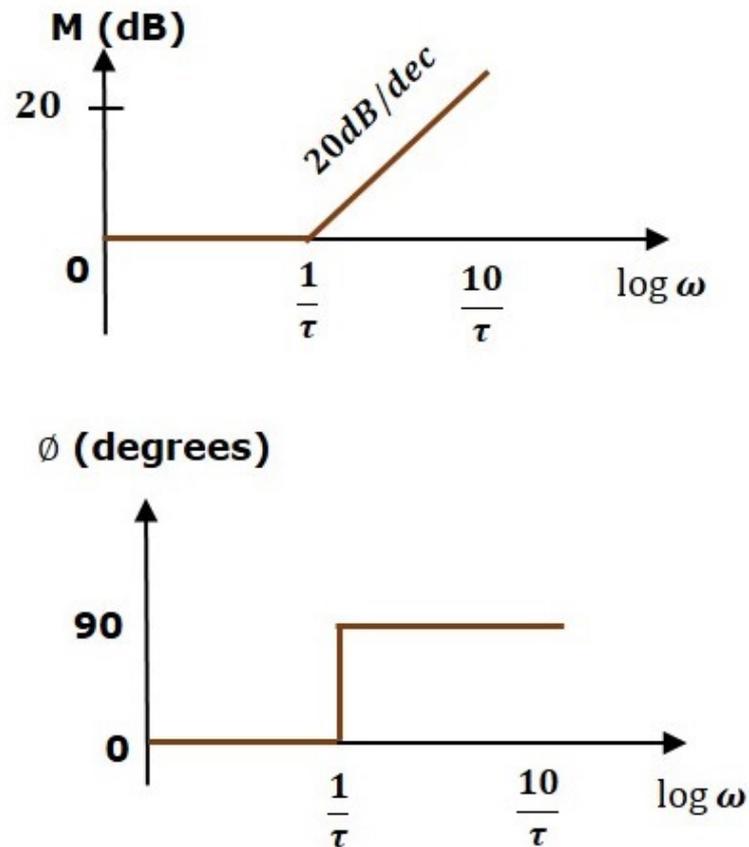


The following figure shows the corresponding Bode plot.



The magnitude plot is having magnitude of 0 dB upto  $\omega = \frac{1}{\tau}$  rad/sec. From  $\omega = \frac{1}{\tau}$  rad/sec, it is having a slope of 20 dB/dec. In this case, the phase plot is having phase angle of 0 degrees up to  $\omega = \frac{1}{\tau}$  rad/sec and from here, it is having phase angle of  $90^\circ$ . This Bode plot is called the **asymptotic Bode plot**.

As the magnitude and the phase plots are represented with straight lines, the Exact Bode plots resemble the asymptotic Bode plots. The only difference is that the Exact Bode plots will have simple curves instead of straight lines.

Similarly, you can draw the Bode plots for other terms of the open loop transfer function which are given in the table.

## Control Systems - Construction of Bode Plots

In this chapter, let us understand in detail how to construct (draw) Bode plots.

### Rules for Construction of Bode Plots

Follow these rules while constructing a Bode plot.

Represent the open loop transfer function in the standard time constant form.

Substitute,  $s = j\omega$  in the above equation.

Find the corner frequencies and arrange them in ascending order.

Choose the starting frequency of the Bode plot as  $1/10^{\text{th}}$  of the minimum corner frequency or  $0.1 \text{ rad/sec}$  whichever is smaller value and draw the Bode plot upto 10

times maximum corner frequency.

Draw the magnitude plots for each term and combine these plots properly.

Draw the phase plots for each term and combine these plots properly.

**Note** – The corner frequency is the frequency at which there is a change in the slope of the magnitude plot.

## Example

Consider the open loop transfer function of a closed loop control system

$$G(s)H(s) = \frac{10s}{(s+2)(s+5)}$$

Let us convert this open loop transfer function into standard time constant form.

$$G(s)H(s) = \frac{10s}{2\left(\frac{s}{2} + 1\right)5\left(\frac{s}{5} + 1\right)}$$
$$\Rightarrow G(s)H(s) = \frac{s}{\left(1 + \frac{s}{2}\right)\left(1 + \frac{s}{5}\right)}$$

So, we can draw the Bode plot in semi log sheet using the rules mentioned earlier.

## Stability Analysis using Bode Plots

From the Bode plots, we can say whether the control system is stable, marginally stable or unstable based on the values of these parameters.

Gain cross over frequency and phase cross over frequency

Gain margin and phase margin

### Phase Cross over Frequency

The frequency at which the phase plot is having the phase of  $-180^\circ$  is known as **phase cross over frequency**. It is denoted by  $\omega_{pc}$ . The unit of phase cross over frequency is **rad/sec**.

### Gain Cross over Frequency

The frequency at which the magnitude plot is having the magnitude of zero dB is known as **gain cross over frequency**. It is denoted by  $\omega_{gc}$ . The unit of gain cross over frequency is **rad/sec**.

The stability of the control system based on the relation between the phase cross over frequency and the gain cross over frequency is listed below.

If the phase cross over frequency  $\omega_{pc}$  is greater than the gain cross over frequency  $\omega_{gc}$  then the control system is **stable**.

If the phase cross over frequency  $\omega_{pc}$  is equal to the gain cross over frequency  $\omega_{gc}$ , then the control system is **marginally stable**.

If the phase cross over frequency  $\omega_{pc}$  is less than the gain cross over frequency  $\omega_{gc}$ , then the control system is **unstable**.

## Gain Margin

Gain margin  $GM$  is equal to negative of the magnitude in dB at phase cross over frequency.

$$GM = 20\log\left(\frac{1}{M_{pc}}\right) = 20\log M_{pc}$$

Where,  $M_{pc}$  is the magnitude at phase cross over frequency. The unit of gain margin (GM) is **dB**.

## Phase Margin

The formula for phase margin  $PM$  is

$$PM = 180^\circ + \phi_{gc}$$

Where,  $\phi_{gc}$  is the phase angle at gain cross over frequency. The unit of phase margin is **degrees**.

The stability of the control system based on the relation between gain margin and phase margin is listed below.

If both the gain margin  $GM$  and the phase margin  $PM$  are positive, then the control system is **stable**.

If both the gain margin  $GM$  and the phase margin  $PM$  are equal to zero, then the control system is **marginally stable**.

If the gain margin  $GM$  and / or the phase margin  $PM$  are/is negative, then the control system is **unstable**.

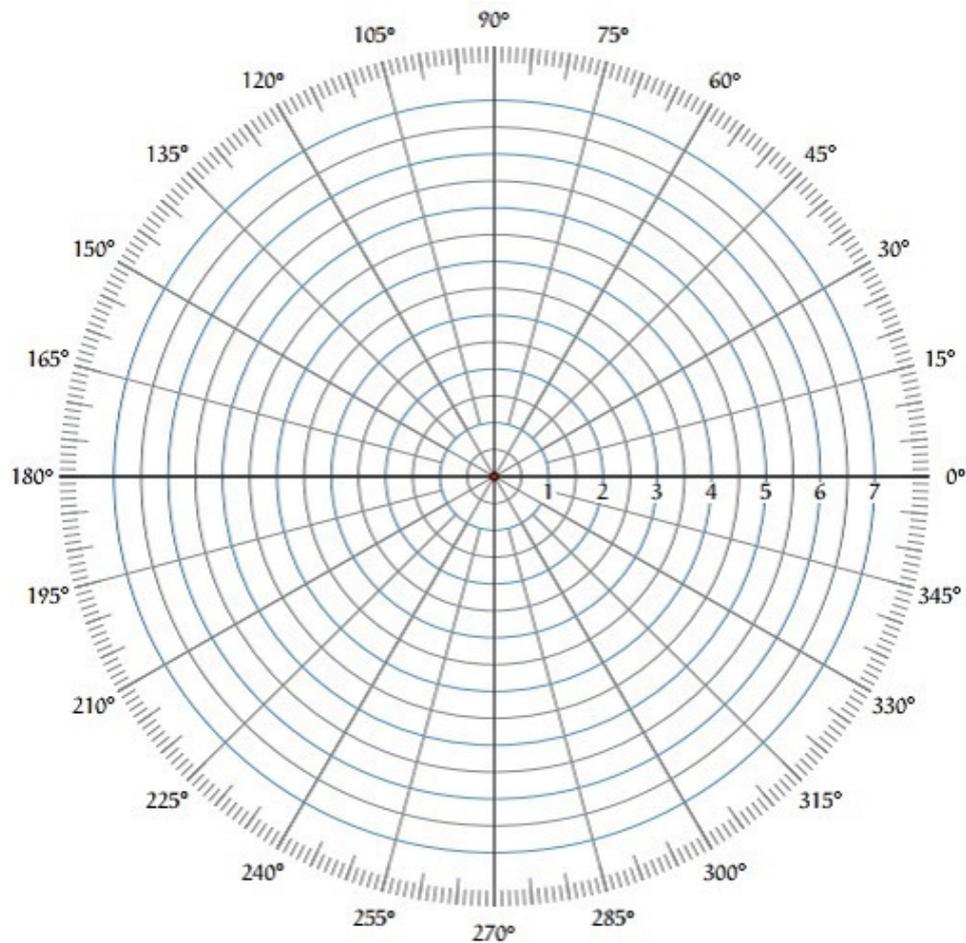
# Control Systems - Polar Plots

In the previous chapters, we discussed the Bode plots. There, we have two separate plots for both magnitude and phase as the function of frequency. Let us now discuss about polar plots. Polar plot is a plot which can be drawn between magnitude and phase. Here, the magnitudes are represented by normal values only.

The polar form of  $G(j\omega)H(j\omega)$  is

$$G(j\omega)H(j\omega) = |G(j\omega)H(j\omega)| \angle G(j\omega)H(j\omega)$$

The **Polar plot** is a plot, which can be drawn between the magnitude and the phase angle of  $G(j\omega)H(j\omega)$  by varying  $\omega$  from zero to  $\infty$ . The polar graph sheet is shown in the following figure.



This graph sheet consists of concentric circles and radial lines. The **concentric circles** and the **radial lines** represent the magnitudes and phase angles respectively. These angles are represented by positive values in anti-clock wise direction. Similarly, we can represent angles with negative values in clockwise direction. For example, the angle  $270^0$  in anti-clock wise direction is equal to the angle  $-90^0$  in clockwise direction.

## Rules for Drawing Polar Plots

Follow these rules for plotting the polar plots.

Substitute,  $s = j\omega$  in the open loop transfer function.

Write the expressions for magnitude and the phase of  $G(j\omega)H(j\omega)$ .

Find the starting magnitude and the phase of  $G(j\omega)H(j\omega)$  by substituting  $\omega = 0$ . So, the polar plot starts with this magnitude and the phase angle.

Find the ending magnitude and the phase of  $G(j\omega)H(j\omega)$  by substituting  $\omega = \infty$ . So, the polar plot ends with this magnitude and the phase angle.

Check whether the polar plot intersects the real axis, by making the imaginary term of  $G(j\omega)H(j\omega)$  equal to zero and find the value(s) of  $\omega$ .

Check whether the polar plot intersects the imaginary axis, by making real term of  $G(j\omega)H(j\omega)$  equal to zero and find the value(s) of  $\omega$ .

For drawing polar plot more clearly, find the magnitude and phase of  $G(j\omega)H(j\omega)$  by

Loading [MathJax]/jax/output/HTML-CSS/jax.js ue(s) of  $\omega$ .

## Example

Consider the open loop transfer function of a closed loop control system.

$$G(s)H(s) = \frac{5}{s(s+1)(s+2)}$$

Let us draw the polar plot for this control system using the above rules.

**Step 1** – Substitute,  $s = j\omega$  in the open loop transfer function.

$$G(j\omega)H(j\omega) = \frac{5}{j\omega(j\omega+1)(j\omega+2)}$$

The magnitude of the open loop transfer function is

$$M = \frac{5}{\omega(\sqrt{\omega^2+1})(\sqrt{\omega^2+4})}$$

The phase angle of the open loop transfer function is

$$\phi = -90^\circ - \tan^{-1}\omega - \tan^{-1}\frac{\omega}{2}$$

**Step 2** – The following table shows the magnitude and the phase angle of the open loop transfer function at  $\omega = 0$  rad/sec and  $\omega = \infty$  rad/sec.

Frequency (rad/sec)	Magnitude	Phase angle(degrees)
0	$\infty$	-90 or 270
$\infty$	0	-270 or 90

So, the polar plot starts at  $(\infty, -90^\circ)$  and ends at  $(0, -270^\circ)$ . The first and the second terms within the brackets indicate the magnitude and phase angle respectively.

**Step 3** – Based on the starting and the ending polar co-ordinates, this polar plot will intersect the negative real axis. The phase angle corresponding to the negative real axis is  $-180^\circ$  or  $180^\circ$ . So, by equating the phase angle of the open loop transfer function to either  $-180^\circ$  or  $180^\circ$ , we will get the  $\omega$  value as  $\sqrt{2}$ .

By substituting  $\omega = \sqrt{2}$  in the magnitude of the open loop transfer function, we will get  $M = 0.83$ . Therefore, the polar plot intersects the negative real axis when  $\omega = \sqrt{2}$  and the polar coordinate is  $(0.83, -180^\circ)$ .

So, we can draw the polar plot with the above information on the polar graph sheet.

## Control Systems - Nyquist Plots

Nyquist plots are the continuation of polar plots for finding the stability of the closed loop control systems by varying  $\omega$  from  $-\infty$  to  $\infty$ . That means, Nyquist plots are used to draw the complete frequency response of the open loop transfer function.

# Nyquist Stability Criterion

The Nyquist stability criterion works on the **principle of argument**. It states that if there are  $P$  poles and  $Z$  zeros are enclosed by the 's' plane closed path, then the corresponding  $G(s)H(s)$  plane must encircle the origin  $P - Z$  times. So, we can write the number of encirclements  $N$  as,

$$N = P - Z$$

If the enclosed 's' plane closed path contains only poles, then the direction of the encirclement in the  $G(s)H(s)$  plane will be opposite to the direction of the enclosed closed path in the 's' plane.

If the enclosed 's' plane closed path contains only zeros, then the direction of the encirclement in the  $G(s)H(s)$  plane will be in the same direction as that of the enclosed closed path in the 's' plane.

Let us now apply the principle of argument to the entire right half of the 's' plane by selecting it as a closed path. This selected path is called the **Nyquist** contour.

We know that the closed loop control system is stable if all the poles of the closed loop transfer function are in the left half of the 's' plane. So, the poles of the closed loop transfer function are nothing but the roots of the characteristic equation. As the order of the characteristic equation increases, it is difficult to find the roots. So, let us correlate these roots of the characteristic equation as follows.

The Poles of the characteristic equation are same as that of the poles of the open loop transfer function.

The zeros of the characteristic equation are same as that of the poles of the closed loop transfer function.

We know that the open loop control system is stable if there is no open loop pole in the the right half of the 's' plane.

$$\text{i.e., } P = 0 \Rightarrow N = -Z$$

We know that the closed loop control system is stable if there is no closed loop pole in the right half of the 's' plane.

$$\text{i.e., } Z = 0 \Rightarrow N = P$$

**Nyquist stability criterion** states the number of encirclements about the critical point  $(1+j0)$  must be equal to the poles of characteristic equation, which is nothing but the poles of the open loop transfer function in the right half of the 's' plane. The shift in origin to  $(1+j0)$  gives the characteristic equation plane.

## Rules for Drawing Nyquist Plots

Follow these rules for plotting the Nyquist plots.

os of open loop transfer function  $G(s)H(s)$  in 's' plane.

Draw the polar plot by varying  $\omega$  from zero to infinity. If pole or zero present at  $s = 0$ , then varying  $\omega$  from  $0^+$  to infinity for drawing polar plot.

Draw the mirror image of above polar plot for values of  $\omega$  ranging from  $-\infty$  to zero ( $0^-$  if any pole or zero present at  $s=0$ ).

The number of infinite radius half circles will be equal to the number of poles or zeros at origin. The infinite radius half circle will start at the point where the mirror image of the polar plot ends. And this infinite radius half circle will end at the point where the polar plot starts.

After drawing the Nyquist plot, we can find the stability of the closed loop control system using the Nyquist stability criterion. If the critical point  $(-1+j0)$  lies outside the encirclement, then the closed loop control system is absolutely stable.

## Stability Analysis using Nyquist Plots

From the Nyquist plots, we can identify whether the control system is stable, marginally stable or unstable based on the values of these parameters.

Gain cross over frequency and phase cross over frequency

Gain margin and phase margin

### Phase Cross over Frequency

The frequency at which the Nyquist plot intersects the negative real axis (phase angle is  $180^\circ$ ) is known as the **phase cross over frequency**. It is denoted by  $\omega_{pc}$ .

### Gain Cross over Frequency

The frequency at which the Nyquist plot is having the magnitude of one is known as the **gain cross over frequency**. It is denoted by  $\omega_{gc}$ .

The stability of the control system based on the relation between phase cross over frequency and gain cross over frequency is listed below.

If the phase cross over frequency  $\omega_{pc}$  is greater than the gain cross over frequency  $\omega_{gc}$  then the control system is **stable**.

If the phase cross over frequency  $\omega_{pc}$  is equal to the gain cross over frequency  $\omega_{gc}$  then the control system is **marginally stable**.

If phase cross over frequency  $\omega_{pc}$  is less than gain cross over frequency  $\omega_{gc}$  then the control system is **unstable**.

### Gain Margin

The gain margin  $GM$  is equal to the reciprocal of the magnitude of the Nyquist plot at the phase cross over frequency.

$$GM = \frac{1}{M_{pc}}$$

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where,  $M_{pc}$  is the magnitude in normal scale at the phase cross over frequency.

## Phase Margin

The phase margin  $PM$  is equal to the sum of  $180^0$  and the phase angle at the gain cross over frequency.

$$PM = 180^0 + \phi_{gc}$$

Where,  $\phi_{gc}$  is the phase angle at the gain cross over frequency.

The stability of the control system based on the relation between the gain margin and the phase margin is listed below.

If the gain margin  $GM$  is greater than one and the phase margin  $PM$  is positive, then the control system is **stable**.

If the gain margin  $GM$  is equal to one and the phase margin  $PM$  is zero degrees, then the control system is **marginally stable**.

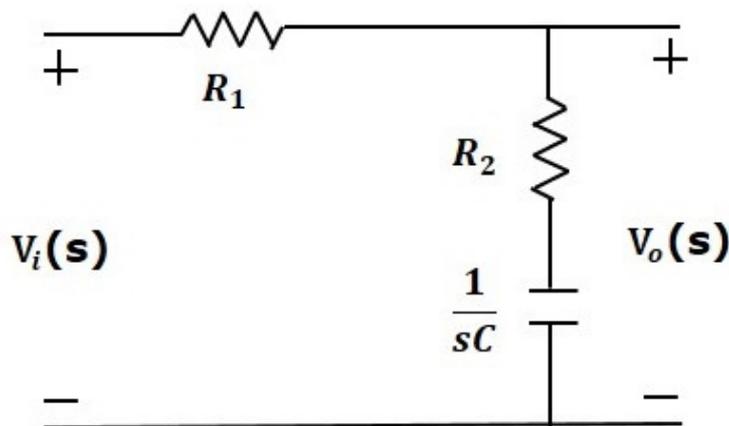
If the gain margin  $GM$  is less than one and / or the phase margin  $PM$  is negative, then the control system is **unstable**.

## Control Systems - Compensators

There are three types of compensators — lag, lead and lag-lead compensators. These are most commonly used.

### Lag Compensator

The Lag Compensator is an electrical network which produces a sinusoidal output having the phase lag when a sinusoidal input is applied. The lag compensator circuit in the 's' domain is shown in the following figure.



Here, the capacitor is in series with the resistor  $R_2$  and the output is measured across this combination.

The transfer function of this lag compensator is -

$$\frac{V_o(s)}{V_i(s)} = \frac{1}{\alpha} \left( \frac{s + \frac{1}{\tau}}{s + \frac{1}{\alpha\tau}} \right)$$

Where,

$$\tau = R_2C$$

$$\alpha = \frac{R_1 + R_2}{R_2}$$

From the above equation,  $\alpha$  is always greater than one.

From the transfer function, we can conclude that the lag compensator has one pole at  $s = -\frac{1}{\alpha\tau}$  and one zero at  $s = -\frac{1}{\tau}$ . This means, the pole will be nearer to origin in the pole-zero configuration of the lag compensator.

Substitute,  $s = j\omega$  in the transfer function.

$$\frac{V_o(j\omega)}{V_i(j\omega)} = \frac{1}{\alpha} \left( \frac{j\omega + \frac{1}{\tau}}{j\omega + \frac{1}{\alpha\tau}} \right)$$

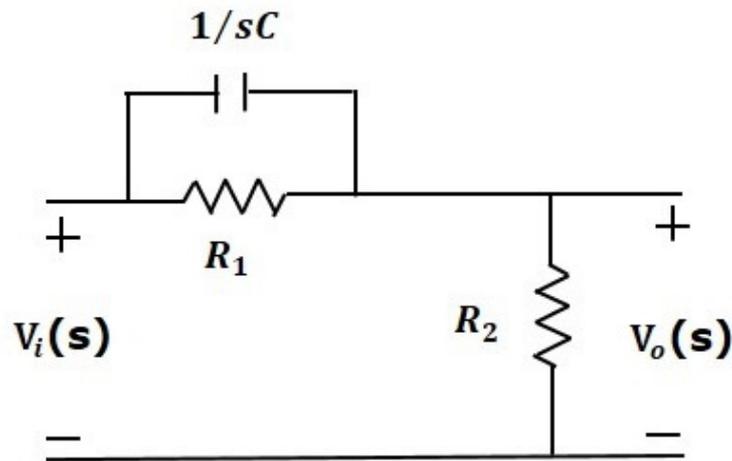
Phase angle  $\phi = \tan^{-1}\omega\tau - \tan^{-1}\alpha\omega\tau$

We know that, the phase of the output sinusoidal signal is equal to the sum of the phase angles of input sinusoidal signal and the transfer function.

So, in order to produce the phase lag at the output of this compensator, the phase angle of the transfer function should be negative. This will happen when  $\alpha > 1$ .

## Lead Compensator

The lead compensator is an electrical network which produces a sinusoidal output having phase lead when a sinusoidal input is applied. The lead compensator circuit in the 's' domain is shown in the following figure.



Here, the capacitor is parallel to the resistor  $R_1$  and the output is measured across resistor  $R_2$ .

The transfer function of this lead compensator is -

$$\frac{V_o(s)}{V_i(s)} = \beta \left( \frac{s\tau + 1}{\beta s\tau + 1} \right)$$

Where,

$$\tau = R_1 C$$

$$\beta = \frac{R_2}{R_1 + R_2}$$

From the transfer function, we can conclude that the lead compensator has pole at  $s = -\frac{1}{\beta\tau}$  and zero at  $s = -\frac{1}{\tau}$ .

Substitute,  $s = j\omega$  in the transfer function.

$$\frac{V_o(j\omega)}{V_i(j\omega)} = \beta \left( \frac{j\omega\tau + 1}{\beta j\omega\tau + 1} \right)$$

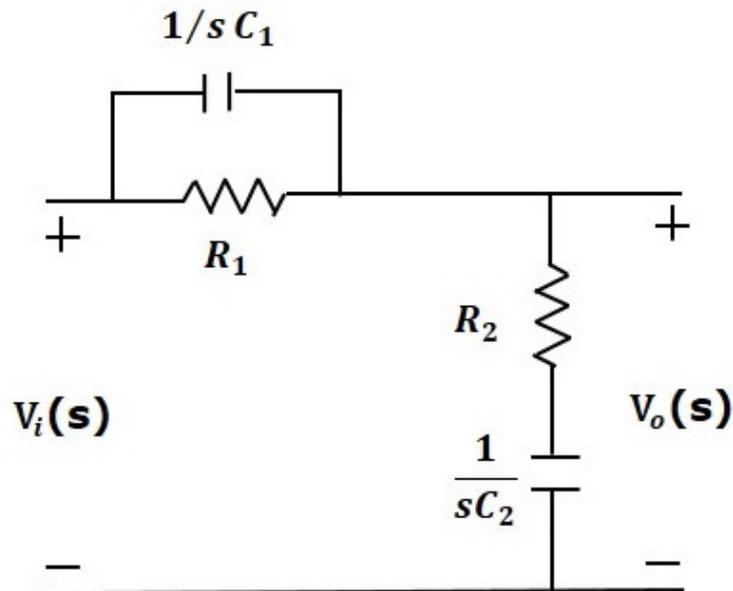
Phase angle  $\phi = \tan^{-1}\omega\tau - \tan^{-1}\beta\omega\tau$

We know that, the phase of the output sinusoidal signal is equal to the sum of the phase angles of input sinusoidal signal and the transfer function.

So, in order to produce the phase lead at the output of this compensator, the phase angle of the transfer function should be positive. This will happen when  $0 < \beta < 1$ . Therefore, zero will be nearer to origin in pole-zero configuration of the lead compensator.

## Lag-Lead Compensator

Lag-Lead compensator is an electrical network which produces phase lag at one frequency region and phase lead at other frequency region. It is a combination of both the lag and the lead compensators. The lag lead compensator circuit in the 's' domain is shown in the



This circuit looks like both the compensators are cascaded. So, the transfer function of this circuit will be the product of transfer functions of the lead and the lag compensators.

$$\frac{V_o(s)}{V_i(s)} = \beta \left( \frac{s\tau_1 + 1}{\beta s\tau_1 + 1} \right) \frac{1}{\alpha} \left( \frac{s + \frac{1}{\tau_2}}{s + \frac{1}{\alpha\tau_2}} \right)$$

We know  $\alpha\beta = 1$ .

$$\Rightarrow \frac{V_o(s)}{V_i(s)} = \left( \frac{s + \frac{1}{\tau_1}}{s + \frac{1}{\beta\tau_1}} \right) \left( \frac{s + \frac{1}{\tau_2}}{s + \frac{1}{\alpha\tau_2}} \right)$$

Where,

$$\tau_1 = R_1 C_1$$

$$\tau_2 = R_2 C_2$$

## Control Systems - Controllers

The various types of controllers are used to improve the performance of control systems. In this chapter, we will discuss the basic controllers such as the proportional, the derivative and the integral controllers.

### Proportional Controller

The proportional controller produces an output, which is proportional to error signal.

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$$u(t) \propto e(t)$$

$$\Rightarrow u(t) = K_P e(t)$$

Apply Laplace transform on both the sides -

$$U(s) = K_P E(s)$$

$$\frac{U(s)}{E(s)} = K_P$$

Therefore, the transfer function of the proportional controller is  $K_P$ .

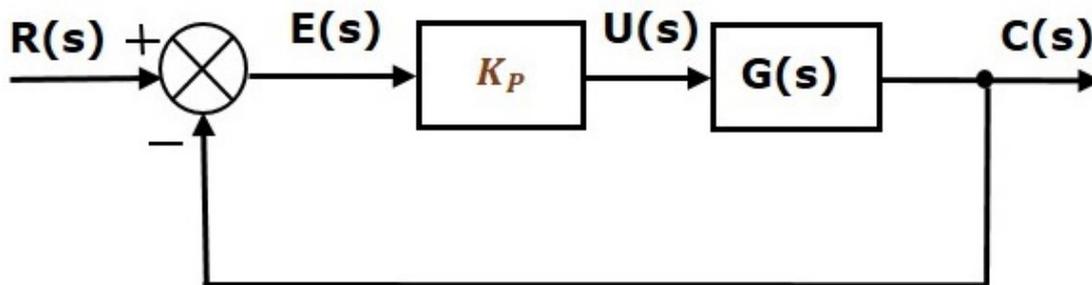
Where,

$U(s)$  is the Laplace transform of the actuating signal  $u(t)$

$E(s)$  is the Laplace transform of the error signal  $e(t)$

$K_P$  is the proportionality constant

The block diagram of the unity negative feedback closed loop control system along with the proportional controller is shown in the following figure.



The proportional controller is used to change the transient response as per the requirement.

## Derivative Controller

The derivative controller produces an output, which is derivative of the error signal.

$$u(t) = K_D \frac{de(t)}{dt}$$

Apply Laplace transform on both sides.

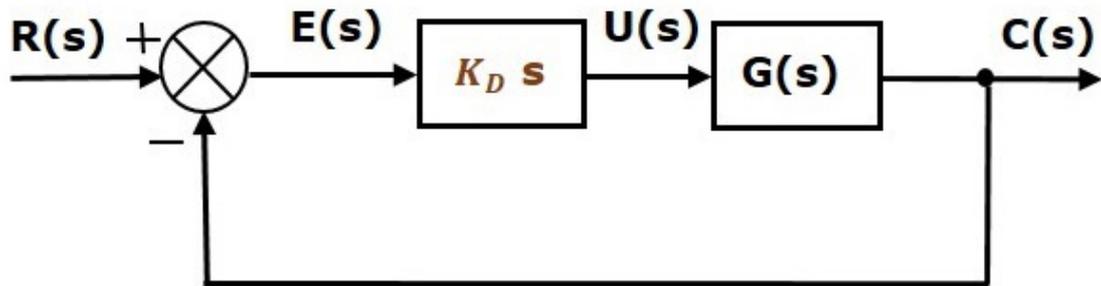
$$U(s) = K_D s E(s)$$

$$\frac{U(s)}{E(s)} = K_D s$$

Therefore, the transfer function of the derivative controller is  $K_D s$ .

Where,  $K_D$  is the derivative constant.

The block diagram of the unity negative feedback closed loop control system along with the derivative controller is shown in the following figure.



The derivative controller is used to make the unstable control system into a stable one.

## Integral Controller

The integral controller produces an output, which is integral of the error signal.

$$u(t) = K_I \int e(t) dt$$

Apply Laplace transform on both the sides -

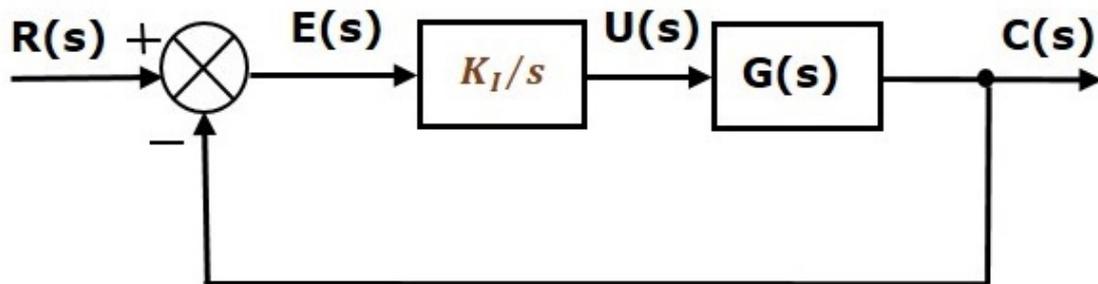
$$U(s) = \frac{K_I E(s)}{s}$$

$$\frac{U(s)}{E(s)} = \frac{K_I}{s}$$

Therefore, the transfer function of the integral controller is  $\frac{K_I}{s}$ .

Where,  $K_I$  is the integral constant.

The block diagram of the unity negative feedback closed loop control system along with the integral controller is shown in the following figure.



The integral controller is used to decrease the steady state error.

Let us now discuss about the combination of basic controllers.

## Proportional Derivative (PD) Controller

The proportional derivative controller produces an output, which is the combination of the outputs of proportional and derivative controllers.

$$u(t) = K_P e(t) + K_D \frac{de(t)}{dt}$$

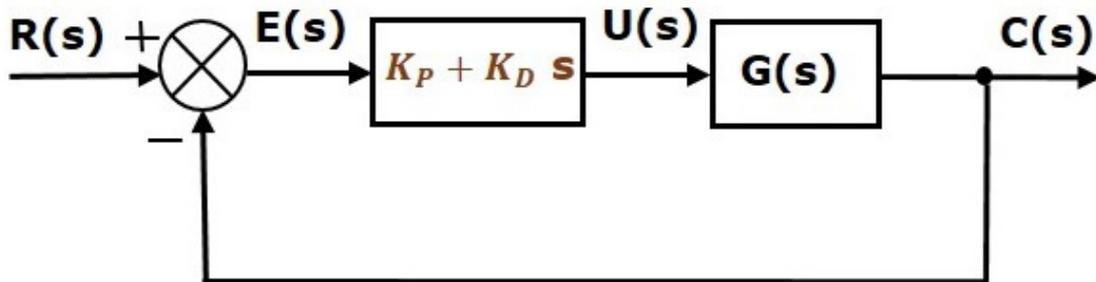
Apply Laplace transform on both sides -

$$U(s) = (K_P + K_D s)E(s)$$

$$\frac{U(s)}{E(s)} = K_P + K_D s$$

Therefore, the transfer function of the proportional derivative controller is  $K_P + K_D s$ .

The block diagram of the unity negative feedback closed loop control system along with the proportional derivative controller is shown in the following figure.



The proportional derivative controller is used to improve the stability of control system without affecting the steady state error.

## Proportional Integral (PI) Controller

The proportional integral controller produces an output, which is the combination of outputs of the proportional and integral controllers.

$$u(t) = K_P e(t) + K_I \int e(t) dt$$

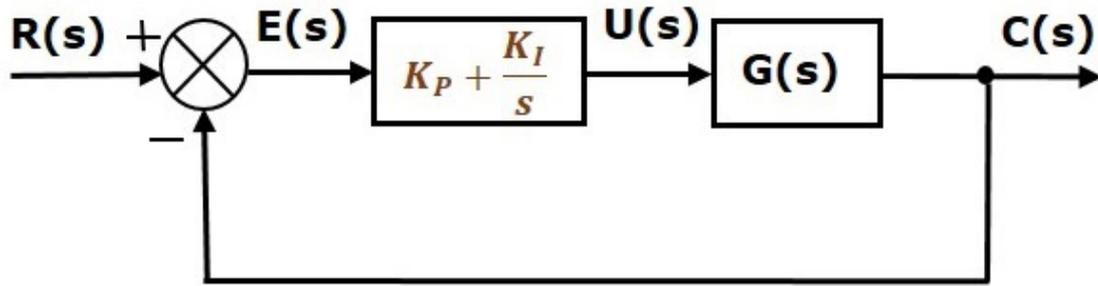
Apply Laplace transform on both sides -

$$U(s) = \left( K_P + \frac{K_I}{s} \right) E(s)$$

$$\frac{U(s)}{E(s)} = K_P + \frac{K_I}{s}$$

Therefore, the transfer function of proportional integral controller is  $K_P + \frac{K_I}{s}$ .

The block diagram of the unity negative feedback closed loop control system along with the proportional integral controller is shown in the following figure.



The proportional integral controller is used to decrease the steady state error without affecting the stability of the control system.

## Proportional Integral Derivative (PID) Controller

The proportional integral derivative controller produces an output, which is the combination of the outputs of proportional, integral and derivative controllers.

$$u(t) = K_P e(t) + K_I \int e(t) dt + K_D \frac{de(t)}{dt}$$

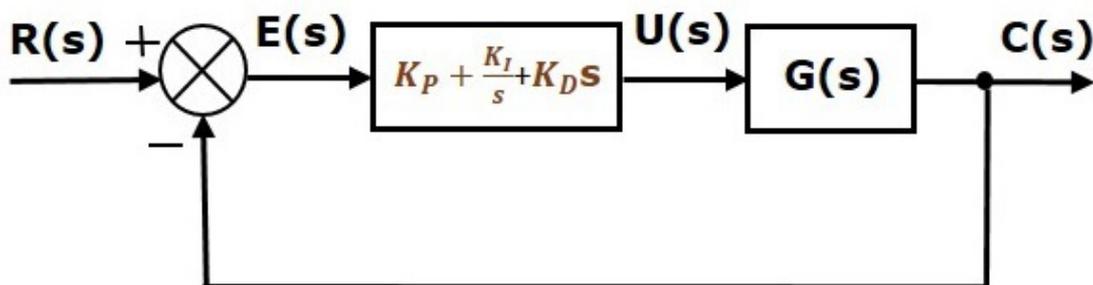
Apply Laplace transform on both sides -

$$U(s) = \left( K_P + \frac{K_I}{s} + K_D s \right) E(s)$$

$$\frac{U(s)}{E(s)} = K_P + \frac{K_I}{s} + K_D s$$

Therefore, the transfer function of the proportional integral derivative controller is  $K_P + \frac{K_I}{s} + K_D s$ .

The block diagram of the unity negative feedback closed loop control system along with the proportional integral derivative controller is shown in the following figure.



The proportional integral derivative controller is used to improve the stability of the control system and to decrease steady state error.

## Control Systems - State Space Model

The **state space model** of Linear Time-Invariant (LTI) system can be represented as,