right)$$

$\Rightarrow$ $\quad c_2 = 0,$

$\therefore$ From (13), $\quad y(x, t) = c_1 c_4 \cos\frac{n\pi ct}{l}\sin\frac{n\pi x}{l}$ ...(14)

$$y(x, 0) = \text{A}\sin\frac{\pi x}{l} = c_1 c_4 \sin\frac{n\pi x}{l}$$

$\Rightarrow$ $\quad c_1 c_4 = \text{A},\ n = 1.$ | Comparing

Hence from (14), $\quad y(x, t) = \text{A}\cos\frac{\pi ct}{l}\sin\frac{\pi x}{l}$

which is the required solution.

Example 2. *Show how the wave equation* $c^2\frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial t^2}$ *can be solved by the method of separation of variables. If the initial displacement and velocity of a string stretched between* $x = 0$ *and* $x = l$ *are given by* $y = f(x)$ *and* $\frac{\partial y}{\partial t} = g(x)$*, determine the constants in the series solution.*

[JNTUK, (Set 1) 2014, (Set 2) 2015]

Sol. The wave equation is $\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2}$ ...(1)

Let $\quad y = \text{XT}$ ...(2)

where X is a function of x only and T is a function of t only.

$$\frac{\partial y}{\partial t} = \frac{\partial}{\partial t}(\text{XT}) = \text{X}\frac{d\text{T}}{dt}$$

$$\frac{\partial^2 y}{\partial t^2} = \text{X}\frac{d^2\text{T}}{dt^2}$$

Similarly, $\quad \frac{\partial^2 y}{\partial x^2} = \text{T}\frac{d^2\text{X}}{dx^2}.$

Substituting in (1), we get

$$\text{X}\frac{d^2\text{T}}{dt^2} = c^2\text{T}\frac{d^2\text{X}}{dx^2} \Rightarrow \text{XT}'' = c^2\text{TX}''$$

$\Rightarrow$ $\quad \frac{1}{c^2}\frac{\text{T}''}{\text{T}} = \frac{\text{X}''}{\text{X}}$ ...(3)

Case I. When $\quad \frac{1}{c^2}\frac{\text{T}''}{\text{T}} = \frac{\text{X}''}{\text{X}} = p^2$ (say)

(i) $\quad \frac{1}{c^2}\frac{\text{T}''}{\text{T}} = p^2 \Rightarrow \frac{d^2\text{T}}{dt^2} - p^2c^2\text{T} = 0.$

Auxiliary equation is

$$m^2 - p^2c^2 = 0$$

$$m = \pm pc$$

$$\text{C.F.} = c_1 e^{pct} + c_2 e^{-pct}$$

$$\text{P.I.} = 0$$

$\therefore$ $\quad \text{T} = \text{C.F.} + \text{P.I.} = c_1 e^{pct} + c_2 e^{-pct}$

(*ii*) $$\frac{X}{X} \quad p^2 \quad \Rightarrow \quad \frac{d^2X}{dx^2} - p^2X = 0.$$

Auxiliary equation is

$$m^2 - p^2 = 0$$

$$m = \pm p$$

$$\text{C.F.} = c_3 e^{px} + c_4 e^{-px}$$

$$\text{P.I.} = 0.$$

$\therefore$ $$X = \text{C.F.} + \text{P.I.} = c_3 e^{px} + c_4 e^{-px}.$$

Hence, the solution is

$$y = XT = (c_1 e^{pct} + c_2 e^{-pct})(c_3 e^{px} + c_4 e^{-px}). \qquad ...(4)$$

Case II. When

$$\frac{1}{c^2}\frac{T}{T} \quad \frac{X}{X} = -p^2 \text{ (say)}$$

(*i*) $$\frac{1}{c^2}\frac{T}{T} = -p^2 \quad \Rightarrow \quad \frac{d^2T}{dt^2} + p^2c^2T = 0.$$

Auxiliary equation is

$$m^2 + p^2c^2 = 0 \quad \Rightarrow \quad m = \pm pci$$

$\therefore$ $$\text{C.F.} = (c_5 \cos pct + c_6 \sin cpt)$$

$$\text{P.I.} = 0.$$

$\therefore$ $$T = \text{C.F.} + \text{P.I.} = c_5 \cos cpt + c_6 \sin cpt$$

(*ii*) $$\frac{X}{X} \quad p^2 \quad \Rightarrow \quad \frac{d^2X}{dx^2} + p^2X = 0.$$

Auxiliary equation is

$$m^2 + p^2 = 0 \quad \Rightarrow \quad m = \pm pi$$

$\therefore$ $$\text{C.F.} = c_7 \cos px + c_8 \sin px$$

$$\text{P.I.} = 0$$

$\therefore$ $$X = c_7 \cos px + c_8 \sin px.$$

Hence, the solution is

$$y = XT = (c_5 \cos cpt + c_6 \sin cpt)(c_7 \cos px + c_8 \sin px) \qquad ...(5)$$

Case III. When, $\frac{1}{c^2}\frac{T}{T} \quad \frac{X}{X} \quad 0$

(*i*) $$\frac{1}{c^2}\frac{T}{T} = 0 \quad \Rightarrow \quad \frac{d^2T}{dt^2} = 0$$

$\Rightarrow$ $$T = c_9 + c_{10}t$$

(*ii*) $$\frac{X}{X} \quad 0 \quad \Rightarrow \quad \frac{d^2X}{dx^2} = 0$$

$\Rightarrow$ $$X = c_{11} + c_{12}x.$$

Hence, the solution is

$$y(x, t) = (c_9 + c_{10}t)(c_{11} + c_{12}x) \qquad ...(6)$$

Of the above three solutions given by (4), (5) and (6), we have to choose the solution which is consistent with the physical nature of the problem. Since, we are dealing with a problem on vibrations, y must be a periodic function of x and t therefore the solution must involve trigonometric terms hence solution (5) is the required solution.

Boundary conditions are

$$y(0, t) = 0, \qquad y(l, t) = 0$$

$$y = f(x) \qquad \text{when } t = 0$$

$$\frac{\partial y}{\partial t} = g(x) \qquad \text{when } t = 0$$

From equation (5), $y(0, t) = (c_5 \cos cpt + c_6 \sin cpt)\, c_7$

$$0 = (c_5 \cos cpt + c_6 \sin cpt)\, c_7$$

$$\Rightarrow \qquad c_7 = 0.$$

Hence from (5), $\quad y(x, t) = (c_5 \cos cpt + c_6 \sin cpt)\, c_8 \sin px$ ...(7)

$$y(l, t) = 0 = (c_5 \cos cpt + c_6 \sin cpt)\, c_8 \sin pl$$

$$\Rightarrow \qquad \sin pl = 0 = \sin n\pi \ (n \in \mathrm{I}) \quad \Rightarrow \quad p = \frac{n\pi}{l}.$$

$\therefore$ From (7), $\quad y(x, t) = \left[c_5 \cos \frac{n\pi ct}{l} + c_6 \sin \frac{n\pi ct}{l}\right] c_8 \sin \frac{n\pi x}{l}$ ...(8)

$$= \left[a_n \cos \frac{n\pi ct}{l} + b_n \sin \frac{n\pi ct}{l}\right] \sin \frac{n\pi x}{l}$$

where $\quad c_5 c_8 = a_n$ and $c_6 c_8 = b_n$

The general solution is

$$y(x, t) = \sum_{n=1}^{\infty} \left[a_n \cos \frac{n\pi ct}{l} + b_n \sin \frac{n\pi ct}{l}\right] \sin \frac{n\pi x}{l} \qquad \text{...(9)}$$

$$y(x, 0) = f(x) = \sum_{1}^{\infty} a_n \sin \frac{n\pi x}{l}$$

where $\quad a_n = \frac{2}{l}\int_0^l f(x) \,.\, \sin \frac{n\pi x}{l}\, dx$ ...(10)

From (9), $\quad \frac{\partial y}{\partial t} = \frac{\pi c}{l}\sum_{1}^{\infty}\left[-n\, a_n \sin \frac{n\pi ct}{l} + n\, b_n \cos \frac{n\pi ct}{l}\right] \sin \frac{n\pi x}{l}$

At $t = 0$, $\quad \left(\frac{\partial y}{\partial t}\right)_{t=0} = g(x) = \frac{\pi c}{l}\sum_{1}^{\infty} n b_n \sin \frac{n\pi x}{l}$

where $\quad \frac{n\pi c}{l} b_n = \frac{2}{l}\int_0^l g(x) \,.\, \sin \frac{n\pi x}{l}\, dx$

$\Rightarrow \quad b_n = \frac{2}{n\pi c}\int_0^l g(x) \,.\, \sin \frac{n\pi x}{l}\, dx.$ ...(11)

Hence, the required solution is

$$y(x, t) = \sum_{n=1}^{\infty} \left[a_n \cos \frac{n\pi ct}{l} + b_n \sin \frac{n\pi ct}{l}\right] \sin \frac{n\pi x}{l}$$

where $\quad a_n = \frac{2}{l}\int_0^l f(x) \,.\, \sin \frac{n\pi x}{l}\, dx$

and $$b_n = \frac{2}{n\pi c}\int_0^l g(x)\sin\frac{n\pi x}{l}\,dx.$$

Example 3. *A tightly stretched string with fixed end points x = 0 and x = l is initially in a position given by* $y = y_0 \sin^3 \frac{\pi x}{l}$. *If it is released from rest from this position, find the displacement y(x, t).* [G.B.T.U., (C.O.) 2011; JNTUK, (Set 2) 2015]

Sol. The equation of the string is

$$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2} \quad ...(1)$$

The solution of eqn. (1) is

$$y(x, t) = (c_1 \cos cpt + c_2 \sin cpt)(c_3 \cos px + c_4 \sin px) \quad ...(2)$$

| Refer sol. of Ex. 1

Boundary conditions are

$$y(0, t) = 0 \quad ...(3)$$

$$y(l, t) = 0 \quad ...(4)$$

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0 \quad ...(5)$$

$$y(x, 0) = y_0 \sin^3\frac{\pi x}{l} \quad ...(6)$$

Applying boundary condition in (2),

$$y(0, t) = 0 = (c_1 \cos cpt + c_2 \sin cpt)\, c_3$$

$\Rightarrow$ $c_3 = 0$

$\therefore$ From (2), $y(x, t) = (c_1 \cos cpt + c_2 \sin cpt)\, c_4 \sin px \quad ...(7)$

Again, $y(l, t) = 0 = (c_1 \cos cpt + c_2 \sin cpt)\, c_4 \sin pl$

$\Rightarrow$ $\sin pl = 0 = \sin n\pi \quad (n \in I)$

$\therefore$ $p = \frac{n\pi}{l}$

Hence, from (7),

$$y(x, t) = \left(c_1 \cos\frac{n\pi ct}{l} + c_2 \sin\frac{n\pi ct}{l}\right) c_4 \sin\frac{n\pi x}{l} \quad ...(8)$$

$$\frac{\partial y}{\partial t} = \frac{n\pi c}{l}\left(-c_1 \sin\frac{n\pi ct}{l} + c_2 \cos\frac{n\pi ct}{l}\right) c_4 \sin\frac{n\pi x}{l}$$

At $t = 0$,

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0 = \frac{n\pi c}{l} c_2 c_4 \sin\frac{n\pi x}{l}$$

$\Rightarrow$ $c_2 = 0.$

$\therefore$ From (8),

$$y(x, t) = c_1 c_4 \sin\frac{n\pi x}{l}\cos\frac{n\pi ct}{l} = b_n \sin\frac{n\pi x}{l}\cos\frac{n\pi ct}{l}$$

Most general solution is

$$y(x, t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}\cos\frac{n\pi ct}{l} \quad ...(9)$$

$$y(x, 0) = y_0 \sin^3\frac{\pi x}{l} = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$\Rightarrow \quad y_0 \left[\frac{3\sin\frac{\pi x}{l} - \sin\frac{3\pi x}{l}}{4}\right] = b_1 \sin\frac{\pi x}{l} + b_2 \sin\frac{2\pi x}{l} + b_3 \sin\frac{3\pi x}{l} + \ldots$$

Comparing, we get

$$b_1 = \frac{3y_0}{4},\ b_2 = 0,\ b_3 = -\frac{y_0}{4},\ b_4 = b_5 = \ldots = 0$$

Hence, from (9),

$$y(x, t) = \frac{3y_0}{4}\sin\frac{\pi x}{l}\cos\frac{\pi ct}{l} - \frac{y_0}{4}\sin\frac{3\pi x}{l}\cos\frac{3\pi ct}{l}.$$

Example 4. *A tightly stretched flexible string has its ends fixed at $x = 0$ and $x = l$. At time $t = 0$, the string is given a shape defined by $F(x) = \mu x(l - x)$, μ is a constant and then released. Find the displacement $y(x, t)$ of any point x of the string at any time $t > 0$.*

(JNTUK, 2015)

Sol. The wave equation is $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$ ...(1)

The solution of equation (1) is

$y(x, t) = (c_1 \cos cpt + c_2 \sin cpt)(c_3 \cos px + c_4 \sin px)$ ...(2) (Refer sol. of Ex. 1)

Boundary conditions are $y(0, t) = 0$ ...(3)

$y(l, t) = 0$ ...(4)

$\frac{\partial y}{\partial t} = 0$ at $t = 0$ ...(5)

and $y(x, 0) = \mu x(l - x)$ ...(6)

From (2), $y(0, t) = 0 = (c_1 \cos cpt + c_2 \sin cpt)c_3$

$\Rightarrow$ $c_3 = 0.$

$\therefore$ From (2), $y(x, t) = (c_1 \cos cpt + c_2 \sin cpt)\, c_4 \sin px$ ...(7)

$y(l, t) = 0 = (c_1 \cos cpt + c_2 \sin cpt)\, c_4 \sin pl$

$\Rightarrow$ $\sin pl = 0 = \sin n\pi \ (n \in I)$

$p = \frac{n\pi}{l}.$

From (7), $y(x, t) = \left(c_1 \cos\frac{n\pi ct}{l} + c_2 \sin\frac{n\pi ct}{l}\right) c_4 \sin\frac{n\pi x}{l}$

...(8)

Now from (7), $\frac{\partial y}{\partial t} = \frac{n\pi c}{l}\left(-c_1 \sin\frac{n\pi ct}{l} + c_2 \cos\frac{n\pi ct}{l}\right).\, c_4 \sin\frac{n\pi x}{l}$

At $t = 0$, $\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0 = \frac{n\pi c}{l} c_2 c_4 \sin\frac{n\pi x}{l}$

$\Rightarrow$ $c_2 = 0.$

$\therefore$ From (8), $y(x, t) = c_1 c_4 \cos\frac{n\pi ct}{l}\sin\frac{n\pi x}{l}$

$\Rightarrow$ $y(x, t) = b_n \cos\frac{n\pi ct}{l}\sin\frac{n\pi x}{l}$ where $c_1 c_4 = b_n$.

The most general solution is

$$y(x, t) = \sum_{n=1}^{\infty} b_n \cos\frac{n\pi ct}{l}\sin\frac{n\pi x}{l} \quad \ldots(9)$$

$$y(x, 0) = \mu(lx - x^2) = \sum_{1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

where $$b_n = \frac{2}{l}\int_0^l \mu(lx - x^2) \sin \frac{n\pi x}{l}\, dx$$

$$= \frac{2\mu}{l}\left\{\left[(lx - x^2)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)\right]_0^l - \int_0^l (l - 2x)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) dx\right\}$$

$$= \frac{2\mu}{l}\left\{\frac{l}{n\pi}\int_0^l (l - 2x)\cos\frac{n\pi x}{l}\, dx\right\}$$

$$= \frac{2\mu}{n\pi}\left\{\left[(l - 2x)\frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right]_0^l - \int_0^l (-2)\frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}}\, dx\right\} = \frac{2\mu}{n\pi}\cdot\frac{2l}{n\pi}\int_0^l \sin\frac{n\pi x}{l}\, dx$$

$$= \frac{4\mu l}{n^2\pi^2}\left[\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right]_0^l = \frac{4\mu l^2}{n^3\pi^3}(-\cos n\pi + 1) = \frac{4\mu l^2}{n^3\pi^3}[1 - (-1)^n].$$

$\therefore$ From (9), $$y(x, t) = \frac{4\mu l^2}{\pi^3}\sum_{1}^{\infty}\frac{[1 - (-1)^n]}{n^3}\cos\frac{n\pi ct}{l}\sin\frac{n\pi x}{l}$$

$$= \frac{8\mu l^2}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}\sin\frac{(2n-1)\pi x}{l}\cos\frac{(2n-1)\pi ct}{l}.$$

Solve $\frac{\partial^2 u}{\partial x^2} = c^2\frac{\partial^2 u}{\partial x^2}, 0 < x < l, t > 0$. *Subject to* $u(0,t) = u(l,t) = 0, u(x,0) = x(l - x), \frac{\partial u}{\partial t}(x,0) = 0$,

(OU 2017)

Solution: In the above problem (4) if we take u = 1, we get

$$u(x,t) = \frac{8l^2}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}\sin\frac{(2n-1)\pi x}{l}\cos\frac{(2n-1)\pi ct}{l}$$

Example 5. *A string is stretched between two fixed points (0, 0) and (l, 0) and released at rest from the initial deflection given by*

and $$f(x) = \begin{cases}\left(\frac{2k}{l}\right)x, & 0 < x < \frac{l}{2}\\ \left(\frac{2k}{l}\right)(l - x), & \frac{l}{2} < x < l\end{cases}$$

Find the deflection of the string at any time.

Sol. The equation for the vibrations of the string is

$$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{dx^2} \quad ...(1)$$

The solution of eqn. (1) is

$y(x, t) = (c_1 \cos cpt + c_2 \sin cpt)(c_3 \cos px + c_4 \sin px)$ ...(2) [Refer sol. of Ex. 1]

Boundary conditions are, $y(0, t) = 0, y(l, t) = 0$

$$\frac{\partial y}{\partial t} = 0 \quad \text{at} \quad t = 0$$

$$y\,(x,\,0) = \begin{cases} \dfrac{2k}{l}\,x, & 0 < x < \dfrac{l}{2} \\[2mm] \dfrac{2k}{l}\,(l - x), & \dfrac{l}{2} < x < l \end{cases}$$

From (2), $\quad y(0,\,t) = (c_1 \cos cpt + c_2 \sin cpt)\, c_3$

$$0 = (c_1 \cos cpt + c_2 \sin cpt)\, c_3$$

$\Rightarrow \quad c_3 = 0.$

$\therefore$ From (2), $\quad y(x,\,t) = (c_1 \cos cpt + c_2 \sin cpt)\, c_4 \sin px \quad \text{...(3)}$

$$y(l,\,t) = 0 = (c_1 \cos cpt + c_2 \sin cpt)\, c_4 \sin pl$$

$\Rightarrow \quad \sin pl = 0 = \sin n\pi;\ n \in \text{I}$

$$p = \frac{n\pi}{l}.$$

$\therefore$ From (3), $\quad y(x,\,t) = \left(c_1 \cos \dfrac{n\pi ct}{l} + c_2 \sin \dfrac{n\pi ct}{l}\right) c_4 \sin \dfrac{n\pi x}{l} \quad \text{...(4)}$

$$\frac{\partial y}{\partial t} = \frac{n\pi c}{l}\left(-c_1 \sin \frac{n\pi ct}{l} + c_2 \cos \frac{n\pi ct}{l}\right) c_4 \sin \frac{n\pi x}{l}$$

At $t = 0$, $\quad \left(\dfrac{\partial y}{\partial t}\right)_{t=0} = 0 = \dfrac{n\pi c}{l}\left(c_2 c_4 \sin \dfrac{n\pi x}{l}\right)$

$\Rightarrow \quad c_2 = 0.$

$\therefore$ From (4), $\quad y(x,\,t) = c_1 c_4 \cos \dfrac{n\pi ct}{l} \sin \dfrac{n\pi x}{l}$

$$= b_n \cos \frac{n\pi ct}{l} \sin \frac{n\pi x}{l} \qquad (\text{where } c_1 c_4 = b_n) \quad \text{...(5)}$$

The most general solution is

$$y(x,\,t) = \sum_{n=1}^{\infty} b_n \cos \frac{n\pi ct}{l} \sin \frac{n\pi x}{l} \quad \text{...(6)}$$

$$y(x,\,0) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \qquad \text{[From (6)]}$$

where $\quad b_n = \dfrac{2}{l}\displaystyle\int_0^l y(x,0) \sin \dfrac{n\pi x}{l}\, dx$

$$= \frac{2}{l}\left[\int_0^{l/2} \frac{2k}{l}\, x \sin \frac{n\pi x}{l}\, dx + \int_{l/2}^{l} \frac{2k}{l}\,(l - x) \sin \frac{n\pi x}{l}\, dx\right]$$

$$= \frac{4k}{l^2}\left[\int_0^{l/2} x \sin \frac{n\pi x}{l}\, dx + \int_{l/2}^{l} (l - x) \sin \frac{n\pi x}{l}\, dx\right]$$

$$= \frac{4k}{l^2}\left[\left\{x\left(\frac{-\cos \frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)\right\}_0^{l/2} - \int_0^{l/2} 1.\left(\frac{-\cos \frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) dx\right.$$

$$+ \left\{ (l - x)\left[\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right]_{l/2}^{l} - \int_{l/2}^{l}(-1)\cdot\left[\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right]dx\right\}$$

$$= \frac{4k}{l^2}\left[\left\{-\frac{l}{n\pi}\cdot\frac{l}{2}\cos\frac{n\pi}{2} + \frac{l}{n\pi}\left[\frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right]_0^{l/2}\right\} + \left\{\frac{l}{2}\cdot\frac{l}{n\pi}\cos\frac{n\pi}{2} - \frac{l}{n\pi}\left[\frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right]_{l/2}^{l}\right\}\right]$$

$$= \frac{4k}{l^2}\left[\frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2} - \frac{l^2}{n^2\pi^2}\left(\sin n\pi - \sin\frac{n\pi}{2}\right)\right]$$

$$= \frac{4k}{l^2}\left(\frac{2l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right) = \frac{8k}{n^2\pi^2}\sin\frac{n\pi}{2}.$$

∴ From (6), $y(x, t) = \frac{8k}{\pi^2}\sum_{1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{2}\cos\frac{n\pi ct}{l}\sin\frac{n\pi x}{l}$.

Problem. Solve the wave equation $\frac{\partial^2 u}{\partial x^2} = c^2\frac{\partial^2 u}{\partial x^2}$, $0 \le x \le l, t \ge 0$ subject to conditions

$$u(0,t) = 0 = u(l,t), u(x,0) = \begin{cases} x \text{ if } 0 \le x \le \frac{l}{2} \\ l - x \text{ if } \frac{l}{2} \le x \le l \end{cases} \text{ and } \frac{\partial u}{\partial t} = 0 \text{ at t = 0} \qquad \text{(OU 2017)}$$

Solution. In the above problem (5) If we take 2k = l we get

$$u(x,t) = \frac{4l}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{2}\cos\frac{n\pi ct}{l}\sin\frac{n\pi x}{l}$$

Example 6. *A tightly stretched violin string of length l and fixed at both ends is plucked at x =* $\frac{l}{3}$ *and assumes initially the shape of a triangle of height a. Find the displacement y at any distance x and any time t after the string is released from rest.*

Sol. One Dimensional wave equation is

$$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2} \qquad ...(1)$$

The solution of eqn. (1) is

$$y(x, t) = (c_1 \cos cpt + c_2 \sin cpt)(c_3 \cos px + c_4 \sin px) \qquad ...(2) \quad \text{(Refer sol. of Ex. 1)}$$

Eqn. of line OC is $y - 0 = \frac{a - 0}{\frac{l}{3} - 0}(x - 0)$

$$y = \frac{3a}{l}x \qquad ...(3)$$

Eqn. of line CA is $y - a = \frac{0 - a}{l - l/3}\left(x - \frac{l}{3}\right)$

$$y - a = -\frac{a}{\left(\frac{2l}{3}\right)}\left(x - \frac{l}{3}\right) = -\frac{3a}{2l}\left(x - \frac{l}{3}\right)$$

$$y - a = -\frac{3ax}{2l} + \frac{a}{2}$$

$$y = -\frac{3ax}{2l} + \frac{3a}{2} = \frac{3a}{2}\left(1 - \frac{x}{l}\right) \qquad \text{...(4)}$$

Hence the boundary conditions are

$$y(0, t) = 0 \qquad \text{...(5)}$$

$$y(l, t) = 0 \qquad \text{...(6)}$$

$$\frac{\partial y}{\partial t} = 0 \text{ at } t = 0 \qquad \text{...(7)}$$

and
$$y(x, 0) = \begin{cases} \dfrac{3ax}{l}, & 0 < x < l/3 \\ \dfrac{3a}{2}\left(1 - \dfrac{x}{l}\right), & \dfrac{l}{3} < x < l \end{cases} \qquad \text{...(8)}$$

From (2), $y(0, t) = \quad 0 = (c_1 \cos cpt + c_2 \sin cpt)\, c_3$

$\Rightarrow$ $c_3 = 0.$

$\therefore$ From (2), $y(x, t) = (c_1 \cos cpt + c_2 \sin cpt)\, c_4 \sin px$ ...(9)

$y(l, t) = 0 = (c_1 \cos cpt + c_2 \sin cpt)\, c_4 \sin pl$

$\Rightarrow$ $\sin pl = 0 = \sin n\pi (n \in \text{I}).$

$\Rightarrow$ $p = \dfrac{n\pi}{l}.$

$$\therefore \quad y(x, t) = \left(c_1 \cos \frac{n\pi ct}{l} + c_2 \sin \frac{n\pi ct}{l}\right) c_4 \sin \frac{n\pi x}{l} \qquad \text{...(10)}$$

$$\frac{\partial y}{\partial t} = \frac{n\pi c}{l}\left(-c_1 \sin \frac{n\pi ct}{l} + c_2 \cos \frac{n\pi ct}{l}\right) c_4 \sin \frac{n\pi x}{l}.$$

At $t = 0$,

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0 = \frac{n\pi c}{l}\left(c_2 c_4 \sin \frac{n\pi x}{l}\right)$$

$\Rightarrow$ $c_2 = 0.$

$$\therefore \quad y(x, t) = c_1 c_4 \cos \frac{n\pi ct}{l} \sin \frac{n\pi x}{l} = b_n \cos \frac{n\pi ct}{l} \sin \frac{n\pi x}{l}.$$

The most general solution is

$$y(x, t) = \sum_{n=1}^{\infty} b_n \cos \frac{n\pi ct}{l} \sin \frac{n\pi x}{l} \qquad \text{...(11)}$$

From (11), $y(x, 0) = \displaystyle\sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$, where

$$b_n = \frac{2}{l}\int_0^l y(x, 0) \sin \frac{n\pi x}{l}\, dx$$

$$= \frac{2}{l}\left[\int_0^{l/3} \frac{3ax}{l} \sin \frac{n\pi x}{l}\, dx + \int_{l/3}^{l} \frac{3a}{2}\left(1 - \frac{x}{l}\right) \sin \frac{n\pi x}{l}\, dx\right]$$

$$= \frac{2}{l}\left[\frac{3a}{l}\int_0^{l/3} x \sin \frac{n\pi x}{l}\, dx + \frac{3a}{2}\int_{l/3}^{l}\left(1 - \frac{x}{l}\right) \sin \frac{n\pi x}{l}\, dx\right]$$

$$= \frac{6a}{l^2}\left[\left\{x\left(\frac{-\cos \frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)\right\}_0^{l/3} - \int_0^{l/3} 1.\left(\frac{-\cos \frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) dx\right.$$

$$+ \frac{3a}{l}\left[\left\{\left(1-\frac{x}{l}\right)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)\right\}_{l/3}^{l} - \int_{l/3}^{l}\left(-\frac{1}{l}\right)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)dx\right]$$

$$= \frac{6a}{l^2}\left[-\frac{l}{n\pi}\cdot\frac{l}{3}\cos\frac{n\pi}{3} + \frac{l}{n\pi}\left\{\frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right\}_0^{l/3}\right] + \frac{3a}{l}\left[\frac{l}{n\pi}\cdot\frac{2}{3}\cos\frac{n\pi}{3} - \frac{1}{n\pi}\cdot\left\{\frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right\}_{l/3}^{l}\right]$$

$$= \frac{6a}{l^2}\left[-\frac{l^2}{3n\pi}\cos\frac{n\pi}{3} + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{3}\right] + \frac{3a}{l}\left[\frac{2l}{3n\pi}\cos\frac{n\pi}{3} - \frac{l}{n^2\pi^2}\left\{0-\sin\frac{n\pi}{3}\right\}\right]$$

$$= \frac{6a}{n\pi}\left[-\frac{1}{3}\cos\frac{n\pi}{3} + \frac{1}{n\pi}\sin\frac{n\pi}{3}\right] + \frac{6a}{n\pi}\left[\frac{1}{3}\cos\frac{n\pi}{3}\right] + \frac{3a}{n^2\pi^2}\sin\frac{n\pi}{3}$$

$$\Rightarrow \qquad b_n = \frac{9a}{n^2\pi^2}\sin\frac{n\pi}{3}$$

$\therefore$ From (11), $$y(x, t) = \frac{9a}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{3}\cos\frac{n\pi ct}{l}\sin\frac{n\pi x}{l}.$$

Example 7. *The points of trisection of a string are pulled aside through the same distance on opposite sides of the position of equilibrium and the string is released from rest. Derive an expression for the displacement of the string at subsequent time and show that the mid-point of the string always remains at rest.*

Sol. The equation for the vibration of the string is

$$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2} \qquad ...(1)$$

The solution of eqn. (1) is

$$y(x, t) = (c_1 \cos cpt + c_2 \sin cpt)(c_3 \cos px + c_4 \sin px) \qquad ...(2)$$

Let l be the length of string (Refer sol. of Ex. 1)

Equation of OB is,

$$y - 0 = \frac{h-0}{\frac{l}{3}-0}(x-0)$$

$$\Rightarrow \qquad y = \frac{3h}{l}x \qquad ...(3)$$

Equation of BC is,

$$y - h = \frac{-h-h}{\frac{2l}{3}-\frac{l}{3}}\left(x-\frac{l}{3}\right)$$

$$= \frac{-2h}{\left(\frac{l}{3}\right)}\left(x-\frac{l}{3}\right) = -\frac{6h}{l}\left(x-\frac{l}{3}\right)$$

$$y - h = -\frac{6hx}{l} + 2h$$

$$y = 3h - \frac{6hx}{l} = 3h\left(1-\frac{2x}{l}\right) \qquad ...(4)$$

Equation of CA is, $y + h = \frac{0+h}{l-\frac{2l}{3}}\left(x-\frac{2l}{3}\right) = \frac{3h}{l}\left(x-\frac{2l}{3}\right) = \frac{3hx}{l} - 2h$

$$y = \frac{3hx}{l} - 3h = 3h\left(\frac{x}{l} - 1\right) \qquad \text{...(5)}$$

Hence, Boundary conditions are

$$y(0, t) = 0, \quad y(l, t) = 0$$

$$\frac{\partial y}{\partial t} = 0 \quad \text{when } t = 0$$

and

$$y(x, 0) = \begin{cases} \dfrac{3h}{l} x, & 0 \le x \le l/3 \\ \dfrac{3h}{l}(l - 2x), & \dfrac{l}{3} \le x \le \dfrac{2l}{3} \\ \dfrac{3h}{l}(x - l), & \dfrac{2l}{3} \le x \le l \end{cases}$$

From (2), $\quad y(0, t) = 0 = (c_1 \cos cpt + c_2 \sin cpt)\, c_3$

$\Rightarrow \quad c_3 = 0.$

$\therefore$ From (2),

$$y(x, t) = (c_1 \cos cpt + c_2 \sin cpt)\, c_4 \sin px \qquad \text{...(6)}$$

$$y(l, t) = 0 = (c_1 \cos cpt + c_2 \sin cpt)\, c_4 \sin pl$$

$\Rightarrow \quad \sin pl = 0 = \sin n\pi \ (n \in \text{I})$

$\therefore \quad p = \dfrac{n\pi}{l}.$

$\therefore$ From (6), $\quad y(x, t) = \left(c_1 \cos \dfrac{n\pi ct}{l} + c_2 \sin \dfrac{n\pi ct}{l}\right) c_4 \sin \dfrac{n\pi x}{l} \qquad \text{...(7)}$

$$\frac{\partial y}{\partial t} = \frac{n\pi c}{l}\left(-c_1 \sin \frac{n\pi ct}{l} + c_2 \cos \frac{n\pi ct}{l}\right) c_4 \sin \frac{n\pi x}{l}.$$

At $t = 0$, $\quad \left(\dfrac{\partial y}{\partial t}\right)_{t=0} = 0 = \dfrac{n\pi c}{l} c_2 c_4 \sin \dfrac{n\pi x}{l}$

$\Rightarrow \quad c_2 = 0$

$\therefore$ From (7), $\quad y(x, t) = c_1 c_4 \cos \dfrac{n\pi ct}{l} \sin \dfrac{n\pi x}{l} = b_n \cos \dfrac{n\pi ct}{l} \sin \dfrac{n\pi x}{l}.$

The most general solution is

$$y(x, t) = \sum_{1}^{\infty} b_n \cos \frac{n\pi ct}{l} \sin \frac{n\pi x}{l} \qquad \text{...(8)}$$

$$y(x, 0) = \sum_{1}^{\infty} b_n \sin \frac{n\pi x}{l}, \text{ where}$$

$$b_n = \frac{2}{l}\int_0^l y(x, 0) \sin \frac{n\pi x}{l}\, dx$$

$$= \frac{2}{l}\left[\int_0^{l/3} \frac{3h}{l} x \sin \frac{n\pi x}{l}\, dx + \int_{l/3}^{2l/3} \frac{3h}{l}(l - 2x) \sin \frac{n\pi x}{l}\, dx + \int_{2l/3}^{l} \frac{3h}{l}(x - l) \sin \frac{n\pi x}{l}\, dx\right]$$

$$= \frac{2}{l} \cdot \frac{3h}{l}\int_0^{l/3} x \sin \frac{n\pi x}{l}\, dx + \frac{2}{l} \cdot \frac{3h}{l}\int_{l/3}^{2l/3} (l - 2x) \sin \frac{n\pi x}{l}\, dx + \frac{2}{l} \cdot \frac{3h}{l}\int_{2l/3}^{l} (x - l) \sin \frac{n\pi x}{l}\, dx$$

$$= \frac{6h}{l^2}\left[\left\{x\left(\frac{-\cos \frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)\right\}_0^{l/3} - \int_0^{l/3} 1 \cdot \left(\frac{-\cos \frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) dx\right]$$

$$+ \frac{6h}{l^2}\left[\left\{(l - 2x)\left(\frac{-\cos \frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)\right\}_{l/3}^{2l/3} - \int_{l/3}^{2l/3} (-2)\left(\frac{-\cos \frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) dx\right]$$

$$= \frac{6h}{l^2}\left[-\frac{l}{n\pi}\cdot\frac{l}{3}\cos\frac{n\pi}{3} + \frac{l}{n\pi}\left\{\frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right\}_0^{l/3}\right]$$

$$+ \frac{6h}{l^2}\left[\frac{l}{3}\cdot\frac{l}{n\pi}\left(-\cos\frac{2n\pi}{3}\right) - \left(-\cos\frac{n\pi}{3}\right)\frac{l}{3}\cdot\frac{l}{n\pi} + \frac{2l}{n\pi}\left\{\frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right\}_{l/3}^{2l/3}\right]$$

$$+ \frac{6h}{l^2}\left[\frac{l}{3}\cdot\frac{l}{n\pi}\cos\frac{2n\pi}{3} - \frac{l}{n\pi}\left\{\frac{\sin n\pi x/l}{n\pi/l}\right\}_{2l/3}^{l}\right]$$

$$= -\frac{2h}{n\pi}\cos\frac{n\pi}{3} + \frac{6h}{n^2\pi^2}\sin\frac{n\pi}{3} - \frac{2h}{n\pi}\cos\frac{2n\pi}{3}$$

$$+ \frac{2h}{n\pi}\cos\frac{n\pi}{3} + \frac{12h}{n^2\pi^2}\left(\sin\frac{2n\pi}{3} - \sin\frac{n\pi}{3}\right) + \frac{2h}{n\pi}\cos\frac{2n\pi}{3} - \frac{6h}{n^2\pi^2}\left(0 - \sin\frac{2n\pi}{3}\right)$$

$$= \frac{18h}{n^2\pi^2}\sin\frac{n\pi}{3} - \frac{18h}{n^2\pi^2}\sin\frac{2n\pi}{3} = \frac{18h}{n^2\pi^2}\sin\frac{n\pi}{3} - \frac{18h}{n^2\pi^2}\sin\left(n\pi - \frac{n\pi}{3}\right)$$

$$= \frac{18h}{n^2\pi^2}\sin\frac{n\pi}{3} + \frac{18h}{n^2\pi^2}\sin\frac{n\pi}{3}\cos n\pi$$

$$= \begin{cases} \dfrac{36h}{n^2\pi^2}\sin\dfrac{n\pi}{3}, & \text{when } n \text{ is even} \\ 0, & \text{when } n \text{ is odd} \end{cases}$$

$\therefore$ From (8), $y(x, t) = \dfrac{36h}{\pi^2}\displaystyle\sum_{n=2,4,\ldots}\dfrac{1}{n^2}\sin\dfrac{n\pi}{3}\cos\dfrac{n\pi ct}{l}\sin\dfrac{n\pi x}{l}$

$$y(x, t) = \frac{9h}{\pi^2}\sum_{m=1,2,\ldots}\frac{1}{m^2}\sin\frac{2m\pi}{3}\cos\frac{2m\pi ct}{l}\sin\frac{2m\pi x}{l} \qquad \text{...(9)}$$

(where $n = 2m$)

Putting $x = \dfrac{l}{2}$ in eqn. (6), we get

$$y\left(\frac{l}{2}, t\right) = \frac{9h}{\pi^2}\sum_{m=1}^{\infty}\sin\left(\frac{2m\pi}{3}\right)\cdot\frac{1}{m^2}\cdot\cos\frac{2m\pi ct}{l}\cdot\sin m\pi = 0.$$

Hence, mid-point of the string is always at rest.

Example 8. *If a string of length l is initially at rest in equilibrium position and each of its points is given the velocity* $\left(\dfrac{\partial y}{\partial t}\right)_{t=0} = b\sin^3\dfrac{\pi x}{l}$, *find the displacement y (x, t).*

Sol. The equation for the vibrations of the string is

$$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2} \qquad \text{...(1)}$$

The solution of equation (1) is

$$y(x, t) = (c_1\cos cpt + c_2\sin cpt)(c_3\cos px + c_4\sin px) \qquad \text{...(2)} \quad \text{[Refer sol. of Ex. 1]}$$

Boundary conditions are, $\quad y(0, t) = 0 \qquad$ ...(3)

$y(l, t) = 0 \qquad$ ...(4)

$$y(x, 0) = 0 \qquad \text{...(5)}$$

$$\left(\frac{\partial y}{\partial t}\right) = b \sin^3 \frac{\pi x}{l} \text{ at } t = 0 \qquad \text{...(6)}$$

From eqn. (2), $y(0, t) = 0 = (c_1 \cos cpt + c_2 \sin cpt)\, c_3$

$\Rightarrow \qquad c_3 = 0.$

$\therefore$ From (2), $y(x, t) = (c_1 \cos cpt + c_2 \sin cpt)\, c_4 \sin px \qquad \text{...(7)}$

$y(l, t) = 0 = (c_1 \cos cpt + c_2 \sin cpt)\, c_4 \sin pl$

$\Rightarrow \qquad \sin pl = 0 = \sin n\pi \; (n \in \text{I})$

$\therefore \qquad p = \dfrac{n\pi}{l}.$

$\therefore$ From (7), $y(x, t) = \left(c_1 \cos \dfrac{n\pi ct}{l} + c_2 \sin \dfrac{n\pi ct}{l}\right) c_4 \sin \dfrac{n\pi x}{l} \qquad \text{...(8)}$

$y(x, 0) = 0 = c_1 c_4 \sin \dfrac{n\pi x}{l}$

$\Rightarrow \qquad c_1 = 0.$

$\therefore$ From (8), $y(x, t) = c_2 c_4 \sin \dfrac{n\pi ct}{l} \sin \dfrac{n\pi x}{l}$

$= b_n \sin \dfrac{n\pi ct}{l} \sin \dfrac{n\pi x}{l}$ where $c_2 c_4 = b_n$

The general solution is

$$y(x, t) = \sum_{1}^{\infty} b_n \sin \frac{n\pi ct}{l} \sin \frac{n\pi x}{l} \qquad \text{...(9)}$$

$$\frac{\partial y}{\partial t} = \sum_{1}^{\infty} b_n \cdot \frac{n\pi c}{l} \cos \frac{n\pi ct}{l} \sin \frac{n\pi x}{l}$$

At $t = 0$, $\left(\dfrac{\partial y}{\partial t}\right)_{t=0} = \displaystyle\sum_{1}^{\infty} b_n \cdot \frac{n\pi c}{l} \sin \frac{n\pi x}{l}$

$$b \sin^3 \frac{\pi x}{l} = \sum_{1}^{\infty} b_n \cdot \frac{n\pi c}{l} \sin \frac{n\pi x}{l}$$

$$\frac{b}{4}\left(3 \sin \frac{\pi x}{l} - \sin \frac{3\pi x}{l}\right) = b_1 \frac{\pi c}{l} \sin \frac{\pi x}{l} + \frac{2 b_2 \pi c}{l} \sin \frac{2\pi x}{l} + 3 b_3 \frac{\pi c}{l} \sin \frac{3\pi x}{l} + \ldots$$

$\Rightarrow \qquad b_1 \dfrac{\pi c}{l} = \dfrac{3b}{4} \quad \Rightarrow \quad b_1 = \dfrac{3bl}{4\pi c}$

$b_2 = 0$ and $\dfrac{3 b_3 \pi c}{l} = -\dfrac{b}{4} \quad \Rightarrow \quad b_3 = -\dfrac{bl}{12\pi c}$

Also, $b_4 = 0 = b_5 = \ldots$ etc.

Hence from (9), $y(x, t) = \dfrac{3bl}{4\pi c} \sin \dfrac{\pi ct}{l} \sin \dfrac{\pi x}{l} - \dfrac{bl}{12\pi c} \sin \dfrac{3\pi ct}{l} \sin \dfrac{3\pi x}{l}$

$= \dfrac{bl}{12\pi c}\left(9 \sin \dfrac{\pi x}{l} \sin \dfrac{\pi ct}{l} - \sin \dfrac{3\pi x}{l} \sin \dfrac{3\pi ct}{l}\right).$

Example 9. *A tightly stretched string with fixed end points x = 0 and x = l is initially at rest in its equilibrium position. If it is set vibrating by giving to each of its points an initial velocity* $\lambda x(l - x)$*, find the displacement of the string at any distance x from one end at any time t.* [M.T.U., (SUM) 2011; JNTUK, (Set 1) 2015]

Sol. Here the boundary conditions are $y(0, t) = y(l, t) = 0$

$$y(x, t) = \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi ct}{l} + b_n \sin\frac{n\pi ct}{l}\right)\sin\frac{n\pi x}{l} \qquad \text{...(1)} \quad |\text{ Refer sol. of Ex. 2}$$

Since the string was at rest initially, $y(x, 0) = 0$

$\therefore$ From (1), $\quad 0 = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{l} \;\Rightarrow\; a_n = 0$

$$\therefore \quad y(x, t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi ct}{l}\sin\frac{n\pi x}{l} \qquad \text{...(2)}$$

and
$$\frac{\partial y}{\partial t} = \sum_{n=1}^{\infty}\frac{n\pi c}{l} b_n \cos\frac{n\pi ct}{l}\sin\frac{n\pi x}{l} = \frac{\pi c}{l}\sum_{n=1}^{\infty} nb_n \cos\frac{n\pi ct}{l}\sin\frac{n\pi x}{l}$$

But $\quad \frac{\partial y}{\partial t} = \lambda x(l - x) \quad$ when $t = 0$

$$\therefore \quad \lambda x(l - x) = \frac{\pi c}{l}\sum_{n=1}^{\infty} nb_n \sin\frac{n\pi x}{l}$$

$$\Rightarrow \quad \frac{\pi c}{l} nb_n = \frac{2}{l}\int_0^l \lambda x(l - x)\sin\frac{n\pi x}{l}\, dx$$

$$= \frac{2\lambda}{l}\left[x(l - x)\left(-\frac{l}{n\pi}\cos\frac{n\pi x}{l}\right) - (l - 2x)\left(-\frac{l^2}{n^2\pi^2}\sin\frac{n\pi x}{l}\right) + (-2)\left(\frac{l^3}{n^3\pi^3}\cos\frac{n\pi x}{l}\right)\right]_0^l$$

$$= \frac{4\lambda l^2}{n^3\pi^3}(1 - \cos n\pi) = \frac{4\lambda l^2}{n^3\pi^3}[1 - (-1)^n]$$

$$= \begin{cases} 0, & \text{when } n \text{ is even} \\ \dfrac{8\lambda l^2}{n^3\pi^3}, & \text{when } n \text{ is odd} \end{cases} \quad i.e., \ \frac{8\lambda l^2}{\pi^3(2m - 1)^3}, \text{ taking } n = 2m - 1$$

$$\Rightarrow \quad b_n = \frac{8\lambda l^3}{c\pi^4(2m - 1)^4}$$

$\therefore$ From (2), the required solution is

$$y(x, t) = \frac{8\lambda l^3}{c\pi^4}\sum_{m=1}^{\infty}\frac{1}{(2m - 1)^4}\sin\frac{(2m - 1)\pi ct}{l}\sin\frac{(2m - 1)\pi x}{l}.$$

Example 10. *Transform the equation* $\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2}$ *to its normal form using the transformation* $u = x + ct$, $v = x - ct$ *and hence solve it. Show that the solution may be put in the form* $y = \frac{1}{2}[f(x + ct) + f(x - ct)]$. (M.T.U., 2013)

Assume initial conditions $y = f(x)$ *and* $(\partial y/\partial t) = 0$ *at* $t = 0$.

Sol. One dimensional wave equation is

$$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2} \qquad \text{...(1)}$$

Let us introduce two new independent variables

$$u = x + ct \qquad \text{...(2)}$$

and
$$v = x - ct \qquad \text{...(3)}$$

so that y becomes a function of u and v.

Then,
$$\frac{\partial y}{\partial x} = \frac{\partial y}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial y}{\partial v}\cdot\frac{\partial v}{\partial x} = \frac{\partial y}{\partial u} + \frac{\partial y}{\partial v} \qquad \text{...(4)} \quad [\text{Using (2) and (3)}]$$

Also, $$\frac{\partial}{\partial x} \equiv \frac{\partial}{\partial u} + \frac{\partial}{\partial v} \qquad ...(5)$$

$\therefore$ $$\frac{\partial^2 y}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial y}{\partial x}\right) = \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)\left(\frac{\partial y}{\partial u} + \frac{\partial y}{\partial v}\right)$$

$$= \frac{\partial^2 y}{\partial u^2} + 2\frac{\partial^2 y}{\partial u\,\partial v} + \frac{\partial^2 y}{\partial v^2} \qquad ...(6)$$

Also, $$\frac{\partial y}{\partial t} = \frac{\partial y}{\partial u}\cdot\frac{\partial u}{\partial t} + \frac{\partial y}{\partial v}\cdot\frac{\partial v}{\partial t} = c\frac{\partial y}{\partial u} + \frac{\partial y}{\partial v}(-c) = c\left(\frac{\partial y}{\partial u} - \frac{\partial y}{\partial v}\right) \qquad ...(7)$$

$\Rightarrow$ $$\frac{\partial}{\partial t} \equiv c\left(\frac{\partial}{\partial u} - \frac{\partial}{\partial v}\right) \qquad ...(8)$$

$\therefore$ $$\frac{\partial^2 y}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial y}{\partial t}\right) = c\left(\frac{\partial}{\partial u} - \frac{\partial}{\partial v}\right) c\left(\frac{\partial y}{\partial u} - \frac{\partial y}{\partial v}\right)$$

$$= c^2\left(\frac{\partial^2 y}{\partial u^2} - 2\frac{\partial^2 y}{\partial u\,\partial v} + \frac{\partial^2 y}{\partial v^2}\right) \qquad ...(9)$$

From (1), (6) and (9), we have

$$c^2\left(\frac{\partial^2 y}{\partial u^2} - 2\frac{\partial^2 y}{\partial u\,\partial v} + \frac{\partial^2 y}{\partial v^2}\right) = c^2\left(\frac{\partial^2 y}{\partial u^2} + 2\frac{\partial^2 y}{\partial u\,\partial v} + \frac{\partial^2 y}{\partial v^2}\right)$$

$\Rightarrow$ $$-4c^2\frac{\partial^2 y}{\partial u\,\partial v} = 0$$

$\Rightarrow$ $$\frac{\partial^2 y}{\partial u\,\partial v} = 0 \qquad ...(10) \quad (\because\ c^2 \neq 0)$$

Integrating eqn. (10) partially, w.r.t. u, we get

$$\frac{\partial y}{\partial v} = f_1(v).$$

Integrating again w.r.t. v partially, we get

$$y = \int f_1(v)\,\partial v + \psi(u) = \phi(v) + \psi(u)$$

$\Rightarrow$ $$y(x, t) = \phi(x - ct) + \psi(x + ct) \qquad ...(11)$$

which is d' Alembert's solution of wave equation.

Applying initial conditions $y = f(x)$ and $\frac{\partial y}{\partial t} = 0$ at $t = 0$ in (11), we get

$$f(x) = \phi(x) + \psi(x) \text{ and } 0 = -\phi'(x) + \psi'(x)$$

Hence, $$\phi(x) = \psi(x) = \frac{1}{2}f(x)$$

$\therefore$ $$y = \frac{1}{2}[f(x + ct) + f(x - ct)].$$

Example 11. *A tightly stretched string with fixed end points $x = 0$ and $x = \pi$ is initially at rest in its equilibrium position. If it is set vibrating by giving to each of its points an initial velocity*

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0.03 \sin x - 0.04 \sin 3x,$$

then find the displacement $y(x, t)$ at any point of string at any time t.

Sol. The equation for the vibrations of a string is

$$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2} \qquad ...(1)$$

Its solution is

$$y(x, t) = (c_1 \cos pct + c_2 \sin pct)(c_3 \cos px + c_4 \sin px) \qquad ...(2)$$

Boundary conditions are

$$y(0, t) = 0 = y(\pi, t)$$

$$y(x, 0) = 0$$

and $$\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0.03 \sin x - 0.04 \sin 3x.$$

From (2), $$y(0, t) = 0 = (c_1 \cos pct + c_2 \sin pct)\, c_3$$

$\Rightarrow$ $$c_3 = 0$$

From (2), $$y(x, t) = (c_1 \cos pct + c_2 \sin pct)\, c_4 \sin px \quad \text{...(3)}$$

$$y(\pi, t) = 0 = (c_1 \cos pct + c_2 \sin pct)\, c_4 \sin p\pi$$

$\Rightarrow$ $$\sin p\pi = 0 = \sin n\pi \ (n \in \text{I})$$

$\Rightarrow$ $$p = n$$

From (3), $$y(x, t) = (c_1 \cos nct + c_2 \sin nct)\, c_4 \sin nx \quad \text{...(4)}$$

$$y(x, 0) = 0 = c_1 c_4 \sin nx$$

$\Rightarrow$ $$c_1 = 0.$$

$\therefore$ From (4), $$y(x, t) = c_2 c_4 \sin nct \sin nx = b_n \sin nct \sin nx \quad \text{...(5)}$$

where $$c_2 c_4 = b_n$$

The most general solution is

$$y(x, t) = \sum_{n=1}^{\infty} b_n \sin nct \sin nx$$

$$\frac{\partial y}{\partial t} = \sum_{n=1}^{\infty} nc\, b_n \cos nct \sin nx$$

At $t = 0$,

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = \sum_{n=1}^{\infty} nc\, b_n \sin nx$$

$$0.03 \sin x - 0.04 \sin 3x = cb_1 \sin x + 2cb_2 \sin 2x + 3cb_3 \sin 3x + \ldots$$

$\Rightarrow$ $$cb_1 = 0.03 \quad \Rightarrow \quad b_1 = \frac{0.03}{c}$$

$$b_2 = 0$$

and $$3cb_3 = -0.04 \quad \Rightarrow \quad b_3 = \frac{-0.0133}{c}.$$

$\therefore$ From (6), $$y(x, t) = \frac{0.03}{c} \sin ct \sin x - \frac{0.0133}{c} \sin 3ct \sin 3x$$

$$= \frac{1}{c}[0.03 \sin x \sin ct - 0.0133 \sin 3x \sin 3ct].$$

EXERCISE

1. Find the deflection $y(x, t)$ of the vibrating string of length π and ends fixed, corresponding to zero initial velocity and initial deflection $f(x) = k(\sin x - \sin 2x)$ given $c^2 = 1$.

2. Solve: $y_{tt} = 4y_{xx}$; $y(0, t) = 0 = y(5, t)$, $y(x, 0) = 0$ $\left(\frac{\partial y}{\partial t}\right)_{t=0} = f(x)$

if (*i*) $f(x) = 5 \sin \pi x$ (*ii*) $f(x) = 3 \sin 2\pi x - 2 \sin 5\pi x$.

3. Find the deflection of the vibrating string which is fixed at the ends $x = 0$ and $x = 2$ and the motion is started by displacing the string into the form $\sin^3\left(\frac{\pi x}{2}\right)$ and releasing it with zero initial velocity at $t = 0$.

(*M.T.U., 2012*)

4. Find the solution of the equation of a vibrating string of length l satisfying the initial conditions:

$$y = F(x) \quad \text{when } t = 0$$

and $$\frac{\partial y}{\partial t} = \phi(x) \quad \text{when } t = 0$$

It is assumed that the equation of a vibrating string is $y_{tt} = a^2 y_{xx}$.

5. The vibrations of an elastic string is governed by the partial differential equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$$

The length of the string is π and ends are fixed. The initial velocity is zero and the initial deflection is $u(x, 0) = 2(\sin x + \sin 3x)$. Find the deflection $u(x, t)$ of the vibrating string at any time t.

6. A tight string of length 20 cms fastened at both ends is displaced from its position of equilibrium by imparting to each of its points an initial velocity given by

$$v = \begin{cases} x; & 0 \le x \le 10 \\ 20 - x; & 10 \le x \le 20 \end{cases};$$

x being the distance from one end. Determine the displacement at any subsequent time.

7. Using d' Alembert's method, find the deflection of a vibrating string of unit length having fixed ends, with initial velocity zero and initial deflection $f(x) = a(x - x^3)$.

8. Reduce the equation $u_{xx} - 2u_{xy} + u_{yy} = 0$ to its normal form using the transformation $v = x, z = x + y$ and solve it.

9. Solve the equation $\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial x \partial y} - 2\frac{\partial^2 u}{\partial y^2} = 0$ using the transformation $v = x + y$, $z = 2x - y$.

10. The ends of a tightly stretched string of length l are fixed at $x = 0$ and $x = l$. The string is at rest with the point $x = b$ drawn aside through a small distance d and released at time $t = 0$. Show that

$$y(x, t) = \frac{2dl^2}{\pi^2 b(l - b)} \sum_{n=1}^{\infty} \frac{1}{n^2} \sin \frac{n\pi b}{l} \sin \frac{n\pi x}{l} \cos \frac{n\pi ct}{l}.$$

11. Find the deflection of the vibrating string of unit length whose end points are fixed if the initial velocity is zero and the initial deflection is given by $u(x, 0) = \begin{cases} 1, & 0 \le x \le \frac{1}{2} \\ -1, & \frac{1}{2} < x \le 1 \end{cases}$ (*G.B.T.U., 2012*)

12. (*i*) Find the deflection $u(x, t)$ of a tightly stretched vibrating string of unit length that is initially at rest and whose initial position is given by

$$\sin \pi x + \frac{1}{3} \sin 3\pi x + \frac{1}{5} \sin 5\pi x, \quad 0 \le x \le 1$$ (*G.B.T.U., 2013*)

(*ii*) A string is stretched and fastened to two points distance l apart. Find the displacement $y(x, t)$ at any point at a distance x from one end at time t given that:

$$y(x, 0) = A \sin\left(\frac{2\pi x}{l}\right)$$ (*M.T.U., 2013*)

Answers

1. $y(x, t) = k(\cos t \sin x - \cos 2t \sin 2x)$

2. (*i*) $y(x, t) = \frac{5}{2} \sin \pi x \sin 2\pi t$ (*ii*) $y(x, t) = \frac{3}{4} \sin 2\pi x \sin 4\pi t - \frac{1}{5} \sin 5\pi x \sin 10\pi t$

3. $y(x, t) = \frac{3}{4} \sin \frac{\pi x}{2} \cos \frac{\pi ct}{2} - \frac{1}{4} \sin \frac{3\pi x}{2} \cos \frac{3\pi ct}{2}$

4. $y(x, t) = \sum_{n=1}^{\infty} \sin \frac{n\pi x}{l} \left[a_n \cos \frac{n\pi at}{l} + b_n \sin \frac{n\pi at}{l}\right]$

where $a_n = \frac{2}{l} \int_0^l F(x) \sin \frac{n\pi x}{l} \, dx$ and $b_n = \frac{2}{n\pi a} \int_0^l \phi(x) \sin \frac{n\pi x}{l} \, dx$

5. $y(x, t) = 2[\cos t \sin x + \cos 3t \sin 3x]$

6. $y(x, t) = \frac{1600}{a\pi^3} \left[\sin \frac{\pi x}{20} \sin \frac{\pi at}{20} - \frac{1}{3^3} \sin \frac{3\pi x}{20} \sin \frac{3\pi at}{20} + \ldots\right]$

7. $y(x, t) = ax(1 - x^2 - 3c^2t^2)$

8. $\frac{\partial^2 u}{\partial v^2} = 0\,; u = xf_1(x + y) + f_2(x + y)$.

9. $\frac{\partial^2 u}{\partial v\, \partial z} = 0\,; u = f_1(x + y) + f_2(2x - y)$

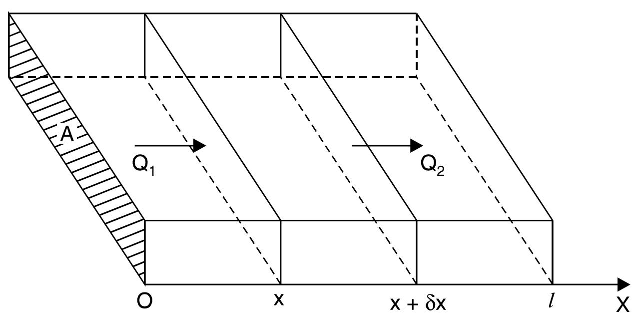
11. $y(x, t) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n}\left[1 + \cos n\pi - 2\cos \frac{n\pi}{2}\right] \sin n\pi x \cos n\pi ct$

12. (*i*) $u(x, t) = \sin \pi x \cos \pi ct + \frac{1}{3} \sin 3\pi x \cos 3\pi ct + \frac{1}{5} \sin 5\pi x \cos 5\pi ct$

(*ii*) $y(x, t) = A \sin \left(\frac{2\pi x}{l}\right) \cos \left(\frac{2\pi ct}{l}\right)$

5.4 SOLUTION OF ONE-DIMENSIONAL HEAT FLOW EQUATION $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$

Consider the flow of heat by conduction in a uniform bar. It is assumed that the sides of the bar are insulated and the loss of heat from the sides by conduction or radiation is negligible. Take one end of the bar as origin and the direction of flow as the positive *x*-axis. The temperature *u* at any point of the bar depends on the distance *x* of the point from one end and the time *t*. Also, the temperature of all points of any cross-section is the same.



The amount of heat crossing any section of the bar per second depends on the area A of the cross-section, the conductivity K of the material of the bar and the temperature gradient $\frac{\partial u}{\partial x}$ *i.e.*, rate of change of temperature w.r.t. distance normal to the area.

∴ Q_1, the quantity of heat flowing into the section at a distance *x*

$$= -KA\left(\frac{\partial u}{\partial x}\right)_x \text{ per sec.}$$

(the negative sign on the right is attached because as *x* increases, *u* decreases).

Q_2, the quantity of heat flowing out of the section at a distance $x + \delta x$

$$= -KA\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} \text{ per sec.}$$

Hence the amount of heat retained by the slab with thickness δx is

$$Q_1 - Q_2 = KA\left[\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial x}\right)_x\right] \text{ per sec} \quad ...(1)$$

But the rate of increase of heat in the slab $= s\rho A\, \delta x \frac{\partial u}{\partial x}$ ...(2)

where s is the specific heat and ρ, the density of the material.

$\therefore$ From (1) and (2), $s\rho A\,\delta x\dfrac{\partial u}{\partial t} = KA\left[\left(\dfrac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\dfrac{\partial u}{\partial x}\right)_x\right]$

or $$s\rho\frac{\partial u}{\partial t} = K\left[\frac{\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial x}\right)_x}{\delta x}\right]$$

Taking the limit as $\delta x \to 0$, we have

$$s\rho\frac{\partial u}{\partial t} = K\frac{\partial^2 u}{\partial x^2} \quad \text{or} \quad \frac{\partial u}{\partial t} = \frac{K}{s\rho}\frac{\partial^2 u}{\partial x^2}$$

or $$\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2}, \text{ where } c^2 = \frac{K}{s\rho}$$

is known as diffusivity of the material of the bar.

Consider the heat equation $$\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2} \quad \text{...(1)}$$

Let $$u = XT \quad \text{...(2)}$$

be a solution of (1), where X is a function of x only and T is a function of t only.

Then $$\frac{\partial u}{\partial t} = XT' \quad \text{and} \quad \frac{\partial^2 u}{\partial x^2} = X''T$$

Substituting in (1), we have $XT' = c^2X''T$

Separating the variables, we get $$\frac{X''}{X} = \frac{1}{c^2}\frac{T'}{T} \quad \text{...(3)}$$

Now, the LHS of (3) is a function of x only and the RHS is a function of t only. Since x and t are independent variables, this equation can hold only when both sides reduce to a constant, say k. Then equation (3) leads to the ordinary differential equations

$$\frac{d^2X}{dx^2} - kX = 0 \quad \text{and} \quad \frac{dT}{dt} - kc^2T = 0 \quad \text{...(4)}$$

Solving equations (4), we get

(*i*) *When k is positive* and = p^2, say

$$X = c_1e^{px} + c_2e^{-px}, \; T = c_3e^{c^2p^2t}$$

(*ii*) *When k is negative* and = $-p^2$, say

$$X = c_1\cos px + c_2\sin px, \; T = c_3e^{c^2p^2t}$$

(*iii*) *When k = 0*

$$X = c_1x + c_2, \; T = c_3.$$

Thus the various possible solutions of the heat equation (1) are:

$$u = (c_1e^{px} + c_2e^{-px}).c_3e^{c^2p^2t}$$

$$u = (c_1\cos px + c_2\sin px)\,.\,c_3e^{c^2p^2t}$$

$$u = (c_1 x + c_2)c_3.$$

Of these three solutions, we have to choose that solution which is consistent with the physical nature of the problem. Since u decreases as time t increases, the only suitable solution of the heat equation is

$$u = (c_1 \cos px + c_2 \sin px)\, e^{-c^2p^2t}.$$

SOLVED PROBLEMS

Example 1. *A rod of length l with insulated sides is initially at a uniform temperature u_0. Its ends are suddenly cooled to 0°C and are kept at that temperature. Find the temperature function u(x, t).* (G.B.T.U., 2011)

Sol. The temperature function $u(x, t)$ satisfies the differential equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

As proved in section (5.3), we have

$$u(x, t) = (c_1 \cos px + c_2 \sin px)e^{-c^2p^2t} \qquad ...(1)$$

Since, the ends $x = 0$ and $x = l$ are cooled to 0°C and kept at that temperature throughout, the boundary conditions are $u(0, t) = u(l, t) = 0$ for all t

Also, $u(x, 0) = u_0$ is the initial condition.

Since $u(0, t) = 0$, we have from (1), $0 = c_1\, e^{-c^2p^2t} \Rightarrow c_1 = 0$

$\Rightarrow$ From (1), $$u(x, t) = c_2 \sin px.\, e^{-c^2p^2t} \qquad ...(2)$$

Since $u(l, t) = 0$, we have from (2), $0 = c_2 \sin pl \,.\, e^{-c^2p^2t}$

$\Rightarrow$ $\sin pl = 0 \Rightarrow pl = n\pi$

$\therefore$ $p = \dfrac{n\pi}{l}$, n being an integer

Solution (2) reduces to $u(x, t) = b_n \sin \dfrac{n\pi x}{l} .\, e^{\frac{-c^2n^2\pi^2t}{l^2}}$ on replacing c_2 by b_n.

The most general solution is obtained by adding all such solutions for $n = 1, 2, 3, ...$

$$\therefore \quad u(x, t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} .\, e^{\frac{-c^2n^2\pi^2t}{l^2}} \qquad ...(3)$$

Since $u(x, 0) = u_0$, we have $u_0 = \displaystyle\sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$

which is half-range sine series for u_0.

$$\therefore \quad b_n = \frac{2}{l}\int_0^l u_0 \sin \frac{n\pi x}{l}\, dx = \begin{cases} 0, & \text{when } n \text{ is even} \\ \dfrac{4u_0}{n\pi}, & \text{when } n \text{ is odd} \end{cases}$$

Hence the temperature function

$$u(x, t) = \frac{4u_0}{\pi} \sum_{n=1,3,5,......} \frac{1}{n} \sin \frac{n\pi x}{l}\, e^{-\frac{c^2n^2\pi^2t}{l^2}}$$

$$= \frac{4u_0}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n-1} \sin \frac{(2n-1)\pi x}{l} e^{-\frac{c^2(2n-1)^2\pi^2 t}{l^2}}.$$

Example 2. *Find the temperature in a bar of length 2 whose ends are kept at zero and lateral surface insulated if the initial temperature is sin* $\frac{\pi x}{2} + 3\sin\frac{5\pi x}{2}$. (M.T.U., 2011)

Sol. Let $u(x, t)$ be the temperature in the bar. The boundary conditions are

$$u(0, t) = 0 = u(2, t) \text{ for any } t. \quad \text{...(1)}$$

The initial condition is

$$u(x, 0) = \sin \frac{\pi x}{2} + 3 \sin \frac{5\pi x}{2} \quad \text{...(2)}$$

One-dimensional heat flow equation is

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{...(3)}$$

Its solution is

$$u(x, t) = (c_1 \cos px + c_2 \sin px)\, c_3\, e^{-c^2p^2t} \quad \text{...(4)}$$

$$u(0, t) = 0 = c_1 c_3 e^{-c^2p^2t} \quad \text{| Using (1)}$$

$$\Rightarrow \quad c_1 = 0$$

$\therefore$ From (4),

$$u(x, t) = c_2 c_3 \sin px\; e^{-c^2p^2t} \quad \text{...(5)}$$

$$u(2, t) = 0 = c_2 c_3 \sin 2p\, e^{-c^2p^2t} \quad \text{| Using (1)}$$

$$\Rightarrow \quad \sin 2p = 0 = \sin n\pi$$

$$\therefore \quad p = \frac{n\pi}{2},\ n \in \mathrm{I}$$

Hence from (5),

$$u(x, t) = b_n \sin \frac{n\pi x}{2}\; e^{-\frac{n^2\pi^2c^2t}{4}} \quad \text{| } \because \ c_2c_3 = b_n$$

The most general solution is

$$u(x, t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{2}\; e^{-\frac{n^2\pi^2c^2t}{4}} \quad \text{...(6)}$$

$$u(x, 0) = \sin\left(\frac{\pi x}{2}\right) + 3\sin\left(\frac{5\pi x}{2}\right) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{2}$$

$$= b_1 \sin\left(\frac{\pi x}{2}\right) + b_2 \sin\left(\frac{2\pi x}{2}\right) + \dots + b_5 \sin\left(\frac{5\pi x}{2}\right) + \dots$$

Comparing, we get

$$b_1 = 1 \text{ and } b_5 = 3$$

Hence from (6),

$$u(x, t) = \sin\left(\frac{\pi x}{2}\right) e^{-\pi^2 c^2 t/4} + 3 \sin\left(\frac{5\pi x}{2}\right) e^{-25\pi^2 c^2 t/4}.$$

Example 3. *An insulated rod of length l has its ends A and B maintained at 0°C and 100°C respectively until steady state conditions prevail. If B is suddenly reduced to 0°C and maintained at 0°C, find the temperature at a distance x from A at time t.*

Find also the temperature if the change consists of raising the temperature of A to 20°C and reducing that of B to 80°C.

Sol. Initial temperature distribution in the rod is

$$u_1 = 0 + \left(\frac{100 - 0}{l}\right) x = \frac{100}{l} x$$

Final temperature distribution (*i.e.*, in steady state) is

$$u_2 = 0 + \left(\frac{0 - 0}{l}\right) x = 0$$

To get u in the intermediate period,

$$u = u_2(x) + u_1(x, t)$$

where $u_2(x)$ is the steady state temperature distribution in the rod. $u_1(x, t)$ is the transient temperature distribution which tends to zero as t increases.

$u_1(x, t)$ satisfies one dimensional heat flow equation

$\therefore \quad u(x, t) = \sum_{n=1}^{\infty} (a_n \cos px + b_n \sin px)\, e^{-c^2 p^2 t}$ ...(1)

In steady state, $\quad u(0, t) = 0 = u(l, t)$ ...(2)

$\therefore$ From (1), $\quad u(0, t) = 0 = \sum_{n=1}^{\infty} a_n e^{-c^2 p^2 t} \Rightarrow a_n = 0$ ...(3)

Also, $\quad u(l, t) = 0 = \sum_{n=1}^{\infty} b_n \sin pl\, e^{-c^2 p^2 t}$ | Using (3)

$\Rightarrow \quad \sin pl = 0 = \sin n\pi,\ n \in I$

or $\quad p = \frac{n\pi}{l}$ ...(4)

$\therefore$ From (1), (3) and (4), $u(x, t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}\, e^{-\left(\frac{n\pi}{l}\right)^2 c^2 t}$ ...(5)

Using initial condition,

$$u(x, 0) = \frac{100}{l} x = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

which is half-range sine series for $\frac{100}{l} x$.

$$\therefore \quad b_n = \frac{2}{l} \int_0^l \frac{100}{l} x \sin \frac{n\pi x}{l}\, dx$$

Hence the temperature function

$$u(x, t) = -\frac{200}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin\frac{n\pi x}{l}\,e^{-\frac{n^2\pi^2c^2t}{l^2}}$$

In the second part, the initial condition remains the same as in first part *i.e.,*

$$u(x, 0) = \frac{100}{l}x.$$

Boundary conditions are $u(0, t) = 20$ and $u(l, t) = 80$ for all values of t then, final temperature distribution is

$$u_2 = 20 + \left(\frac{80-20}{l}\right)x = 20 + \frac{60}{l}x$$

Then, $$u = u_2(x) + u_1(x, t)$$

$$u = 20 + \frac{60}{l}x + \sum_{n=1}^{\infty}(a_n\cos px + b_n\sin px)\,e^{-c^2p^2t} \qquad \text{...(6)}$$

$$u(0, t) = 20 = 20 + \sum_{n=1}^{\infty}a_n e^{-c^2p^2t} \qquad \text{| From (6)}$$

$\Rightarrow$ $$a_n = 0$$

$\therefore$ From (6), $$u = 20 + \frac{60}{l}x + \sum_{n=1}^{\infty}b_n\sin px\,e^{-c^2p^2t} \qquad \text{...(7)}$$

$$u(l, t) = 80 = 20 + \frac{60}{l}l + \sum_{n=1}^{\infty}b_n\sin pl\,e^{-c^2p^2t} \qquad \text{| From (7)}$$

$\Rightarrow$ $$0 = \sum_{n=1}^{\infty}b_n\sin pl\,e^{-c^2p^2t}$$

$$\sin pl = 0 = \sin n\pi,\ n \in \text{I}$$

$\therefore$ $$p = \frac{n\pi}{l} \qquad \text{...(8)}$$

From (7) and (8), $$u = 20 + \frac{60}{l}x + \sum_{n=1}^{\infty}b_n\sin\frac{n\pi x}{l}\,e^{-\left(\frac{n\pi c}{l}\right)^2 t} \qquad \text{...(9)}$$

Using initial condition,

$$u(x, 0) = \frac{100}{l}x = 20 + \frac{60}{l}x + \sum_{n=1}^{\infty}b_n\sin\frac{n\pi x}{l}$$

$\Rightarrow$ $$\frac{40}{l}x - 20 = \sum_{n=1}^{\infty}b_n\sin\frac{n\pi x}{l}$$

where $$b_n = \frac{2}{l}\int_0^l\left(\frac{40}{l}x - 20\right)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\left\{\left[\left(\frac{40}{l}x - 20\right)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)\right]_0^l - \int_0^l \frac{40}{l}\cdot\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)dx\right\}$$

$$= \frac{2}{l}\left\{-\frac{20l}{n\pi}\cos n\pi - \frac{20l}{n\pi} + \frac{40}{n\pi}\left[\frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right]_0^l\right\}$$

$$= \frac{-40}{n\pi}(1+\cos n\pi) = \begin{cases} 0, & \text{when } n \text{ is odd} \\ -\dfrac{80}{n\pi}, & \text{when } n \text{ is even}\end{cases}$$

Hence equation (9) becomes,

$$u(x, t) = 20 + \frac{60}{l}x - \frac{80}{\pi}\sum_{\substack{n=2,4,\ldots \\ (n \text{ is even})}} \frac{1}{n}\sin\frac{n\pi x}{l}\, e^{-\left(\frac{n\pi c}{l}\right)^2 t}$$

$$= 20 + \frac{60}{l}x - \frac{40}{\pi}\sum_{m=1}^{\infty}\frac{1}{m}\sin\frac{2m\pi x}{l}\, e^{-\frac{4c^2m^2\pi^2 t}{l^2}}. \qquad (\text{taking } n = 2m)$$

Example 4. *The ends A and B of a rod of length 20 cm are at temperatures 30°C and 80°C until steady state prevails. Then the temperature of the rod ends are changed to 40°C and 60°C respectively. Find the temperature distribution function u(x, t). The specific heat, density and the thermal conductivity of the material of the rod are such that the combination* $\frac{k}{s\rho} = c^2 = 1$.

Sol. Initial temperature distribution in the rod is

$$u_1 = 30 + \left(\frac{80-30}{20}\right)x = 30 + \frac{5}{2}x$$

Final temperature distribution (*i.e.*, in steady state) is

$$u_2 = 40 + \left(\frac{60-40}{20}\right)x = 40 + x$$

To get u in the intermediate period,

$$u = u_1(x, t) + u_2(x)$$

where $u_2(x)$ is the steady state temperature distribution in the rod $u_1(x, t)$ is the transient temperature distribution which tends to zero as t increases.

$\because$ $u_1(x, t)$ satisfies one dimensional heat flow equation.

$$\therefore \qquad u = 40 + x + \sum_{n=1}^{\infty}(a_n \cos px + b_n \sin px)\, e^{-p^2 t} \qquad ...(1)$$

In steady state,

$$u(0, t) = 40 \qquad ...(2)$$
$$u(20, t) = 60 \qquad ...(3)$$

From (1), and (2), $u(0, t) = \quad 40 = 40 + \sum_{n=1}^{\infty} a_n e^{-p^2 t}$ | From (2)

$$0 = \sum_{n=1}^{\infty} a_n e^{-p^2 t} \Rightarrow a_n = 0 \qquad \text{...(4)}$$

$\therefore$ From (1),

$$u = 40 + x + \sum_{n=1}^{\infty} b_n \sin px \, e^{-p^2 t}$$

Again, $\quad u(20, t) = 60 = 60 + \sum_{n=1}^{\infty} b_n \sin 20p \, e^{-p^2 t}$

$\Rightarrow \quad 0 = \sum_{n=1}^{\infty} b_n \sin 20p \, e^{-p^2 t}$

$$\sin 20p = 0 = \sin n\pi, \; n \in \mathrm{I}$$

$\Rightarrow \quad p = \dfrac{n\pi}{20}$

$\therefore \quad u = 40 + x + \sum_{n=1}^{\infty} b_n \sin \dfrac{n\pi x}{20} e^{-\left(\frac{n\pi}{20}\right)^2 t} \qquad \text{...(5)}$

Using initial condition,

$$u(x, 0) = 30 + \frac{5}{2}x \quad \text{in eqn. (5), we get}$$

$\Rightarrow \quad 30 + \dfrac{5}{2}x = 40 + x + \sum_{n=1}^{\infty} b_n \sin \dfrac{n\pi x}{20}$

$\Rightarrow \quad \dfrac{3}{2}x - 10 = \sum_{n=1}^{\infty} b_n \sin \dfrac{n\pi x}{20}$

where $\quad b_n = \dfrac{2}{20}\displaystyle\int_0^{20}\left(\frac{3}{2}x - 10\right)\sin\frac{n\pi x}{20}\,dx = -\frac{20}{n\pi}\,[2(-1)^n + 1]$

From (5), $\quad u(x, t) = 40 + x - \dfrac{20}{\pi}\displaystyle\sum_{n=1}^{\infty}\left[\frac{2(-1)^n + 1}{n}\right]\sin\frac{n\pi x}{20}\,e^{-\left(\frac{n\pi}{20}\right)^2 t}.$

Example 5. *The temperature distribution in a bar of length π which is perfectly insulated at ends x = 0 and x = π is governed by partial differential equation*

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

Assuming the initial temperature distribution as u(x, 0) = f(x) = cos 2x, find the temperature distribution at any instant of time. (M.T.U., 2011)

Sol. $$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \qquad \text{...(1)}$$

Its solution is $\quad u(x, t) = c_1 e^{-p^2 t}(c_2 \cos px + c_3 \sin px) \qquad$ ...(2)

Since ends of bar are insulated, no heat can pass from either sides and boundary conditions are

$$\frac{\partial u}{\partial x} = 0 \quad \text{at } x = 0 \qquad \text{...(3)}$$

and $$\frac{u}{x} = 0 \quad \text{at } x = \pi \qquad ...(4)$$

From (2), $$\frac{u}{x} \quad c_1 e^{\quad p^2 t} (-pc_2 \sin px + pc_3 \cos px)$$

At $x = 0$,

$$0 = c_1 e^{\quad p^2 t} \; pc_3 \quad \Rightarrow \quad c_3 = 0$$

$\therefore$ From (2), $$u(x, t) = c_1 c_2 \, e^{\quad p^2 t} \cos px \qquad ...(5)$$

Again $$\frac{u}{x} = -pc_1 c_2 \, e^{\quad p^2 t} \sin px$$

At $x = \pi$,

$$0 = -pc_1 c_2 e^{\quad p^2 t} \sin p\pi$$

$\Rightarrow$ $$\sin p\pi = 0 = \sin n\pi \; (n \in \mathrm{I})$$

$$p\pi = n\pi \quad \Rightarrow \quad p = n$$

$\therefore$ From (5), $u(x, t) = b_n \, e^{\quad n^2 t} \cos nx$, where $c_1 c_2 = b_n$

Most general solution is

$$u(x, t) = \sum_{n \quad 1} b_n \, e^{\quad n^2 t} \cos nx \qquad ...(6)$$

$$u(x, 0) = \cos 2x = \sum_{n \quad 1} b_n \cos nx$$

Comparing, we get $b_2 = 1$ and $n = 2$. All other b_i's are zero.

$\therefore$ From (6), $u(x, t) = e^{-4t} \cos 2x$.

Example 6. *Solve the equation* $\frac{u}{t} \quad \frac{^2 u}{x^2}$ *with boundary condition u(x, 0) = 3 sin nπx, u(0, t) = 0, u (l, t) = 0, where 0 < x < l, t > 0.* [JNTUK, (Set 3) 2014]

Sol. The solution to the equation

$$\frac{u}{t} \quad \frac{^2 u}{x^2} \qquad ...(1)$$

is given by $$u(x, t) = c_1 e^{\quad p^2 t} (c_2 \cos px + c_3 \sin px) \qquad ...(2)$$

From (2), $$u(0, t) = c_1 c_2 e^{\quad p^2 t}$$

$\Rightarrow$ $$0 = c_1 c_2 e^{\quad p^2 t}$$

$\Rightarrow$ $$c_2 = 0.$$

$\therefore$ From (2), $$u(x, t) = c_1 c_3 e^{\quad p^2 t} \sin px \qquad ...(3)$$

$$u(l, t) = 0 = c_1 c_3 e^{\quad p^2 t} \sin pl$$

$\Rightarrow$ $$\sin pl = 0 = \sin n\pi (n \in \mathrm{I})$$

$\therefore$ $$p = \frac{n}{l}.$$

From (3), $\quad u(x, t) = c_1 c_3 e^{-\frac{n^2\pi^2}{l^2}t} \sin \frac{n\pi x}{l} = b_n e^{-\frac{n^2\pi^2 t}{l^2}} \sin \frac{n\pi x}{l}$

The most general solution is

$$u(x, t) = \sum_{n=1}^{\infty} b_n e^{-(n^2\pi^2 t/l^2)} \sin \frac{n\pi x}{l} \qquad \text{...(4)}$$

$\therefore$ From (4), $\quad u(x, 0) = \sum_{1}^{\infty} b_n \sin \frac{n\pi x}{l}$

$\Rightarrow \quad 3 \sin n\pi x = \sum_{1}^{\infty} b_n \sin \frac{n\pi x}{l}.$

Comparison gives, $\quad b_n = 3,\ l = 1.$

Hence from (4), the required solution is

$$u(x, t) = 3 \sum_{n=1}^{\infty} e^{-n^2\pi^2 t} \sin n\pi x.$$

Example 7. *A bar with insulated sides is initially at a temperature 0°C throughout. The end x = 0 is kept at 0°C, and heat is suddenly applied at the end x = l so that* $\frac{\partial u}{\partial x}$ *= A for x = l, where A is a constant. Find the temperature function u(x, t).*

Sol. One dimensional heat flow equation is

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad \text{...(1)}$$

Its solution is

$$u(x, t) = c_1 e^{-p^2c^2t} (c_2 \cos px + c_3 \sin px)$$

or $$u(x, t) = (A_1 \cos px + B \sin px)\, e^{-p^2c^2t} \qquad \text{...(2)}$$

Applying the zero end conditions as,

$$u(0, t) = 0 = A_1 e^{-p^2c^2t}$$

$\Rightarrow \quad A_1 = 0.$

$\therefore$ From (2), $\quad u(x, t) = B \sin px\, e^{-p^2c^2t} \qquad \text{...(3)}$

From (3), $\quad \frac{\partial u}{\partial x} = pB \cos px\, e^{-p^2c^2t}.$

At $x = l$, $\quad \left(\frac{\partial u}{\partial x}\right)_{x=l} = 0 = pB \cos pl\, e^{-p^2c^2t}$

$\Rightarrow \quad \cos pl = 0 = \cos\left(n\pi - \frac{\pi}{2}\right);\ n \in I \quad$ or $\quad pl = (2n - 1)\frac{\pi}{2}$

$\Rightarrow \quad p = (2n - 1)\frac{\pi}{2l}.$

$\therefore$ From (3), $\quad u(x, t) = B \sin px\, e^{-p^2c^2t} \qquad \text{...(4)} \quad$ where $p = (2n-1)\frac{\pi}{2l}$

The most general solution is

$$u(x, t) = \sum_{n=1}^{\infty} B_n \sin px\, e^{-p^2c^2t} \qquad \text{...(5)} \quad \text{where } p = (2n-1)\frac{\pi}{2l}.$$

The end conditions given for this problem are

(*i*) $u = 0$ at $x = 0$ $\qquad$ (*ii*) $\frac{\partial u}{\partial x} = A$ at $x = l$ $\qquad$ ...(6)

These conditions are different from the zero end conditions. So we add to (5) the solution

$$u = A_1 x + B$$

Choosing A_1 and B so that (6) is satisfied.

This gives, $\quad 0 = B$ and $A_1 = A$

$$\therefore \qquad u(x, t) = Ax + \sum_{n=1}^{\infty} B_n \sin px \, e^{-p^2c^2t} \qquad \ldots(7) \quad \text{where } p = (2n-1)\frac{\pi}{2l}.$$

Applying the condition that $u = 0$ at $t = 0$, we have

$$0 = Ax + \sum_{n=1}^{\infty} B_n \sin px$$

or

$$-Ax = \sum_{n=1}^{\infty} B_n \sin px$$

where

$$B_n = \frac{2}{l}\int_0^l (-Ax)\sin px \, dx, \text{ where } p = (2n-1)\frac{\pi}{2l}$$

$$= -\frac{2A}{l}\left[\left\{x\left(-\frac{\cos px}{p}\right)\right\}_0^l - \int_0^l 1.\left(-\frac{\cos px}{p}\right)dx\right]$$

$$= -\frac{2A}{l}\left[-\frac{l\cos pl}{p} + \frac{1}{p}\left(\frac{\sin px}{p}\right)_0^l\right]$$

$$= -\frac{2A}{l}\left[-\frac{l\cos pl}{p} + \frac{1}{p^2}\sin pl\right] = -\frac{2A(2l)^2}{l(2n-1)^2\pi^2}\sin(2n-1)\frac{\pi}{2}$$

$$(\because \quad \cos pl = 0)$$

$$= \frac{8Al}{\pi^2(2n-1)^2}\sin\left(n\pi - \frac{\pi}{2}\right) = \frac{8Al}{\pi^2(2n-1)^2}(-1)^n \qquad \ldots(8)$$

$$\therefore \quad \text{From (7),} \quad u(x, t) = Ax + \frac{8Al}{\pi^2}\sum_{n=1}^{\infty}\left\{\frac{(-1)^n}{(2n-1)^2}\right\}\sin(2n-1)\frac{\pi x}{2l}\, e^{-\left\{\frac{(2n-1)^2\pi^2c^2t}{4l^2}\right\}}$$

Example 8. *Solve:* $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$ *under the conditions*

(i) $u \neq \infty$ *if* $t \to \infty$ *(ii)* $\frac{\partial u}{\partial x} = 0$ *for* $x = 0$ *and* $x = l$

(iii) $u = lx - x^2$ *for* $t = 0$ *between* $x = 0$ *and* $x = l$.

Sol. Solution to $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$ is

$$u(x, t) = c_1 e^{-c^2kt}(c_2 \cos cx + c_3 \sin cx) \qquad \ldots(1)$$

Eqn. (1) satisfies the condition $u \neq \infty$ if $t \to \infty$

Applying $\frac{\partial u}{\partial x} = 0$ for $x = 0$ and $x = l$ to (1), we get

$$c_3 = 0$$

and

$$c = \frac{n\pi}{l}, \; n \in I$$

$$\therefore \qquad u = c_1c_2\, e^{-\left(\frac{n^2\pi^2kt}{l^2}\right)}\cos\frac{n\pi x}{l} = a_n\cos\frac{n\pi x}{l}\, e^{-\left(\frac{n^2\pi^2kt}{l^2}\right)} \qquad \ldots(2)$$

Again, the second possible solution is

$$u = c_1(c_2x + c_3) \qquad \ldots(3) \quad | \text{ if } c^2 = 0$$

Applying $\frac{\partial u}{\partial x} = 0$ for $x = 0$ and $x = l$ to (3), we get $c_2 = 0$

$\therefore$ $$u = c_1 c_3 = \frac{a_0}{2} \text{ (say)} \qquad \text{...(4)} \quad | \text{ From (3)}$$

$\therefore$ The general solution is the sum of solutions (2) and (4) for various n.

$\therefore$ $$u(x, t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l} e^{-\left(\frac{n^2\pi^2 kt}{l^2}\right)} \qquad \text{...(5)}$$

Now applying $u = lx - x^2$ for $t = 0$ to eqn. (5), we get

$$lx - x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l}$$

Here, $$a_0 = \frac{2}{l} \int_0^l (lx - x^2)\, dx = \frac{l^2}{3}$$

$$a_n = \frac{2}{l} \int_0^l (lx - x^2) \cos \frac{n\pi x}{l}\, dx$$

$$= \begin{cases} -\dfrac{4l^2}{n^2\pi^2}; & \text{when } n \text{ is even} \\ 0; & \text{when } n \text{ is odd} \end{cases} \qquad | \text{ On simplification}$$

$\therefore$ $$u = \frac{l^2}{6} - \frac{4l^2}{\pi^2} \sum_{n=2,4,\ldots} \frac{1}{n^2} \cos \frac{n\pi x}{l} e^{-\left(\frac{n^2\pi^2 kt}{l^2}\right)}$$

Put $n = 2m$, we get

$$u(x, t) = \frac{l^2}{6} - \frac{l^2}{\pi^2} \sum_{m=1}^{\infty} \frac{1}{m^2} \cos \frac{2m\pi x}{l} e^{-\left(\frac{4m^2\pi^2 kt}{l^2}\right)}$$

Show that $e^{at}\sin bx$ is a solution of one dimensional heat equation.

(OU July 2014, Dec 2011)

Solution. Given $u(x,t) \quad e^{at}\sin bx$ ...(1)

Now we have to prove equation (1) is solution of one dimensional heat equation.

$$\frac{u}{t} \quad ae^{at}\sin bx \qquad ...(2)$$

$$\frac{u}{x} \quad be^{at}\cos bx$$

$$\frac{{}^2u}{x^2} \quad b^2e^{at}\sin bx$$

from (2) & (3)

$$\frac{{}^2u}{x^2} \quad b^2 \quad \frac{1}{a}\frac{u}{t}$$

$$\frac{{}^2u}{x^2} \quad \frac{b^2}{a}\frac{u}{t}$$

(or) $$\frac{u}{t} \quad \frac{a}{b^2}\frac{{}^2u}{x^2}$$

This represent one dimensional heat equation of the form

$$\frac{u}{t} \quad c^2\frac{{}^2u}{x^2} \qquad \text{here } c^2 \quad \frac{a}{b^2}$$

Hence equation (1) is a solution of one dimensional heat equation.

Slip Q.3

A homogeneous rod of conducting material of length 100cm has its ends kept at zero temperature and the temperature initially is

$$u(x,0) \quad \begin{matrix} x, \quad 0 \quad x \quad 50 \\ 100 \quad x, \; 50 \quad x \quad 100 \end{matrix}$$

Find the temperature u(t,x) at any time. (OU 2012)

Solution. Consider one dimensional heat equation

$$\frac{u}{t} \quad c^2\frac{{}^2u}{x^2}$$

We know that the solution of (1) is given by ...(1)

$$u(x,t) \quad (c_1\cos px \quad c_2\sin px)e^{c^2p^2t} \qquad ...(2)$$

Now given,

$$\begin{matrix} u(0,t) \quad 0 \\ u(100,t) \quad 0 \end{matrix} \quad \text{Boundary conditions} \qquad ...(3)$$

and $$u(x,0) \quad \begin{matrix} x, \quad 0 \quad x \quad 50 \\ 100 \quad x, \; 50 \quad x \quad 100 \end{matrix} \qquad ...(4)$$

Using (2) and (3),

$$u(0,t) = c_1 e^{-c^2 p^2 t} = 0 \Rightarrow c_1 = 0$$

Also $u(100,t) = c_2 \sin 100p\, e^{-c^2 p^2 t} = 0$

$$\Rightarrow \sin 100p = 0$$

$c_2 \neq 0$, otherwise, from (2), we have $u(x,t) = 0$, which is meaningless

$$\Rightarrow p = \frac{n\pi}{100}, \quad n \in Z$$

$\sin\theta = 0 \Rightarrow \theta = n\pi, n \in Z$

Therefore (2) reduces to

$$u(x,t) = c_2 \sin\frac{n\pi}{100} x\, e^{-\frac{c^2 n^2 \pi^2}{(100)^2} t}$$

$$= a_n \sin\frac{n\pi}{100} x\, e^{-\frac{c^2 n^2 \pi^2}{(100)^2} t} \qquad \text{(Replacing } c_2 \text{ by } a_n\text{)} \qquad ...(5)$$

Giving n the values 1, 2, 3, ... in (5) and adding all the solutions, we have

$$u(x,t) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi}{100} x\, e^{-\frac{c^2 n^2 \pi^2}{(100)^2} t} \qquad ...(6)$$

Using (4), $u(x,0) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi}{100} x$

$$u(x,0) = \begin{cases} x, & 0 \le x \le 50 \\ 100 - x, & 50 \le x \le 100 \end{cases}$$

which is a Fourier half-range sine series in (0, 100) and hence a_n is given by

$$a_n = \frac{2}{100}\int_0^{100} u(x,0)\sin\frac{n\pi}{100}x\,dx$$

$$= \frac{1}{50}\left[\int_0^{50} x\sin\frac{n\pi}{100}x\,dx + \int_{50}^{100}(100-x)\sin\frac{n\pi}{100}dx\right] \quad \text{Integrating by parts}$$

$$= \frac{1}{50}\left[\left\{ x\,\frac{-\cos\frac{n\pi}{100}x}{\frac{n\pi}{100}} - (1)\frac{-\sin\frac{n\pi}{100}x}{\left(\frac{n\pi}{100}\right)^2}\right\}_0^{50}\right.$$

$$+\left[\frac{(100-x)}{50}\cdot\frac{-\cos\frac{n\pi x}{100}}{\frac{n\pi}{100}}-\frac{1}{50}\cdot\frac{-\sin\frac{n\pi x}{100}}{\frac{n^2\pi^2}{100}}\right]_{50}^{100}$$

$$=\frac{1}{50}\left[-\frac{5000}{n\pi}\cos\frac{n\pi}{2}+\frac{(100)^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right]+\left[0+\frac{100}{n\pi}.\cos\frac{n\pi}{2}+\frac{1}{50}.\frac{(100)^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right]$$

$$=\frac{1}{50}\left[2.\frac{(100)^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right]=\frac{400}{n^2\pi^2}\sin\frac{n\pi}{2},\ n\neq 0$$

$$=\begin{cases}0 & \text{, When } n \text{ is even}\\ \frac{400}{n^2\pi^2}\sin\frac{n\pi}{2}, & \text{When } n \text{ is odd}\end{cases}$$

as n varies from 1 *to* ∞.
(other terms get cancelled)

$$=\begin{cases}0 & \text{, When } n \text{ is even}\\ \frac{400}{(2m-1)^2\pi^2}\sin\frac{(2m-1)\pi}{2}, & \text{When } n \text{ is odd}\end{cases}$$

As n *odd*, take $n = 2m-1$,
$m = 0, 1, 2, \ldots$

$$=\begin{cases}0 & \text{, When } n \text{ is even}\\ \frac{400}{(2m-1)^2\pi^2}(-1)^m, & m = 0,1,2,\ldots\end{cases}$$

$\sin(n\pi - \theta) = (-1)^n \sin\theta$

Here $\sin(2m-1)\frac{\pi}{2}$

$= \sin\left(m\pi - \frac{\pi}{2}\right)$

$= (-1)^m \sin\frac{\pi}{2}$

$= (-1)^m$

Therefore, from (6) the required solution is given by

$$u(x,t)=\sum_{m=0}^{\infty}\frac{400(-1)^m}{(2m-1)^2\pi^2}\sin\frac{(2m-1)\pi}{100}x\,e^{-\left(\frac{(2m-1)c\pi}{100}\right)^2 t}$$

EXERCISE

1. (*i*) Solve: $\frac{\partial u}{\partial t}=\alpha\frac{\partial^2 u}{\partial x^2}$; α constant, subject to the boundary conditions $u(0, t) = 0$, $u(\pi, t) = 0$ and the initial condition $u(x, 0) = \sin 2x$.

(*ii*) Solve: $\frac{\partial u}{\partial t}=a^2\frac{\partial^2 u}{\partial x^2}$ given that (*a*) $u = 0$ when $x = 0$ and $x = l$ for all t (*b*) $u = 3 \sin \frac{x}{l}$ when $t = 0$ for all x.

2. (*i*) Determine the solution of one-dimensional heat equation $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$ where the boundary conditions are $u(0, t) = 0$, $u(l, t) = 0$ $(t > 0)$ and the initial condition $u(x, 0) = x$: l being the length of the bar. [*JNTUK, (Set 1) 2016; M.T.U., 2013*]

(*ii*) Find the temperature distribution in a rod of length 2 m whose end points are fixed at temperature zero and the initial temperature distribution is $f(x) = 100x$.

(*iii*) Solve $\frac{\partial^2 z}{\partial x^2} - \frac{\partial z}{\partial y} = 0$ using method of separation of variables to obtain the solution that tends to zero as $y \to \infty$ for all x.

3. The heat flow in a bar of length 10 cm of homogeneous material is governed by partial diff. eqn. $u_t = c^2 u_{xx}$. The ends of the bar are kept at temp. 0°C and initial temp. is $f(x) = x(10 - x)$. Find the temperature in the bar at any instant of time.

4. Find the temperature $u(x, t)$ in a homogeneous bar of heat conducting material of length L cm. with its ends kept at zero temperature and initial temperature given by $\frac{x(L - x)\, d}{L^2}$.

5. A homogeneous rod of conducting material of length 100 cm has its ends kept at zero temperature and the temperature initially is $u(x, 0) = \begin{cases} x, & 0 \le x \le 50 \\ 100 - x, & 50 \le x \le 100 \end{cases}$

Find the temperature $u(x, t)$ at any time.

6. Find the temperature $u(x, t)$ in a slab whose ends $x = 0$ and $x = L$ are kept at zero temperature and whose initial temperature $f(x)$ is given by

$$f(x) = \begin{cases} k, & \text{when } 0 < x < \frac{1}{2}L \\ 0, & \text{when } \frac{1}{2}L < x < L \end{cases}.$$

7. Solve: $u_t = a^2 u_{xx}$ under the conditions $u_x(0, t) = 0 = u_x(\pi, t)$ and $u(x, 0) = x^2 (0 < x < \pi)$.

8. Find the temperature in a thin metal rod of length L with both ends insulated (so that there is no passage of heat through the ends) and with initial temperature $\sin \frac{x}{L}$ in the rod.

]

[**Hint.** $(u_x)_{x=0} = 0 = (u_x)_{x=L}$; $u(x, 0) = \sin \frac{x}{L}$]

9. (*i*) The temperature of a bar 50 cm long with insulated sides is kept at 0° at one end and 100° at the other end until steady conditions prevail. The two end are then suddenly insulated so that the temperature gradient is zero at each end thereafter. Find the temperature distribution.

(*ii*) A bar 10 cm long, with insulated sides, has its ends A and B maintained at temperatures 50°C and 100°C respectively, until steady-state conditions prevail. The temperature at A is suddenly raised to 90°C and at the same time that at B is lowered to 60°C. Find the temperature distribution in the bar at time t.

10. A homogeneous rod of conducting material of length '1' has its ends kept at zero temperature. The temperature at the centre is T and falls uniformly to zero at the two ends. Find the temperature distribution.

$$\left[\textbf{Hint. } u(x, 0) = \begin{cases} 2Tx, & 0 \le x \le \frac{1}{2} \\ 2T(1 - x), & \frac{1}{2} \le x \le 1 \end{cases} \right]$$

11. Solve $\frac{\partial \theta}{\partial t} = k \frac{\partial^2 \theta}{\partial x^2}$, such that

(*i*) θ is finite when $t \to \infty$, (*ii*) $\frac{\partial \theta}{\partial x} = 0$ when $x = 0$ and $\theta = 0$ when $x = l$ for all t,

(*iii*) $\theta = \theta_0$ when $t = 0$ for all values of x between 0 and l.

12. Find a solution of the heat conduction equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ such that

(*i*) u is finite when $t \to \infty$, (*ii*) $u = 100$ when $x = 0$ or π for all values of t,

(*iii*) $u = 0$ when $t = 0$ for all values of x between 0 and π.

(Here, the initially ice-cold rod has its ends in boiling water.)

Answers

1. (*i*) $u(x, t) = \sin 2x\, e^{-4\alpha t}$ (*ii*) $u(x, t) = 3 \sin \frac{\pi x}{l} e^{-(a^2 \pi^2 t/l^2)}$

2. (*i*) $u(x, t) = -\frac{2l}{\pi} \sum_{n=1}^{\infty} \frac{\cos n\pi}{n} \sin \frac{n\pi x}{l} e^{-\left(\frac{c^2 n^2 \pi^2 t}{l^2}\right)}$

(*ii*) $u(x, t) = -\frac{400}{\pi} \sum_{n=1}^{\infty} \frac{\cos n\pi}{n} \sin \frac{n\pi x}{2} e^{-\left(\frac{c^2 n^2 \pi^2 t}{4}\right)}$

(*iii*) $z(x, y) = (c_1 \cos px + c_2 \sin px)\; c_3 e^{-c^2 p^2 y}$

3. $u(x, t) = \frac{800}{\pi^3} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^3} \sin \frac{(2n-1)\pi x}{10} e^{-\left(\frac{(2n-1)^2 \pi^2 c^2 t}{100}\right)}$.

4. $u(x, t) = \frac{8d}{\pi^3} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^3} \sin \frac{(2n-1)\pi x}{L} e^{-\frac{(2n-1)^2 \pi^2 c^2 t}{L^2}}$.

5. $u(x, t) = \frac{400}{\pi^2} \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)^2} \sin \frac{(2m+1)\pi x}{100} e^{-\left(\frac{(2m+1)\pi c}{100}\right)^2 t}$

6. $u(x, t) = \frac{4k}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin^2 \frac{n\pi}{4} \sin \frac{n\pi x}{L} e^{-\left(\frac{c^2 n^2 \pi^2 t}{L^2}\right)}$

7. $u(x, t) = \frac{\pi^3}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx\, e^{-a^2 n^2 t}$

8. $u(x, t) = \frac{2}{\pi} - \frac{4}{\pi} \sum_{m=1}^{\infty} \frac{1}{(4m^2-1)} \cos\left(\frac{2m\pi x}{L}\right) e^{-\frac{4m^2 \pi^2 c^2 t}{L^2}}$

9. (*i*) $u(x, t) = \frac{200}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin \frac{n\pi x}{50} e^{-\left(\frac{n^2 \pi^2 kt}{2500}\right)}$

(*ii*) $u(x, t) = 90 - 3x - \frac{80}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{n\pi x}{5} e^{-\left(\frac{c^2 n^2 \pi^2 t}{25}\right)}$

10. $u(x, t) = \frac{8T}{\pi^2} \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{(2m-1)^2} \sin (2m-1)\pi x\, e^{-[(2m-1)^2 \pi^2 c^2 t]}$

11. $\theta = \frac{4\theta_0}{\pi}\left[e^{-(\pi/2l)^2 kt} \cos \frac{\pi x}{2l} - \frac{1}{3} e^{-\left(\frac{3\pi}{2l}\right)^2 kt} \cos \frac{3\pi x}{2l} + \frac{1}{5} e^{-\left(\frac{5\pi}{2l}\right)^2 kt} \cos \frac{5\pi x}{2l} - \ldots\right]$

12. $u(x, t) = 100 - \frac{400}{\pi} \sum_{m=1}^{\infty} \frac{\sin (2m-1)x}{2m-1} e^{-(2m-1)^2 t}$

5.5 SOLUTION OF LAPLACE EQUATION IN TWO DIMENSIONS $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$

Consider the flow of heat in a metal plate, in the XOY plane. If the temperature at any point is independent of the z-coordinate and depends on x, y and t only, then the flow is called two dimensional and the heat-flow lies in the plane XOY only and is zero along the normal to the plane XOY.

Take a rectangular element of the plate with sides δx and δy and thickness α. As discussed in the one-dimensional heat flow along a bar, the quantity of heat that enters the plate per second from the sides AB and AD is given by

$$-k\alpha\,\delta x\left(\frac{\partial u}{\partial y}\right)_y \quad \text{and} \quad -k\alpha\,\delta y\left(\frac{\partial u}{\partial x}\right)_x$$

respectively and that which flows out through the sides CD and BC per second is

$$-k\alpha\,\delta x\left(\frac{\partial u}{\partial y}\right)_{y+\delta y} \quad \text{and} \quad -k\alpha\,\delta y\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} \quad \text{respectively.}$$

Therefore, the total gain of heat by the rectangular plate ABCD per second

$$= -k\alpha\delta x\left(\frac{\partial u}{\partial y}\right)_y - k\alpha\delta y\left(\frac{\partial u}{\partial x}\right)_x + k\alpha\delta x\left(\frac{\partial u}{\partial y}\right)_{y+\delta y} + k\alpha\delta y\left(\frac{\partial u}{\partial x}\right)_{x+\delta x}$$

$$= k\alpha\delta x\delta y\left[\frac{\left(\frac{\partial u}{\partial x}\right)_{x+\delta x}-\left(\frac{\partial u}{\partial x}\right)_x}{\delta x}+\frac{\left(\frac{\partial u}{\partial y}\right)_{y+\delta y}-\left(\frac{\partial u}{\partial y}\right)_y}{\delta y}\right] \quad \text{...(1)}$$

The rate of gain of heat by the plate is also given by

$$s\rho\delta x\delta y\,\frac{\partial u}{\partial t} \quad \text{...(2)}$$

where s = specific heat and ρ = density of the metal plate.

Equating (1) and (2), we obtain

$$k\alpha\delta x\delta y\left[\frac{\left(\frac{\partial u}{\partial x}\right)_{x+\delta x}-\left(\frac{\partial u}{\partial x}\right)_x}{\delta x}+\frac{\left(\frac{\partial u}{\partial y}\right)_{y+\delta y}-\left(\frac{\partial u}{\partial y}\right)_y}{\delta y}\right] = s\rho\delta x\delta y\,\frac{\partial u}{\partial t}$$

Dividing both sides by $\alpha\delta x\delta y$ and taking the limit as $\delta x \to 0$, $\delta y \to 0$, we get

$$k\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right) = s\,\frac{\partial u}{\partial t}$$

or

$$c^2\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right) = \frac{\partial u}{\partial t} \qquad \text{where } c^2 = \frac{k}{s} \quad \text{...(3)}$$

Equation (3) gives the temperature distribution of the plate in the transient state.

Note 1. In steady state, u is independent of t, so that $\frac{\partial u}{\partial t} = 0$ and the above equation reduces to

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2} = 0 \quad \text{...(4)}$$

which is known as **Laplace's Equation in two dimensions**

Consider the Laplace's equation in two dimensions as:

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2} = 0 \quad \text{...(1)}$$

Let $u = XY$ be a solution of (1), ...(2)

where X is a function of x only and Y is a function of y only

Then $\frac{\partial^2 u}{\partial x^2} = X''Y$ and $\frac{\partial^2 u}{\partial y^2} = XY''$

Substituting in (1), we have $X''Y + XY'' = 0$ or $\frac{X''}{X} = -\frac{Y''}{Y}$ ...(3)

Now the LHS of (3) is a function of x only and the RHS is a function of y only. Since x and y are independent variables, this equation can hold only when both sides reduce to a constant, say k. Then equation (3) leads to the ordinary differential equations

$$\frac{d^2X}{dx^2} - kX = 0 \quad \text{and} \quad \frac{d^2Y}{dy^2} + kY = 0 \quad ...(4)$$

Solving equations (4), we get

(*i*) *When k is positive* and $= p^2$, say

$$X = c_1 e^{px} + c_2 e^{-px},\ Y = c_3 \cos py + c_4 \sin py$$

(*ii*) *When k is negative* and $= -p^2$, say

$$X = c_1 \cos px + c_2 \sin px,\ Y = c_3 e^{py} + c_4 e^{-py}$$

(*iii*) *When k = 0*

$$X = c_1 x + c_2,\ Y = c_3 y + c_4$$

Thus, the various possible solutions of Laplace's equation (1) are:

$$u = (c_1 e^{px} + c_2 e^{-px})(c_3 \cos py + c_4 \sin py) \quad ...(5)$$

$$u = (c_1 \cos px + c_2 \sin px)(c_3 e^{py} + c_4 e^{-py}) \quad ...(6)$$

$$u = (c_1 x + c_2)(c_3 y + c_4) \quad ...(7)$$

Of these three solutions, we have to choose that solution which is consistent with the physical nature of the problem and the given boundary conditions. Solution (6) is the required solution.

$$u(x, y) = (c_1 \cos px + c_2 \sin px)\,(c_3 e^{py} + c_4 e^{-py}).$$

SOLVED PROBLEMS

Example 1. *Use separation of variables method to solve the equation*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

subject to the boundary conditions $u(0, y) = u(l, y) = u(x, 0) = 0$ *and* $u(x, a) = \sin \frac{n\pi x}{l}$.

Sol. The given equation is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad ...(1)$$

Let $u = XY$ ...(2)

where X is a function of x only and Y is a function of y only then,

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2}{\partial x^2}(XY) = Y\frac{d^2X}{dx^2} \quad \text{and} \quad \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2}{\partial y^2}(XY) = X\frac{d^2Y}{dy^2}$$

$\therefore$ From (1), $\quad YX'' + XY'' = 0$

$\Rightarrow \quad \frac{X}{X} \quad \frac{Y}{Y} = 0$

Case I. When $\quad \frac{X}{X} \quad \frac{Y}{Y} = p^2$ (say)

(i) $\quad \frac{X}{X} \quad p^2$

$$X'' - p^2 X = 0$$

Auxiliary equation is $\quad m^2 - p^2 = 0$

$$m = \pm p$$

$\therefore \quad \text{C.F.} = c_1 e^{px} + c_2 e^{-px}$

$$\text{P.I.} = 0$$

$\therefore \quad X = c_1 e^{px} + c_2 e^{-px}$

(ii) $\quad \frac{Y}{Y} \quad p^2 \quad \Rightarrow \quad Y'' + p^2 Y = 0$

Auxiliary equation is $\quad m^2 + p^2 = 0 \quad \Rightarrow \quad m = \pm pi$

$\therefore \quad \text{C.F.} = c_3 \cos py + c_4 \sin py$

$$\text{P.I.} = 0$$

$\therefore \quad y = c_3 \cos py + c_4 \sin py$

Now, $\quad X(0) = 0$

$\Rightarrow \quad c_1 + c_2 = 0 \quad \Rightarrow \quad c_2 = -c_1$

$$X(l) = 0$$

$\Rightarrow \quad c_1 e^{pl} + c_2 e^{-pl} = 0 \quad \Rightarrow \quad c_1 (e^{pl} - e^{-pl}) = 0$

$\Rightarrow \quad c_1 = 0 \qquad$ | Since $e^{pl} - e^{-pl} \neq 0$ (as $p \neq 0 \neq l$)

$\therefore \quad c_2 = 0$

$\therefore \quad X = 0 \quad \Rightarrow \quad u = XY = 0$, which is impossible

Hence we reject case I.

Case II. When $\quad \frac{X}{X} \quad \frac{Y}{Y} = 0$ (say)

(i) $\quad \frac{X}{X} = 0$

$\Rightarrow \quad X'' = 0 \quad \Rightarrow \quad X = c_5 x + c_6$

(ii) $\quad \frac{Y}{Y} \quad 0$

$\Rightarrow \quad Y'' = 0 \quad \Rightarrow \quad Y = c_7 y + c_8$

Now, $\quad X(0) = 0 \quad \Rightarrow \quad c_6 = 0$

$$X(l) = 0$$

$\Rightarrow \quad c_5 l + c_6 = 0 \quad \Rightarrow \quad c_5 l = 0$

$\Rightarrow \quad c_5 = 0 \qquad$ (Since $l \neq 0$)

$\therefore \quad X = 0$

$\therefore \qquad u = XY = 0$, which is impossible

Hence we also reject case II.

Case III. When $\frac{X}{X} \quad \frac{Y}{Y} = -p^2$ (say)

$(i) \qquad \frac{X}{X} = -p^2$

$\Rightarrow \qquad X'' + p^2X = 0 \quad \Rightarrow \quad \frac{d^2X}{dx^2} \quad p^2X = 0.$

Auxiliary equation is $\quad m^2 + p^2 = 0 \quad \Rightarrow \quad m = \pm pi$

$$\text{C.F.} = c_9 \cos px + c_{10} \sin px$$

$$\text{P.I.} = 0$$

$$X = c_9 \cos px + c_{10} \sin px$$

$(ii) \qquad -\frac{Y}{Y} = -p^2$

$\Rightarrow \qquad \frac{Y}{Y} = p^2 \quad \Rightarrow \quad \frac{d^2Y}{dy^2} - p^2Y = 0.$

Auxiliary equation is

$$m^2 - p^2 = 0$$

$$m = \pm p.$$

$\therefore \qquad \text{C.F.} = c_{11}e^{py} + c_{12}e^{-py}$

$$\text{P.I.} = 0$$

Hence, $\qquad Y = c_{11}\, e^{py} + c_{12}\, e^{-py}.$

Now, $\qquad X(0) = 0 \quad \Rightarrow \quad c_9 = 0$

$\therefore \qquad X = c_{10} \sin px$

$$X(l) = 0$$

$$c_{10} \sin pl = 0$$

$\Rightarrow \qquad \sin pl = 0 = \sin n\pi,\ n \in I$

$\therefore \qquad p = \frac{n}{l}$

$\therefore \qquad X = c_{10} \sin \frac{n\ x}{l}$ ...(3)

Again, $\qquad Y(0) = 0$

$\Rightarrow \qquad c_{11} + c_{12} = 0 \quad \Rightarrow \quad c_{12} = -c_{11}$

$$Y = c_{11}(e^{py} - e^{-py}) = c_{11}\left(e^{\frac{n\ y}{l}} \quad e^{\frac{n\ y}{l}}\right) \qquad \text{...(4)}$$

$\therefore \qquad u = XY = c_{10}c_{11} \sin \frac{n\ x}{l}\, [e^{(n\pi y/l)} - e^{(-n\pi y/l)}]$

or $\qquad u(x, y) = b_n \sin \frac{n\ x}{l}\, [e^{(n\pi y/l)} - e^{(-n\pi y/l)}]$ ...(5)

Now, $\qquad u(x, a) = \sin \frac{n\ x}{l} = b_n \sin \frac{n\ x}{l}\, [e^{(n\pi a/l)} - e^{-(n\pi a/l)}]$

$\Rightarrow \qquad b_n = \frac{1}{e^{\frac{n\ a}{l}} \quad e^{\frac{n\ a}{l}}} = \frac{1}{2 \sin h\left(\frac{n\ a}{l}\right)}$

$$\therefore \qquad u(x, y) = \frac{e^{(n\pi y/l)} - e^{-(n\pi y/l)}}{2\sinh\left(\frac{n\pi a}{l}\right)} \sin\frac{n\pi x}{l} = \frac{\sinh\left(\frac{n\pi y}{l}\right)}{\sinh\left(\frac{n\pi a}{l}\right)} \sin\frac{n\pi x}{l}.$$

Example 2. *A rectangular plate with insulated surfaces is 8 cm wide and so long compared to its width that it may be considered infinite in length without introducing an appreciable error. If the temperature along one short edge y = 0 is given by*

$$u(x, 0) = 100 \sin\frac{\pi x}{8}, \; 0 < x < 8$$

while the two long edges x = 0 and x = 8 as well as the other short edge are kept at 0°C, show that the steady state temperature at any point of the plate is given by

$$u(x, y) = 100 e^{-\frac{\pi y}{8}} \sin\frac{\pi x}{8}.$$

Sol. Let $u(x, y)$ be the temperature at any point P of the plate.

Two dimensional heat flow equation in steady state is given by

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad ...(1)$$

Its solution is $\quad u(x, y) = (c_1 \cos px + c_2 \sin px)(c_3 e^{py} + c_4 e^{-py}) \qquad ...(2)$

Boundary conditions are

$$u(0, y) = 0 = u(8, y)$$

$$\underset{y \to \infty}{\text{Lt}}\; u(x, y) = 0$$

$$u(x, 0) = 100 \sin\frac{\pi x}{8}, \quad 0 < x < 8$$

From (2),

$$u(0, y) = 0 = c_1 (c_3 e^{py} + c_4 e^{-py})$$

$\Rightarrow \qquad c_1 = 0.$

$\therefore$ From (2),

$$u(x, y) = c_2 \sin px\, (c_3 e^{py} + c_4 e^{-py}) \qquad ...(3)$$

$$u(8, y) = 0 = c_2 \sin 8p(c_3 e^{py} + c_4 e^{-py})$$

$\Rightarrow \qquad \sin 8p = 0 = \sin n\pi$

$\Rightarrow \qquad p = \frac{n\pi}{8} \quad (n \in \text{I})$

$\therefore$ From (3),

$$u(x, y) = c_2 \sin\frac{n\pi x}{8}\left(c_3 e^{\frac{n\pi y}{8}} + c_4 e^{-\frac{n\pi y}{8}}\right) \qquad ...(4)$$

$$\underset{y \to \infty}{\text{Lt}}\; u(x, y) = 0 = c_2 \sin\frac{n\pi x}{8} \lim_{y \to \infty}\left(c_3 e^{\frac{n\pi y}{8}} + c_4 e^{-\frac{n\pi y}{8}}\right)$$

which is satisfied only when

$$c_3 = 0.$$

$\therefore$ From (4), $\quad u(x, y) = c_2 c_4 \sin\frac{n\pi x}{8} e^{-\frac{n\pi y}{8}} = b_n \sin\frac{n\pi x}{8} e^{-\frac{n\pi y}{8}} \qquad ...(5)$

From (5), $\quad u(x, 0) = 100 \sin\frac{\pi x}{8} = b_n \sin\frac{n\pi x}{8}$

$\Rightarrow \qquad b_n = 100, n = 1.$

$\therefore$ From (5), $u(x, y) = 100 \sin\left(\frac{\pi x}{8}\right) e^{-(\pi y/8)}$

which is the required steady state temperature at any point of the plate.

Example 3. *An infinitely long plane uniform plate is bounded by two parallel edges and an end at right angles to them. The breadth is π. This end is maintained at temperature u_0 at all points and the other edges are at zero temperature. Determine the temperature at any point of the plate in the steady state.* (G.B.T.U., 2012)

Sol. In steady state, two dimensional heat flow equation is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad ...(1)$$

Boundary conditions are,

$$u(0, y) = 0 = u(\pi, y)$$

$$\underset{y \to \infty}{\text{Lt}}\ u(x, y) = 0 \ (0 < x < \pi)$$

and $$u(x, 0) = u_0 \ (0 < x < \pi)$$

Solution to equation (1) is

$u(x, t) = (c_1 \cos px + c_2 \sin px)(c_3 e^{py} + c_4 e^{-py})$ ...(2)

From (2), $u(0, y) = 0 = c_1(c_3 e^{py} + c_4 e^{-py})$

$\Rightarrow$ $c_1 = 0.$

From (2), $u(x, y) = c_2 \sin px(c_3 e^{py} + c_4 e^{-py})$

$u(\pi, y) = 0 = c_2 \sin p\pi(c_3 e^{py} + c_4 e^{-py})$

$\Rightarrow$ $\sin p\pi = 0 = \sin n\pi \ (n \in \text{I})$ ...(3)

$\therefore$ $p = n.$

$\therefore$ From (3), $u(x, y) = c_2 \sin nx \ (c_3 e^{ny} + c_4 e^{-ny})$ ...(4)

$$\lim_{y \to \infty} u(x, y) = 0 = c_2 \sin nx \lim_{y \to \infty} (c_3 e^{ny} + c_4 e^{-ny})$$

which is satisfied only when $c_3 = 0.$

$\therefore$ From (4), $u(x, y) = c_2 c_4 e^{-ny} \sin nx = b_n e^{-ny} \sin nx,$ where $c_2 c_4 = b_n$

The most general solution is

$$u(x, y) = \sum_{n=1}^{\infty} b_n e^{-ny} \sin nx \quad ...(5)$$

$$u(x, 0) = u_0 = \sum_{n=1}^{\infty} b_n \sin nx$$

where $$b_n = \frac{2}{\pi}\int_0^{\pi} u_0 \sin nx \, dx$$

$$= \frac{2u_0}{\pi}\left[\frac{-\cos nx}{n}\right]_0^{\pi} = \frac{2u_0}{n\pi}\{1 - (-1)^n\} = \begin{cases} \frac{4u_0}{n\pi}; & \text{if } n \text{ is odd} \\ 0; & \text{if } n \text{ is even} \end{cases}$$

$\therefore$ From (5), $$u(x, y) = \frac{4u_0}{\pi}\sum_{n=1,3,5,...}^{\infty} \frac{\sin nx}{n} e^{-ny} \quad (n \text{ is odd})$$

or $$u(x, y) = \frac{4u_0}{\pi}\sum_{n=1}^{\infty} \frac{1}{(2n-1)} \sin (2n-1)\, x\, e^{-(2n-1)y}.$$

Example 4. *A rectangular plate with insulated surfaces is 10 cm wide and so long compared to its width that it may be considered infinite in length without introducing an appreciable error. If the temperature along the short edge y = 0 is given by* *and* $$u(x, y) = \begin{cases} 20x, & 0 < x \le 5 \\ 20(10 - x), & 5 < x < 10 \end{cases}$$

and the two long edges x = 0 and x = 10 as well as other short edge are kept at 0°C. Find the temperature u at any point P(x, y).

Sol. In steady state, two dimensional heat flow equation is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad ...(1)$$

Its solution is

$$u(x, y) = (c_1 \cos px + c_2 \sin px)(c_3 e^{py} + c_4 e^{-py}) \qquad ...(2)$$

Boundary conditions are $u(0, y) = 0$

$$u(10, y) = 0$$

$$\lim_{y \to \infty} u(x, y) = u(x, \infty) = 0$$

and $$u(x, 0) = \begin{cases} 20x, & 0 < x \le 5 \\ 20(10 - x), & 5 < x \le 10 \end{cases}$$

From (2), $u(x, y) = 0 = c_1(c_3 e^{py} + c_4 e^{-py}) \quad \Rightarrow \quad c_1 = 0$

From (2), $$u(x, y) = c_2 \sin px\,(c_3 e^{py} + c_4 e^{-py}) \qquad ...(3)$$

$$u(10, y) = 0 = c_2 \sin 10p\,(c_3 e^{py} + c_4 e^{-py})$$

$\Rightarrow$ $\sin 10p = 0 = \sin n\pi$

or $10p = n\pi \ (n \in \mathrm{I})$

$\Rightarrow$ $p = \frac{n\pi}{10}.$

$\therefore$ From (3), $$u(x, y) = c_2 \sin \frac{n\pi x}{10}\left(c_3 e^{\frac{n\pi y}{10}} + c_4 e^{-\frac{n\pi y}{10}}\right) \qquad ...(4)$$

$$\lim_{y \to \infty} u(x, y) = c_2 \sin \frac{n\pi x}{10} \lim_{y \to \infty}\left(c_3 e^{\frac{n\pi y}{10}} + c_4 e^{-\frac{n\pi y}{10}}\right)$$

which is satisfied only when $c_3 = 0$.

Hence from (4), $$u(x, y) = c_2 c_4 \sin \frac{n\pi x}{10} e^{-\frac{n\pi y}{10}} = b_n \sin \frac{n\pi x}{10} e^{-\frac{n\pi y}{10}} \qquad ...(5)$$

The most general solution is

$$u(x, y) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{10} e^{-\frac{n\pi y}{10}} \qquad ...(6)$$

$$u(x, 0) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{10},$$

where $b_n = \frac{2}{10}\int_0^{10} u(x, 0) \sin \frac{n\pi x}{10}\, dx$

$$= \frac{1}{5}\left[\int_0^5 20x \sin \frac{n\pi x}{10}\, dx + \int_5^{10} 20\,(10 - x) \sin \frac{n\pi x}{10}\, dx\right]$$

$$= 4\left[\left\{x\left(\frac{-\cos \frac{n\pi x}{10}}{\frac{n\pi}{10}}\right)\right\}_0^5 - \int_0^5 1\left(\frac{-\cos \frac{n\pi x}{10}}{\frac{n\pi}{10}}\right) dx + \left\{(10 - x)\left(\frac{-\cos \frac{n\pi x}{10}}{\frac{n\pi}{10}}\right)\right\}_5^{10}\right.$$

$$- \int_5^{10} (-1)\left(\frac{-\cos\frac{n\pi x}{10}}{\frac{n\pi}{10}}\right) dx \Bigg]$$

$$= 4\left[\frac{10}{n\pi}(-5)\cos\frac{n\pi}{2} + \frac{10}{n\pi}\left[\frac{\sin\frac{n\pi x}{10}}{\frac{n\pi}{10}}\right]_0^5 + \frac{50}{n\pi}\cos\frac{n\pi}{2} - \frac{10}{n\pi}\left[\frac{\sin\frac{n\pi x}{10}}{\frac{n\pi}{10}}\right]_5^{10}\right]$$

$$= 4\left[-\frac{50}{n\pi}\cos\frac{n\pi}{2} + \frac{100}{n^2\pi^2}\sin\frac{n\pi}{2} + \frac{50}{n\pi}\cos\frac{n\pi}{2} - \frac{100}{n^2\pi^2}\left(0 - \sin\frac{n\pi}{2}\right)\right]$$

$$= \frac{800}{n^2\pi^2}\sin\frac{n\pi}{2}.$$

From (6), $\quad u(x, y) = \frac{800}{\pi^2}\sum_{n=1}^{\infty}\frac{\sin n\pi/2}{n^2}\sin\frac{n\pi x}{10}\, e^{-\frac{n\pi y}{10}}.$

Example 5. *Solve* $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$, *$0 < x < \pi$, $0 < y < \pi$, which satisfies the conditions :*

$u(0, y) = u(\pi, y) = u(x, \pi) = 0$ and $u(x, 0) = \sin^2 x$. (U.K.T.U., 2011)

Sol. The given equation is $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ ...(1)

Its solution consistent with boundary conditions is

$$u(x, y) = (c_1 \cos px + c_2 \sin px)(c_3 e^{py} + c_4 e^{-py}) \quad ...(2)$$

From (2), $\quad u(0, y) = 0 = c_1(c_3 e^{py} + c_4 e^{-py})$

$\Rightarrow \quad c_1 = 0.$

$\therefore$ From (2), $\quad u(x, y) = c_2 \sin px\,(c_3 e^{py} + c_4 e^{-py})$ ...(3)

$u(\pi, y) = 0 = c_2 \sin p\pi(c_3 e^{py} + c_4 e^{-py})$

$\Rightarrow \quad \sin p\pi = 0 = \sin n\pi \; (n \in I)$

$\therefore \quad p = n.$

Hence from (3), $\quad u(x, y) = c_2 \sin nx\,(c_3 e^{ny} + c_4 e^{-ny}) = \sin nx\,(A e^{ny} + B e^{-ny})$ ...(4)

where $\quad c_2 c_3 = A$ and $c_2 c_4 = B$.

From (4), $\quad u(x, \pi) = \sin nx(A e^{n\pi} + B e^{-n\pi})$

$0 = \sin nx(A e^{n\pi} + B e^{-n\pi})$

$\Rightarrow \quad 0 = A e^{n\pi} + B e^{-n\pi}$

$\Rightarrow \quad A e^{n\pi} = -B e^{-n\pi} = -\frac{1}{2}B_n$ (say)

then (4) becomes, $\quad u(x, y) = \sin nx\left[-\frac{1}{2}B_n e^{-n\pi} e^{ny} + \frac{1}{2}B_n e^{n\pi} e^{-ny}\right]$

$$= \frac{1}{2}B_n[e^{n(\pi-y)} - e^{-n(\pi-y)}]\sin nx = B_n \sin hn\,(\pi - y)\sin nx.$$

The most general solution is

$$u(x, y) = \sum_{n=1}^{\infty} B_n \sin hn\,(\pi - y)\sin nx \quad ...(5)$$

$$u(x, 0) = \sin^2 x = \sum_{n=1}^{\infty} B_n \sinh n\pi \sin nx$$

where
$$B_n \sinh n\pi = \frac{2}{\pi}\int_0^{\pi} \sin^2 x \sin nx\, dx$$
$$= \frac{1}{\pi}\int_0^{\pi} (1 - \cos 2x)\sin nx\, dx$$
$$= \frac{1}{\pi}\int_0^{\pi} \left[\sin nx - \frac{1}{2}\{\sin (n+2)x + \sin (n-2)x\}\right] dx$$
$$= \frac{1}{\pi}\left[-\frac{\cos nx}{n} + \frac{\cos (n+2)x}{2(n+2)} + \frac{\cos (n-2)x}{2(n-2)}\right]_0^{\pi}$$
$$= \frac{1}{2\pi}\left[\frac{1}{n+2} + \frac{1}{n-2} - \frac{2}{n}\right]\{(-1)^n - 1\}, \text{ when } n \neq 2$$
$$B_n \sinh n\pi = \begin{cases} \dfrac{-8}{\pi n(n^2-4)}, & \text{when } n \text{ is odd} \\ 0, & \text{when } n \text{ is even and} \neq 2 \end{cases}$$

when $n = 2$,
$$B_2 \sinh 2\pi = \frac{2}{\pi}\int_0^{\pi} \sin^2 x \sin 2x\, dx$$
$$= \frac{1}{\pi}\int_0^{\pi} (1 - \cos 2x)\sin 2x\, dx = \frac{1}{\pi}\int_0^{\pi} \left(\sin 2x - \frac{1}{2}\sin 4x\right) dx$$
$$= \frac{1}{\pi}\left[-\frac{\cos 2x}{2} + \frac{1}{8}\cos 4x\right]_0^{\pi} = 0$$
$$\therefore \quad B_2 = 0.$$

Hence the solution (5) becomes,
$$u(x, y) = \frac{-8}{\pi}\sum_{n=1,3,5,\ldots}^{\infty} \frac{\sin nx \sinh n(\pi - y)}{n(n^2-4)\sinh n\pi}$$
or
$$u(x, y) = -\frac{8}{\pi}\sum_{m=1,2,3,\ldots}^{\infty} \frac{\sin (2m-1)x \sinh (2m-1)(\pi - y)}{(2m-1)\{(2m-1)^2 - 4\}\sinh (2m-1)\pi}.$$

Example 6. *Solve* $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$*, with the rectangle* $0 \le x \le a$, $0 \le y \le b$ *; given that* $u(x, b) = u(0, y) = u(a, y) = 0$ *and* $u(x, 0) = x(a - x)$.

Sol. The equation is
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \ldots(1)$$

Its solution is
$$u(x, y) = (c_1 \cos px + c_2 \sin px)(c_3 e^{py} + c_4 e^{-py}) \qquad \ldots(2)$$
$$u(0, y) = 0 = c_1(c_3 e^{py} + c_4 e^{-py})$$
$$\Rightarrow \quad c_1 = 0.$$
$$\therefore \text{ From (2)}, \quad u(x, y) = c_2 \sin px\,(c_3 e^{py} + c_4 e^{-py}) \qquad \ldots(3)$$
$$u(a, y) = 0 = c_2 \sin ap\,(c_3 e^{py} + c_4 e^{-py})$$
$$\Rightarrow \quad \sin ap = 0 = \sin n\pi \ (n \in I)$$
$$\Rightarrow \quad ap = n\pi \quad \text{or} \quad p = \frac{n\pi}{a}.$$

$\therefore$ From (3), $u(x, y) = c_2 \sin \frac{n\ x}{a}(c_3 e^{\frac{n\ y}{a}} \quad c_4 e^{\frac{n\ y}{a}})$

$$u(x, y) = \sin \frac{n\ x}{a}(A\, e^{\frac{n\ y}{a}} \quad B\, e^{\frac{n\ y}{a}}) \qquad ...(4)$$

where $c_2 c_3 = A$ and $c_2 c_4 = B$

$$u(x, b) = \sin \frac{n\ x}{a}(A\, e^{\frac{n\ b}{a}} \quad B\, e^{\frac{n\ b}{a}})$$

$$0 = \sin \frac{n\ x}{a}(A\, e^{\frac{n\ b}{a}} \quad B\, e^{\frac{n\ b}{a}})$$

$\Rightarrow$ $A\, e^{\frac{n\ b}{a}} \quad B\, e^{\frac{n\ b}{a}} = 0$

$$A\, e^{\frac{n\ b}{a}} \qquad B\, e^{\frac{n\ b}{a}} \qquad \frac{1}{2} B_n \text{ (say).}$$

Then (4) becomes,

$$u(x, y) = \sin \frac{n\ x}{a}\left[\frac{1}{2} B_n e^{\frac{n\ b}{a}} e^{\frac{n\ y}{a}} \quad \frac{1}{2} B_n e^{\frac{n\ b}{a}} e^{\frac{n\ y}{a}}\right]$$

$$= \frac{1}{2} B_n \sin \frac{n\ x}{a} [e^{\frac{n}{a}(b \quad y)} \quad e^{\frac{n}{a}(b \quad y)}]$$

$$= \frac{1}{2} B_n \sin \frac{n\pi x}{a} . 2 \sinh \frac{n\pi}{a} (b - y) = B_n \sin \frac{n\ x}{a} \sinh \frac{n}{a} (b - y).$$

The most general solution is

$$u(x, y) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{a} \sinh \frac{n\pi}{a} (b - y) \qquad ...(5)$$

Applying to this the condition $u(x, 0) = x(a - x)$, we get

From (5), $u(x, 0) = \sum_{n=1}^{\infty} B_n \sinh \frac{n\pi b}{a} \sin \frac{n\pi x}{a}$

$\Rightarrow$ $x(a - x) = \sum_{n=1}^{\infty} B_n \sinh \frac{n\pi b}{a} \sin \frac{n\pi x}{a}$

where $B_n \sinh \frac{n}{a} b \quad \frac{2}{a}\int_0^a x\,(a \quad x) \sin \frac{n}{a}\ x\, dx$

$$= \frac{2}{a}\left\{\left[(ax \quad x^2)\left(\frac{\cos \frac{n}{a} x}{\frac{n}{a}}\right)\right]_0^a \quad \int_0^a (a \quad 2x) . \left(\frac{\cos \frac{n}{a} x}{\frac{n}{a}}\right) dx\right\}$$

$$= \frac{2}{a} . \frac{a}{n} \int_0^a (a \quad 2x) . \cos \frac{n}{a} x\, dx$$

$$= \frac{2}{n}\left\{\left[(a \quad 2x) \frac{\sin \frac{n}{a} x}{\frac{n}{a}}\right]_0^a \quad \int_0^a (\quad 2)\left(\frac{\sin \frac{n}{a} x}{\frac{n}{a}}\right) dx\right\}$$

$$= \frac{4}{n} . \frac{a}{n}\left[\frac{-\cos \frac{n}{a} x}{\frac{n}{a}}\right]_0^a = \frac{4a}{n^2\ ^2} . \frac{a}{n} (1 - \cos n\pi)$$

$$= \frac{4a^2}{n^3\pi^3}[1-(-1)^n] = \begin{cases} \dfrac{8a^2}{n^3\pi^3}, & \text{when } n \text{ is odd} \\ 0, & \text{when } n \text{ is even} \end{cases}$$

$$\therefore \qquad B_n = \begin{cases} \dfrac{8a^2}{\sinh\left(\dfrac{n\pi}{a}b\right)(n^3\pi^3)}, & \text{when } n \text{ is odd} \\ 0, & \text{when } n \text{ is even} \end{cases}$$

$\therefore$ From (5),
$$u(x, y) = \frac{8a^2}{\pi^3}\sum_{n=1, 3, 5, \ldots}^{\infty} \frac{\sin \frac{n\pi x}{a}}{n^3 \sinh \frac{n\pi}{a} b} \cdot \sinh \frac{n\pi}{a}(b-y)$$

(n is odd)

or
$$u(x, y) = \frac{8a^2}{\pi^3}\sum_{n=0}^{\infty} \frac{1}{(2n+1)^3} \sin (2n+1)\frac{\pi x}{a} \cdot \frac{\sinh \frac{(2n+1)\pi}{a}(b-y)}{\sinh \frac{(2n+1)\pi}{a} b}$$

Example 7. *A thin rectangular plate whose surface is impervious to heat flow has at t = 0 an arbitrary distribution of temperature f(x, y). Its four edges x = 0, x = a, y = 0, y = b are kept at zero temperature. Determine the temperature at a point of a plate as t increases.*

Sol. Two dimensional heat flow equation is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{c^2}\frac{\partial u}{\partial t}. \qquad \ldots(1)$$

Boundary conditions are

$$u(0, y, t) = 0 = u(a, y, t)$$
$$u(x, 0, t) = 0 = u(x, b, t)$$
and
$$u(x, y, t) = f(x, y) \text{ at } t = 0.$$

Let the solution be $u = XYT$

where X is a function of x only, Y is a function of y only and T is a function of t only.

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial t}(XYT) = XY\frac{dT}{dt}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2}{\partial x^2}(XYT) = YT\frac{d^2X}{dx^2}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2}{\partial y^2}(XYT) = XT\frac{d^2Y}{dy^2}.$$

From (1), $YT X'' + XTY'' = \frac{1}{c^2}(XYT')$

$$\Rightarrow \qquad \frac{X''}{X} + \frac{Y''}{Y} = \frac{T'}{c^2T} \qquad \ldots(2)$$

There are three possibilities :

(i) $\dfrac{X''}{X} = 0, \qquad \dfrac{Y''}{Y} = 0, \qquad \dfrac{T'}{c^2T} = 0$

(ii) $\dfrac{X''}{X} = K_1^2, \qquad \dfrac{Y''}{Y} = K_2^2, \qquad \dfrac{T'}{c^2T} = K^2$

$$(iii) \qquad \frac{X''}{X} = -K_1^2, \qquad \frac{Y''}{Y} = -K_2^2, \qquad \frac{T'}{c^2 T} = -K^2$$

where $\quad K^2 = K_1^2 + K_2^2.$

Of these three solutions, we have to select the solution which is consistent with the physical nature of the problem.

The solution satisfying the given boundary conditions will be given by (*iii*).

Then,
$$X = c_1 \cos K_1 x + c_2 \sin K_1 x$$
$$Y = c_3 \cos K_2 y + c_4 \sin K_2 y$$
$$T = c_5 e^{\ c^2 K^2 t}$$

$\therefore \qquad u = XYT$

$$\Rightarrow \qquad u(x, y, t) = (c_1 \cos K_1 x + c_2 \sin K_1 x)(c_3 \cos K_2 y + c_4 \sin K_2 y)(c_5 e^{\ c^2K^2t}) \qquad ...(3)$$

$$u(0, y, t) = 0 = c_1 (c_3 \cos K_2 y + c_4 \sin K_2 y) c_5 e^{\ c^2K^2t}$$

$\Rightarrow \qquad c_1 = 0.$

$\therefore$ From (3),
$$u(x, y, t) = c_2 \sin K_1 x (c_3 \cos K_2 y + c_4 \sin K_2 y)(c_5 e^{\ c^2K^2t})$$
$$= c_6 \sin K_1 x\, (c_3 \cos K_2 y + c_4 \sin K_2 y)(e^{\ c^2K^2t}) \qquad ...(4)$$

where $\qquad c_2 c_5 = c_6$

From (4),
$$u(a, y, t) = 0 = c_6 \sin K_1 a (c_3 \cos K_2 y + c_4 \sin K_2 y) e^{\ c^2K^2t}$$

$\Rightarrow \qquad \sin K_1 a = 0 = \sin n\pi \ (n \in I)$

$$\therefore \qquad K_1 = \frac{n}{a}.$$

From (4),
$$u(x, y, t) = c_6 \sin \frac{n \;\; x}{a} \ (c_3 \cos K_2 y + c_4 \sin K_2 y)\ (e^{\ c^2K^2t}) \qquad ...(5)$$

$$u(x, 0, t) = 0 = c_6 \sin \frac{n \;\; x}{a} . c_3\, e^{\ c^2K^2t}$$

$\Rightarrow \qquad c_3 = 0.$

$\therefore$ From (5),
$$u(x, y, t) = c_6 c_4 \sin \frac{n \;\; x}{a} \sin K_2 y\, e^{\ c^2K^2t} \qquad ...(6)$$

$$u(x, b, t) = 0 = c_6 c_4 \sin \frac{n \;\; x}{a} \sin K_2 b\, e^{\ c^2K^2t}$$

$\Rightarrow \qquad \sin K_2 b = 0 = \sin m\pi \ (m \in I)$

$$K_2 b = m\pi$$

$$\Rightarrow \qquad K_2 = \frac{m}{b}.$$

$\therefore$ From (6),
$$u(x, y, t) = c_6 c_4 \sin \frac{n \;\; x}{a} \sin \frac{m \;\; y}{b}\, e^{\ c^2K^2t}$$
$$= A_{mn} \sin \frac{n \;\; x}{a} \sin \frac{m \;\; y}{b}\, e^{\ c^2K^2t} \qquad ...(7) \qquad | \text{ where } \quad c_6 c_4 = A_{mn}$$

But,
$$K^2 = K_1{}^2 + K_2{}^2 = \frac{n^2 \;\; ^2}{a^2} \quad \frac{m^2 \;\; ^2}{b^2}$$

or $$K^2_{mn} = \pi^2 \left(\frac{n^2}{a^2} + \frac{m^2}{b^2} \right).$$

By using K_{mn}, equation (7) becomes,

$$u(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin \frac{n\pi x}{a} \sin \frac{m\pi y}{b} e^{-c^2 K^2_{mn} t} \qquad ...(8)$$

which is the most general solution.

$$u(x, y, 0) = f(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin \frac{n\pi x}{a} \sin \frac{m\pi y}{b}$$

which is the double Fourier half-range sine series for $f(x, y)$.

where $$A_{mn} = \frac{2}{a} \cdot \frac{2}{b} \int_{x=0}^{a} \int_{y=0}^{b} \sin \frac{n\pi x}{a} \sin \frac{m\pi y}{b} f(x, y) dx\, dy.$$

Example 8. *Solve the Laplace equation* $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ *in a rectangle in the xy-plane with* $u(x, 0) = 0$, $u(x, b) = 0$, $u(0, y) = 0$ *and* $u(a, y) = f(y)$ *parallel to y-axis.*

Sol. The given equation is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad ...(1)$$

Let $$u = XY \qquad ...(2)$$

where X is a function of x only and Y is a function of y only. Then,

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2}{\partial x^2}(XY) = YX''$$

and $$\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2}{\partial y^2}(XY) = XY''$$

$\therefore$ From (1), $YX'' + XY'' = 0 \Rightarrow \frac{Y''}{Y} = -\frac{X''}{X}$ ...(3)

Case I. When $\frac{Y''}{Y} = -\frac{X''}{X} = p^2$ (say)

(*i*) $\frac{Y''}{Y} = p^2$

$\Rightarrow$ $Y'' - p^2 Y = 0$

Auxiliary equation is

$$m^2 - p^2 = 0 \Rightarrow m = \pm p$$

$\therefore$ C.F. $= c_1 e^{py} + c_2 e^{-py}$

P.I. $= 0$

$\therefore$ $Y = c_1 e^{py} + c_2 e^{-py}$

(*ii*) $-\frac{X''}{X} = p^2$

$\Rightarrow$ $X'' + p^2 X = 0$

Auxiliary equation is

$$m^2 + p^2 = 0 \quad \Rightarrow \quad m = \pm pi$$

$\therefore$ $$\text{C.F.} = c_3 \cos px + c_4 \sin px$$

$$\text{P.I.} = 0$$

$\therefore$ $$X = c_3 \cos px + c_4 \sin px$$

Now, $$Y(0) = 0$$

$\Rightarrow$ $$c_1 + c_2 = 0 \quad \Rightarrow \quad c_2 = -c_1$$

$$Y(b) = 0$$

$\Rightarrow$ $$c_1 e^{pb} + c_2 e^{-pb} = 0$$

$\Rightarrow$ $$c_1(e^{pb} - e^{-pb}) = 0$$

$\Rightarrow$ $$c_1 = 0$$ | Since $e^{pb} \quad e^{\quad pb} \quad 0$ (as $p \quad 0 \quad b$)

$\therefore$ $$Y = 0 \quad \Rightarrow \quad u = XY = 0, \text{ which is impossible.}$$

Hence, we reject case I.

Case II. When $$\frac{Y}{Y} = -\frac{X}{X} = 0 \text{ (say)}$$

(*i*) $$\frac{Y}{Y} = 0$$

$\Rightarrow$ $$Y'' = 0 \quad \Rightarrow \quad Y = c_5 + c_6 y$$

(*ii*) $$-\frac{X''}{X} = \quad 0$$

$\Rightarrow$ $$X'' = 0 \quad \Rightarrow \quad X = c_7 + c_8 x$$

Now, $$Y(0) = 0 \quad \Rightarrow \quad c_5 = 0$$

$$Y(b) = 0 \quad \Rightarrow \quad c_6 b = 0 \quad \Rightarrow \quad c_6 = 0$$ | $\because \quad b \neq 0$

$\therefore$ $$Y = 0$$

$\therefore$ $$u = XY = 0, \text{ which is impossible}$$

Hence, we also reject case II.

Case III. When $$\frac{Y''}{Y} = -\frac{X''}{X} = -p^2 \text{ (say)}$$

(*i*) $$\frac{Y''}{Y} = -p^2$$

$\Rightarrow$ $$Y'' + p^2 Y = 0$$

Auxiliary equation is

$$m^2 + p^2 = 0 \quad \Rightarrow \quad m = \pm pi$$

$\therefore$ $$\text{C.F.} = c_9 \cos py + c_{10} \sin py$$

$$\text{P.I.} = 0$$

$\therefore$ $$Y = c_9 \cos py + c_{10} \sin py$$

(*ii*) $$-\frac{X''}{X} = -p^2 \quad \Rightarrow \quad X'' - p^2 X = 0$$

Auxiliary equation is

$$m^2 - p^2 = 0 \quad \Rightarrow \quad m = \pm p$$

$\therefore$ $$\text{C.F.} = c_{11} e^{px} + c_{12} e^{-px}$$

$$\text{P.I.} = 0$$

$\therefore$ $$X = c_{11}\, e^{px} + c_{12}\, e^{-px}$$

Now, $$Y(0) = 0 \quad \Rightarrow \quad c_9 = 0$$

$$Y(b) = 0 \quad \Rightarrow \quad c_{10} \sin bp = 0$$

$\therefore$ $$\sin bp = 0 = \sin n\pi,\ n \in I$$

$$p = \frac{n\pi}{b}$$

Hence, $$u = XY = c_{10} \sin \frac{n\pi y}{b} \left(c_{11}\, e^{\frac{n\pi x}{b}} + c_{12} e^{-\frac{n\pi x}{b}} \right) \qquad ...(4)$$

Now, $$u(0, y) = 0 = c_{10} \sin \frac{n\pi y}{b}\, (c_{11} + c_{12})$$

$\Rightarrow$ $$c_{11} + c_{12} = 0 \quad \Rightarrow \quad c_{12} = -\, c_{11}$$

$\therefore$ From (4), $$u(x, y) = c_{10} c_{11} \sin \frac{n\pi y}{b} \left(e^{\frac{n\pi x}{b}} - e^{-\frac{n\pi x}{b}} \right)$$

$$= b_n \sin \frac{n\pi y}{b} \sinh \frac{n\pi x}{b} \qquad ...(5)$$

| where $b_n = 2\, c_{10}\, c_{11}$

Most general solution is

$$u(x, y) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi y}{b} \sinh \frac{n\pi x}{b} \qquad ...(6)$$

Now, $$u(a, y) = f(y) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi y}{b} \sinh \frac{n\pi a}{b}$$

where $$\left(\sinh \frac{n\pi a}{b} \right) b_n = \frac{2}{b} \int_0^b f(y) \sin \frac{n\pi y}{b}\, dy$$

$\Rightarrow$ $$b_n = \frac{2}{b \sinh \left(\frac{n\pi a}{b} \right)} \int_0^b f(y) \sin \frac{n\pi y}{b}\, dy. \qquad ...(7)$$

EXERCISE

1. A long rectangular plate of width a cm with insulated surface has its temperature v equal to zero on both the long sides and one of the short sides so that $v(0, y) = 0$, $v(a, y) = 0$,

$$\lim_{y} v(x, y) = 0 \quad \text{and} \quad v(x, 0) = kx$$

Show that the steady-state temperature within the plate is

$$v(x, y) = \frac{2ak}{\ } \sum_{n\ 1} \frac{(\ 1)^{n\ 1}}{n} e^{\left(\frac{n\ y}{a}\right)} \sin \frac{n\ x}{a}.$$

2. A square plate is bounded by the lines $x = 0$, $y = 0$, $x = 20$ and $y = 20$. Its faces are insulated. The temperature along the upper horizontal edge is given by $u(x, 20) = x(20 - x)$ when $0 < x < 20$ while other three edges are kept at 0°C. Find the steady state temperature in the plate.

3. A rectangular plate has sides a and b. Let the side of length a be taken along OX and that of length b along OY and the other sides along $x = a$ and $y = b$. The sides $x = 0$, $x = a$ and $y = b$ are insulated and the edge $y = 0$ is kept at temperature $u_0 \cos \frac{x}{a}$. Find the steady-state temperature at any point (x, y).

[**Hint.** Boundary conditions are $(u_x)_{x=0} = 0$, $(u_x)_{x=a} = 0$, $(u_y)_{y=b} = 0$ and $u(x, 0) = u_0 \cos(\pi x/a)$]

4. The temperature u is maintained at 0° along three edges of a square plate of length 100 cm and the fourth edge is maintained at 100° until steady-state conditions prevail. Find an expression for the temperature u at any point (x, y).

Hence, show that the temperature at the centre of the plate

$$= \frac{200}{\pi}\left[\frac{1}{\cosh\frac{\pi}{2}} - \frac{1}{3\cosh\frac{3\pi}{2}} + \frac{1}{5\cosh\frac{5\pi}{2}} - \ldots\right].$$

5. A rectangular plate is bounded by the lines $x = 0$, $y = 0$, $x = a$, $y = b$. Its surfaces are insulated and the temperature along the upper horizontal edge is 100°C while the other three edges are kept at 0°C. Find the steady state temperature function $u(x, y)$ and also the temperature at the point $\left(\frac{1}{2}a, \frac{1}{2}b\right)$.

6. Solve the following Laplace equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ in a rectangle with $u(0, y) = 0$, $u(a, y) = 0$, $u(x, b) = 0$ and $u(x, 0) = f(x)$ along x-axis.

7. Solve the boundary value problem:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,\ 0 \le x \le a,\ 0 \le y \le b$$

with the boundary conditions:

$$u_x(0, y) = u_x(a, y) = u_y(x, 0) = 0 \text{ and } u_y(x, b) = f(x)$$)

8. Solve $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ subject to the conditions $u(0, y) = 0$, $u(x, 0) = 0$, $u(1, y) = 0$ and $u(x, 1) = 100 \sin \pi x$.

(*G.B.T.U., 2013*)

Answers

2. $u(x, y) = \frac{3200}{\pi^3} \sum_{n=1}^{\infty} \frac{\sin\frac{(2n-1)\pi x}{20} \sin\frac{(2n-1)\pi y}{20}}{(2n-1)^3 \sinh(2n-1)\pi}$

3. $u(x, y) = u_0 \cos\frac{\pi x}{a} \cosh\frac{\pi}{a}(b - y) \operatorname{sech}\frac{\pi b}{a}$

5. $u(x, y) = \frac{400}{\pi} \sum_{n=1}^{\infty} \frac{\sin\left\{(2m-1)\frac{\pi x}{a}\right\} \sinh\left\{(2m-1)\frac{\pi y}{a}\right\}}{(2m-1)\sinh\left\{(2m-1)\frac{\pi b}{a}\right\}}$;

$u\left(\frac{1}{2}a, \frac{1}{2}a\right) = \frac{200}{\pi}\left[\frac{1}{\cosh\left(\frac{\pi b}{2a}\right)} - \frac{1}{3\cosh\left(\frac{3\pi b}{2a}\right)} + \ldots\right]$

6. $u(x, y) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{a}\right) \sinh\frac{n\pi}{a}(b - y)$, where $B_n = \frac{2}{a \sinh\left(\frac{n\pi b}{a}\right)} \int_0^a f(x) \sin\frac{n\pi x}{a}\, dx$

7. $u(x, y) = \sum_{n=1}^{\infty} b_n \cos\frac{n\pi x}{a}\left(e^{\frac{n\pi y}{a}} - e^{-\frac{n\pi y}{a}}\right)$ where, $b_n = \frac{1}{n\pi \cosh\frac{n\pi}{a}b} \int_0^a f(x) \cos\frac{n\pi x}{a}\, dx$

8. $u(x, y) = 100 \sin \pi x \left(\frac{\sinh \pi y}{\sinh \pi}\right)$.

5.6 OBJECTIVE TYPE QUESTIONS

1. A solution of $y^3 \frac{\partial z}{\partial x} + x^2 \frac{dz}{\partial y} = 0$, $z(x, 0) = e^{4x^3}$, is ...

(*a*) $z = e^{4x^3 - 3y^3}$ (*b*) $z = e^{4x^3 - 2y^2}$ (*c*) $z = e^{4x^3 - 3y}$ (*d*) $z = e^{4x^3 - 3y^4}$

2. A one dimensional wave equation is

(*a*) $\frac{\partial y}{\partial t} = \frac{\partial^2 y}{\partial x^2}$ (*b*) $\frac{\partial^2 y}{\partial t^2} + \frac{\partial^2 y}{\partial x^2} = 0$ (*c*) $\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial x^2}$ (*d*) $\frac{\partial^2 y}{\partial t^2} = \frac{\partial y}{\partial x}$

3. The general solution of $\frac{\partial^2 z}{\partial x^2} = 0$ is ....

(*a*) $z = ax$ (*b*) $z = ax + b$ (*c*) $z = ax^2$ (*d*) $z = ax^2 + b$

4. The solution of wave equation is ....

(*a*) exponential (*b*) logarithmic (*c*) hyperbolic (*d*) periodic

5. The initial displacement of a string which is initially at rest in equilibrium position, is ....

(*a*) positive (*b*) negative (*c*) zero (*d*) non zero

6. A suitable solution of the wave equation $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$ is ....

(*a*) $y(x, t) = (A \cos px + B \sin px)(C e^{pct} + D e^{-pct})$

(*b*) $y(x, t) = (A e^{px} + B e^{-px})(C \cos pct + D \sin pct)$

(*c*) $y(x, t) = (A e^{px} + B e^{-px})(Ce^{pct} + De^{-pct})$

(*d*) $y(x, t) = (A \cos px + B \sin px)(C \cos pct + D \sin pct)$

7. The solution of the wave equation $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$ with displacement as zero, is ....

(*a*) $y(x, t) = \sum_{n=1}^{\infty} C_n \cos\left(\frac{n\pi ct}{l}\right) \sin\left(\frac{n\pi x}{l}\right)$ (*b*) $y(x, t) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi ct}{l}\right) \sin\left(\frac{n\pi x}{l}\right)$

(*c*) $y(x, t) = \sum_{n=1}^{\infty} C_n \cos\left(\frac{n\pi ct}{l}\right) \sin\left(\frac{n\pi x}{l}\right)$ (*d*) $y(x, t) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi ct}{l}\right) \cos\left(\frac{n\pi x}{l}\right)$

8. One dimensional heat equation is ....

(*a*) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ (*b*) $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ (*c*) $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$ (*d*) $\frac{\partial u}{\partial t} = c^2 \frac{\partial u}{\partial x}$

9. Under the steady state condition, the heat equation $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$ reduces to .....

(*a*) $\frac{\partial u}{\partial x} = 0$ (*b*) $\frac{\partial u}{\partial x} = 1$ (*c*) $\frac{\partial^2 u}{\partial x^2} = 0$ (*d*) $\frac{\partial^2 u}{\partial x^2} = 1$

10. The steady state solution of the heat equation $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$, is of the form ....

(*a*) $u = ax + b$ (*b*) $u = at + b$ (*c*) $u = ax + bt$ (*d*) $u = axt + b$

11. The trivial solution of the heat equation $\frac{\partial u}{\partial t} = 2\frac{\partial^2 u}{\partial x^2}$ is ....

(*a*) $u(x, t) = 1$ (*b*) $u(x, t) = 0$ (*c*) $u(x, t) = 2$ (*d*) $u(x, t) = x$

12. The general solution of the heat equation $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$ with boundary conditions as $u(0, t) = 0$, $u(l, t) =$ for all t, is given by $u(x, t) =$ ....

(*a*) $\frac{A_0}{2}+\sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{l}\right) . e^{-\frac{n^2\pi^2c^2t}{l^2}}$ (*b*) $\sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{l}\right) e^{-\frac{n^2\pi^2c^2t}{l^2}}$

(*c*) $\sum_{n=1}^{\infty} A_n \cos(\lambda nx)\, e^{-c^2\lambda_n^2 t}, \lambda_n = \frac{(2n-1)\pi}{2l}$ (*d*) $\sum_{n=1}^{\infty} A_n \sin(\lambda_n x)\, e^{-c^2\lambda_n^2 t}, \lambda_n = \frac{(2n-1)\pi}{2l}$

13. If $u(x, t) = e^{k(x+t)}$ is a solution of the heat equation $\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}$, then the value of k is ....

(*a*) 0 or 1 (*b*) –1 or 1 (*c*) 0 or –1 (*d*) –1

14. A two dimensional Laplace equation is ....

(*a*) $\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 u}{\partial y^2}$ (*b*) $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$ (*c*) $\frac{\partial u}{\partial x}=\frac{\partial^2 u}{\partial y^2}$ (*d*) $\frac{\partial u}{\partial x}+\frac{\partial^2 u}{\partial y^2}=0$

15. A solution of Laplace equaton $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$ is ....

(*a*) $u = x^2 + y^2$ (*b*) $u = x^2 - y^2$ (*c*) $u = x^2y^2$ (*d*) $u = \frac{x^2}{y^2}$

16. If $u = kx^2 + y^2$ is a solution of Laplace equation $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$. Value of k is :

(*a*) 2 (*b*) –2 (*c*) 1 (*d*) –1

17. The two dimensional steady state heat equation is ....

(*a*) $\frac{\partial^2 u}{\partial x^2}=0$ (*b*) $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$

(*c*) $\frac{\partial^2 u}{\partial x^2}-\frac{\partial^2 u}{\partial y^2}=0$ (*d*) $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=0$

18. The trivial solution of the Laplace equation $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$ is ....

(*a*) $u = 1$ (*b*) $u = x$ (*c*) $u = 0$ (*d*) $u = y$

Answers

1. (*d*)	**2.** (*c*)	**3.** (*b*)	**4.** (*d*)	**5.** (*c*)	**6.** (*d*)
7. (*b*)	**8.** (*c*)	**9.** (*c*)	**10.** (*a*)	**11.** (*b*)	**12.** (*b*)
13. (*a*)	**14.** (*b*)	**15.** (*b*)	**16.** (*d*)	**17.** (*b*)	**18.** (*c*)

110. The one dimensional heat conduction partial differential equation $\frac{T}{t} \quad \frac{{}^{2}T}{x^{2}}$ is (GATE-96)

(a) parabolic (b) hyperbolic
(c) elliptic (d) mixed

111. The number of boundary conditions required to solve the differential equation $\frac{{}^{2}}{x^{2}} \quad \frac{{}^{2}}{y^{2}} \quad 0$ is (GATE-2001)

(a) 2 (b) 0
(c) 4 (d) 1

115. The type of the partial differential equation $\frac{f}{t} \quad \frac{{}^{2}f}{x^{2}}$ is (GATE-2013)

(a) Parabolic (b) Elliptic
(c) Hyperbolic (d) Nonlinear

116. The type of the partial differential equation $\frac{{}^{2}p}{x^{2}} \quad \frac{{}^{2}p}{y^{2}} \quad 3\frac{{}^{2}p}{x\ y} \quad 2\frac{p}{x} \quad \frac{p}{y} \quad 0$ is (GATE-2016)

(a) elliptic (b) parabolic
(c) hyperbolic (d) none of these

117. The solution of the partial differential equation $\frac{u}{t} \quad \frac{{}^{2}u}{x^{2}}$ is of the form (GATE-2016)

(a) $C\cos(kt)\ C_1 e^{\sqrt{k/\ }\ x} \quad C_2 e^{\ \sqrt{k/\ }\ x}$

(b) $C e^{kt}\ C_1 e^{\sqrt{k/\ }\ x} \quad C_2 e^{\ \sqrt{k/\ }\ x}$

(c) $C e^{kt}\ C_1 \cos \sqrt{k/\ }\ x \quad C_2 \sin \ \sqrt{k/\ }\ x$

(d) $C\sin(kt)\ C_1 \cos \sqrt{k/\ }\ x \quad C_2 \sin \ \sqrt{k/\ }\ x$

126. Consider the following partial differential equation: $3\frac{{}^{2}}{x^{2}} \quad B\frac{{}^{2}}{x\ y} \quad 3\frac{{}^{2}}{y^{2}} \quad 4 \quad 0$

For this equation to be classified as parabolic, the value of B^2 must be _______ (GATE-2017)

127. Consider the following partial differential equation for u(x, y) with the constant c > 1: $\frac{u}{y} \quad c\frac{u}{x} \quad 0$ Solution of this equation is (GATE-2017)

(a) $u(x,y) \quad f(x \quad cy)$ (b) $u(x,y) \quad f(x \quad cy)$
(c) $u(x,y) \quad f(cx \quad y)$ (d) $u(x,y) \quad f(cx \quad y)$

Answers

110.	(a)	111.	(c)	115.	(a)	116.	(c)
117.	(b)	126.	(36)	127.	(b).		