UNIT 3

Fourier Series

3.0 INTRODUCTION

Fourier Series, named after the French Mathematician and Physicist Jean-Baptiste Joseph Fourier (1768–1830), has several interesting applications in Engineering. He introduced Fourier Series in 1822, while he was dealing with the problems of heat conduction along a bar. The main idea behind the theory of Fourier Series is to represent a function $f(x)$ into harmonic components. Fourier Series can be used to represent a continuous function, discontinuous function and periodic functions. Whereas Taylor's Series and Maclaurin's Series expansion is valid only for functions which are continuous and differentiable. This Fourier Series became a very important tool in mathematical physics and also had a considerable influence on the further development of mathematics. In this chapter, we discuss the basic concepts of Fourier Series.

3.1 PERIODIC FUNCTION

A function $f : \mathrm{R} \to \mathrm{R}$ is said to be periodic if there exists a positive number T such that $f(x + \mathrm{T}) = f(x)$ for all real numbers x and T is called a period of f. If a periodic function has a smallest positive period T, then T is called the primitive period of f. If T is a period of $f(x)$, then

$$f(x) = f(x + \mathrm{T}) = f(x + 2\mathrm{T}) = \ldots = f(x + n\mathrm{T}) = \ldots$$

Also, $$f(x) = f(x - \mathrm{T}) = f(x - 2\mathrm{T}) = \ldots = f(x - n\mathrm{T}) = \ldots$$

$\therefore$ $$f(x) = f(x \pm n\mathrm{T}), \text{ where } n \text{ is a positive integer.}$$

Thus, $f(x)$ repeats itself after periods of T.

Examples :

1. sin x, cos x, cosec x, sec x are periodic functions with period 2p.

2. tan x, cot x are periodic functions with period p.

3. sin nx, cos nx, cosec nx, sec nx are periodic functions with period $\dfrac{2\pi}{n}$.

4. tan nx, cot nx are periodic functions with period $\dfrac{\pi}{n}$.

5. The constant function $f(x) = c$ is a periodic function with every positive real number is a period of f and hence this periodic function has no primitive period.

Note. Let f be a periodic function with period T. If the values of $f(x)$ are known in an interval of length T, then by periodicity of $f(x)$, $f(x)$ can be determined for all x. Hence the graph of a periodic function is obtained by periodic repetition of its graph in any interval of length T.

Examples : 1. The graph of the periodic function $f(x) = \sin x$ is given in figure (3.1)

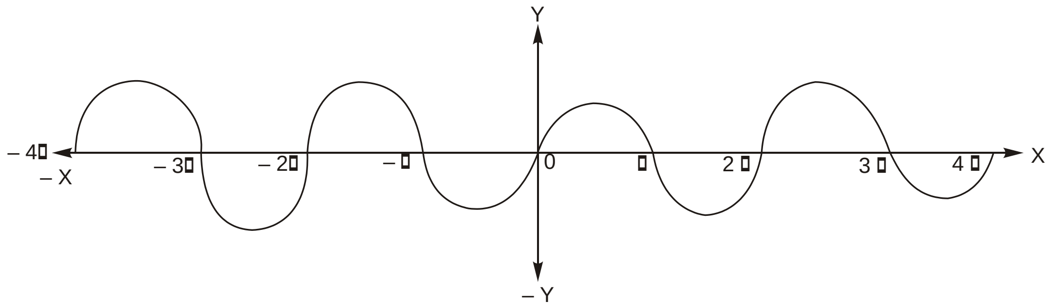

Fig. 3.1

2. The graph of the periodic function $f(x)$ defined as $f(x) = -1$ if $-p \leq x < 0$

$= 1$ if $0 \leq x < p$

and $f(x + 2p) = f(x)$ is given by in figure (3.2.).

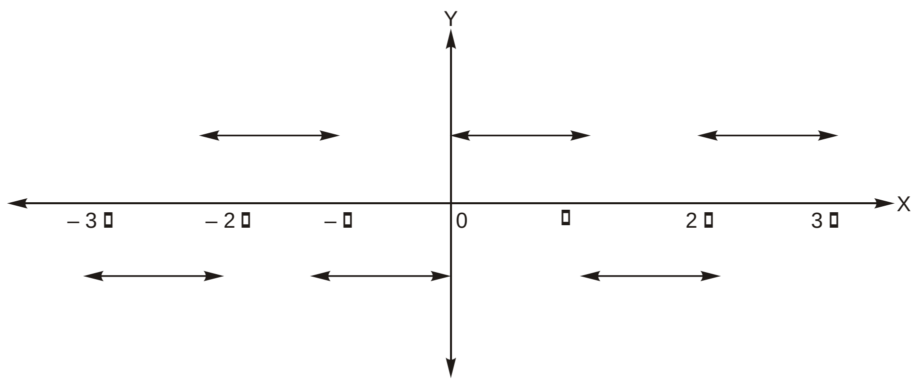

Fig. 3.2

Properties of Periodic Functions :

(1) If T is a period of $f(x)$, then nT, where n is a positive integer, is also a period of $f(x)$.

(2) If $f(x)$ be a periodic function with period T, then for any positive real number a, $f(ax)$ is a periodic function with period $\dfrac{T}{a}$.

(3) If $f(x)$ and $g(x)$ are periodic functions with period T, then $c_1 f(x) + c_2 g(x)$ is also a periodic function with period T. Where c_1 and c_2 are real numbers.

3.2 FOURIER SERIES

If $f(x)$ be a periodic function with period 2p, then the series of the form

$$f(x) = \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + \ldots + b_1 \sin x + b_2 \sin 2x + \ldots$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

is known as Fourier Series of the function $f(x)$ and the constants a_0, a_n and b_n are known as Fourier coefficients of $f(x)$ or Euler's coefficients of $f(x)$.

Notes. To determine a_0, a_n and b_n, we shall need the following results. (here m and n are integers)

1. $$\int_c^{c+2\pi} \sin nx\,dx = -\left[\frac{\cos nx}{n}\right]_c^{c+2\pi} = 0, n \neq 0$$

2. $$\int_c^{c+2\pi} \cos nx\,dx = \left[\frac{\sin nx}{n}\right]_c^{c+2\pi} = 0, n \neq 0$$

3. $$\int_c^{c+2\pi} \sin mx \cos nx\,dx = \frac{1}{2}\int_c^{c+2\pi} [\sin(m+n)x + \sin(m-n)x]\,dx$$

$$= -\frac{1}{2}\left[\frac{\cos(m+n)x}{m+n} + \frac{\cos(m-n)x}{m-n}\right]_c^{c+2\pi} = 0, m \neq n$$

$\therefore$

4. $$\int_c^{c+2\pi} \cos mx \cos nx\,dx = 0, m \neq n$$

5. $$\int_c^{c+2\pi} \sin mx \sin nx\,dx = \frac{1}{2}\int_c^{c+2\pi} [\cos(m-n)x - \cos(m+n)x]\,dx$$

$$= \frac{1}{2}\left[\frac{\sin(m-n)x}{m-n} + \frac{\sin(m+n)x}{m+n}\right]_c^{c+2\pi} = 0, m \neq n$$

6. $$\int_c^{c+2\pi} \cos^2 nx\,dx = \left[\frac{x}{2} + \frac{\sin 2nx}{4n}\right]_c^{c+2\pi} = \pi, n \neq 0$$

7. $$\int_c^{c+2\pi} \sin^2 nx\,dx = -\left[\frac{x}{2} + \frac{\sin 2nx}{4n}\right]_c^{c+2\pi} = \pi, n \neq 0$$

8. $$\int_c^{c+2\pi} \sin nx \cos nx\,dx = \frac{1}{2}\int_c^{c+2\pi} \sin 2nx\,dx = \frac{1}{2}\left[\frac{\cos 2nx}{2n}\right]_c^{c+2x} = 0,\ n \neq 0$$

9. $\sin n\pi = 0$ and $\sin n\pi = (-1)^n$

$$\sin\left(n+\frac{1}{2}\right)\pi = (-1)^n \text{ and } \cos\left(n+\frac{1}{2}\right)\pi = 0$$

$$\sin(n+1)\pi = 0 \text{ and } \cos(n+1)\pi = 1$$

$$\sin(n+c)\pi = (-1)^n \sin c\pi$$

$$\sin(n-c)\pi = (-1)^{n+1} \sin c\pi \quad \text{where n is an integer.}$$

10. $$\int e^{ax}\sin(bx+c)dx = \frac{e^{ax}}{a^2+b^2}[a\sin(bx+c]-b\cos(bx+c)]$$

$$\int e^{ax}\cos(bx+c)dx = \frac{e^{ax}}{a^2+b^2}[a\cos(bx+c]-b\sin(bx+c)]$$

11. Rule of integration by parts :

If u and v are functions of x, then

$$\int uv dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + ...$$

where 'dashes' represents differentiation and the 'subscripts' represents integration with respect to x.

3.3 DETERMINATION OF EULER'S COEFFICIENTS

The Fourier Series for the function $f(x)$ in the interval $c < x < c + 2p$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad ...(1)$$

where a_0, a_n and b_n are known as Euler's coefficients or Euler's formulae.

In finding the coefficients a_0, a_n and b_n, we assume that the series on the right hand side of (1) is uniformly convergent for $c < x < c + 2p$ and it can be integrated term by term in the given interval.

(i) **To find a_0 :** Integrate both sides of (1) with respect to x between the limits c to $c + 2p$.

$$\int_c^{c+2x} f(x)dx = \frac{a_0}{2}\int_c^{c+2x} dx + \int_c^{c+2x}\left(\sum_{n=1}^{\infty} a_n \cos nx\right)dx + \int_c^{c+2x}\left(\sum_{n=1}^{\infty} b_n \sin nx\right)dx$$

$$= \frac{a_0}{2}(c + 2\pi - c) + 0 + 0 \quad \text{[by formulae (1) and (2) above]}$$

$$= a_0\pi$$

$$\Rightarrow \quad a_0 = \frac{1}{\pi}\int_c^{c+2x} f(x)\,dx + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

(*ii*) **To find a_n :** Multiply both sides of (1) by cos *nx* and integrate w.r.t. to *x* between the limits *c* to $c + 2\pi$, we have

$$\int_c^{c+2x} f(x)\cos nx\,dx = \frac{a_0}{2}\int_c^{c+2x}\cos nx\,dx + \int_c^{c+2x}\left(\sum an\cos nx\right)\cos nx\,dx +$$

$$\int_c^{c+2x}\left(\sum_{n=1}^{\infty} b_n \sin nx\right)\cos nx\,dx$$

$= 0 + a_n\pi + 0$ [by formulae (2), (3), (4), (6) and (8)]

$= a_n\,\pi$

$$\Rightarrow \qquad a_n = \frac{1}{\pi}\int_c^{c+2x} f(x)\cos nx\;dx.$$

(*iii*) **To find b_n :** Multiply both sides of (1) by sin *nx* and integrate w.r.t. to *x* between the limits *c* to $c + 2\pi$, we have

$$\int_c^{c+2x} f(x)\sin nx\,dx = \frac{a_0}{2}\int_c^{c+2x}\sin nx\,dx + \int_c^{c+2x}\left(\sum an\cos nx\right)\sin nx\,dx +$$

$$\int_c^{c+2x}\left(\sum_{n=1}^{\infty} b_n \sin nx\right)\sin nx\,dx$$

$= 0 + 0 + b_n\pi$ [by formulae (1), (3), (5), (7) and (8)].

$= b_n\pi$

$$\Rightarrow \qquad b_n = \frac{1}{\pi}\int_c^{c+2x} f(x)\sin nx\;dx$$

Hence

$$\left.\begin{aligned} a_0 &= \frac{1}{\pi}\int_c^{c+2x} f(x)\;dx \\ a_n &= \frac{1}{\pi}\int_c^{c+2x} f(x)\cos nx\,dx \\ b_n &= \frac{1}{\pi}\int_c^{c+2x} f(x)\sin nx\,dx \end{aligned}\right\} \qquad ...(1)$$

These values of a_0, a_n and b_n are known as Euler's coefficients or Euler's formulae.

Corollary 1. If $c = 0$, the interval becomes $0 < x < 2\pi$, and the formulae (I) reduces to

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\ dx$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\, dx$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\, dx$$

Corollary 2. If $c = -\pi$, the interval becomes $-\pi < x < \pi$, and the formulae (I) reduces to

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\ dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx$$

SOLVED PROBLEMS

Problem 1. *Obtain Fourier Series for $f(x) = e^{-x}$ in the interval $0 < x < 2\pi$.*

Solution. Given $f(x) = e^{-x}$, $0 < x < 2\pi$.

Fourier Series of $f(x)$ is given by

$$f(x) = e^{-x} = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \qquad \ldots(1)$$

Now, $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\ dx = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\ dx$

$$= \frac{1}{\pi}\left[-e^{-x}\right]_0^{2\pi} = \frac{1-e^{-2\pi}}{\pi} \qquad \ldots(2)$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\, dx = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\cos nx\, dx$$

$$= \frac{1}{\pi}\left[\frac{e^{-x}}{1+n^2}(-\cos nx + n\sin nx)\right]_0^{2\pi} \quad \left[\because \int e^{ax}\cos bx\ dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)\right]$$

$$= \frac{1-e^{-2\pi}}{\pi(1+n^2)} \qquad \text{...(3)}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\, dx = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\sin nx\, dx$$

$$= \frac{1}{\pi}\left[\frac{e^{-x}}{1+n^2}(-\sin nx + n\cos nx)\right]_0^{2\pi} = \frac{n}{1+n^2}\left(\frac{1-e^{-2\pi}}{\pi}\right) \qquad \text{...(4)}$$

$$\left[\because \int e^{ax}\sin bx\ dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)\right]$$

Put the values of a_0, a_n and b_n in equation (1), we get

$$\backslash\therefore \quad e^{-x} = \frac{1-e^{-2\pi}}{\pi}\sum_{n=1}^{\infty}\frac{\cos nx}{1+n^2} + \frac{1-e^{-2\pi}}{\pi}\sum_{n=1}^{\infty}\frac{\sin nx}{1+n^2}$$

$$= \frac{1-e^{-2\pi}}{\pi}\left[\frac{1}{2} + \left(\frac{1}{2}\cos x + \frac{1}{5}\cos 2x + \frac{1}{10}\cos 3x + \ldots\right) + \left(\frac{1}{2}\sin x + \frac{2}{5}\sin 2x + \frac{3}{10}\sin 3x + \ldots\right)\right].$$

Problem 2. *Expand $f(x) = x \sin x$, $0 < x < 2\pi$ as a Fourier Series.*

Solution. Given $f(x) = x \sin x$, $0 < x < 2\pi$.

The Fourier Series of the function $f(x) = x \sin x$ is given by,

$$f(x) = x\sin x = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx\, dx + \sum_{n=1}^{\infty} b_n \sin nx\, dx \qquad \text{...(1)}$$

Now, $\quad a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x) dx$

$$= \frac{1}{\pi}\int_0^{2\pi} x\sin dx$$

$$= \frac{1}{\pi}\left[x(-\cos x)-(1).(-\sin x)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}[-2\pi] = -2 \qquad \text{...(2)}$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x\sin x\cos nx\,dx = \frac{1}{2\pi}\int_0^{2\pi} x(2\cos nx\sin x)\,dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x[\sin(n+1)x-\sin(n-1)x]\,dx$$

$$= \frac{1}{2\pi}\left[x\left\{-\frac{\cos(n+1)x}{n+1}+\frac{\cos(n-1)x}{n-1}\right\}-(1)\left\{-\frac{\sin(n+1)x}{n+1}+\frac{\sin(n-1)x}{n-1}\right\}\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[2\pi\left\{-\frac{\cos 2(n+1)\pi}{n+1}+\frac{\cos 2(n-1)\pi}{n-1}\right\}\right]$$

$$\backslash\therefore\quad a_n = -\frac{1}{n+1}+\frac{1}{n-1} = \frac{2}{n^2-1},\ n \neq 1 \qquad \text{...(3)}$$

when $n = 1$, we have

$$a_1 = \frac{1}{\pi}\int_0^{2\pi} x\sin x\cos nx\,dx = \frac{1}{2\pi}\int_0^{2\pi} x\sin 2x\,dx$$

$$= \frac{1}{2\pi}\left[x\left(-\frac{\cos 2x}{2}\right)-1\left(-\frac{\sin 2x}{4}\right)\right]_0^{2\pi} \qquad \text{...(4)}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x\sin x\sin nx\,dx \; b_n = \frac{1}{2\pi}\int_0^{2\pi} x(2\sin nx\sin x)dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x[\cos(n-1)x - \cos(n+1)x)]_0^{2\pi}dx$$

$$= \frac{1}{2\pi}\left[x\left\{-\frac{\sin(n-1)x}{n-1} - \frac{\sin(n+1)x}{n+1}\right\} - 1.\left\{-\frac{\cos(n-1)x}{n-1} + \frac{\cos(n+1)x}{n+1}\right\}\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[\frac{\cos 2(n-1)\pi}{(n-1)^2} - \frac{\cos 2(n+1)\pi}{(n+1)^2} - \frac{1}{(n-1)^2} + \frac{1}{(n+1)^2}\right]$$

$$= \frac{1}{2\pi}\left[\frac{1}{(n-1)^2} - \frac{1}{(n-1)^2} - \frac{1}{(n-1)^2} + \frac{1}{(n+1)^2}\right] \qquad ...(5)$$

$\therefore \; b_n = 0, \; n \neq 1$

When $n = 1$, we have

$$b_1 = \frac{1}{\pi}\int_0^{2\pi} x\sin x\sin x\,dx = \frac{1}{2\pi}\int_0^{2\pi} x(1-\cos 2x)dx$$

$$= \frac{1}{2\pi}\left[x\left(x - \frac{\sin 2x}{2}\right) - (1).\left(\frac{x^2}{2} + \frac{\cos 2x}{4}\right)\right]_0^{2\pi} = \frac{1}{2\pi}\left[2\pi(2\pi) - \frac{4\pi^2}{2}\frac{1}{4} + \frac{1}{4}\right]$$

$$\therefore \; b_1 = \frac{1}{2\pi}(2\pi^2) = \pi \qquad ...(6)$$

Put the values of a_0, a_n, a_1, b_n and b_1 in equation (1), we get

$$f(x) = x\sin x = \frac{a_0}{2} + a_1\cos x + b_1\sin x + \sum_{n=1}^{\infty} a_n\cos nx + \sum_{n=1}^{\infty} b_n\sin nx$$

$$= -1 - \frac{1}{2}\cos x + \pi\sin x + \sum_{n=1}^{\infty}\frac{2}{n^2+1}\cos nx + 0$$

\ $$f(x) = x\sin x = -1 + \pi\sin x - \frac{1}{2}\cos x + \frac{2}{2^2-1}\cos 2x + \frac{2}{3^2-1}\cos 3x + ...$$

Problem 3. *Find the Fourier series expansion of the function* $f(x) = \frac{1}{4}(\pi - x)^2$*, in the interval* $0 < x < 2\pi$.

Solution. Given $f(x) = \frac{1}{4}(\pi - x)^2$, $0 < x < 2\pi$. Fourier Series of the function $f(x)$ is given by

$$f(x) = \frac{1}{4}(\pi - x)^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \qquad \text{...(1)}$$

Now, $$\frac{a_0}{2} = \frac{1}{\pi}\int_0^{2\pi} f(x)dx = \frac{1}{\pi}\int_0^{2\pi}\frac{1}{4}(\pi - x)^2 dx$$

$$= \frac{1}{4\pi}\left[\frac{(\pi - x)^3}{-3}\right]_0^{2\pi} = -\frac{1}{12\pi}\left[-\pi^3 - \pi^3\right] = \frac{\pi^2}{6} \qquad \text{...(2)}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\, dx = \frac{1}{\pi}\int_0^{2\pi}\frac{1}{4}(\pi - x)^2 \cos nx\, dx$$

$$= \frac{1}{4\pi}\left[(\pi - x)^2 \frac{\sin nx}{n} - \{-2(\pi - x)\}\left(-\frac{\cos nx}{n^2}\right) + 2\left(-\frac{\sin nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{1}{4\pi}\left[\left(0 + \frac{2\pi\cos 2n\pi}{n^2} + 0\right) - \left(0 - \frac{2\pi\cos 0}{n^2} + 0\right)\right]$$

$$= \frac{1}{4\pi}\left[\frac{2\pi}{n^2} + \frac{2\pi}{n^2}\right] = \frac{1}{n^2} \qquad \text{...(3)}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\, dx = \frac{1}{\pi}\int_0^{2\pi}\frac{1}{4}(\pi - x)^2 \sin nx\, dx$$

$$= \frac{1}{4\pi}\left[(\pi - x)^2\left(-\frac{\cos nx}{n}\right) - \{-2(\pi - x)\}\left(-\frac{\sin nx}{n^2}\right) + 2\left(\frac{\cos nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{1}{4\pi}\left[\left(-\frac{\pi^2 \cos 2n\pi}{n} - 0 + \frac{2\cos 2n\pi}{n^3}\right) - \left(\frac{\pi^2}{n} - 0 + \frac{2\cos 0}{n^3}\right)\right]$$

$$= \frac{1}{4\pi}\left[\left(-\frac{\pi^2}{n} + \frac{2}{n^3}\right) - \left(-\frac{\pi^2}{n} + \frac{2}{n^3}\right)\right]$$

$$\therefore \; b_n = 0 \qquad \text{...(4)}$$

Put the values of a_0, a_n and b_n in (1), we get,

$$f(x) = \frac{1}{4}(\pi - x)^2 = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{\cos nx}{n^3}$$

$$\therefore \qquad f(x) = \frac{\pi^2}{12} + \frac{\cos x}{1^2} + \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} + \ldots$$

Problem 4: Find the fourier series for f(x) = e^x in $[0.2\pi]$ (OU Dec 2014)

Sol. Given f(x) = e^x in $0 \le x \le 2\pi$
Fourier series of the function f(x) is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos_{nx} + \sum_{n=1}^{\infty} b_n \sin_{nx} \text{......(1)}$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} e^x dx = \frac{1}{\pi}\left(e^x\right)_0^{2\pi}$$

Now $$= \frac{1}{\pi}\left[e^{2\pi} - e^0\right]$$

$$= \frac{1}{\pi}\left[e^{2\pi} - 1\right] \text{......(2)}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx dx$$

and

$$= \frac{1}{\pi}\int_0^{2\pi} e^x \cos nx dx$$

we have

$$\int e^{ax}\cos bx dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)$$

$$\frac{1}{\pi}\left[\frac{e^{ax}}{1+n^2}(\cos nx + n\sin nx)\right]_0^{2\pi}$$

$$= \frac{1}{\pi(1+n^2)}\left[e^{2\pi}(\cos 2n\pi + n\sin 2n\pi) - e^0(\cos 0 + n\sin 0)\right]$$

$$= \frac{1}{\pi(1+n^2)}\left[e^{2\pi}[(-1)^{2n}+0] - 1[1+0]\right]$$

$$= \frac{1}{\pi(1+n^2)}(e^{2\pi}-1)(\because \sin 2n\pi = 0, \cos 2n\pi = 1)$$

$$= \frac{e^{2\pi}-1}{\pi(1+n^2)}$$

Now $b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx dx$

$$= \frac{1}{\pi}\int_0^{2\pi} e^x \sin nx dx$$

$$\int e^{ax}\sin bx dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)$$

$$= \frac{1}{\pi}\left[\frac{e^x}{1+n^2}(\sin nx - n\cos nx)\right]_0^{2\pi}$$

$$= \frac{1}{\pi(1+n^2)}\left[e^{2\pi}(\sin 2n\pi - n\cos 2n\pi) - e^0(\sin 0 - n\cos 0)\right]$$

$$= \frac{1}{\pi(1+n^2)}\left[e^{2\pi}(0 - n(-1)^{2n}) - 1(0-n)\right]$$

$$= \frac{1}{\pi(1+n^2)}\left[-ne^{2\pi} + n\right] = \frac{-n}{\pi(1+n^2)}(e^{2\pi}-1)$$

Substiute a_0, $a_{n.}$, b_n in (1)

$$e^x = \frac{e^{2\pi}-1}{\pi} + \sum_{n=1}^{\infty}\frac{e^{2\pi}-1}{\pi(1+n^2)}\cos nx + \sum_{n=1}^{\infty}\frac{(-n)(e^{2\pi}-1)}{\pi(1+n^2)}\sin nx.$$

Problem 5. *Obtain the Fourier Series to represent* $x - x^2$ *in* $(-\pi, \pi)$ *and deduce that*

$$\frac{\pi^2}{12} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} \ldots$$

Solution. Let $f(x) = x - x^2$

The Fourier Series for the function $f(x) = x - x^2$, in the interval $-\pi < x < \pi$ is given by,

$$f(x) = x - x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \qquad \ldots(1)$$

Now, $$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi}(x - x^2)dx = \frac{1}{\pi}\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_{-\pi}^{\pi}$$

$$=\frac{1}{\pi}\left[\left(\frac{\pi^2}{2}-\frac{\pi^3}{3}\right)-\left(\frac{\pi^2}{2}+\frac{\pi^3}{3}\right)\right]=-\frac{2\pi^3}{3} \qquad \ldots(2)$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\; dx=\frac{1}{\pi}\int_{-\pi}^{\pi}(x-x^2)\cos nx\; dx$$

$$=\frac{1}{\pi}\int_{-\pi}^{\pi}\left[(x-x^2)\frac{\sin nx}{n}-(1-2x)\left(\frac{\cos nx}{n^2}\right)+(-2)\left(\frac{\sin nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$=\frac{1}{\pi}\left[(1-2\pi)\frac{\cos n\pi}{n^2}-(1+2\pi)\frac{\cos n\pi}{n^2}\right]=\frac{1}{\pi}\left(-4\pi\frac{\cos n\pi}{n^2}\right)$$

$$=-4\frac{(-1)^n}{n^2} \qquad \ldots(3)\ [\because \cos n\pi=(-1)^n]$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\; dx=\frac{1}{\pi}\int_{-\pi}^{\pi}(x-x^2)\sin nx\; dx$$

$$=\frac{1}{\pi}\int_{-\pi}^{\pi}\left[(x-x^2)\left(\frac{\cos nx}{n}\right)-(1-2x)\left(-\frac{\sin nx}{n^2}\right)+(-2)\left(\frac{\cos nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$=\frac{1}{\pi}\left[(\pi^2-\pi)\frac{\cos n\pi}{n}-2\frac{\cos n\pi}{n^3}+(-\pi-\pi^2)\frac{\cos n\pi}{n}+2\frac{\cos n\pi}{n^3}\right]$$

$$=\frac{1}{\pi}\left(-2\pi\frac{\cos n\pi}{n}\right)=-2\frac{(-1)^n}{n} \qquad \ldots(4)$$

Put a_0, a_n and b_n values in (1), we get

$$x-x^2=-\frac{\pi^2}{3}-4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx-2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin nx$$

$$=-\frac{\pi^2}{3}-4\left[\frac{\cos x}{1^2}+\frac{\cos 2x}{2^2}+\frac{\cos 3x}{3^2}+\ldots\right]-2\left[\frac{\sin x}{1}+\frac{\sin 2x}{2}+\frac{\sin 3x}{3}+\ldots\right]$$

$$\therefore \quad x - x^2 + 4\left[\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} + \ldots\right] + 2\left[\frac{\sin x}{1} + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \ldots\right] \qquad \ldots(5)$$

Deduction : Put $x = 0$ in (5), we get

$$0 = -\frac{\pi^2}{3} - 4\left(\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} + \ldots\right)$$

$$\Rightarrow \frac{\pi}{12} - 4\left(\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} \ldots\right)$$

3.3.1 DIRICHLET'S CONDITIONS

The sufficient conditions for the uniform convergence of a Fourier Series are called Dirichlet's conditions. All the functions that normally arise in engineering problems satisfy these conditions and hence they can be expressed as a Fourier Series.

Any function $f(x)$ can be expressed as a Fourier Series of the form,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

where a_0, a_n and b_n are constants, provided

(*i*) $f(x)$ is periodic, single valued and finite,

(*ii*) $f(x)$ has a finite number of finite discontinuities in any one period,

(*iii*) $f(x)$ has a finite number of maxima and minima.

Where these conditions are satisfied, the Fourier Series converges to $f(x)$ at every point of continuity.

At a point of discontinuity, the sum of the series is equal to the mean of the limits on the right and left.

i.e.,
$$\frac{1}{2}\,[f(x + 0) + f(x - 0)]$$

where $f(x + 0)$ and $f(x - 0)$ represent the limit on the right and the limit on the left respectively.

3.4 FOURIER SERIES FOR DISCONTINUOUS FUNCTIONS

We derived Euler's formulae for a_0, a_n and b_n on the assumption that $f(x)$ is continuous in $(c, c + 2p)$.

However, if $f(x)$ has finitely many points of finite discontinuity, even then it can be expressed as a Fourier Series.

Let $f(x)$ be defined by

$f(x) = f_1(x),\ c < x < x_0$

$\quad = f_2(x),\ x_0 < x < c + 2\pi$

where x_0 is the point of discontinuity in the interval $(c, c + 2\pi)$. Then the values of Euler's coefficients a_0, a_n and b_n are given by

$$a_0 = \frac{1}{\pi}\left[\int_c^{x_0} f_1(x)\ dx + \int_{x_0}^{c+2x} f_2(x)\ dx\right]$$

$$a_n = \frac{1}{\pi}\left[\int_c^{x_0} f_1(x)\cos nx\ dx + \int_{x_0}^{c+2x} f_2(x)\cos nx\ dx\right]$$

$$b_n = \frac{1}{\pi}\left[\int_c^{x_0} f_1(x)\sin nx\ dx + \int_{x_0}^{c+2x} f_2(x)\sin nx\ dx\right]$$

Problem 1. *Find the Fourier Series expansion of*

$$f(x) = \begin{cases} 0\ ; -\pi \le x \le 0 \\ x^2;\ \ 0 \le x \le \pi \end{cases}.$$

Solution. Given $f(x) = 0$, – p £ x £ 0

$\quad = x^2$; 0 £ x £ p

The Fourier Series of $f(x)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \qquad \ldots(1)$$

Now $$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\,dx + \int_0^{\pi} f(x)dx\right] = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\,dx + \int_0^{\pi} x^2 dx\right]$$

$$= \frac{1}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi} = \frac{\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\cos nx\,dx + \int_{0}^{\pi} f(x)\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} 0.\cos nx\,dx + \int_{0}^{\pi} x^2\cos nx\,dx\right] = \frac{1}{\pi}\int_{0}^{\pi} x^2\cos nx\,dx$$

$$= \frac{1}{\pi}\left[x^2.\frac{\sin nx}{n} - (2x).\left(-\frac{\cos nx}{n^2}\right) + (2).\left(-\frac{\sin nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{1}{\pi}.\frac{2\pi}{n^2}\cos n\pi = \frac{2}{n^2}(-1)^n, n = 1, 2, 3... \qquad \textbf{...(3)}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\sin nx\,dx + \int_{0}^{\pi} f(x)\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} 0.\sin nx\,dx + \int_{0}^{\pi} x^2\sin nx\,dx\right] = \frac{1}{\pi}\int_{0}^{\pi} x^2\sin nx\,dx$$

$$= \frac{1}{\pi}\left[x^2\left(-\frac{\cos nx}{n}\right) - (2x)\left(-\frac{\sin nx}{n^2}\right) + (2).\left(\frac{\cos nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[-\frac{\pi^2}{n}\cos n\pi + \frac{2}{n^3}(\cos n\pi - 1)\right]$$

$$= \frac{1}{\pi}(-1)^n + \frac{2}{\pi n^3}\left[(-1)^n - 1\right] \qquad \textbf{...(4)}$$

Substituting the values of (2), (3) and (4) in (1)

$$f(x)=\frac{\pi^2}{6}+2\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx+\sum_{n=1}^{\infty}\left\{\frac{\pi}{n}(-1)^{n+1}+\frac{2}{\pi n^3}\left[(-1)^n-1\right]\right\}\sin nx$$

Problem 2. *Expand* $f(x)=\begin{cases}-\pi, & -\pi<x<0\\ x, & 0<x<\pi\end{cases}$

as a Fourier Series and hence deduce that $\frac{\pi^2}{8}=\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots$

Solution. Given $f(x)=\begin{cases}-\pi, & -\pi<x<0\\ x, & 0<x<\pi\end{cases}$

The Fourier Series of $f(x)$ is given by

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx+\sum_{n=1}^{\infty}b_n\sin nx \qquad ...(1)$$

Now, $a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\;dx$ \

$$=\frac{1}{\pi}\left[\int_{-\pi}^{0}f(x)\;dx+\int_{0}^{\pi}f(x)\;dx\right]=\frac{1}{\pi}\left[\int_{-\pi}^{0}(-\pi)\;dx+\int_{0}^{\pi}x\;dx\right]$$

$$=\frac{1}{\pi}\left[(-\pi x)_{-\pi}^{0}+\left(\frac{x^2}{2}\right)_0^{\pi}\right]=\frac{1}{\pi}\left[-\pi^2+\frac{\pi^2}{2}\right]$$

$$\therefore \quad a_0=-\frac{\pi}{2} \qquad ...(2)$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\;dx$$

$$=\frac{1}{\pi}\left[\int_{-\pi}^{0}f(x)\cos nx\;dx+\int_{0}^{\pi}f(x)\cos nx\;dx\right]$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0}(-\pi)\cos nx \; dx + \int_{0}^{\pi} x\cos nx \;\; dx\right]$$

$$= \frac{1}{\pi\, n}\left[\left(-\frac{\pi \sin nx}{n}\right)_{-\pi}^{0} + \left\{x, \frac{\sin nx}{n} - (1)\left(-\frac{\cos nx}{n^2}\right)\right\}_{0}^{\pi}\right]$$

$$= \frac{1}{\pi\, n^2}\left[\cos n\pi - \cos 0\right] = \frac{1}{\pi\, n^2}\left[(-1)^n - 1\right]$$

$$\therefore \qquad a_n = \begin{cases} 0; & \text{if } n \text{ is even} \\ \dfrac{2}{\pi\, n^2}; & \text{if } n \text{ is odd} \end{cases}$$

$$\therefore \qquad a_{2n-1} = \frac{2}{\pi\, n^2} \qquad \text{...(3)}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx \;\; dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\sin nx \; dx + \int_{0}^{\pi} f(x)\sin nx \;\; dx\right]$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} -\pi \sin nx \; dx + \int_{0}^{\pi} x\cos nx \;\; dx\right]$$

$$= \frac{1}{\pi}\left[\left(\frac{\pi \cos nx}{n}\right)_{-\pi}^{0} + \left\{x.\left(-\frac{\cos nx}{n}\right) - (1)\left(-\frac{\sin nx}{n^2}\right)\right\}_{0}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{n}(1 - \cos n\pi) - \frac{1}{n}(\pi \cos n\pi - 0) + 0\right] = \frac{1}{\pi}.\frac{\pi}{n}\left[1 - \cos n\pi - \cos n\pi\right]$$

$$\therefore \qquad b_n = \frac{1}{n}\left[1 - 2\cos n\pi\right] \qquad \text{...(4)}$$

Substitute the values of a_0, a_n and b_n in (1), we have

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_{2n-1} \cos(2n-1)x + \sum_{n=1}^{\infty} b_n \sin nx$$

$$\therefore \quad f(x) = -\frac{\pi}{4} + \sum_{n=1}^{\infty} \frac{-2}{\pi(2n-1)^2} \cos(2n-1)x + \sum_{n=1}^{\infty} \left(\frac{1-2\cos n\pi}{n}\right) \times \sin nx$$

$$\Rightarrow \quad f(x) = -\frac{\pi}{4} - \frac{2}{\pi} + \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \cos(2n-1)x + \sum_{n=1}^{\infty} \left(\frac{1-2\cos n\pi}{n}\right) \sin nx \qquad \ldots(5)$$

Deduction : Put $x = 0$, in (5),

$$f(0) = -\frac{\pi}{4} - \frac{2}{\pi} + \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$$

but $\quad f(x)$ is not defined at $x = 0$.

$\Rightarrow \quad f(x)$ is discontinuous at $x = 0$.

Therefore, the value of $f(x)$ at $x = 0$ is given by

$$f(0) = \frac{1}{2}[f(0-0) + f(0+0)] = \frac{1}{2}[-\pi + 0]$$

$$\Rightarrow \quad f(0) = -\frac{\pi}{2}$$

Put this value in above equation, we have

$$-\frac{\pi}{2} = -\frac{\pi}{4} - \frac{2}{\pi} + \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$$

$$\Rightarrow \quad -\frac{\pi}{2} + \frac{\pi}{4} = -\frac{2}{\pi} + \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$$

$$\Rightarrow \quad \frac{\pi}{4} = \frac{2}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\right]$$

$$\Rightarrow \quad \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots = \frac{\pi^2}{8}$$

Problem 3. *Obtain the Fouirer Series to represent the function $f(x)$ given by*

$$f(x) = \begin{cases} x, & 0 \le x \le \pi \\ 2\pi - x, & \pi \le x \le 2\pi \end{cases}$$

and hence deduce that $\dfrac{\pi^2}{8} = \dfrac{1}{1^2} + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \ldots$

Solution. Given $$f(x) = \begin{cases} x, & 0 \le x \le \pi \\ 2\pi - x, & \pi \le x \le 2\pi \end{cases}$$

Fourier Series of the function $f(x)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \qquad \ldots(1)$$

Now, $$a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x)\, dx$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} f(x)\, dx + \int_{\pi}^{2\pi} f(x)\, dx\right] = \frac{1}{\pi}\left[\int_0^{\pi} x\, dx + \int_{\pi}^{2\pi} (2\pi - x)\, dx\right]$$

$$= \frac{1}{\pi}\left[\left(\frac{x^2}{2}\right)_0^{\pi} + \left(2\pi x - \frac{x^2}{2}\right)_{\pi}^{2\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi^2}{2} + (4\pi^2 - 2\pi^2) - \left(2\pi^2 - \frac{\pi^2}{2}\right)\right] = \pi \qquad \ldots(2)$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx\, dx$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} f(x) \cos nx\, dx + \int_{\pi}^{2\pi} f(x) \cos nx\, dx\right]$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} x \cos nx\, dx + \int_{\pi}^{2\pi} (2\pi - x) \cos nx\, dx\right]$$

$$= \frac{1}{\pi}\left[\left(x.\frac{\sin nx}{n} - (1)\left(-\frac{\cos nx}{n^2}\right)\right)_0^{\pi} + \left\{(2\pi - x)\left(x.\frac{\sin nx}{n} - (1)\left(-\frac{\cos nx}{n^2}\right)\right)_{\pi}^{2\pi}\right\}\right]$$

$$= \frac{1}{\pi}\left[\left(\frac{\cos n\pi}{n^2} - \frac{1}{n^2}\right) + \left(-\frac{\cos 2n\pi}{n^2} + \frac{\cos n\pi}{n^2}\right)\right]$$

$$= \frac{1}{n\pi^2}\left[(-1)^n - 1 - 1 + (-1)^n\right] = \frac{2}{n\pi^2}\left[(-1)^n - 1\right]$$

$$= \begin{cases} -\dfrac{4}{n\pi^2}, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even} \end{cases} \qquad \text{...(3)}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \sin nx \, dx$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} f(x) \sin nx \, dx + \int_{\pi}^{2\pi} f(x) \sin nx \, dx\right]$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} x \sin nx \, dx + \int_{\pi}^{2\pi} (2\pi - x) \sin nx \, dx\right]$$

$$= \frac{1}{\pi}\left[\left\{(x) - \frac{\cos nx}{n} - (1)\left(-\frac{\sin nx}{n^2}\right)\right\}_0^{\pi} + \left\{(2\pi - x)\left(-\frac{\cos nx}{n} - (1)\left(-\frac{\sin nx}{n^2}\right)\right)_{\pi}^{2\pi}\right\}\right]$$

$$= \frac{1}{\pi}\left[-\frac{\pi \cos n\pi}{n} + \frac{\pi \cos n\pi}{n}\right]$$

$$= 0 \qquad \text{...(4)}$$

Put the values of (2), (3) and (4) in (1), we get

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots\right) \qquad \text{...(5)}$$

Deduction : Put $x = 0$ in (5), we get

$$0 = \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\right)$$

$$\Rightarrow \frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots$$

Problem 4. *Show that the Fourier Series expansion of the function*

$$f(x) = \begin{cases} 0\,; & -\pi \le x \le 0 \\ \sin x; & 0 \le x \le \pi \end{cases}$$

is
$$f(x) = \frac{1}{\pi} + \frac{1}{2}\sin x - \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\cos 2nx}{4n^2 - 1}$$

and hence deduce that,

(i) $\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} + \ldots = \frac{1}{2}$ (ii) $\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \ldots = \frac{\pi - 2}{4}$

Sol. Given $f(x) = \begin{cases} 0\,; & -\pi \le x \le 0 \\ \sin x; & 0 \le x \le \pi \end{cases}$

The fourier series of the function f(x) is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos_{nx} + \sum_{n=1}^{\infty} b_n \sin_{nx} \quad \ldots\ldots(1)$$

Now, $$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\, dx + \int_{0}^{\pi} f(x)\, dx\right]$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} (0)\, dx + \int_{0}^{\pi} \sin nx\, dx\right]$$

$$= \frac{2}{\pi} \quad \ldots(2)$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\cos nx\, dx + \int_{0}^{\pi} \cos nx\, dx\right]$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} (0)\cos nx\, dx + \int_{0}^{\pi} \sin x \cos nx\, dx\right]$$

$$= \frac{1}{2\pi}\int_{0}^{\pi} 2\cos nx \sin x\, dx$$

$$= \frac{1}{2\pi}\int_{0}^{\pi} [\sin(n+1)x - \sin(n-1)x]dx$$

$$= \frac{1}{2\pi}\left[\left(-\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right)_0^{\pi}, n \neq 1\right]_0^{\pi}$$

$$= \frac{1}{2\pi}\left[-\frac{\cos(n+1)\pi}{n+1} + \frac{\cos(n-1)\pi}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right]$$

$$= \frac{1}{2\pi}\left[-\frac{(-1)^{n+1}}{n+1} + \frac{(-1)^{n-1}}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right]$$

$$= \begin{cases} \frac{1}{2}\left(-\frac{1}{n+2} + \frac{1}{n-2} + \frac{1}{n+1} - \frac{1}{n-1}\right); \text{ when } n \text{ is odd} \\ \frac{1}{2}\left(-\frac{1}{n+1} - \frac{1}{n-1} + \frac{1}{n+1} - \frac{1}{n+1}\right) \; ; \text{ when } n \text{ is even} \end{cases}$$

$$= \begin{cases} 0; \text{ when } n \text{ is odd i.e., } n = 3, 5\ldots \\ -\frac{2}{\pi(n^2-1)} \; ; \text{ when } n \text{ is even} \end{cases} \qquad \ldots(3)$$

when n = 1, we have

$$a_1 = \frac{1}{\pi}\int_{0}^{\pi} \sin x \cos x\, dx$$

$$= \frac{1}{2\pi}\int_0^{\pi} \sin 2x \, dx$$

$$= \frac{1}{2\pi}\left[-\frac{\cos 2x}{2}\right]_0^{\pi} = 0 \qquad ...(4)$$

Now, $$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx \, dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\sin nx \, dx + \int_0^{\pi} f(x)\sin nx \, dx\right]$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} (0)\sin nx \, dx + \int_0^{\pi} \sin x \sin nx \, dx\right]$$

$$= \frac{1}{2\pi}\int_0^{\pi} 2\sin nx \sin x \, dx$$

$$= \frac{1}{2\pi}\int_0^{\pi} [\cos(n-1)x - \cos(n+1)x] dx$$

$$= \frac{1}{2\pi}\left[\frac{\sin(n-1)x}{n-1} - \frac{\sin(n+1)x}{n+1}\right]_0^{\pi}$$

$$= 0, \ n \neq 1 \qquad ...(5)$$

When n = 1, we have $$b_1 = \frac{1}{\pi}\int_0^{\pi} \sin x \sin x \, dx = \frac{1}{2\pi}\int_0^{\pi} (1-\cos 2x) \, dx$$

$$= \frac{1}{2\pi}\left[x - \frac{\sin 2x}{2}\right]_0^{\pi} = \frac{1}{2} \qquad ...(6)$$

Substitute the values of (2), (3), (4), (5) and (6) in (1), we get

$$f(x) = \frac{1}{\pi} - \frac{2}{\pi}\left[\frac{\cos 2x}{2^2-1} + \frac{\cos 4x}{4^2-1} + \frac{\cos 6x}{6^2-1} + ...\right] + \frac{1}{2}\sin x$$

$$= \frac{1}{\pi} + \frac{2}{\pi}\sin x - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\cos 2nx}{(2n)^2 - 1} \qquad ...(7)$$

Deductions : (i) Put x = 0 in (7), we have

$$0 = \frac{1}{\pi} - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2 - 1}$$

$$\Rightarrow \quad \frac{1}{2} = \sum_{n=1}^{\infty}\frac{1}{4n^2 - 1}$$

$$\Rightarrow \quad \frac{1}{2} = \sum_{n=1}^{\infty}\frac{1}{(2n-1)(2n+1)}$$

$$\frac{1}{2} = \frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + ...$$

(ii) Put $x = \frac{\pi}{2}$ in (7), we have

$$1 = \frac{1}{\pi} + \frac{1}{2} - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\cos n\pi}{4n^2 - 1}$$

$$\Rightarrow \quad \frac{1}{2} - \frac{1}{\pi} = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{4n^2 - 1}$$

$$\Rightarrow \quad \frac{\pi - 2}{4} = \sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)(2n+1)} = -\left(-\frac{1}{1.3} + \frac{1}{3.5} - ...\right)$$

$$\Rightarrow \quad \frac{\pi - 2}{4} = \frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} + ...$$

Example 6: Obtain the fourier series for the function f(x) is given by

$$f(x) = \begin{cases} 1 + \frac{2x}{\pi}; & -\pi \le x \le 0 \\ 1 - \frac{2x}{\pi}\ ; & 0 \le x \le \pi \end{cases}$$

and hence deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + ... = \frac{\pi^2}{8}$.

Sol. Given $f(x)=\begin{cases} 1+\dfrac{2x}{\pi};\ -\pi \le x \le 0 \\ 1-\dfrac{2x}{\pi}\ ;\ \ 0 \le x \le \pi \end{cases}$

Now, $f(-x)=1+\dfrac{2}{\pi}(-x)$ in $[-\pi,0]$ = $=1-\dfrac{2x}{\pi}=f(x)$ in $[0,\pi]$

and $f(-x)=1+\dfrac{2}{\pi}(-x)$ in $[0,\pi]$ = $f(x)=1+\dfrac{2x}{\pi}$ in $[-\pi,0]$

$\therefore$ The function f(x) is even and hence the Fourier Series of f(x) is given by

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos nx \qquad ...(1)$$

Now, $$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx=\frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

$$=\frac{2}{\pi}\int_0^{\pi}\left(1-\frac{2x}{\pi}\right)dx=\frac{2}{\pi}\left[x-\frac{2}{\pi}\frac{x^2}{2}\right]_0^{\pi}$$

$$=\frac{2}{\pi}\left[x-\frac{x^2}{2}\right]_0^{\pi}=0$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$=\frac{2}{\pi}\int_0^{\pi}\left(1-\frac{2x}{\pi}\right)\cos nx\,dx$$

$$=\frac{2}{\pi}\left[\left(1-\frac{2x}{\pi}\right)\frac{\sin nx}{n}-\left(-\frac{2}{\pi}\right)\left(-\frac{\cos nx}{n^2}\right)\right]_0^{\pi}$$

$$=\frac{2}{\pi}\left(-\frac{2}{\pi n^2}\right)[\cos n\pi-\cos 0]$$

$$= -\frac{4}{\pi^2 n^2}\left[(-1)^n - 1\right]$$

$$\therefore \qquad a_n = \begin{cases} 0, & \text{if } n \text{ is even} \\ \dfrac{8}{n^2\pi^2}, & \text{if } n \text{ is odd} \end{cases}$$

$$\therefore \qquad a_{2n-1} = \frac{8}{\pi^2(2n-1)^2} \text{ for all n.}$$

Substituting the values a_0, a_n in (1), we have

$$f(x) = \sum_{n=1}^{\infty} \frac{8}{\pi^2(2n-1)^2}\cos nx$$

$$\Rightarrow \qquad f(x) = \frac{8}{\pi^2}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}\cos nx$$

Deduction : Put x = 0 in (4),

$$f(0) = \frac{8}{\pi^2}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$$

$$\Rightarrow \qquad 1 = \frac{8}{\pi^2}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\right]$$

$$\Rightarrow \qquad \frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots$$

3.5 Even and Odd Functions

Even Function : A function $f(x)$ is said to be an even function if

$$f(-x) = f(x) \text{ for all } x.$$

Example : x^2, $\cos x$, $\sin^2 x$ are even functions.

Notes : 1. The sum of two even functions is even.

2. The product of two even functions is even.

3. If $f(x)$ is an even function, the

$$\int_{-c}^{c} f(x)\,dx = 2\int_{0}^{c} f(x)\,dx$$.

4. The graph of an even function is symmetrical about Y-axis.

Odd Function : A function $f(x)$ is said to be an odd function if

$$f(-x) = -f(x), \text{ for all } x.$$

Example : x, x^3, $\sin x$, $\tan^3 x$ are even functions.

Notes : **1.** The sum of two odd function, is an odd function.

2. The product of two odd functions is an even function.

3. The product of an even function and an odd function is an odd function.

4. If $f(x)$ is an odd function, then $\int_{-c}^{c} f(x)\,dx = 0$.

5. The graph of odd function is symmetrical about origin.

Remark : There are some functions which are neither odd nor even.

Example : $x^2 + \sin x$, $x - x^2$, e^x

Corollary 3. (*a*) **When *f* (*x*) is an even function,**

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)dx$$

Since cos *nx* is an even function, therefore $f(x)$ cos *nx* is also an even function.

$$\therefore \qquad a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos nx\,dx$$

Since sin *nx* is an odd function, therefore $f(x)$ sin *nx* is also an odd function.

$$\therefore \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = 0$$

Hence, if a periodic function $f(x)$ is an even function, its fourier expansion contains only cosine terms,

i.e.,
$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

where
$$a_0 = \frac{2}{\pi}\int_{0}^{\pi} f(x)\,dx,$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

(*b*) **When *f* (*x*) is an odd function,**

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = 0$$

Since cos *nx* is an even function, therefore, *f* (*x*) cos *nx* is an odd function.

$$\therefore \qquad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = 0$$

Since sin *nx* is an odd function, therefore, *f* (*x*) sin *nx* is an even function.

$$b_n = \frac{2}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = 0$$

$$= \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$

Hence, if a periodic function *f*(*x*) is an odd function,its fourier expansion contains only sine terms,

i.e., $$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

where $$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx.$$

Problem 1. *Find the Fourier Series of f (x) = x, in –* $\pi < x < \pi$.

Solution. Given $f(x) = x.$

Since $f(-x) = -x = -f(x)$

\ $f(x)$ is an odd function and hence $a_0 = a_n = 0$.

\ The Fourier Series of $f(x) = x$ is given by

$$f(x) = x = \sum_{n=1}^{\infty} b_n \sin nx \qquad \text{...(1)}$$

Now, $$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin nx\, dx$$

$$= \frac{1}{\pi} \left[x\left(-\frac{\pi \cos n\pi}{n} \right) - (1)\left(-\frac{\sin nx}{n^2} \right) \right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi} \left[\left(-\frac{\pi \cos n\pi}{n} + 0 \right) - \left(\frac{\pi \cos n\pi}{n} + 0 \right) \right]_{-\pi}^{\pi}$$

$$= -\frac{2\pi \cos n\pi}{n\pi} = \frac{2 \cos n\pi}{n}$$

$\therefore$ $$b_n = \frac{-2(-1)^n}{n} \qquad \text{...(2)} \quad (\because \cos n\pi = (-1)^n)$$

Put the value of b_n in (1), we get

$$x = \sum_{n=1}^{\infty} \frac{-2(-1)^n}{n} \sin nx$$

$\Rightarrow$ $$x = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin nx.$$

Problem 2. *Obtain the Fourier Series for the function f (x) = x^2, $-\pi < x < \pi$. Hence show that*

(i) $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + ... = \frac{\pi^2}{6}$

(ii) $\frac{1}{1^2} - \frac{1}{2^2} - \frac{1}{3^2} - \frac{1}{4^2} ... = \frac{\pi^2}{12}$

(iii) $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + ... = \frac{\pi^2}{8}$

Solution. Given $f(x) = x^2$

Since $f(-x) = (-x)^2 = x^2 = f(x)$

$\therefore$ $f(x)$ is an even function and hence $b_n = 0$.

$\therefore$ The Fourier Series of $f(x) = x^2$ is given by

$$f(x) = x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \qquad \text{...(1)} \ (\because b_n = 0)$$

Now, $$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} f(x)\,dx \qquad (\because \text{ since } f(x) \text{ is even function})$$

$$\therefore \quad a_0 = \frac{2}{\pi} x^2 dx = \frac{2}{\pi}\left(\frac{x^3}{3}\right)_0^{\pi} = \frac{2}{\pi} \times \frac{\pi^3}{3} = \frac{2}{3}\pi^2 \qquad \text{...(2)}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx\,dx$$

$$= \frac{2}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) - (2x)\left(-\frac{\cos nx}{n^2}\right) + 2\left(-\frac{\sin nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\left(0 + \frac{2\pi\cos n\pi}{n^2} - 0\right) - (0 + 0 - 0)\right]$$

$$= \frac{2}{\pi}\left[2\pi\frac{\cos n\pi}{n^2}\right]$$

$$\therefore \quad a_n = 4\frac{(-1)^n}{n^2} \qquad \text{...(3) } [\because \cos np = (-1)^n]$$

Put a_0 and a_n values in (1), we have

$$x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx$$

$$= \frac{\pi^2}{3} - 4\left[\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \ldots\right] \qquad \text{...(4)}$$

Deductions :

Case (*i*). Put $x = \pi$ in (4), we get

$$\pi^2 = \frac{\pi^2}{3} - 4\left[\frac{-1}{1^2} - \frac{1}{2^2} - \frac{1}{3^2} - \frac{1}{4^2}\ldots\right]$$

$$= \frac{\pi^2}{3} + 4\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\right]$$

$$\Rightarrow \quad \pi^2 - \frac{\pi^2}{3} = 4\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\right]$$

$$\Rightarrow \quad \frac{2\pi^2}{12} = \frac{1}{1^2} + \frac{2}{2^2} + \frac{3}{3^2} + \ldots$$

$$\Rightarrow \qquad \frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots \qquad \ldots(5)$$

Case (*ii*). Put $x = 0$ in (4), we get

$$0 = \frac{\pi^2}{3} - 4\left[\frac{-1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \ldots\right]$$

$$\Rightarrow \qquad \frac{\pi^2}{12} = \frac{1}{1^2} - \frac{2}{2^2} + \frac{3}{3^2} + \ldots. \qquad \ldots(6)$$

Case (*iii*). Adding (5) and (6), we get

$$\frac{\pi^2}{4} = 2\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\right)$$

$$\Rightarrow \qquad \frac{\pi^2}{8} = \frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} + \ldots \qquad \ldots(7)$$

Problem 3. *Expand the function f (x) = x sin x as a Fourier Series in the interval – p £ x £ p and deduce that* $\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \frac{1}{7.9} \ldots = \frac{\pi - 2}{4}$.

Solution. Given $f(x) = x \sin x$

Since $f(-x) = (-x)\sin(-x) = x \sin x = f(x)$, $f(x) = x \sin x$ is an even function and hence $b_n = 0$.

∴ The Fourier Series of $f(x) = x \sin x$ is given by

$$f(x) = x \sin x \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \qquad \ldots(1)$$

Now,

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx$$

$$= \frac{2}{\pi} \int_{0}^{\pi} f(x)\, dx \qquad (\because f(x) \text{ is even function})$$

$$= \frac{2}{\pi} \int_{0}^{\pi} x \sin x\, dx$$

$$= \frac{2}{\pi}\left[x(-\cos x) - (1).(-\sin x)\right]_0^\pi$$

$$\therefore \qquad a_0 = \frac{2}{\pi}\left[-\pi\cos\pi\right] = 2 \qquad \ldots(2)$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x\sin x\cos nx\, dx \qquad (\because \text{ even } \times \text{ even} = \text{even})$$

$$= \frac{1}{\pi}\int_0^{\pi} x(2\sin x\cos nx)\, dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x\left[\sin(n+1)x - \sin(n-1)x)\right]dx$$

$$= \frac{1}{\pi}\left[x\left\{-\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right\} - (1)\left\{-\frac{\sin(n+1)x}{(n+1)^2} + \frac{\sin(n-1)x}{(n-1)^2}\right\}\right]_0^\pi$$

$$= \frac{1}{\pi}\left[\pi\left\{-\frac{\cos(n+1)\pi}{n+1} + \frac{\cos(n-1)\pi}{n-1}\right\}\right], \quad n \neq 1.$$

$$\therefore \qquad a_n = \frac{\cos(n-1)\pi}{n-1} - \frac{\cos(n+1)\pi}{n+1}, \; n \neq 1.$$

When n is odd, $n \neq 1$, $n - 1$ and $n + 1$ are even,

$$\therefore \qquad a_n = \frac{1}{n-1} - \frac{1}{n+1} = \frac{2}{n^2-1} \qquad \ldots(3)$$

When n is even, $n - 1$ and $n + 1$ are odd,

$$\therefore \qquad a_n = \frac{-1}{n-1} + \frac{1}{n+1} = \frac{-2}{n^2-1} \qquad \ldots(4)$$

When $n = 1$, we have

$$a_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos x\,dx = \frac{2}{\pi}\int_0^{\pi} x\sin x\cos x\,dx = \frac{1}{\pi}\int_0^{\pi} x\sin 2x\,dx$$

$$= \frac{1}{\pi}\left[x\left(-\frac{\cos 2x}{2}\right) - (1)\left(-\frac{\sin 2x}{2}\right)\right]_0^{\pi}$$

$$\therefore \qquad a_1 = \frac{1}{\pi}\left[-\frac{\pi\cos 2\pi}{2}\right] = -\frac{1}{2} \qquad \ldots(5)$$

Put the values of (2), (3), (4), (5) in (1), we have

$$x\sin x = 1 - \frac{1}{2}\cos x - 2\left(\frac{\cos 2x}{2^2-1} - \frac{\cos 3x}{3^2-1} + \frac{\cos 4x}{4^2-1} - \ldots\right)$$

Deduction : Put $x = \frac{\pi}{2}$, we get

$$\frac{\pi}{2} = 1 - 2\left(\frac{1}{2^2-1} + \frac{1}{4^2-1} - \frac{1}{6^2-1} - \ldots\right)$$

$$\Rightarrow \qquad \frac{\pi}{2} - 1 = 2\left(\frac{1}{3} - \frac{1}{15} + \frac{1}{35} - \ldots\right)$$

$$\Rightarrow \qquad \frac{\pi-2}{4} = \frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \ldots$$

Problem 4. *Express $f(x) = |x|, -\pi < x < \pi$ as a Fourier Series and hence show that*

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots.$$

Solution. Given $f(x) = |x|$.

Since $f(-x) = |-x| = |x| = f(x)$, $\therefore$ $f(x)$ is an even function and hence $b_n = 0$.

The Fourier Series of $f(x) = |x|$ in $-\pi < x < \pi$ is given by

$$f(x) = |x| = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \qquad \ldots(1)$$

Now, $$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx$$

$$= \frac{2}{\pi}|x|\, dx = \frac{2}{\pi}\int_0^{\pi} x\, dx$$

$$= \frac{2}{\pi}\left(\frac{x^2}{2}\right)_0^{\pi} = \frac{2}{\pi}.\frac{\pi^2}{2} = \pi \qquad \text{...(2)}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx$$

$$= \frac{2}{\pi}\int_0^{\pi}|x|\cos nx\, dx = \frac{2}{\pi}\int_0^{\pi} x\cos nx\, dx$$

$$= \frac{2}{\pi}\left[x\left(\frac{\sin nx}{n}\right) - (1)\left(-\frac{\cos nx}{n^2}\right)\right]_0^{\pi} = \frac{2}{\pi}\left[\frac{\cos n\pi}{n^2} - \frac{1}{n^2}\right]$$

$$= \frac{2}{\pi n^2}\left[(-1)^n - 1\right]$$

$$= \begin{cases} 0, & \text{if } n \text{ is even} \\ -\dfrac{4}{\pi n^2}, & \text{if } n \text{ is odd} \end{cases} \qquad \text{...(3)}$$

Put the values of (2) and (3) in (1), we get

$$|x| = \frac{\pi}{2} - \frac{4}{\pi}\left[\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \dots\right] \qquad \text{...(4)}$$

Deduction : Put $x = 0$ in (4), we get

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots$$

Problem 5. *Show that for* $-\pi < x < \pi$,

$$\sin ax = \frac{2\sin a\pi}{\pi}\left[\frac{\sin x}{1^2-a^2} - \frac{2\sin 2x}{2^2-a^2} + \frac{3\sin ax}{3^2-a^2} - \dots\right].$$

Solution. Let $f(x) = \sin ax, -\pi < x < \pi$.

Since $f(x) = \sin ax$ is an odd function, $a_0 = a_n = 0$.

$\therefore$ The Fourier Series of $f(x) = \sin ax, -\pi < x < \pi$ is given by

$$f(x) = \sin ax = \sum_{n=1}^{\infty} b_n \sin nx \qquad \dots(1)$$

Now
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} \sin ax \sin nx\,dx \qquad (\because \text{odd} \times \text{odd} = \text{even})$$

$$= \frac{1}{\pi}\int_0^{\pi} [\cos(n-a)x - \cos(n+a)x]\,dx$$

$$= \frac{1}{\pi}\left[\frac{\sin(n-a)x}{n-a} - \frac{\sin(n+a)x}{n+a}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{\sin(n-a)\pi}{n-a} - \frac{\sin(n+a)\pi}{n+a}\right]$$

$$= \frac{1}{\pi}\left[\frac{(-1)^n(-\sin a\pi)}{n-a} - \frac{(-1)^n \sin a\pi}{n+a}\right]$$

$$= \frac{(-1)^n \sin a\pi}{\pi}\left[\frac{1}{n-a} + \frac{1}{n+a}\right]$$

$\therefore$
$$b_n = (-1)^{n+1}\frac{2n\sin a\pi}{\pi(\pi^2-a^2)} \qquad \dots(2)$$

Put the value of b_n in (1), we have

$$\therefore \quad \sin ax = \frac{2\sin a\pi}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2-a^2}\sin nx$$

$$= \frac{2\sin a\pi}{\pi}\left[\frac{\sin x}{1^2-a^2} - \frac{2\sin 2x}{2^2-a^2} + \frac{3\sin 3x}{3^2-a^2} - \ldots\right]$$

Slip 3

Problem 6. Express f(x) = | sin x| as a fourier series in the interval $-\pi < x < \pi$

Sol. Given f(x) = | sin x| in $-\pi < x < \pi$

Now f(-x) = |sin(-x)| = |sin x| = |sin x| = f(x)

$\Rightarrow$ f(x) is an even function and here $b_n = 0$

$\therefore$ The fourier series of f(x) = |sin x| is given by

$$f(x) = |\sin x| = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \qquad (1)$$

Now $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}|\sin x|dx$

$$= \frac{2}{\pi}\int_0^{\pi}\sin x\, dx \qquad (\because f(x) \text{ is an even function})$$

$$= \frac{2}{\pi}[-\cos x]_0^{\pi} = \frac{-2}{\pi}[\cos\pi - \cos 0]$$

$$= \frac{-2}{\pi}[-1-1] = \frac{4}{\pi}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cosh xdx$$

$$= \frac{2}{\pi}\int_0^{\pi}\sin x\cosh xdx$$

$$= \frac{1}{\pi}\int_0^{\pi}[\sin(1+n)x + \sin(1-n)x]dx$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} |\sin x| dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} \sin x \, dx \qquad (\because f(x) \text{ is an even function})$$

$$= \frac{2}{\pi}[-\cos x]_0^{\pi} = \frac{-2}{\pi}[\cos\pi - \cos 0]$$

$$= \frac{-2}{\pi}[-1-1] = \frac{4}{\pi}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cosh xdx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} \sin x \cosh xdx$$

$$= \frac{1}{\pi}\int_{0}^{\pi} [\sin(1+n)x + \sin(1-n)x]dx$$

$$[\because 2\sin A \cos B = \sin(A+B) + \sin(A-B)]$$

$$= \frac{1}{\pi}\left[-\frac{\cos(1+n)x}{1+n} - \frac{\cos(1-n)x}{1-n}\right]_0^{\pi} \qquad [1\text{-}n \pm 0]$$

$$= \frac{1}{\pi}\left[\left(\frac{\cos(1+n)x}{1+n} + \frac{\cos(1-n)x}{1-n}\right) - \left(\frac{1}{1+n} - \frac{1}{1-n}\right)\right] \qquad [1\text{-}n \pm 0]$$

$$= \frac{1}{\pi}\left[\frac{(-1)^{1+n}}{1+n} + \frac{(-1)^{1-n}}{1-n} - \frac{1}{1+n} - \frac{1}{1-n}\right]$$

$$\frac{-1}{\pi}\left[\frac{1}{1+n}\left[(-1)^{1+n} - 1\right] + \frac{1}{1+n}\left[(-1)^{1+n} - 1\right]\right]$$

$$= \frac{-1}{\pi}\left[\frac{1}{1+n}\left[(-1)^{1+n} - 1\right] + \frac{1}{1-n}\left[(-1)^{n+1} - 1\right]\right]$$

$$= \frac{-1}{\pi}\left[(-1)^{1+n} - 1\right]\left\{\frac{1}{1+n} + \frac{1}{1-n}\right\} = \frac{-1}{\pi}\left\{\frac{(1-n)+(1+n)}{(1-n^2)}\right\}(-1)^{n+1}$$

$$= \frac{-1}{\pi}\left\{\frac{2}{1+n^2}\right\}\left[(-1)^{1+n} - 1\right]$$

$$= \frac{-2}{\pi(1-n^2)}\left[(-1)^{n+1} - 1\right]$$

Substitute a_0, a_n valesin (1), we get

$$|\sin x| = \frac{2}{\pi} + \sum_{n=1}^{\infty} \frac{-2}{\pi(1-n^2)}\left[(-1)^{1+n} - 1\right]\cos nx$$

Problem 7. If f(x) = |cos x|. Expand f(x) as a fourier series in the interval $(-\pi, \pi)$

Sol. Since f(-x) = |cos(-x)| = |cos x| = f(x)

$\therefore f(x)$ is an even function and hence $b_n = 0$

$$f(x) = |\cos x| = \begin{cases} \cos x & 0 < x < \frac{\pi}{2} \\ -\cos x & \frac{\pi}{2} < x < \pi \end{cases}$$

The fourier series of f(x) is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad .....(1)$$

where

$$a_0 = \frac{2}{\pi}\int_0^{\pi} |\cos x|\, dx$$

$$= \frac{2}{\pi}\left[\int_0^{\frac{\pi}{2}} \cos x dx + \int_{\frac{\pi}{2}}^{\pi} -\cos x dx\right]$$

$$= \frac{2}{\pi}\left[(\sin x)_0^{\frac{\pi}{2}} - (\sin x)_{\frac{\pi}{2}}^{\pi}\right]$$

$$= \frac{2}{\pi}\left[(1-0)-(0-1)\right]$$

$$= \frac{2}{\pi}[1+1] = \frac{4}{\pi}$$

$$a_0 = \frac{4}{\pi}$$

and

$$a_n = \frac{2}{\pi}\int_0^{\pi} |\cos x| \cos nx dx$$

$$= \frac{2}{\pi}\left[\int_0^{\frac{\pi}{2}} \cos x \cos nx dx + \int_{\frac{\pi}{2}}^{\pi} -\cos x \cos nx dx\right]$$

$$= [\because 2\cos A \cos B = \cos(A+B) + \cos(A-B)]$$

$$= \frac{1}{\pi}\left[\int_0^{\frac{\pi}{2}} [\cos(1+n)x + \cos(1-n)x] dx - \int_{\frac{\pi}{2}}^{\pi} \cos(1+n)x + \cos(1-n)x] dx\right]$$

we can re write as

$$=\frac{1}{\pi}\left(\int_0^{\frac{\pi}{2}}[\cos(n+1)x+\cos(n-1)x]dx-\int_{\frac{\pi}{2}}^{\pi}[\cos(n+1)x+\cos(n-1)x]dx\right)\quad(\because \cos(-x)=\cos x)$$

$$\therefore\ |\cos x| = \frac{2}{\pi}+\sum_{n=1}^{\infty}\frac{-4}{\pi(n^2-1)}\cos\left(\frac{n\pi}{2}\right).\cos nx$$

Hence $|\cos x| = \frac{2}{\pi}-\frac{4}{\pi}\left\{\frac{1}{3}\cos 2x+\frac{1}{15}\cos 4x+\frac{1}{35}\cos 6x\ldots.\right\}$

$$=\frac{1}{\pi}\left\{\left(\frac{\sin(n+1)\frac{\pi}{2}}{(n+1)}+\frac{\sin(n-1)\frac{\pi}{2}}{n-1}-(0+0)\right)-\left(\frac{\sin(n+1)\pi}{n+1}+\frac{\sin(n-1)\pi}{n-1}\right)-\frac{\sin(n+1)\frac{\pi}{2}}{(n+1)}+\frac{\sin(n-1)\frac{\pi}{2}}{n-1}\right\}$$

$$=\frac{1}{\pi}\left\{\left(\frac{\cos n\frac{\pi}{2}}{(n+1)}-\frac{\cos n\frac{\pi}{2}}{n-1}\right)-\left(\frac{\cos n\frac{\pi}{2}}{(n+1)}-\frac{\cos n\frac{\pi}{2}}{n-1}\right)\right\}$$

$$=\frac{1}{\pi}\left(\frac{2\cos n\frac{\pi}{2}}{(n+1)}-\frac{2\cos n\frac{\pi}{2}}{n-1}\right)=\frac{2\cos\left(\frac{n\pi}{2}\right)}{\pi}\left[\frac{1}{n+1}-\frac{1}{n-1}\right]$$

$$=\frac{2}{\pi}\cos\left(\frac{n\pi}{2}\right)\left(\frac{(n-1)-(n+1)}{n^2-1}\right)=\frac{-4}{\pi(n^2-1)}\cos\left(\frac{n\pi}{2}\right)$$

$$a_n=\frac{-4}{\pi(n^2-1)}\cos\left(\frac{n\pi}{2}\right)\ \text{for}\ n\neq 1.$$

If $n=1,\ a_1=\frac{2}{\pi}\int_0^{\pi}\cos x\cos x dx$

$$=\frac{2}{\pi}\left(\int_0^{\frac{\pi}{2}}\cos^2 xdx+\int_{\frac{\pi}{2}}^{\pi}\cos^2 xdx\right)$$

$$=\frac{2}{\pi}(0+0)=0$$

$$\therefore |\cos x| = \frac{2}{\pi} + \sum_{n=1}^{\infty} \frac{-4}{\pi(n^2-1)} \cos\left(\frac{n\pi}{2}\right).\cos nx$$

$$\text{Hence } |\cos x| = \frac{2}{\pi} - \frac{4}{\pi}\left\{\frac{1}{3}\cos 2x + \frac{1}{15}\cos 4x + \frac{1}{35}\cos 6x....\right\}$$

EXERCISE

1. Expand in a Fourier Series the function $f(x) = x$ in the interval $0 < x < 2$.

2. If $f(x) = \left(\frac{\pi - x}{2}\right)^2$ in the interval $0 < x < 2\pi$, show that $f(x) = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{\cos nx}{n^2}$ and hence obtain the following relations :

(a) $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + ... = \frac{\pi^2}{6}$

(b) $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + ... = \frac{\pi^2}{12}$

(c) $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + ... = \frac{\pi^2}{8}$

3. Obtain the Fourier Series for the function

$f(x) = \frac{1}{2}(\pi - x)$ in the interval $0 < x < 2$p.

4. Prove that in the interval $-\pi < x < \pi$,

$$x\cos x = -\frac{1}{2}\sin x + 2\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2-1}\sin nx$$

5. Expand $f(x) = |\cos x|$ as a Fourier Series in the interval $-\pi < x < \pi$.

6. Find the Fourier Series for $f(x) = \pi + x$ in $(-\pi, \pi)$

7. If $f(x) = \sqrt{1 - \cos x}$ in $(0, 2\pi)$, then show that $f(x) = \frac{2\sqrt{2}}{\pi} + \sum_{n=1}^{\infty} \frac{-4\sqrt{2}}{(4n^2-1)}\cos nx$ and hence show that

$$\frac{1}{2} = \frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + ...$$

8. Expand $x \cos x$ as a Fourier Series in $(0, 2\pi)$

9. Find the Fourier Series for $f(x)$ in the interval (– p, p) when

$$f(x) = \begin{cases} \pi + x, & -\pi < x < 0 \\ \pi - x, & 0 < x < \pi \end{cases}$$

10. Find the Fouirer Series to represent the function

$$f(x)=\begin{cases}-k, & -\pi<x<0\\ k, & 0<x<\pi\end{cases}$$

and hence deduce that $\dfrac{\pi}{4}=1-\dfrac{1}{3}+\dfrac{1}{5}-\dfrac{1}{7}+\ldots$

11**.** Find the Fourier Series expansion for

$$f(x)=\begin{cases}-\pi, & -\pi<x<0\\ x, & 0<x<\pi\end{cases}$$

and hence deduce $\dfrac{1}{1^2}+\dfrac{1}{3^2}+\dfrac{1}{5^2}+\ldots=\dfrac{\pi^2}{8}$

12. Find the Fourier Series expansion of

$$f(x)=\begin{cases}0, & -\pi<x<0\\ x, & 0<x<\pi\end{cases}$$

13. An alternating current after passing through rectifier has the form $i=\begin{cases}I_0\sin x, & 0\le x<\pi\\ 0, & \pi\le x<2\pi\end{cases}$, where

I_0 is the maximum current and the period is 2p. Express i as a Fourier Series.

ANSWER

1. $f(x)=\pi-2\sum_{n=1}^{\infty}\dfrac{\sin nx}{n}$

3. $f(x)=\sum_{n=1}^{\infty}\dfrac{\sin nx}{n}$

5. $|\cos x|=\dfrac{2}{\pi}+\dfrac{4}{\pi}\left(\dfrac{\cos 2x}{3}-\dfrac{\cos 4x}{15}+\ldots.\right)$

6. $\pi+x=\dfrac{\pi}{2}-2\sum_{n=1}^{\infty}\dfrac{(-1)^n}{n}\sin nx$

8. $x\cos x=\pi\cos x-\dfrac{1}{2}\sin x-2\sum_{n=2}^{\infty}\dfrac{1}{n^2-1}\sin nx$

1. $f(x) = \frac{\pi}{2} + \frac{4}{\pi}\left(\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots\right)$

2. $f(x) = \frac{4k}{\pi}\left(\sin x + \frac{\sin 3x}{3^2} + \frac{\sin 5x}{5^2} + \ldots\right)$

3. $f(x) = \frac{\pi}{2} - \frac{2}{\pi}\left(\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots\right) + \left(3\sin x - \frac{\sin 2x}{2} + \sin 3x - \frac{\sin 4x}{4} + \ldots\right)$

4. $f(x) = \frac{\pi}{4} + \frac{1}{\pi}\sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{n^2}\right]\cos nx - \sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin nx$

5. $i = \frac{I_0}{2} + \frac{I_0}{2}\sin x - \frac{2I_0}{\pi}\left(\frac{\cos 2x}{2^2 - 1} + \frac{\cos 4x}{4^2 - 1} + \frac{\cos 6x}{6^2 - 1} + \ldots\right)$

3.6 HALF-RANGE FOURIER SERIES

In several engineering and physical applications it is required to obtain the Fourier Series expansion of a function $f(x)$ in the range $(0, l)$ in a Fourier Series of period $2l$ or more generally in the range (0, p) in a Fourier Series of period 2p. Where l or p is the half the period of Fourier Series. Such an expression is called **half-range Fourier Series.**

Half-Range Sine Series :

Suppose $f(x)$ is defined in the interval $(0, l)$.

Now we define a new function as follows :

$$F(x) = \begin{cases} f(x); & \text{if } \quad 0 \le x \le 1 \\ -f(-x); & \text{if } \; -1 \le x \le 0 \end{cases}$$

It is clear from the definition that F (x) is an odd function defined in the interval $[-l, l]$. Hence the Fourier Series of F (x) contains only sine terms. Further in the interval $[0, l]$, F $(x) = f(x)$ and hence the sine series of F (x) gives the required sine series of $f(x)$ in $[0, l]$.

Thus, $$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

where $$b_n = \frac{2}{l}\int_0^l f(x)\left(\frac{n\pi x}{l}\right)dx$$

Note. If $f(x)$ is defined in the interval (0, p) then half-range sine series is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

where $$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\, dx.$$

Half-Range Cosine Series :

Suppose $f(x)$ is defined in the interval $(0, l)$. Now we define a new function as follows :

$$F(x) = \begin{cases} f(x); & \text{if} \quad 0 \le x \le 1 \\ f(-x); & \text{if} \quad -1 \le x \le 0 \end{cases}$$

Clearly, F (x) is an even function defined in the interval $[-l, l]$. Hence the Fourier Series of F (x) contains only cosine terms. Further in the interval $[0, l]$, F $(x) = F(x)$ and hence the cosine series of F(x) gives the cosine series of $f(x)$ in $[0, l]$.

Thus, $$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$$

where $$a_0 = \frac{2}{l}\int_0^{l} f(x)\, dx$$

$$a_n = \frac{2}{l}\int_0^{l} f(x)\cos\frac{n\pi x}{l}\, dx$$

Note. If $f(x)$ is defined in $(0, \pi)$ then the half-range cosine series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

where $$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\, dx$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\, dx.$$

SOLVED PROBLEMS

Problem 1. *Prove that the function f (x) = x can be expanded in a series of cosines in 0 < x <* π *as* $x=\frac{\pi}{2}-\frac{4}{\pi}\left[\frac{\cos x}{1^2}+\frac{\cos 3x}{3^2}+\frac{\cos 5x}{5^2}+\ldots\right]$ *and hence show that* $\frac{\pi^2}{8}=\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots$

Solution. Given $f(x) = x,\ 0 < x < \pi$.

The half-range cosine series is given by

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx \quad \ldots(1)$$

Now

$$a_0=\frac{2}{\pi}\int_0^{\pi}f(x)\,dx$$

$$=\frac{2}{\pi}\int_0^{\pi}x\,dx=\frac{2}{\pi}\left(\frac{x^2}{2}\right)_0^{\pi}=\pi \quad \ldots(2)$$

$$a_n=\frac{2}{\pi}\int_0^{\pi}f(x)\cos nx\,dx$$

$$=\frac{2}{\pi}\int_0^{\pi}x.\cos nx\,dx$$

$$=\frac{2}{\pi}\left[x.\left(\frac{\sin nx}{n}\right)-(1)\left(-\frac{\cos nx}{n^2}\right)\right]_0^{\pi}=\frac{2}{\pi}\left[\frac{\cos n\pi}{n^2}-\frac{1}{n^2}\right]$$

$$=\frac{2}{n^2\pi}\left[(-1)^n-1\right]$$

$$\therefore \quad a_n=\begin{cases}0\ ; & \text{if } n \text{ is even}\\ -\frac{4}{\pi n^2}; & \text{if } n \text{ is odd}\end{cases} \quad \ldots(3)$$

Substitute the values of a_0 and a_n in (1), we have

$$f(x) = x = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{-4}{\pi(2n-1)^2} \cos nx$$

i.e., $$f(x) = x = \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots\right] \qquad \ldots(4)$$

Deduction : Put $x = 0$ in (4), we get

$$f(0) = \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{\cos 0}{1^2} + \frac{\cos 0}{3^2} + \frac{\cos 0}{5^2} + \ldots\right]$$

Since $x = 0$ is a point of discontinuity,

$$\therefore \qquad f(0) = \frac{1}{2}\left[\underset{x \to 0}{Lt} f(x) + \underset{x \to 0}{Lt} f(x)\right]$$

$$= \frac{1}{2}(0+0) = 0.$$

$$\therefore \qquad 0 = \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\right]$$

$$\Rightarrow \qquad \frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots$$

Problem 2. *Find the half-range cosine series for the function $f(x) = x^2$ in $0 \le x \le \pi$ and hence find the sum of the series*

$$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots$$

Solution. Given $f(x) = x^2,\ 0 \le x \le \pi$

The half-range cosine series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \qquad ..(1)$$

Now, $$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\ dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x^2\ dx = \frac{2}{\pi}\left(\frac{x^3}{3}\right)_0^{\pi} = \frac{2\pi^2}{3} \qquad \text{...(2)}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\ dx = \frac{2}{\pi}\int_0^{\pi} x^2\cos nx\ dx$$

$$= \frac{2}{\pi}\left[x^2.\frac{\sin nx}{n} - (2x)\left(-\frac{\cos nx}{n^2}\right) + 2\left(-\frac{\sin nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{2}{n^2}(\pi\cos n\pi - 0.\{\cos 0\})\right] = \frac{4}{\pi n^2}.\pi(-1)^n$$

$\therefore$ $$a_n = \frac{(-1)^n.4}{n^2} \qquad \text{...(3)}$$

Substituting the values of a_0, a_n in (1),

$$f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4(-1)^n}{n^2}\cos nx$$

$\therefore$ $$x^2 = \frac{\pi^2}{3} + 4\left[-\frac{\cos x}{1^2} + \frac{\cos 2x}{2^2} - \frac{\cos 3x}{3^2} + ...\right]$$

$\Rightarrow$ $$x^2 = \frac{\pi^2}{3} - 4\left[\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - ...\right] \qquad \text{...(4)}$$

Deduction : Put $x = 0$ in (4), we have

$$0 = \frac{\pi^2}{3} - 4\left[\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - ...\right]$$

$\Rightarrow$ $$-\frac{\pi^2}{3} = -4\left[\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - ...\right]$$

$\Rightarrow$ $$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + ... = \frac{\pi^2}{12}.$$

Problem 3. *Find the half-range sine series for f (x) = x (π – x), in 0 < x < π .*

Deduce $\frac{1}{1^3}-\frac{1}{3^3}+\frac{1}{5^3}-\frac{1}{7^3}+\ldots=\frac{\pi^2}{32}$.

Solution. Given $f(x) = x(\pi - x) = \pi x - x^2,\ 0 < x < \pi$.

The half-range sine series is given by

$$f(x)=\sum_{n=1}^{\infty} b_n \sin nx \qquad \ldots(1)$$

Now, $b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx = \frac{2}{\pi}\int_0^{\pi} x(\pi - x)\sin nx\,dx = \frac{2}{\pi}\int_0^{\pi}(\pi x - x^2)\sin nx\,dx$

$$=\frac{2}{\pi}\left[(\pi x-x^2)\left(-\frac{\cos nx}{n}\right)-(\pi-2x)\left(-\frac{\sin nx}{n^2}\right)+(-2)\left(\frac{\cos n\pi}{n^3}\right)+\frac{2\cos n(0)}{n^3}\right]$$

$$=\frac{2}{\pi}\left[(\pi^2-\pi^2)\left(-\frac{\cos n\pi}{n}\right)-(\pi-2\pi)\left(-\frac{\sin n\pi}{n^2}\right)+(-2)\left(\frac{\cos n\pi}{n^3}\right)+\frac{2\cos n(0)}{n^3}\right]$$

$$=\frac{4}{\pi n^3}\left[1-\cos n\pi\right]=\frac{4}{\pi}\left[\frac{1-(-1)^n}{n^3}\right] \qquad \ldots(2)$$

Substituting the value of (2) in (1), we get

$$f(x)=\sum_{n=1}^{\infty}\frac{4}{\pi}\left[\frac{1-(-1)^n}{n^3}\right]\sin nx$$

$$=\frac{4}{\pi}\left[\frac{2\sin x}{1^3}+\frac{2\sin 3x}{3^3}+\frac{2\sin 5x}{5^3}+\ldots\right]$$

$$\therefore \qquad f(x)=\frac{8}{\pi}\left[\frac{\sin x}{1^3}+\frac{\sin 3x}{3^3}+\frac{\sin 5x}{5^3}+\ldots\right] \qquad \ldots(3)$$

Deduction : Put $x = \frac{\pi}{2}$ in (3),

$$f\left(\frac{\pi}{2}\right)=\frac{8}{\pi}\left[\frac{\sin\frac{\pi}{2}}{1^3}+\frac{\sin\frac{3\pi}{2}}{3^3}+\frac{\sin\frac{5\pi}{2}}{5^3}+\dots\right]$$

$$\frac{\pi}{2}\left(\pi-\frac{\pi}{2}\right)=\frac{8}{\pi}\left[\frac{1}{1^3}-\frac{1}{3^3}+\frac{1}{5^3}-\dots\right]$$

$$\Rightarrow \quad \frac{\pi^2}{4}=\frac{8}{\pi}\left[\frac{1}{1^3}-\frac{1}{3^3}+\frac{1}{5^3}-\dots\right]$$

$$\Rightarrow \quad \frac{\pi^3}{32}=\left[\frac{1}{1^3}-\frac{1}{3^3}+\frac{1}{5^3}-\dots\right].$$

Problem 4. *Obtain half-range cosine series for the function f (x) = x sin x in the interval 0 < x < π and hence deduce,*

$$1+\frac{2}{1.3}-\frac{2}{3.5}+\frac{2}{5.7}-\dots=\frac{\pi}{2}.$$

Solution. Given $f(x) = x \sin x,\ 0 < x < \pi$.

The half-range cosine series of $f(x)$ is given by $f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx$ ...(1)

Now, $a_0=\frac{2}{\pi}\int_0^{\pi}f(x)\,dx$

$$=\frac{2}{\pi}\int_0^{\pi}x\sin x\,dx=\frac{2}{\pi}\left[x(-\cos x)-(1)(-\sin x)\right]_0^{\pi}$$

$$=\frac{2}{\pi}\left[-\pi\cos\pi\right]=2 \qquad \text{...(2)}$$

$$a_n=\frac{2}{\pi}\int_0^{\pi}f(x)\cos nx\,dx=\frac{2}{\pi}\int_0^{\pi}x\sin x\cos nx\,dx=\frac{1}{\pi}\int_0^{\pi}x(2\cos nx\sin nx)\,dx$$

$$=\frac{1}{\pi}\int_0^{\pi}x\left[\sin(n+1)x-\sin(n-1)x\right]dx$$

$$= \frac{1}{\pi}\left[x\left\{ -\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1} \right\} - (1)\left\{ -\frac{\sin(n+1)x}{(n+1)^2} - \frac{\sin(n-1)x}{(n-1)^2} \right\} \right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[-\frac{\pi\cos(n+1)\pi}{(n+1)} + \frac{\pi\cos(n-1)\pi}{(n-1)} \right], n \neq 1.$$

$$= -\frac{(-1)^{n+1}}{n+1} + \frac{(-1)^{n-1}}{n-1} = (-1)^{n-1}\left[\frac{1}{n-1} - \frac{1}{n+1} \right]$$

$$= \frac{2(-1)^{n-1}}{(n-1)(n+1)};\ n \neq 1 \qquad \text{...(3)}$$

When $n = 1$, we have

$$a_1 = \frac{2}{\pi}\int_0^{\pi} x \sin x \cos x \, dx = \frac{1}{\pi}\int_0^{\pi} x \sin 2x \, dx$$

$$= \frac{1}{\pi}\left[x.\left(-\frac{\cos 2x}{2} \right) - (1)\left(-\frac{\sin 2x}{2^2} \right) \right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[-\frac{\pi\cos 2\pi}{2} \right] = -\frac{1}{2} \qquad \text{...(4)}$$

Substituting the values of (2), (3) and (4) in (1), we have,

$$f(x) = x \sin x = 1 - \frac{1}{2}\cos x - 2\left(\frac{\cos 2x}{1.3} - \frac{\cos 3x}{2.4} + \frac{\cos 4x}{3.5} - \dots \right) \qquad \text{...(5)}$$

Deduction : Put $x = \frac{\pi}{2}$ in (5),

$$\frac{\pi}{2}.\sin\left(\frac{\pi}{2}\right) = 1 - 2\left(\frac{-1}{1.3} + \frac{1}{3.5} - \frac{1}{5.7} + \dots \right)$$

Þ $\quad 1 + \frac{2}{1.3} - \frac{2}{3.5} + \frac{2}{5.7} + \dots = \frac{\pi}{2}.$

Example 1: Find the half range fourier cosine series of the function

$$f(x) = \begin{cases} x/2 & \text{if} \quad 0 < x < \pi/2 \\ \text{x - x/2} & \text{if} \quad \pi/2 < x < \pi \end{cases} \qquad \text{(OU Dec 2017)}$$

Sol. Given $f(x) = \begin{cases} x/2 & if \quad 0 < x < \pi/2 \\ \text{x - x/2} & \text{if} \quad \pi/2 < x < \pi \end{cases}$

The half range cosine series is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx ....(1)$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx$$

$$= \frac{2}{\pi}\left\{\int_0^{\pi/2} x/2 \ dx + \int_{\pi/2}^{\pi} \pi - x/2 \, dx\right\}$$

$$= \frac{2}{\pi}\left\{\frac{x^2}{4}\int_0^{\pi/2} + \pi \ x - \frac{x^2}{4}\int_{\pi/2}^{\pi}\right]$$

$$= \frac{2}{\pi}\left[\frac{\pi^2}{16} + \pi^2 - \frac{\pi^2}{4} - \left(\frac{\pi^2}{2} - \frac{\pi^2}{16}\right)\right]$$

$$= \frac{2}{\pi}\left[\frac{2\pi^2}{16} + \frac{\pi^2}{4}\right]$$

$$= \frac{2}{\pi}\left[\frac{6\pi^2}{16}\right]$$

$$= \frac{3\pi}{4}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} \cos nx\, dx$$

$$= \frac{2}{\pi}\left\{\int_0^{\pi/2} f(x)\cos nx\, dx + \int_{\pi/2}^{\pi} f(x)\cos nx\, dx\right\}$$

$$= \frac{2}{\pi}\left\{\int_0^{\pi/2} \frac{x}{2}\cos nx\, dx + \int_{\pi/2}^{\pi} (\pi - x/2)\cos nx\, dx\right\}$$

$$= \frac{2}{\pi}\left\{\frac{1}{2}\left[x\left(\frac{\sin nx}{n}\right) + \frac{\cos nx}{n^2}\right]_0^{\pi/2} + \left[\left(\pi - \pi/2\right)\left(\frac{\sin nx}{n}\right) - \left(\frac{-1}{2}\right)\left(\frac{\cos nx}{n^2}\right)\right]_{\pi/2}^{\pi}\right\}$$

$$\because \int \text{U V dx} = \text{uv}_1 - u'v_2 + u''v_3$$

$$= \frac{2}{\pi}\left\{\frac{1}{2}\left[\frac{\pi}{2n}\sin\frac{n\pi}{2} + \frac{1}{n^2}\cos\frac{n\pi}{2} - \frac{1}{n^2}\right] + \left[0 - \frac{1}{2n^2}\cos n\pi - \left(\frac{3\pi}{4n}\right)\sin\frac{n\pi}{2} + \frac{1}{2n^2}\cos\frac{n\pi}{2}\right]\right\}$$

$$= \frac{2}{\pi}\left\{\frac{\pi}{4n}\sin\frac{n\pi}{2} + \frac{1}{2n^2}\cos\frac{n\pi}{2} - \frac{1}{2n^2} - \frac{1}{2n^2}\cos n\pi - \frac{3\pi}{4n}\sin\frac{n\pi}{2} + \frac{1}{2n^2}\cos\frac{n\pi}{2}\right\}$$

$$= \frac{2}{\pi}\left\{\frac{-\pi}{2n}\sin\frac{n\pi}{2} + \frac{1}{n^2}\cos\frac{n\pi}{2} - \frac{1}{2n^2}\cos n\pi - \frac{1}{2n^2}\right\}$$

sub a_0, a_n *in* (1)

$$f(x) = \frac{3\pi}{8} + \sum_{n=1}^{\infty}\frac{2}{\pi}\left\{\frac{-\pi}{2n}\sin\frac{n\pi}{2} + \frac{1}{n^2}\cos\frac{n\pi}{2} - \frac{1}{2n^2}\cos n\pi - \frac{1}{2n^2}\right\}\cos nx.$$

Find the half range fourier sine series for

$$f(x) = \begin{cases} x & 0 < x < \frac{\pi}{2} \\ 0 & \frac{\pi}{2} < x < \pi \end{cases}$$

Sol. Given $f(x) = \begin{cases} x \;;& 0 < x < \frac{\pi}{2} \\ 0 \;;& \frac{\pi}{2} < x < \pi \end{cases}$

The half range sine series of f(x) is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx \quad ......(1)$$

Now, $b_n = \dfrac{2}{\pi}\displaystyle\int_0^{\pi} f(x)\sin nx\, dx$

$$= \frac{2}{\pi}\left[\int_0^{\frac{\pi}{2}} f(x)\sin nx\, dx + \int_{\frac{\pi}{2}}^{0} f(x)\sin nx\, dx\right]$$

$$= \frac{2}{\pi}\left[\int_0^{\frac{\pi}{2}} x\sin nx\, dx + \int_{\frac{\pi}{2}}^{0} 0.\sin nx\, dx\right]$$

$$= \frac{2}{\pi}\int_0^{\frac{\pi}{2}} x\sin nx\, dx$$

$$\int u\, v\, dx = uv_1 - U'V_2 + U''V_3$$

$$= \frac{2}{\pi}\left[x\left(\frac{-\cos nx}{n}\right)...(1)\left(\frac{-\sin x}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\left(\frac{-\pi}{n}\cos n\pi + 0\right) - (0+0)\right] = \frac{-2}{n}\cos n\pi = \frac{-2}{n}(-1)^n .....(2)$$

Substitute in (1), *we get*

$$f(x) = \sum_{n=1}^{\infty} \frac{2}{n}(-1)^{n+1}\sin nx$$

EXERCISE

2. Expand $f(x) = \cos x$, $0 < x <$ p in half-range sine series.

3. Obtain a cosine series for $f(x) = e^x$ in $0 < x <$ px.

4. Show that when $0 < x <$ p,

$$\pi - x = \frac{\pi}{2} + \frac{\sin 2x}{1} + \frac{\sin 4x}{2} + \frac{\sin 6x}{3} + \ldots$$

6. Prove that the function $f(x) = x$ can be expanded as a series of sines in 0 £ x £ p as $x = 2\left[\frac{\sin x}{1} + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \ldots\right]$ and hence show that $1 - \frac{1}{3} + \frac{1}{5} - \ldots = \frac{\pi}{4}$.

7. Find a Fourier sine series of $f(x) = k$ in $0 < x <$ p.

12. Expand $f(x) = \begin{cases} x; & 0 < x < \frac{\pi}{2} \\ \pi - x; & \frac{\pi}{2} < x < \pi \end{cases}$ as a Fourier sine and cosine series.

ANSWER

2. $f(x) = \sum_{n=1}^{\infty} \frac{4(2n)\sin(2nx)}{\pi\left[(2n)^2 - 1\right]}$

3. $e^x = \frac{e^{\pi - 1}}{\pi} + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{\left[e^{\pi}(-1)^n - 1\right]\cos nx}{n^2 + 1}$

7. $f(x) = \frac{4k}{\pi}\left[\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \ldots\right]$.

12. (*a*) $f(x) = \frac{4}{\pi}\left[\sin x - \frac{\sin 3x}{3^2} + \frac{\sin 5x}{5^2} - \ldots\right]$

(*b*) $f(x) = \frac{\pi}{4} - \frac{2}{\pi}\left[\frac{\cos 2x}{1^2} + \frac{\cos 6x}{3^2} + \frac{\cos 10x}{5^2} + \ldots\right]$.

3.7 CHANGE OF INTERVAL

In the previous article, we have expanded a function $f(x)$ defined in an interval of length 2p in a Fourier Series of period 2p.

In many engineering problems, it is desired to expand a function in a Fourier Series over an interval of length $2l$ and not 2p. This can be achieved by a transformation of the variable.

Consider a periodic function $f(x)$ defined in the interval $c < x < c + 2l$.

To change the interval into one of length 2p,

we put, $$\frac{x}{l} = \frac{z}{\pi} \Rightarrow z = \frac{\pi x}{l}$$

so that when $x = c$, $$z = \frac{\pi c}{l} = d \text{ (say)}$$

and when $x = c + 2l$, $$z = \frac{\pi(c+2l)}{l}$$

$$= \frac{\pi c}{l} + 2\pi = d + 2\pi\,.$$

Thus the function $f(x)$ of period $2l$ in $(c, c + 2l)$ is transformed to the function $f\left(\frac{lz}{\pi}\right) = F(z)$, say, of period 2p in $(d, d + 2p)$ and the later function can be expanded as the Fourier Series of the form,

$$f(z) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nz + \sum_{n=1}^{\infty} b_n \sin nz \qquad ...(1)$$

where $$a_0 = \frac{1}{\pi}\int_d^{d+2\pi} F(z)\,dz$$

$$\left.\begin{aligned} a_n &= \frac{1}{\pi}\int_d^{d+2\pi} F(z)\cos nz\,dz \\ b_n &= \int_d^{d+2\pi} F(z)\sin nz\,dz \end{aligned}\right\} \qquad ...(2)$$

Now making the inverse substitution

$$z = \frac{\pi\,x}{l},\; dz = \frac{\pi}{l}\,dx$$

When $z = d$, $x = c$ and

When $z = d$ + 2p, $x = c + 2l$

The expression (1) becomes

$$F(z) = F\left(\frac{\pi\,x}{l}\right) = f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi\,x}{l} + \sum_{n=1}^{\infty} b_n \cos\frac{n\pi\,x}{l}$$

and the coefficients a_0, a_n and b_n from (2) reduces to

$$a_0 = \frac{1}{l}\int_{d}^{c+2\pi} f(x)\,dx$$

$$a_n = \frac{1}{l}\int_{d}^{c+2\pi} f(x)\cos\frac{n\pi\,x}{l}\,dx$$

$$b_n = \frac{1}{l}\int_{c}^{c+2\pi} f(x)\sin\frac{n\pi\,x}{l}\,dx$$

Hence the Fourier Series of a function $f(x)$ in the interval $c < x < c + 2l$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi\,x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi\,x}{l}$$

where

$$a_0 = \frac{1}{l}\int_{c}^{c+2\pi} f(x)\,dx$$

$$a_n = \frac{1}{l}\int_{c}^{c+2\pi} f(x)\cos\frac{n\pi\,x}{l}\,dx$$

$$b_n = \frac{1}{l}\int_{c}^{c+2\pi} f(x)\sin\frac{n\pi\,x}{l}\,dx$$

Corollary 1 : Put $c = 0$, the interval becomes $0 < x < 2l$ and the above results reduces to

$$a_0 = \frac{1}{l}\int_0^{2l} f(x)\,dx$$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}\,dx$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}\,dx$$

Corollary 2 : Put $c = -l$, the interval becomes $-l < x < l$ and the above results reduces to

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx$$

Corollary 3 : (*a*) If *f* (*x*) is an even function, we have

$$a_0 = \frac{2}{l}\int_0^{l} f(x)\,dx$$

$$a_n = \frac{2}{l}\int_0^{l} f(x)\cos\frac{n\pi x}{l}\,dx \quad \text{and}$$

$$b_n = 0$$

(*b*) **If *f* (*x*) is an odd function,** we have

$$a_0 = 0,\ a_n = 0 \text{ and}$$

$$b_n = \frac{2}{l}\int_0^{l} f(x)\sin\frac{n\pi x}{l}\,dx.$$

SOLVED PROBLEM

PROBLEM 1. IF $f(x) = x^2, -1 \le x \le 1$, obtain the Fourier Series and deduce that

$$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \ldots$$

Solution. Given $f(x) = x^2, -1 \le x \le 1$

TH

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \qquad \ldots(1)$$

NOW, $a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx\, b_n = \frac{1}{l}\int_{-l}^{l} x^2 dx = \frac{2}{l}\int_{-l}^{l} x^2 dx$

$$= \frac{2}{l}\left[\frac{x^3}{3}\right]_0^l = \frac{2l^2}{3} \qquad \ldots(2)$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l} = \int_{-l}^{l} x^2 \cos\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\int_{0}^{l} x^2 \cos\frac{n\pi x}{l}\,dx \qquad \left(\because x^2.\cos\frac{n\pi x}{l} \text{ IS EVEN FUNCTION}\right)$$

$$= \frac{2}{l}\left[x^2.\left(\sin\frac{n\pi x}{l}\right)\left(\frac{l}{n\pi}\right) - (2x)\left(-\cos\frac{n\pi x}{l}\right)\left(\frac{l^2}{n^2\pi^2}\right) + (2)\left(\frac{-\sin n\pi x}{l}\right)\left(\frac{l^3}{n^3\pi^3}\right)\right]_0^l$$

$$= \frac{2}{l}.\frac{2l^2}{n^2\pi^2}\left[x\cos\frac{n\pi x}{l}\right]_0^l = \frac{4l}{n^2\pi^2}\left[l\cos n\pi - 0.\cos 0\right]$$

$$\therefore \quad a_n = \frac{4l^2}{n^2\pi^2}(-1)^n \qquad \ldots(3)$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx = \frac{1}{l}\int_{-l}^{l} x^2 \sin\frac{n\pi x}{l}\,dx$$

$\therefore \qquad b_n = 0 \quad \text{(IS ODD FUNCTION)}$ ...(4)

Substituting the values of a_0, a_n and b_n in (1), we have

$$f(x) = x^2 = \frac{l^2}{3} + \sum_{n=1}^{\infty} \frac{(-1)^n 4l^2}{n^2\pi^2} \cos\frac{n\pi x}{l}$$

$$= \frac{l^2}{3} - \frac{4l^2}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} \cos\frac{n\pi x}{l} \qquad \text{...(5)}$$

Deduction : Put $x = 0$ in (5), we have

$$f(0) = \frac{l^2}{3} - \frac{4l^2}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$$

Þ $$0 = \frac{l^2}{3} - \frac{4l^2}{\pi^2}\left[\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots\right]$$

Þ $$\frac{1}{3} = \frac{4}{\pi^2}\left[1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots\right]$$

Þ $$\frac{\pi^2}{12} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots$$

Problem 3. *Find the Fourier expansion of the function* $f(x) = x - x^2$ *in the interval* $-1 < x < 1$.

Solution. Given $\quad f(x) = x - x^2, \ -1 < x < 1$

Here $2l = 1 - (-1) = 2$ Þ $l = 1$

The Fourier Series of $f(x)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x + \sum_{n=1}^{\infty} b_n \sin n\pi x \qquad \text{...(1)}$$

Now, $\qquad a_n = \frac{1}{l}\int_{-l}^{l} f(x)\,dx$

$$a_n = \frac{1}{l}\int_{-l}^{l}(x - x^2)\,dx = \left(\frac{x^2}{2} - \frac{x^3}{2}\right)_{-1}^{1} = -\frac{2}{3} \qquad \ldots(2)$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi\, x}{l}\,dx = \frac{1}{l}\int_{-l}^{l}(x - x^2)\cos n\pi\, dx$$

$$= \int_{-l}^{l} x\cos n\pi\, x - \int_{-l}^{l} x^2 \cos n\pi\, x\,dx$$

$$= 0 - 2\int_{0}^{l} x^2 \cos n\pi\, x\,dx$$

$$= -2\left[x^2.\left(\sin\frac{n\pi\, x}{n\pi}\right) - (2x)\left(\frac{-\cos n\pi\, x}{n^2\pi^2}\right) + (2)\left(-\frac{\sin n\pi\, x}{n^3\pi^3}\right)\right]_0^l$$

$$= -2\left[\frac{2\cos n\pi}{n^2\pi^2}\right] = -\frac{4(-1)^n}{n^2\pi^2} = \frac{4(-1)^{n+1}}{n^2\pi^2} \qquad \ldots(3)$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi\, x}{l}\,dx$$

$$= \frac{1}{l}\int_{-l}^{l}(x - x^2)\sin n\pi\, dx$$

$$= \int_{-1}^{1} x\sin n\pi\, x\,dx - \int_{-1}^{1} x^2 \sin n\pi\, x\,dx$$

$$= 2\int_{-1}^{1} x\sin n\pi\, x\,dx - 0$$

$$= 2\int_{-1}^{1} x\sin n\pi\, x\,dx - 0 = 2\left[x.\left(\frac{\cos n\pi\, x}{n\pi}\right) - (1)\left(-\frac{\sin n\pi\, x}{n^3\pi^3}\right)\right]_0^1$$

$$= 2\left[-\frac{\cos n\pi\, x}{n\pi}\right] = \frac{-2(-1)^n}{n\pi} = \frac{2(-1)^{n+1}}{n\pi}$$

Substituting the values of (2), (3) and (4) in (1), we have

$f(x) = x - x^2$

$$= -\frac{1}{3} + \frac{4}{\pi^2}\left(\frac{\cos \pi x}{1^2} - \frac{\cos 2\pi x}{2^2} + \frac{\cos 3\pi x}{3^2} - \dots\right) + \frac{2}{\pi}\left(\frac{\sin \pi x}{1} - \frac{\sin 2\pi x}{2} + \frac{\sin 3\pi x}{3} - \dots\right)$$

Slip 7a

Ex 3. Find the fourier series of the function f(x) = 1 + x on $-1 \le x \le 1$ and hence deduce that

$1 - \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + ... = \frac{\pi}{4}$ (OU Dec 2017)

Sol. Given f(x) = 1 + x in $-1 \le x \le 1$

here l = 1 and f(x) is neither even and nor odd .

The fourier series of f(x) is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right)\right) \text{......(1)}$$

where $a_0 = \frac{1}{l}\int_{-l}^{l} f(x)dx$

$$= \frac{1}{1}\int_{-1}^{1}(1+x)dx$$

$$= \left(x + \frac{x^2}{2}\right)_{-1}^{1}$$

$$= \left[\left(1 + \frac{1}{2}\right) - \left(-1 + \frac{1}{2}\right)\right]$$

$$= \frac{3}{2} + \frac{1}{2}$$

$a_0 = 2$

Now

$$1+x=\frac{2}{2}+\sum_{n=1}^{\infty}0.\cos n\pi x+\frac{2(-1)^{n+1}}{n\pi}\sin(n\pi x)$$

$$1+x=1+\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin n\pi x$$

$$Put\ x=\frac{1}{2}$$

$$1+\frac{1}{2}=1+\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\left(\frac{n\pi}{2}\right)$$

$$\frac{\pi}{4}=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\left(\frac{n\pi}{2}\right)$$

$$\frac{\pi}{4}=\frac{1}{1}\sin\left(\frac{\pi}{2}\right)-\frac{1}{2}\sin\pi+\frac{1}{3}\sin\left(\frac{3\pi}{2}\right)-\frac{1}{4}\sin(2\pi)+\frac{1}{5}\sin\left(\frac{5\pi}{2}\right)-\frac{1}{6}\sin(3\pi)+\frac{1}{7}\sin\left(\frac{7\pi}{2}\right)+\ldots$$

$$\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots..$$

$$b_n=\frac{1}{l}\int_{-l}^{l}f(x)\sin\left(\frac{n\pi l}{l}\right)dx$$

$$=\frac{1}{1}\int_{-1}^{1}(1+x)\sin\left(\frac{n\pi l}{1}\right)dx$$

$$=\frac{1}{1}\int_{-1}^{1}(1+x)\sin n\pi x\,dx$$

$$\left[\int u\,v\,dx=uv_1-U'v_2+U''v_3-U'''v_4\right]$$

$$=\left[(1+x)\left(\frac{-\cos n\pi x}{n\pi}\right)\ldots(1)\left(\frac{-\sin n\pi x}{n^2\pi^2}\right)\right]_{-1}^{1}$$

$$=\left[(1+1)\left(\frac{\cos n\pi}{n\pi}\right)+\left(\frac{\sin n\pi}{n^2\pi^2}\right)\right]-\left[-(1-1)\left(\frac{\cos n\pi)}{n\pi}\right)+\left(\frac{\sin n\pi)}{n^2\pi^2}\right)\right]$$

we know that $\cos n\pi=(-1)^n,\ \ \sin n\pi=0$ $\qquad\because\sin(-\theta)=-\sin\theta$

$$=\left[\left(\frac{-2(-1)^n}{n\pi}+0\right)-(0-0)\right]$$

$$b_n=\frac{2(-1)^{n+1}}{n\pi}$$

Substitute a0,an,bn values in (1)

Here equation (1) will become

$$1+x=\frac{2}{2}+\sum_{n=1}^{\infty}0.\cos n\pi x+\frac{2(-1)^{n+1}}{n\pi}\sin(n\pi x)$$

$$1+x=1+\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin n\pi x$$

$$Put\ \ x=\frac{1}{2}$$

$$1+\frac{1}{2}=1+\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\left(\frac{n\pi}{2}\right)$$

$$\frac{\pi}{4}=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\left(\frac{n\pi}{2}\right)$$

$$\frac{\pi}{4}=\frac{1}{1}\sin\left(\frac{\pi}{2}\right)-\frac{1}{2}\sin\pi+\frac{1}{3}\sin\left(\frac{3\pi}{2}\right)-\frac{1}{4}\sin(2\pi)+\frac{1}{5}\sin\left(\frac{5\pi}{2}\right)-\frac{1}{6}\sin(3\pi)+\frac{1}{7}\sin\left(\frac{7\pi}{2}\right)+\ldots$$

$$\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots..$$

Problem: Find the fourier expansion of the function f(x) = x^2 in the interval (-2,2)

Sol. Given f(x) = x^2, -2 < x < 2

Hence l = 2

The fourier series of f(x) is given by

$$f(x)=\sum_{n=1}^{\infty}b_n\sin n\pi x\ \ldots..(1)$$

$$b_n=\frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$=\frac{2}{1}\int_0^1(x+x^2)\sin(n\pi x)dx.\ \ (\because l=1)$$

$$\left[\int u\, v\, dx = uv_1 - U'v_2 + U''v_3 \ldots\right]$$

$$= 2\left\{(x+x^2)\left[\frac{-\cos n\pi x}{n\pi}\right] - (1+2x)\left[\frac{-\sin n\pi x}{(n\pi)^2}\right] + (2)\left[\frac{\cos n\pi x}{(n\pi)^3}\right]\right\}_0^1$$

$$= 2\left\{\left[-2\frac{\cos n\pi x}{n\pi}\right] - 0 + \frac{2}{(n\pi)^3}\cos n\pi\right\} - \left\{0 - 0 + \frac{2}{(n\pi)^3}\right\}$$

$$= 2\left[\frac{-2}{n\pi}(-1)^n + \frac{2}{(n\pi)^3}(-1)^n - \frac{2}{(n\pi)^3}\right]$$

$$= 2\left[\frac{-2}{n\pi}(-1)^n\right] + \frac{4}{(n\pi)^3}\left[(-1)^n - 1\right]$$

$$b_n = \frac{-4}{n\pi}(-1)^n + \frac{4}{(n\pi)^3}\left\{(-1)^n - 1\right\}$$

Substitute b_n value in (1)

$$x + x^2 = \sum_{n=1}^{\infty}\left\{\frac{-4}{n\pi}(-1)^n + \frac{4}{(n\pi)^3}(-1)^n - 1\right\}\sin n\pi x.$$

Problem 2. *Obtain the Fourier Series expansion of* $f(x) = \left(\frac{\pi - x}{2}\right)$ *in 0 < x < 2.*

Solution. Given $f(x) = \left(\frac{\pi - x}{2}\right)$, $0 < x < 2$

Here $2l = 2 - 0 = 2$ Þ $l = 1$.

The Fourier Series of $f(x)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x + \sum_{n=1}^{\infty} b_n \sin n\pi x \qquad \ldots(1)$$

Now,
$$a_0 = \frac{1}{l}\int_0^{2l} f(x)dx = \frac{1}{l}\int_0^2 \left(\frac{\pi - x}{2}\right)\cos\frac{n\pi x}{l}dx$$

$$= \frac{1}{2}\left[\pi x - \frac{x^2}{2}\right]_0^2 = (\pi - 1) \qquad \ldots(2)$$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}dx = \frac{1}{l}\int_0^{2}\left(\frac{\pi - x}{2}\right)\cos\frac{n\pi x}{l}dx$$

$$= \frac{1}{2}\left[(\pi - x)\left(\frac{\sin n\pi x}{n\pi}\right) - (-1)\left(-\frac{\cos n\pi x}{n^2\pi^2}\right)\right]_0^2$$

$$= \frac{1}{2n^2\pi^2}[\cos 2n\pi - \cos 0]$$

$$= \frac{1}{2n^2\pi^2}[1-1] = 0 \qquad (\text{Q} \cos 2np = 0) \quad ...(3)$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}dx = \frac{1}{l}\int_0^{2}\left(\frac{\pi - x}{2}\right)\sin\frac{n\pi x}{l}dx$$

$$= \frac{1}{2}\left[(\pi - x)\left(-\frac{\cos n\pi x}{n\pi}\right) - (-1).\left(-\frac{\sin n\pi x}{n^2\pi^2}\right)\right]_0^2$$

$$= -\frac{1}{2n\pi}\left[(\pi - 2)\cos 2n\pi - \pi\cos 0\right] = \frac{1}{n\pi} \qquad ...(4)$$

\ Substitute these values in (1), we get

$$f(x) = \frac{\pi - x}{2} = \frac{\pi - 1}{2} + \sum_{n=1}^{\infty}\frac{1}{n\pi}\sin n\pi x.$$

Slip 10

Prob. Find the half-range sine series and cosine series for the function f(x) = 1 in 0 < x < 2.

Sol. The half-range fourier cosine series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) .....(1)$$

here l = 2

$$a_0 = \frac{2}{l}\int_0^l f(x)dx = \frac{2}{2}\int_0^2 1dx = \int_0^2 1dx = (x)_0^2 = 2$$

nd $a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx$

$$= \frac{2}{2}\int_0^2 1\cos\left(\frac{n\pi x}{2}\right)dx$$

$$= \int_0^2 1\cos\left(\frac{n\pi x}{2}\right)dx$$

$$= \left[\frac{\sin\left(\frac{n\pi x}{2}\right)}{\left(\frac{n\pi}{2}\right)}\right]_0^2$$

$$= \frac{2}{n\pi}\left[\sin\left(\frac{n\pi x}{2}\right)\right]_0^2$$

$$= \frac{2}{n\pi}\left[\sin(n\pi) - \sin 0\right]$$

$$= 0 \qquad\qquad a_n = 0$$

$$\Rightarrow f(x) = \frac{2}{2} + 0$$

$$f(x) = 1$$

II The half-range Fourier series is given by

$$f(x) = 1 = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) \ldots\ldots(1)$$

where $b_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx$

Here l = 2

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{2}{4}\int_0^4 f(x)\sin\frac{n\pi x}{4}dx \quad (\because l = 4)$$

$$= \frac{1}{2}\left[\int_0^2 x\sin\left(\frac{n\pi x}{4}\right)dx + \int_2^4 2\sin\left(\frac{n\pi x}{4}\right)dx\right]$$

$$\left[\int u\, v\, dx = uv_1 - U'v_2 + U''v_3....\right]$$

$$= \frac{1}{2}\left\{\left[x\left(-\cos\frac{n\pi x}{4}\right)\left(\frac{4}{n\pi}\right)....(1)\left(-\sin\frac{n\pi x}{4}\right)\left(\frac{16}{n^2\pi^2}\right)\right]_0^2 + 2\left(-\cos\left(\frac{n\pi x}{4}\right)\left(\frac{4}{n\pi}\right)\right)_2^4\right\}$$

$$= \frac{1}{2}\left\{\left[\frac{-8}{n\pi}\cos\left(\frac{n\pi}{2}\right)\frac{16}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right) - (0-0)\right]\frac{-8}{n\pi}[\cos n\pi - \cos n\pi]\right\}$$

$$= \frac{1}{2}\left[\frac{16}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right) - \frac{8}{n\pi}\cos n\pi\right] = \frac{4}{n\pi}\left[\frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right) - \cos n\pi\right]$$

$$b_n = \frac{-2}{n\pi}[\cos(n\pi) - \cos 0]$$

$$b_n = \frac{-2}{n\pi}[(-1)^n - 1]$$

Substitute b_n in (2)

$$f(x) = \sum_{n=1}^{\infty}\left(\frac{-2}{n\pi}\right)\left((-1)^n - 1\right)\sin\left(\frac{n\pi}{2}\right)$$

Problem 5. *Obtain half-range sine series for $f(x) = e^x$ in $0 < x < 1$.*

Solution. Given $f(x) = e^x$, $0 < x < 1$.

The half-range sine series of $f(x)$ is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin n\pi x \qquad \text{(since } l = 1\text{)}$$

Now,
$$b_n = \frac{2}{\pi}\int_0^l f(x)\sin\frac{n\pi x}{l}dx \qquad ...(1)$$

$$= 2\int_0^l e^x \sin n\pi\, x\, dx \qquad (\because l = 1)$$

$$= 2\left[\frac{e^x}{1+(n\pi)^2}(\sin n\pi\, x - n\pi \cos n\pi\, x)\right]_0^l$$

$$\left[\because \int e^{ax} \sin bx\, dx = \frac{e^x}{a^2+b^2}(a \sin bx - b \cos bx)\right]$$

$$= 2\left[\frac{e^x}{1+(n\pi)^2}(-n\pi \cos n\pi) - \frac{1}{\left[1+(n\pi)^2\right]}(-n\pi)\right]$$

$$= \frac{2}{1+n^2\pi^2}\left[-en\pi(-1)^n + n\pi\right] = \frac{2}{1+n^2\pi^2}\left[1-e(-1)^n\right]$$

Substituting this value in (1), we get

$$e^x = 2\pi\sum_{n=1}^{\infty}\frac{n\left[1-e(-1)^n\right]}{1+n^2\pi^2}$$

$$= 2\pi\left[\frac{1+e}{1+\pi^2}\sin \pi\, x + \frac{2(1-e)}{1+4\pi^2}\sin 2\pi\, x + \frac{3(1+e)}{1+9\pi^2}\sin 3\pi\, x + \ldots\right]$$

Problem 6. *Expand f (x) = x as a half-range cosine series in 0 < x < 2.*

Solution. Given $f(x) = x$, $0 < x < 2$.

The half-range cosine series of $f(x)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \frac{\cos n\pi\, x}{2} \qquad \ldots(1)$$

Now, $$a_0 = \frac{2}{l}\int_0^l f(x)dx = \frac{2}{2}\int_0^2 x\, dx \qquad (\because l = 2)$$

$$= \left(\frac{x^2}{2}\right)_0^2 = 2 \qquad \ldots(2)$$

$$a_n = \frac{2}{l}\int_0^l f(x)\frac{\cos n\pi x}{2}dx = \frac{2}{2}\int_0^2 x\frac{\cos n\pi x}{2}dx$$

$$= \left[x.\left(\sin\frac{n\pi x}{2}\right).\left(\frac{2}{n\pi}\right) - (1)\left(-\cos\frac{n\pi x}{2}\right)\left(\frac{4}{n^2\pi^2}\right)\right]_0^2$$

$$= \left[\left(\frac{\cos n\pi x}{2}\right).\left(\frac{4}{n^2\pi^2}\right)\right]_0^2 = \frac{4}{n^2\pi^2}[\cos n\pi - 1]$$

$$= \frac{4}{n^2\pi^2}[(-1)^n - 1]$$

$$= \begin{cases} 0 \ ; \text{ if } n \text{ is even} \\ -\dfrac{8}{n^2\pi^2}; \text{ if } n \text{ is odd} \end{cases} \qquad \ldots(3)$$

Substituting the values of a_0 and a_n in (1), we get

$$f(x) = 1 + \sum_{n=1}^{\infty}\frac{4}{n^2\pi^2}[(-1)^n - 1]\cos\frac{n\pi x}{2}$$

$$= 1 - \frac{8}{\pi^2}\left(\frac{\cos\frac{\pi x}{2}}{1^2} + \frac{\cos\frac{3\pi x}{2}}{3^2} + \frac{\cos\frac{5\pi x}{2}}{5^2} + \ldots\right)$$

EXERCISE

1. Find the half-range cosine series for the function $f(x) = (x - 1)^2$ in the interval $0 < x < 1$ and hence show that

(a) $\dfrac{\pi^2}{6} = \dfrac{1}{1^2} + \dfrac{1}{2^2} + \dfrac{1}{3^2} + \ldots$ (b) $\dfrac{\pi^2}{8} = \dfrac{1}{1^2} + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \ldots$

(c) $\dfrac{\pi^2}{12} = \dfrac{1}{1^2} - \dfrac{1}{2^2} + \dfrac{1}{3^2} - \dfrac{1}{4^2} + \ldots$ **[JNTU 2005/S, Set No. 2 and 4)**

5. Find half-range Fourier sine series of $f(x) = x$, (0, 2).

8. Find a Fourier Series for $f(x) = ax + b$ in $0 < x < l$. **[JNTU 2004, Set No. 2]**

9. Show that in the interval (0, 1)

$$\cos\pi x = \frac{8}{\pi}\sum_{n=1}^{\infty}\frac{n}{4n^2-1}\sin 2n\pi x\,.$$

10. Find half-range sine series for $f(x) = \begin{cases} \frac{1}{4} - x;\ 0 < x < \frac{1}{2} \\ x - \frac{3}{4};\frac{1}{2} < x < 1 \end{cases}$.

11. Obtain half-range sine series for the function

$$f(x) = \begin{cases} \sin x;\ 0 \le x \le \frac{\pi}{4} \\ \cos x; \frac{\pi}{4} \le x \le \frac{\pi}{2} \end{cases}.$$

Answers

1. $f(x) = \frac{1}{3} + \frac{1}{\pi^2}\left[\frac{\cos 2\pi x}{2^2} + \frac{\cos 3\pi x}{3^2} + \ldots\right]$

5. $f(x) = -\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin\left(\frac{n\pi x}{2}\right)$

8 $ax + b = \frac{2}{\pi}\left[\frac{b-(al+b)(-1)^n}{n}\right]\sin\left(\frac{n\pi x}{l}\right)$.

10. $f(x) = \left(\frac{1}{\pi} - \frac{4}{\pi^2}\right)\sin\pi x + \left(\frac{1}{3\pi} - \frac{4}{3^2\pi^2}\right)\sin 3\pi x + \left(\frac{1}{5\pi} - \frac{4}{5^2\pi^2}\right)\sin 5\pi x + \ldots$

11. $f(x) = \frac{8}{\pi}\cos\frac{\pi}{4}\left[\frac{\sin 2x}{1.2} - \frac{\sin 6x}{5.7} + \frac{\sin 10x}{9.11} + \ldots\right]$