Unit-4

Vibrations

→ A body is said to vibrate if it has a to & fro motion.

Ex: The swinging of a pendulum is example of vibration.

→ When ever a elastic body such as beam, shaft, or spring is displaced from its equilibrium position by application of external forces, work is done on elastic constraints of the forces on the body & is stored as internal energy in the form of elastic or strain energy.

→ Now when the body is released, this stored energy causes the body to move towards its equilibrium position, the whole of elastic or strain energy is converted to K.E. over which the body passes through the mean position in opposite direction, again the K.E is converted to strain energy, due to which the body tends to move towards equilibrium position.

→ In this way vibratory motion is repeated.

In study of mechanical vibrations the bodies are treated as elastic bodies instead of rigid bodies.

→ Vibration is undesirable in most eng. systems & desirable in few cases.

Definitions:

1) Free (natural vibrations)

Vibrations in which there are no friction & external forces after initial release of body are called free or natural vibrations.

2) Damped vibrations:- when the energy of vibrating system is gradually dissipated by friction & other resistances, the vibrations are said to be damped vibrations. The vibrations gradually decrease & system rests in its equilibrium position.

3) Forced vibrations:- vibrations that take place under the excitation of external forces is called forced vibration. The forced vibrations can take place at different forced frequencies or external frequencies.

4) Time Period:- It is time taken to complete one cycle (or) it is time taken by motion to repeat itself & is measured in seconds.

5) Cycle:- It is the motion completed during one time period.

Amplitude:- max displacement of vibrating body from mean position

reavem
compl
uni

Frequency: It is number of cycles of motion completed in one sec.

Units: ~~Heet~~ Hertz (Hz)

7) Resonance: When the frequency of external force is same as that of natural frequency of the system, a state of resonance is said to have been reached.

Types of vibrations:

Consider a vibrating body, e.g. a rod, shaft, & spring. Figure shows a massless shaft one end fixed, other end carrying a heavy disc. The system can execute the following types of vibrations

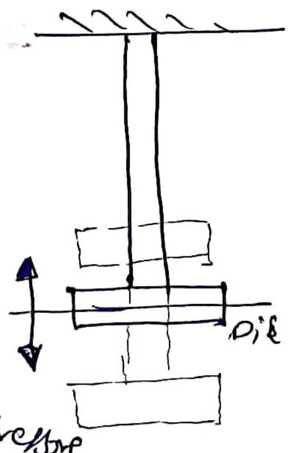
1) Longitudinal vibrations

2) Transverse vibration

3) Torsional vibration

Longitudinal vibration:

[The particles of body move parallel to axis of shaft]

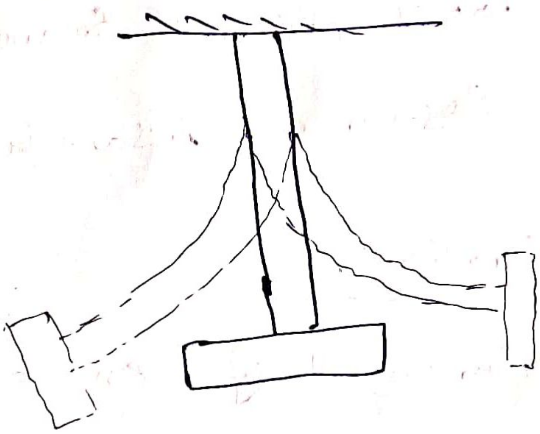


If the shaft is elongated & shortened so that mass moves up & down, resulting tensile, compressive stress in shaft, the vibrations are said to be longitudinal.

Transverse vibrations:

when the shaft is bent alternately and tensile & compressive stress due to bending result, the vibrations are said to be transverse.

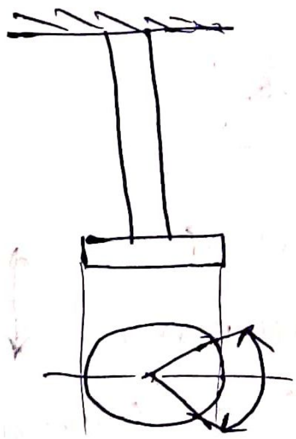
(The particle of the body move perpendicular to axis of shaft)



Torsional vibrations

when the shaft is twisted & untwisted alternatively torsional shear stress are induced, the vibrations are said to be torsional vibrations.

The particle of body move in circle about axis of shaft



uses of vibration

unbalanced Rotating, Reciprocating Machine parts

1) Loose transmission of belts & chains

3) without teeth of gears for power transmission

4) External Exciting force acting on the systems.

5) Loose fastening of the moving parts

Effects of vibrations

1) They produce Undesirable noise

2) " " Unwanted stresses in M/c parts

3) It reduce performance of Machine

4) fatigue failure of M/c components occur

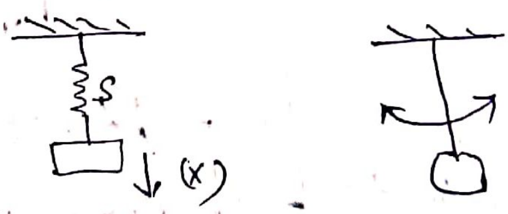
5) poor surface finish on the production machinery.

Degree of freedom

The ~~min~~ Minimum number of independent co-ordinates required to specify the motion of a systems at any instant is called D.O.F

D.O.F vary from 0 to ∞

1) Single degree of freedom



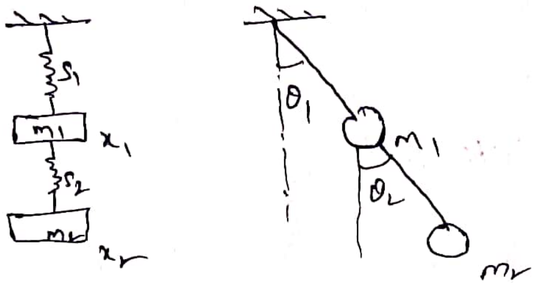
Here one co-ordinate is sufficient to specify the position of mass at any time is called Single D.O.F

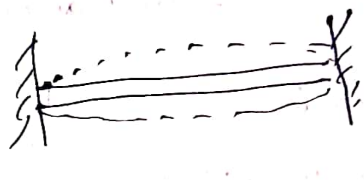
Ex 1) Spring mass system

2) Simple pendulum

Two degree of freedom

To describe the motion, the system requires two independent co-ordinates is called two degree of freedom

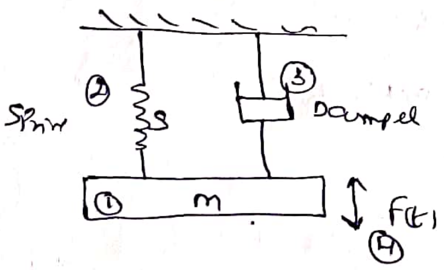




Infinite degree of freedom

String stretched between two supports has infinite DOF

Basic Elements of a vibrating system



1) Mass of Inertia Elements
2) Spring
3) Damper
4) Excitation force F(t)

Classification of vibration

- Vibration According to actuating force
  - Free (or) Natural Vibration
    - Damped
    - Undamped
  - Forced Vibration
    - Damped
    - Undamped

- Vibration according to stress in the supporting medium
  - Longitudinal vibration
  - Transverse vibration
  - Torsional vibration

Scanned by CamScanner

Longitudinal vibrations :- Natural frequency of longitudinal vibrations

→ The natural frequency of a vibrating system may be found from any of the following methods:

1) **Equilibrium method:**

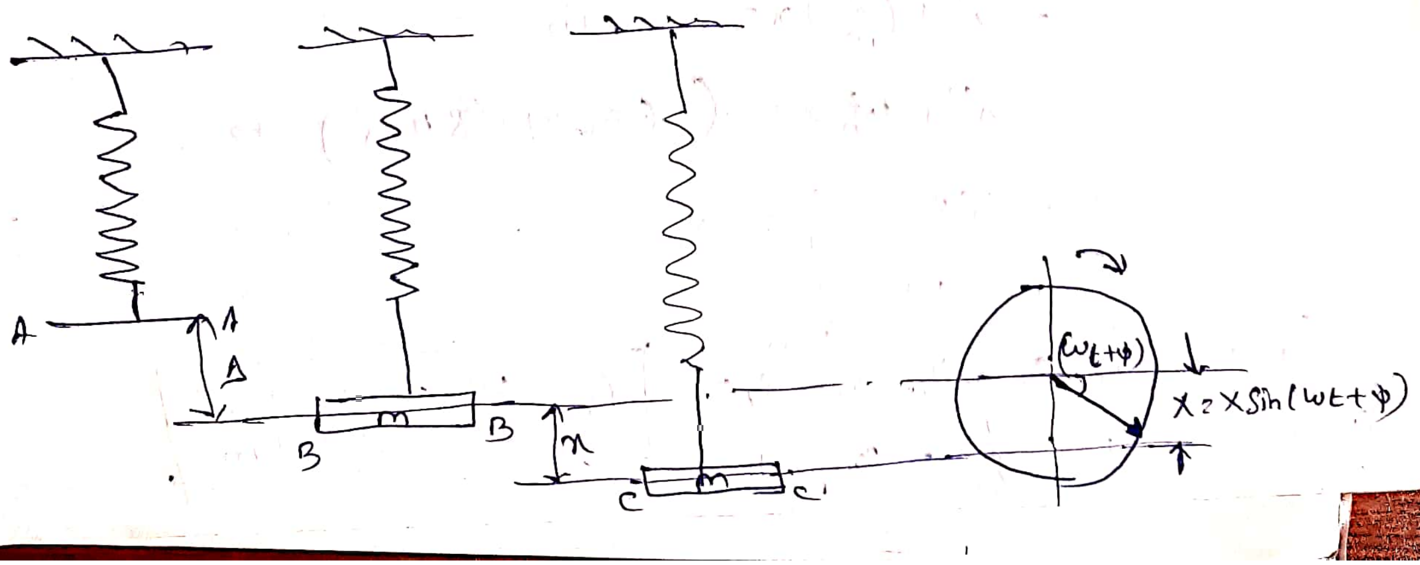
→ It is based on principle that whenever a vibratory system is in equilibrium, the algebraic sum of forces & moments acting on it is zero. This is in accordance with D'Alembert principle that sum of inertia force & external force on a body in equilibrium must be zero.

→ Consider a spring suspended vertically from a rigid support with its free end at A-A

→ If mass 'm' is attached from free end, the spring is stretched by a distance Δ & B-B becomes equilibrium position

→ Thus 'Δ' is static deflection of spring under mass 'm'

Let 's' is the stiffness of spring (force required for unit deflection)



In static equilibrium

Upward force = down ward force

$S\Delta = mg$ $\quad \frac{S}{m} = \frac{g}{\delta}$

Now if the mass 'm' pulled farther down through a distance (x) the forces acting on the mass will be

Inertia force = $m\ddot{x}$ ($\downarrow$)

Spring force (restoring force) = Sx ($\uparrow$)

As per D'alembert principle the inertia force & external force on the body in any direction is to be zero.

$\therefore m\ddot{x} + Sx = 0$

If the mass is released it will start oscillating above & below the equilibrium position. The oscillation will continue for ever if there is no frictional resistance to the motion.

$\therefore m\ddot{x} + Sx = 0$

$= \ddot{x} + \left(\frac{S}{m}\right)x = 0$ — (1)

$= \ddot{x} + \omega_n^2 x = 0$ (Eq of SHM) (2)

from 1 & 2 $\frac{S}{m} = \omega_n^2$

$\omega_n = \sqrt{\frac{S}{m}}$

frequency of vibrating system $\boxed{f_n = \frac{\omega_n}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{S}{m}}}$

$\boxed{\text{Time period } T = \frac{1}{f_n} = 2\pi\sqrt{m/S}}$

Energy method

In a conservative system (system with no damping)

The total mechanical energy remain constant.

i.e sum of KE + PE = constant

$$\frac{d}{dt}(KE + PE) = 0$$

$KE = \frac{1}{2} m \dot{x}^2$ $\quad \dot{x}$ = velocity

PE = mean force × displacement

$= \left(\frac{0 + Sx}{2}\right) \times x = \frac{Sx^2}{2}$

$$\frac{d}{dt}\left(\frac{1}{2} m \dot{x}^2 + \frac{Sx^2}{2}\right) = 0$$

$$\frac{1}{2} m \, 2\dot{x}\ddot{x} + \frac{1}{2} S \, 2x \cdot \dot{x} = 0$$

$$m\ddot{x} + Sx = 0 \quad \text{--- (1)} \qquad \ddot{x} + \left(\frac{S}{m}\right)x = 0$$

$$\ddot{x} + \omega_n^2 x = 0 \quad \text{--- (2)}$$

$$\boxed{\omega_n = \sqrt{S/m}}$$

$$f_n = \frac{\omega_n}{2\pi} = \frac{1}{2\pi}\sqrt{S/m}$$

$$T = \frac{1}{f_n} = 2\pi\sqrt{\frac{m}{S}}$$

3) Rayleigh's method

In this method $(K.E)_{max}$ at mean position = $(P.E)_{max}$ at Extreme position

i.e $(K.E)$ at mean position = $(P.E)$ at Extreme position

let the motion be SHM.

$\therefore\ x = X \sin\omega_n t$

'X' is Max displacement from mean to Extreme position

$\dot{x} = \omega_n X \cos\omega_n t$

$(\dot{x})_{max} = \omega_n X$

$(K.E)_{max}$ at Mean position = $(P.E)_{max}$ at Extreme position

$$\frac{1}{2} m (\dot{x})^2 = \frac{1}{2} S X^2$$

$$\frac{1}{2} m (\omega_n X)^2 = \frac{1}{2} S X^2$$

$$m \omega_n^2 \cancel{X^2} = S \cancel{X^2}$$

$$\omega_n^2 = \frac{S}{m}$$

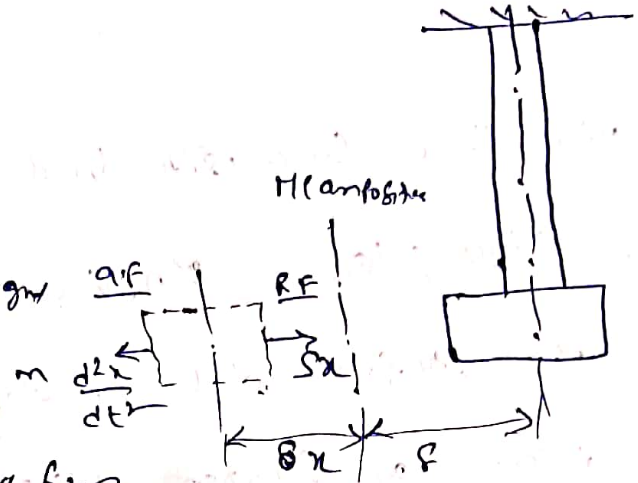
$$\boxed{\omega_n = \sqrt{S/m}}$$

Natural frequency of free transverse vibrations

Let w = weight of the body

s = stiffness of shaft

δ = static deflection due to weight of the body



x = displacement of the body from ~~mean~~ mean position

$\therefore$ Restoring force $= -sx$ — (1)

Acceleration force $= m\frac{d^2x}{dt^2}$ — (2)

Equate 1 & 2

$-sx = m\frac{d^2x}{dt^2}$

$\therefore m\frac{d^2x}{dt^2} + sx = 0$

$\frac{d^2x}{dt^2} + \frac{s}{m}x = 0$

$\omega_n^2 = \frac{s}{m}$ $\quad \omega_n = \sqrt{\frac{s}{m}}$

Time period $t_p = \frac{2\pi}{\omega_n} = \frac{2\pi}{\sqrt{\frac{m}{s}}}$

$$s \cdot \delta = mg$$
$$\frac{m}{s} = \frac{\delta}{g}$$
$$\frac{s}{m} = \frac{g}{\delta}$$

Natural frequency $= f_n = \frac{\omega_n}{2\pi}$

$\boxed{f_n = \frac{1}{2\pi}\sqrt{\frac{s}{m}}}$

Time period $(t_p) = \frac{1}{f_n}$

$= \frac{1}{f_n} = \frac{1}{2\pi\sqrt{s/m}}$

$f_n = \frac{1}{2\pi}\sqrt{\frac{m}{s}}$

$\boxed{f_n = \frac{1}{2\pi}\sqrt{\frac{g}{\delta}}}$

Natural frequency of free transverse vibrations for shafts & beams

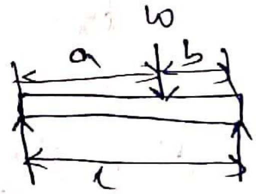
1) Shaft carrying single concentrated load

In case of shafts & beams of negligible mass carrying a concentrated mass, the force is $\propto$ to the deflection of mass from equilibrium position and relation derived for natural frequency holds good

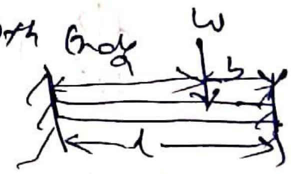
$$\therefore f_n = \frac{1}{2\pi}\sqrt{\frac{g}{\delta}}$$

where $\delta = \dfrac{WL^3}{3EI}$ for cantilever supporting a concentrated mass at free end.

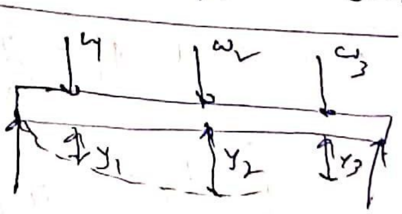
$\delta = \dfrac{Wa^2b^2}{3EIL}$ for simply supported beam



$\delta = \dfrac{Wa^3b^3}{3EIl^3}$ for beam fixed at both ends



Natural frequency of free transverse vibration for shaft carrying no. of loads



let W_1, W_2, W_3 point loads

y_1, y_2, y_3 total deflection under loads W_1, W_2, W_3

$$\text{Max } PE = \frac{1}{2}W_1y_1 + \frac{1}{2}W_2y_2 + \frac{1}{2}W_3y_3 = \frac{1}{2}\Sigma Wy \quad (1) \qquad m = \frac{W}{g}$$

$$\text{Max } KE = \frac{1}{2}\frac{W_1}{g}(\omega y_1)^2 + \frac{1}{2}\frac{W_2}{g}(\omega y_2)^2 + \frac{1}{2}\frac{W_3}{g}(\omega y_3)^2$$

$$= \frac{\omega^2}{2g}\left[W_1y_1^2 + W_2y_2^2 + W_3y_3^2\right] = \frac{\omega^2}{2g}\Sigma Wy^2 \quad (2)$$

from 1 & 2

$$\frac{\omega^2}{2g}\Sigma Wy^2 = \frac{1}{2}\Sigma Wy \Rightarrow \omega^2 = g\,\Sigma Wy / \Sigma Wy^2$$

Tor

v_f + Carrying several loads

There are two methods to find the Natural frequency of System 1) Energy or (Rayleigh's) Method
2) Dunkerley's method.

Dunkerley's method

Let $W_1\ W_2\ W_3 \ldots$ concentrated loads on shaft due to Masses $m_1\ m_2\ m_3 \ldots$, $\delta_1\ \delta_2\ \delta_3 \ldots$ static deflection of shaft under each load.

Let shaft carry a U.D.L/m length over its whole span. Static deflection at mid span due to load of this mass is δ_S.

Also let f_n = Natural frequency of transverse vibration of whole system

f_{ns} = Natural frequency with distributed load acting alone (δ) (due to mass of shaft)

f_{n1}, f_{n2}, f_{n3} = Natural frequency of transverse vibration when each $W_1\ W_2\ W_3$ act alone

∴ As per Dunkerley's formula

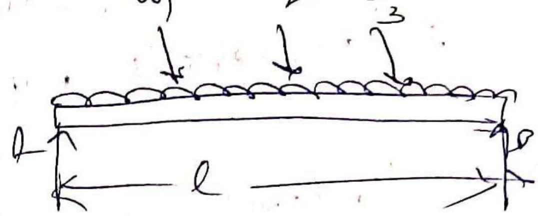
$$\frac{1}{(f_n)^2} = \frac{1}{(f_{n1})^2} + \frac{1}{(f_{n2})^2} + \frac{1}{(f_{n3})^2} + \cdots \frac{1}{(f_{ns})^2}$$

Scanned by CamScanner

$$f_{n1} = \frac{1}{2\pi}\sqrt{\frac{g}{\delta_1}} = \frac{0.4985}{\sqrt{\delta_1}}$$

$$f_{n2} = \frac{1}{2\pi}\sqrt{g/\delta_2} = \frac{0.4985}{\sqrt{\delta_2}}$$

$$f_{n3} = \frac{1}{2\pi}\sqrt{g/\delta_3} = 0.4985/\sqrt{\delta_3}$$



$$f_{ns} = \frac{0.5614}{\sqrt{\delta_s}} \qquad \delta_s = \frac{5\,wL^4}{384\,EI}$$

$$\therefore \frac{1}{f_n^2} = \frac{1}{(0.4985)^2}[\delta_1 + \delta_2 + \delta_3 + \cdots) + \frac{1}{(0.5614)^2}\delta_s.$$

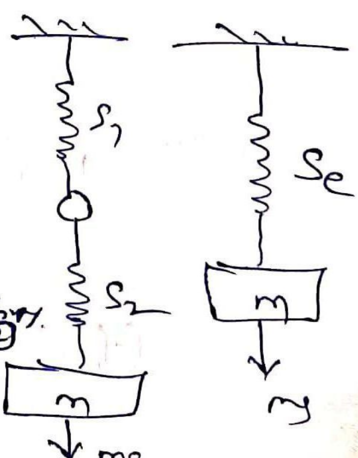
$$\boxed{f_n = \frac{0.4985}{\sqrt{\delta_1 + \delta_2 + \delta_3 + \cdots + \frac{\delta_s}{1.27}}}}$$

Equivalent system of combination of springs

⇒ The following are two types of equivalent system of combination of springs

1) Springs in series 2) Spring in parallel

Springs in series



Let Se is stiffness of equivalent spring Se

deflection of equivalent spring = deflection of spring (1) + deflection of spring (2)

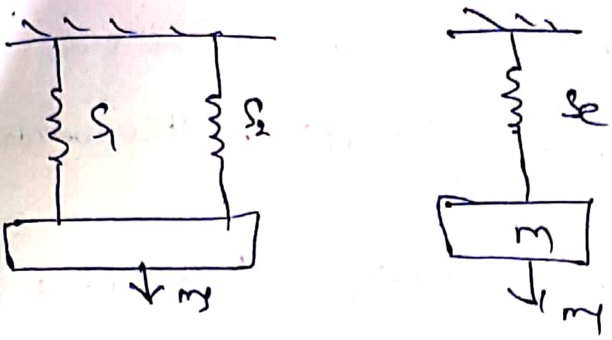
$$\delta = \delta_1 + \delta_2$$

But $mg = Se\,\delta$ | $\delta_1 = \frac{mg}{S_1}$ | $\frac{mg}{Se} = \frac{mg}{S_1} + \frac{mg}{S_2}$

$\therefore \delta = \frac{mg}{Se}$ | $\delta_2 = \frac{mg}{S_2}$ | $\frac{1}{Se} = \frac{1}{S_1} + \frac{1}{S_2}$

$$\boxed{S = \frac{S_1 S_2}{S_1 + S_2}}$$

Springs in Parallel:



$$mg = m_1 g + m_2 g$$

$$S_e \delta = S_1 \delta + S_2 \delta$$

$$S_e = S_1 + S_2 \qquad S_e = \frac{S_1 S_2}{S_1 + S_2}$$

Whirling of Rotating Shafts

→ An interesting feature of rotating shaft is that they tend to bow out at certain speeds and whirl in complicated manner. The rotational speed at which shaft tends to bow is called whirling or critical or whipping speed of shaft.

→ Critical speed occurs when the speed of shaft is equals to natural frequency of lateral vibration of shafts at this speed shaft start vibrate violently in transverse direction.

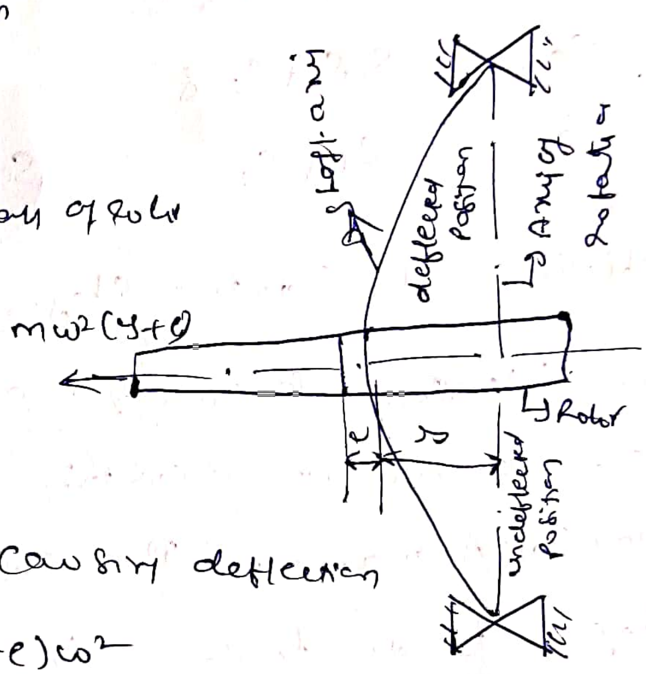
→ Whirling motion is defined as rotation of the plane created by shaft bend.

→ Whirling of shaft can take place in same or opposite direction to the rotating shaft.

→ Whirling speed may or may not equal to rotation

→ The excessive vibration associated with critical speed may cause permanent deformation resulting in structural dam.

Consider a shaft supported by two bearings & carrying a rotor at the middle of shaft



Let m = Mass of rotor

e = Initial eccentricity of center of mass of rotor

s = Stiffness of shaft

y = deflection of rotor due to C.F.

ω = angular velocity of shaft

∴ Cf acting radially outwards causing deflection is given by $Cf = m(y+e)\omega^2$

force resisting the deflection $= S \cdot y$

for equilibrium

$$S \cdot y = m(y+e)\omega^2$$

$$Sy - my\omega^2 = m\omega^2 e$$

$$y(s - m\omega^2) = m\omega^2 e$$

$$y = \frac{m\omega^2 e}{s - m\omega^2} = \frac{e}{\frac{s}{m\omega^2} - 1}$$

where $\omega_n = \sqrt{\frac{s}{m}}$ circular frequency

$$y = \frac{e}{\left(\frac{\omega_n}{\omega}\right)^2 - 1}$$

$$y = \frac{\omega^2 e}{\omega_n^2 - \omega^2}$$

vibration ... when $\omega_c = \omega_n$, the deflection y is infinite

speed ... and speed ω_c is called critical speed or whirling speed

random ...

$$\omega_c = \omega_n = \sqrt{s/m}$$

$$= \sqrt{\frac{s \cdot g}{W}} \quad ; \quad W = s \cdot \delta, \quad s = \frac{W}{\delta}$$

$$\omega_c = \omega_n = \sqrt{\frac{g}{\delta}}$$

when $\omega > \omega_n$ & $\left(\frac{\omega_n}{\omega}\right)^2 < 1$ the 'y' is −ve

This means that shaft deflects in opposite direction

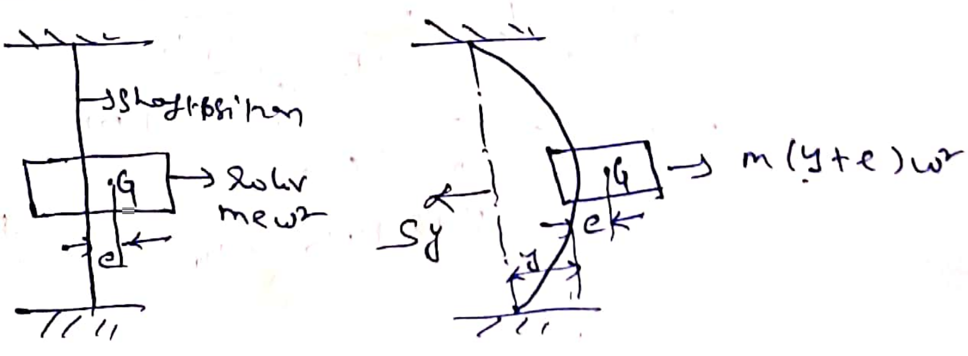
If c.g of rotor lies between centre line of shaft & centre line of bearing e is taken as −ve otherwise e is +ve.

let N_c critical whirling speed in rps

$$\therefore 2\pi N_c = \sqrt{g/\delta}$$

$$N_c = \frac{1}{2\pi}\sqrt{\frac{g}{\delta}}$$

$$\boxed{\therefore N_c = f_n = \frac{1}{2\pi}\sqrt{\frac{g}{\delta}}}$$ f_n = natural frequency



Scanned by CamScanner

Problems on Natural frequency of vibrations

(1) fig shows a vibrating System. Determine Equivalent Spring stiffness and natural frequency of system when

a) when mass is suspended to a spring.

b) " " " at bottom of two springs

c) " " " fixed between two springs

d) " " fixed midpoint of a spring.

Sol: $S_1 = 5N/mm$ $S_2 = 8N/mm$ $m = 10kg$ Take

a)

Natural frequency $f_n = \frac{1}{2\pi}\sqrt{S/m}$

$$= \frac{1}{2\pi}\sqrt{\frac{S_1}{m}} = \frac{1}{2\pi}\sqrt{\frac{5\times10^3}{10}} \quad \frac{N/m}{kg}$$

$= 3.56 Hz.$

b)

Here in both springs the spring force is same but deflection is different

Spring in series

$$\frac{mg}{S} = \frac{mg}{S_1} + \frac{mg}{S_2} \Rightarrow S = \frac{S_1 S_2}{S_1 + S_2}$$

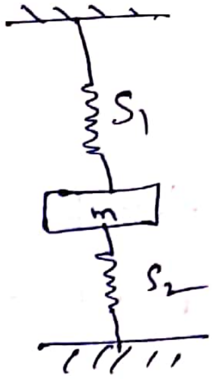
$$f_n = \frac{1}{2\pi}\sqrt{S/m} = \frac{1}{2\pi}\sqrt{\frac{S_1 S_2}{(S_1+S_2)\,m}}$$

$$= \frac{1}{2\pi}\sqrt{\frac{(5\times10^3)(8\times10^3)}{(5+8)\times10^3\times10}} = 2.79 Hz$$

Scanned by CamScanner

9

are spring forces are different but deflection are same in both springs & mass is 'm'



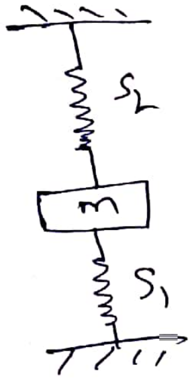
δ = deflection of spring & of the mass 'm'

$\delta_S = \delta S_1 + \delta S_2$

$S = S_1 + S_2 = 5 + 8 = 13\ N/mm$

$f_n = \frac{1}{2\pi}\sqrt{S/m} = \frac{1}{2\pi}\sqrt{\frac{13\times10^3}{10}} = 5.74\ Hz.$

d)



Stiffness of spring on each side

$\frac{S_1}{\frac{1}{2}} = 2S_1, \quad \frac{S_2}{1/2} = 2S_2$

Total stiffness $S = 2S_1 + 2S_2 = 4S_1$

$f_n = \frac{1}{2\pi}\sqrt{\frac{4S_1}{m}} = \frac{1}{2\pi}\sqrt{\frac{4\times5\times10^3}{10}} = 7.12\ Hz$

(2) find Natural frequency of a cantilever beam if a point load 'w' acts at the end of beam & δ is static deflection.

Sol: Natural frequency $f_n = \frac{1}{2\pi}\sqrt{\frac{g}{\delta}}$

$= \frac{1}{2\pi}\sqrt{\frac{g}{wl^3/3EI}}$

$\delta = \frac{wl^3}{3EI}$ for cantilever beam

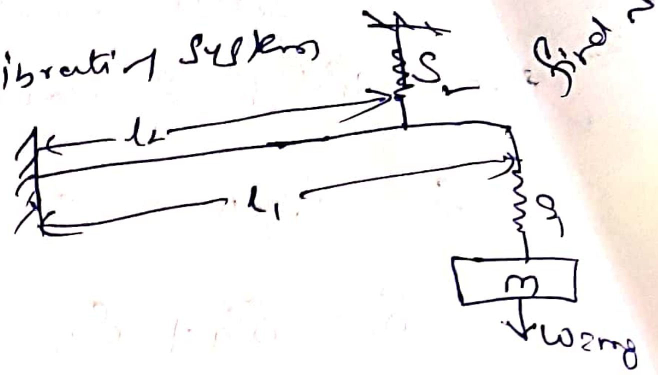
$\cancel{f_n = \frac{1}{2\pi}\sqrt{\frac{wl^3 g}{3EI}}}$

$= \frac{1}{2\pi}\sqrt{\cancel{\frac{g}{3EI/wl^3}}}$

$\boxed{f_n = \frac{1}{2\pi}\sqrt{\frac{3EI\,g}{wl^3}}}$

Scanned by CamScanner

(3) find natural frequency of a vibrating system



Sol:

force in spring (1) = $F_1 = W$

force in spring (2) = $F_2 = W \frac{l_1}{l_2}$

deflection of mass

$\delta =$ deflection of spring (1) + $\frac{l_1}{l_2}$ deflection of spring (2)

$$\delta = \frac{W}{S_1} + \frac{l_1}{l_2}\left(\frac{W \frac{l_1}{l_2}}{S_2}\right)$$

$$\delta = W\left[\frac{1}{S_1} + \frac{\left(\frac{l_1}{l_2}\right)^2}{S_2}\right]$$

$$\delta = W\left[\frac{S_2 + S_1\left(\frac{l_1}{l_2}\right)^2}{S_1 S_2}\right]$$

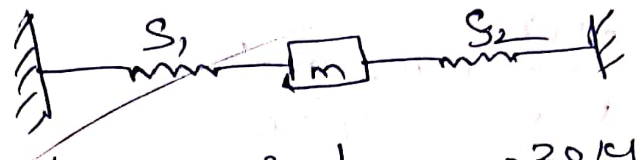
Natural frequency $f_n = \frac{1}{2\pi}\sqrt{g/\delta}$

$$= \frac{1}{2\pi}\sqrt{\frac{g}{mg\left(\frac{S_2 + S_1\left(\frac{l_1}{l_2}\right)^2}{S_1 S_2}\right)}}$$

$$\boxed{f_n = \frac{1}{2\pi}\sqrt{\frac{S_1 S_2}{\left(S_2 + S_1\left(\frac{l_1}{l_2}\right)^2\right) \times m}}}$$

Scanned by CamScanner

Find natural frequency of vibrating system



$S_1 = 20 N/m$ $S_2 = 30 N/m$ $m = 20 kg$

$$f_n = \frac{1}{2\pi}\sqrt{\frac{S}{m}}$$

$$S = S_1 + S_2 = 20 + 30 = 50 N/m$$

$$f_n = \frac{1}{2\pi}\sqrt{\frac{50}{20}} = 0.2516 \text{ Hz}$$

(3) A shaft supported freely at the ends has mass 120 kg placed 250 mm from one end. Determine natural transverse vibration, if the length of shaft is 700 mm. Take $E = 200 \text{ GN/m}^2$, dia of shaft = 40 mm

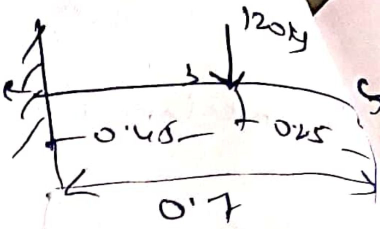
Sol: $m = 120 kg$. $a = 0.25 m$ $b = 0.7 - 0.25 = 0.45 m$

∵ length of shaft = 0.7 m

$l = 0.7 m$ $E = 200 \times 10^9 N/m^2$ $d = 0.04 m$

Moment of Inertia $I = \pi/64 \, d^4$

$$= \pi/64 (0.04)^4 = 0.1256 \times 10^{-6} m^4$$



Static deflection at load point

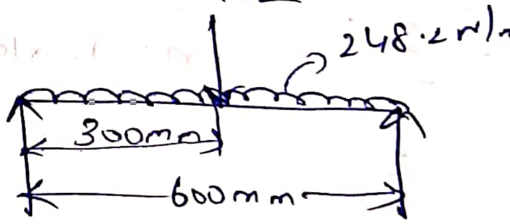
$$\delta = \frac{W a^2 b^2}{3EIL} = \frac{mg\, a^2 b^2}{3EIL}$$

$$= \frac{120 \times 120 \times 9.81 \times 0.25^2 \times 0.45^2}{3 \times 200 \times 10^9 \times 0.1256 \times 10^{-6} \times 0.7} = 0.282 \times 10^{-3}\ m$$

$$f_n = \frac{1}{2\pi}\sqrt{g/\delta} = \frac{1}{2\pi}\sqrt{\frac{9.81}{0.282 \times 10^{-3}}} = 29.68\ Hz.$$

Whirling speed of shaft

(6) A shaft is simply supported at the ends & is 20mm dia & 600mm in length. The shaft carries a load of 19.62 N at its center. weight of shaft per meter length is 248.2 N find critical speed of shaft. Take E = 200 GN/m²



Sol:

Dia (d) = 20 mm = 0.02m

Length (L) = 600mm = 0.6m

W (load at center) = 19.62 N

weight of shaft / m length = 248.2 N

E = 200 × 10⁹ N/m²

frequency of transverse vibration for shaft carrying several loads is given as

$$f_n = \frac{0.4985}{\sqrt{\delta_1 + \delta_2 + \cdots \frac{\delta_s}{1.27}}}$$

for given problem $f_n = \dfrac{0.4985}{\sqrt{\delta_1 + \frac{\delta_s}{1.27}}}$

$\delta_1 =$ Static deflection due to point load at center

$$\delta_1 = \frac{Wl^3}{48EI} \quad (\text{for a SSB with point load } \{ W = mg)$$

$$I = \pi/64\, d^4 = \pi/64\,(0.02)^4 = 7.855 \times 10^{-9}\, m^4$$

$$\delta_1 = \frac{19.62 \times 0.6^3}{48 \times (200 \times 10^9) \times 7.855 \times 10^{-9}} = 0.000056\, m$$

$\delta_s =$ Static deflection at the center due to weight of shaft

$$\delta_s = \frac{5}{384}\frac{wl^4}{EI} \quad (\text{for SSB with UDL})$$

$$= \frac{5}{384} \times \frac{248.2 \times 0.6^4}{200 \times 10^9 \times 7.855 \times 10^{-9}} = 0.000266\, m$$

$$f_n = \frac{0.4985}{\sqrt{0.000058 + \frac{0.000266}{1.27}}} = 30.6\, Hz$$

Critical speed of shaft in rps = frequency of trans. vibration.

$N_c = f_n = 30.6$ rps.

$= 30.6 \times 60$ rpm $= 1836$ rpm

$$\boxed{N_c = 1836 \text{ rpm}}$$

(7) Calculate whirling speed of shaft 20mm dia & 0.6m length carrying a mass of 1 kg at its mid point. ρ of shaft material is 40000 kg/m³ & E = 200GPa. Shaft is freely supported.

Sol: $m = 1kg$ $d = 0.02m$ $\rho = 40,000\ kg/m^3$ $E = 200 \times 10^9\ N/m^2$

Weight of shaft $(\rho) = \frac{m}{V}$

$W = m \times g$

$m = \rho \times V$

$\frac{W}{g} = \rho \times V \Rightarrow W = \rho \times V \times g$ $V = a \times l$

$= 40,000 \times \pi/4\ (0.02)^2 \times 0.6 \times 9.81$

$= 73.98\ N$

Static deflection $\delta = \frac{Wl^3}{48EI}$ $I = \pi/64\ d^4 = \pi/64\ (0.02)^4$

$= 7.85 \times 10^{-9}\ m^4$

$\delta = \frac{73.96 \times 0.6^3}{48 \times 200 \times 10^9 \times 7.85 \times 10^{-9}} = \frac{15.97}{75360} = 2.11 \times 10^{-4}$

$\omega_c = \sqrt{\frac{g}{\delta}} = \sqrt{\frac{9.81}{2.11 \times 10^{-4}}} = 215.1$ rad/sec

$\omega = \frac{2\pi N}{60}$

$\omega_c = 2054$ rpm

(12)

A shaft of dia 10mm carries at its centre mass of 12kg it is supported by two bearing, the centre distance is 400mm find whirling speed $\rho = 7500 kg/m^3$

1) considered mass of shaft 2) neglect mass of shaft

Sol:

$d = 10mm = 0.01m$ $l = 400mm = 0.4m$ $W = 12 \times 9.81$ N $= 118$ N

$\rho = 7500 kg/m^3$ $\rho = \frac{m}{V}$

Mass of shaft / m length $m_s = \rho \times V$

$= \rho \times a \times l$

$= 7500 \times \pi/4 (0.01)^2 \times 1$

$m_s = 0.589$ kg/m $w_s = 5.77$ kg.

Moment of Inertia $I = \pi/64 d^4 = \pi/64 (0.01)^4$

$= 4.9 \times 10^{-10} m^4$

Static deflection of load 12 kg at centre

$$\delta = \frac{Wl^3}{48EI} = \frac{12 \times 9.81 \times (0.4)^3}{48 \times 200 \times 10^9 \times 4.9 \times 10^{-10}} = 1.6 \times 10^{-3} m$$

Static deflection due to weight of shaft

$$\delta_s = \frac{5 w l^4}{384 EI} = \frac{5 \times 0.589 \times 9.81 \times 0.4^4}{384 \times 200 \times 10^9 \times 4.9 \times 10^{-10}}$$

$$= \frac{0.7395}{37632} = 1.96 \times 10^{-5} m$$

Considering mass of shaft

$$f_n = \frac{4.985}{\sqrt{\delta + \frac{\delta_s}{1.27}}}$$

$$= \frac{4.985}{\sqrt{1.6\times10^{-3} + \frac{1.96\times10^{-5}}{1.27}}} = 124 \text{ cycles/sec}$$

Whirling speed of shaft N_c = 124 r.p.s = 124×60

$$\boxed{N_c = 7440 \text{ rpm}}$$

2) Neglect mass of shaft

$$f_n = \frac{4.985}{\sqrt{\delta}} = \frac{4.985}{\sqrt{1.6\times10^{-3}}} = 124.6 \text{ cycle/sec}$$

N_c = 124.6 rps

= 124.6×60 rpm

$$\boxed{N_c = 7447 \text{ rpm}}$$

[D]amped Vibrations

If damper is provided in vibrating system, the motion of system will be opposed by it & energy of system will be dissipated in friction is called Damped vibrations.

Differential Equation of Damped vibration

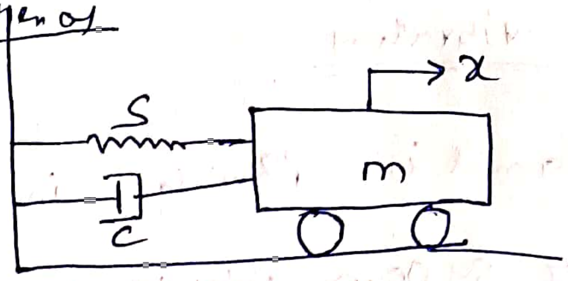
Following are the differential Eq of damped vibration

1) Vibrations with viscous damping | Single degree of freedom with viscous damping

2) Critical damping constant & damping ratio.

Vibrations with viscous damping | Single degree of freedom

Various forces can be written as



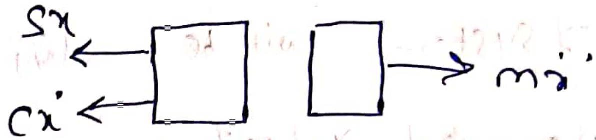
1) Damping force

$= c\frac{dx}{dt}$

$= c\dot{x}$

2) Acceleration force

$= m\frac{d^2x}{dt^2} = m\ddot{x}$



FBD

3) Spring force $= Sx$.

Eq of motion can be written as

$$m\ddot{x} + c\dot{x} + Sx = 0$$

This is called characteristic eq of system. This is differential eq of second order in x

Assuming a solution of the form

$x = e^{kt}$ where k is constant to be determined

$\therefore \dot{x} = k e^{kt}$

$\ddot{x} = k^2 e^{kt}$

$$m k^2 e^{kt} + c k e^{kt} + S e^{kt} = 0$$

$$k^2 + \frac{ck}{m} + \frac{S}{m} = 0.$$ on solving

$$k = \frac{-\frac{c}{m} \pm \sqrt{\frac{c^2}{m^2} - 4\times 1\times\frac{S}{m}}}{2}$$

$$K = \frac{-c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \left(\frac{s}{m}\right)}$$

$$K_1 = \frac{-c}{2m} + \sqrt{\left(\frac{c}{2m}\right)^2 - \left(\frac{s}{m}\right)}$$

$$K_2 = \frac{-c}{2m} - \sqrt{\left(\frac{c}{2m}\right)^2 - \left(\frac{s}{m}\right)}$$

Now the solution of eqn

$m\ddot{x} + c\dot{x} + sx = 0$ can be written as

$$x = c_1 e^{u_1 t} + c_2 e^{u_2 t} = c_1 e^{k_1 t} + c_2 e^{k_2 t}$$

where c_1 c_2 are two arbitrary constants

K_1 K_2 are two roots

$$\boxed{\therefore x = c_1 e^{\left[\frac{c}{2m} + \sqrt{\left(\frac{c}{2m}\right)^2 - \left(\frac{s}{m}\right)}\right]t} + c_2 e^{\left[\frac{c}{2m} - \sqrt{\left(\frac{c}{2m}\right)^2 - \left(\frac{s}{m}\right)}\right]t}}$$

Critical damping constant & Damping ratio (C_c)

The critical damping (C_c) is defined as the value of damping coeff (c) for which the mathematical term of

$$\left(\frac{c}{2m}\right)^2 - \left(\frac{s}{m}\right) = 0$$

$$\left(\frac{C_c}{2m}\right)^2 - \left(\frac{s}{m}\right) = 0$$

$$\frac{C_c}{2m} = \sqrt{\frac{s}{m}} \Rightarrow C_c = 2m\sqrt{\frac{s}{m}}$$

$$C_c = 2\,m\,\omega_n \Rightarrow \frac{C_c}{2m} = \omega_n \quad (1)$$

The ratio of $\frac{c}{C_c}$ is called damping ratio & it is denoted by ε

$$\therefore \varepsilon = \frac{c}{C_c} \quad -(2)$$

Consider the term $\frac{c}{2m}$ x, ÷ with C_c

$$\frac{c}{C_c} \times \frac{C_c}{2m} \quad \text{from eq (1) \& (2)} \quad \therefore \frac{c}{2m} = \varepsilon\omega$$

$$\frac{C_c}{C_c} \times \frac{C_c}{2m} = \varepsilon \times \omega_n$$

The solution of eq can be written as

$$x = A_1 e^{[-\varepsilon + \sqrt{\varepsilon^2 - 1}]\omega t} + A_2 e^{[\varepsilon - \sqrt{\varepsilon^2 - 1}]\omega t}$$

The nature of system depends upon the value of damping, damping ratio (ε), value of Aeng

Types of damping system

1) over damped system

2) Under damped system

3) critical damped system.

over damping system (when roots are real).

damping ratio > 1
($\xi > 1$) over damped system

When damping is present. the motion of Eqn of spring mass system is given by

$$U^2 + \frac{c}{m} U + \frac{s}{m} = 0 \quad (or) \quad K^2 + \frac{c}{m} K + \frac{s}{m} = 0$$

Solution for above differential Eqn is

$$x = A_1 e^{K_1 t} + A_2 e^{K_2 t}$$

A_1, A_2 are two arbitrary constant & can be determine from initial condition

i.e $\xi > 1$ the system is overdamped.

$$\therefore \left(\frac{c}{2m}\right)^2 - \frac{s}{m} = 0$$

$$\left(\frac{c}{2m}\right)^2 > \frac{s}{m}$$ This is case of over damping.

& roots K_1 K_2 are real but negative

$$x = A_1 e^{K_1 t} + A_2 e^{K_2 t}$$

$$x = A_1 e^{\left[\left(-\frac{c}{2m} + \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{s}{m}}\right]t} + A_2 e^{\left[\frac{-c}{2m} - \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{s}{m}}\right]t}$$

In this case of overdamping, the mass moves slowly to equilibrium & it is periodic motion.

Underdamping (when roots are complex conjugate) ($\xi <$

when $\xi < 1$ system is underdamped i.e

i.e $\left(\frac{s}{m}\right) > \left(\frac{c}{2m}\right)^2$ the roots U_1, U_2 are complex conjugate

$i = \sqrt{-1}$

$$K_1 = U_1 = -\frac{c}{2m} + i\sqrt{\left(\frac{s}{m}\right) - \left(\frac{c}{2m}\right)^2} = \left(-\xi + i\sqrt{1-\xi^2}\right)\omega_n$$

$$K_2 = U_2 = -\frac{c}{2m} - i\sqrt{\left(\frac{s}{m}\right) - \left(\frac{c}{2m}\right)^2} = \left(-\xi - i\sqrt{1-\xi^2}\right)\omega_n$$

let $a = \frac{c}{2m}$ $\quad \frac{s}{m} = \omega_n^2$

and $\sqrt{\frac{s}{m} - \left(\frac{c}{2m}\right)^2} = \omega_d = \sqrt{\omega_n^2 - a^2}$

$\therefore$ roots may be written as

$K_1 = U_1 = -a + i\omega_d$

$K_2 = U_2 = -a - i\omega_d$

$\therefore$ General solution of differential eq. $\quad A_1, A_2 = C_1, C_2$

$$x = A_1 e^{U_1 t} + A_2 e^{U_2 t}$$

$$= A_1 e^{(-a + i\omega_d)t} + A_2 e^{(-a - i\omega_d)t}$$

$$= e^{-at}\left[A_1 e^{i\omega_d t} + A_2 e^{-i\omega_d t}\right] \quad \text{---(1)}$$

According to Euler's theorem

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$e^{-i\theta} = \cos\theta - i\sin\theta$

Eq (1) may be written as

$$x = e^{-at}\left[A_1(\cos\omega_d t + i\sin\omega_d t)\right] + A_2(\cos\omega_d t - i\sin\omega_d t]$$

$$= e^{-at}\left((A_1+A_2)\cos\omega_d t + i(A_1 - A_2)\sin\omega_d t\right)$$

let $A = A_1 + A_2$

$B = i(A_1 - A_2)$

$$\therefore x = e^{-at}\left[A\cos\omega_d t + iB\sin\omega_d t\right] \quad (2)$$

initial condition
let $A = C\cos\theta$ $\quad B = C\sin\theta$ $\quad \tan\theta = \frac{B}{A}$ Eq (2) becomes

$$x = e^{-at}\left[C\cos\theta\cos\omega_d t + iC\sin\theta\sin\omega_d t\right]$$

$$x = C\cdot e^{-at}\cos(\omega_d t - \theta).$$

from initial condition when $\theta = 0$ $A = C$

Eq (2) written as

$$x = e^{-at}\left[A\cos\omega_d t + iB\sin(0)\right]$$

$$x = Ae^{-at}\cos\omega_d t$$

where $\omega_d = \sqrt{\left(\frac{s}{m}\right) - \left(\frac{c}{2m}\right)^2} = \sqrt{\omega_n^2 - a^2}$

$$= \sqrt{\omega_n^2 - (\varepsilon \omega_n)^2}$$

$$= \omega_n \sqrt{1 - \varepsilon^2}$$

∴ Time period $t_p = \frac{2\pi}{\omega}$

$$= \frac{2\pi}{\sqrt{\left(\frac{s}{m}\right) - \left(\frac{c}{2m}\right)^2}}$$

$$= \frac{2\pi}{\sqrt{\omega_n^2 - (\varepsilon \omega_n)^2}}$$

$$T_p = \frac{2\pi}{\omega_n \sqrt{1 - \varepsilon^2}}$$

Frequency $f_n = \frac{1}{T_p} = \frac{\omega_d}{2\pi}$

$$= \frac{1}{2\pi} \sqrt{\left(\frac{s}{m}\right) - \left(\frac{c}{2m}\right)^2}$$

$$\boxed{f_n = \frac{1}{2\pi} \omega_n \sqrt{1 - \varepsilon^2}}$$

ritical damping :- (when roots are equal)

When $\xi = 1$. The system is critically damped i.e

if $\left(\frac{c}{2m}\right)^2 = \frac{s}{m}$ Then two roots u_1 & u_2 are equal

$\therefore$ frequency of damped vibration is zero

for critical damping

$$x = A_1 e^{u_1 t} + A_2 e^{u_2 t}$$

$$\boxed{x = (A_1 + A_2) e^{-\frac{c}{2m} t}}$$

Above eqn of motion is aperiodic.

critical damping coeff under critical damping conditions

$$\frac{c_c}{2m} = \sqrt{\frac{s}{m}}$$

$$c_c = 2m \sqrt{\frac{s}{m}} = 2\sqrt{\frac{sm^2}{m}}$$

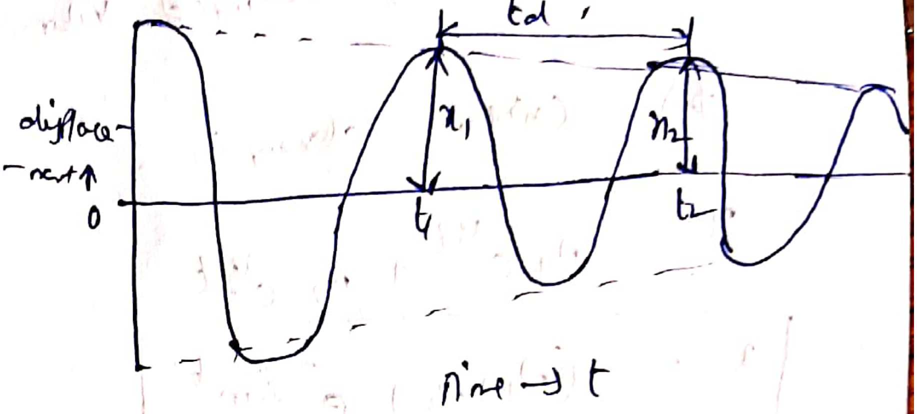
$$c_c = \cancel{2m\sqrt{sm}} = 2m\sqrt{\frac{s}{m}}$$

$$\boxed{c_c = 2m\omega_n}$$ critical damping coeff

Logarithmic decrement

Logarithmic decrement is defined as the natural logarithm of any two successive amplitudes on the same side of the mean position in an under damped system.

It is denoted by δ.



∴ Max amplitude = x

$$x = X e^{-\varepsilon \omega_n t}$$

∴ $x_1 = X e^{-\varepsilon \omega_n t_1}$

$x_2 = X e^{-\varepsilon \omega_n t_2}$.

Hence ratio of two successive amplitudes

$$\frac{x_1}{x_2} = \frac{X e^{-\varepsilon \omega_n t_1}}{X e^{-\varepsilon \omega_n t_2}} = \frac{e^{-\varepsilon \omega_n t_1}}{e^{-\varepsilon \omega_n t_2}}$$

$$= e^{-\varepsilon \omega_n t_1 - (-\varepsilon \omega_n t_2)}$$

$$= e^{-\varepsilon \omega_n t_1 + \varepsilon \omega_n t_2}$$

$$= e^{\varepsilon \omega_n (t_2 - t_1)}$$

where $t_2 - t_1 = T_d$ = time period $= \frac{2\pi}{\omega_d}$

∴ $\frac{x_1}{x_2} = e^{\varepsilon \omega_n \left(\frac{2\pi}{\omega_d}\right)}$ $\omega_d = \omega_n \sqrt{1-\varepsilon^2}$

∴ $\frac{x_1}{x_2} = e^{\varepsilon \omega_n \frac{2\pi}{\omega_n \sqrt{1-\varepsilon^2}}} = e^{\frac{\varepsilon 2\pi}{\sqrt{1-\varepsilon^2}}}$ $\boxed{\therefore \frac{x_1}{x_2} = e^{\frac{2\pi\varepsilon}{\sqrt{1-\varepsilon^2}}}}$

Similarly $\frac{x_1}{x_2} = \frac{x_2}{x_3} = \frac{x_3}{x_4} = \cdots = e^{\frac{2\pi\varepsilon}{\sqrt{1-\varepsilon^2}}}$

$$\ln\left(\frac{x_1}{x_2}\right) = \ln e^{\frac{2\pi\varepsilon}{\sqrt{1-\varepsilon^2}}} = \frac{2\pi\varepsilon}{\sqrt{1-\varepsilon^2}}$$

$\therefore$ logarithmic decrement $= \delta = \ln\left(\frac{x_1}{x_2}\right) = \frac{2\pi\varepsilon}{\sqrt{1-\varepsilon^2}}$.

damping factor $= \boxed{\delta = \frac{2\pi\varepsilon}{\sqrt{1-\varepsilon^2}}}$ $\quad \delta \simeq 2\pi\varepsilon$ (if ε is very small)

Problems on damped vibrations.

1) In a single degree damped vibrating system, the suspended mass of 4 kg makes 24 oscillations in 20 seconds. The amplitude decreases to 0.3 of initial value after 4 ~~seconds~~ oscillations. Find the stiffness of spring, logarithmic decrement, damping factor, damping coeff

Sol: Suspended mass $m = 4$ kg

No of oscillations in 20 seconds $= N = 24$

$$f_n = \frac{\text{No of oscillations}}{\text{time}} = \frac{24}{20} = 1.2\,Hz.$$

$$\omega_n = 2\pi f_n = 2\pi \times 1.2 = 7.54 \text{ rad/sec}$$

1) WKT stiffness of spring $\omega_n = \sqrt{\frac{s}{m}}$

$$7.54 = \sqrt{\frac{s}{4}}$$

$$s = 227.4 \text{ N/m.}$$

2) Logarithmic decrement

$$\frac{x_0}{x_4} = \frac{x_0}{x_1} \times \frac{x_1}{x_2} \times \frac{x_2}{x_3} \times \frac{x_3}{x_4}$$

$$= \left(\frac{x_0}{x_1}\right)^4 = \frac{x_0}{x_1} = \frac{x_1}{x_2} = \frac{x_2}{x_3} = \frac{x_3}{x_4}$$

$$\frac{x_0}{x_1} = \left(\frac{x_0}{x_4}\right)^{\frac{1}{4}} = \left(\frac{1}{0.3}\right)^{\frac{1}{4}}$$

$$= 1.35$$

$\therefore$ Logarithmic decrement $= \ln\left(\frac{x_0}{x_4}\right)$

$$= \ln(1.35) = \delta = 0.300$$

3) Damping factor

$$\delta = \frac{2\pi\varepsilon}{\sqrt{1-\varepsilon^2}}$$

$$\sqrt{1-\varepsilon^2} = \frac{2\pi\varepsilon}{\delta} = \frac{2\pi\varepsilon}{0.300} = 20.94\,\varepsilon$$

$$1-\varepsilon^2 = (20.94\,\varepsilon)^2$$

$$1-\varepsilon^2 = 438\,\varepsilon^2$$

$$1 = 439\,\varepsilon^2 \quad \varepsilon^2 = \frac{1}{439} = 0.0477$$

4) Damping Coeff:

$$C = 2\,m\,\omega_n\,\varepsilon$$

$$= 2 \times 4 \times 7.54 \times 0.0477$$

$$= 2.877\ N/m/s.$$

A vibrating system consist of mass of 8kg, spring stiffness 5.6N/mm & a dashpot of damping coeff 40N/m/sec find 1) Damping factor 2) Logarithmic decrement 3) Ratio of two successive amplitudes

Sol:- given $m = 8kg$ $\quad s = 5.6 N/mm = 5600 N/m$

damping coeff $c = 40 N/m/sec$

1) Damping factor

WKT $\quad \omega_n = \sqrt{s/m} = \sqrt{\frac{5600}{8}} = 26.45 \text{ rad/sec}$

Damping coeff $c = 2 m \omega_n \varepsilon$

$40 = 2 \times 8 \times 26.45 \times \varepsilon$

Damping coeff $\varepsilon = \frac{40}{2 \times 8 \times 26.45}$ $\quad \varepsilon = 0.0945$

2) Logarithmic decrement $\quad \delta = \frac{2\pi \varepsilon}{\sqrt{1-\varepsilon^2}}$

$= \frac{2\pi \times 0.0945}{\sqrt{1-0.0945^2}}$ $\quad \delta = 0.6$

3) Ratio of two consecutive Amplitudes

$\frac{x_n}{x_{n+1}} = e^{\delta}$

$= e^{0.6}$

$= 1.822$

(3) A Mass of 5 kg is suspended on a spring of stiffness 4000 N/m. The system is fitted with a damper with damping ratio 0.2. The mass is pulled down 50 mm & released. Find 1) damped frequency
2) displacement, velocity, acceleration after 0.3 sec

Sol: Mass $m = 5$ kg $S = 4000$ N/m $\xi = 0.2$ (damping ratio)

1) Damping frequency

WKT $f_n = \frac{\omega_n}{2\pi} = \frac{1}{2\pi}\sqrt{S/m} = \frac{1}{2\pi}\sqrt{\frac{4000}{5}} = 4.5$ Hz

$\omega =$ ∴ Damped frequency $f = \omega_n\sqrt{1-\xi^2}$

$= \cancel{400\sqrt{1-0.2}}$

$= \sqrt{\frac{S}{m}}\sqrt{1-\xi^2} = \sqrt{\frac{4000}{5}} \times \sqrt{1-0.2^2}$

$\omega = f = 27.71$ rad/sec

2) A ~~Mass of 5 kg~~

Displacement, velocity, acceleration after 0.3 sec

initial displacement is down 50 mm

∴ $x = -50$ mm $x = X e^{-\xi\omega_n t}\cos\theta$ $\theta = \omega t$

∴ $x = X e^{-\xi\omega_n t}\cos(\omega t)$

$= 50\, e^{-0.2 \times 28.28 \times 0.3} \cos(27.7 \times t)$

$v = \frac{dx}{dt} = -204.7$ mm/s

$a = \frac{dv}{dt} = 5554$ mm/s²

(28)

A mass of 5 kg is suspended on a spring & set oscillating. It is observed that the amplitude reduces to 5% of its initial value after 2 oscillations. It takes 0.5 seconds to do this. Find 1) damping ratio 2) natural frequency 3) actual frequency 4) spring stiffness 5) critical damping coeff 6) actual damping coeff.

Sol: Given $m = 5$ kg

1) Damping ratio: for two oscillations

$$\frac{x_1}{x_2} = \frac{1}{5\%} = \frac{1}{5/100} = \frac{100}{5} = 20$$

$$\ln\left(\frac{x_1}{x_2}\right) = \frac{2\pi\varepsilon}{\sqrt{1-\varepsilon^2}}$$

$$\ln(20) = \frac{2\pi\varepsilon}{\sqrt{1-\varepsilon^2}} \Rightarrow \varepsilon = 0.232$$

2) Natural frequency $f_n = \frac{\text{No of oscillations}}{\text{time}} = \frac{2}{0.5} = 4\,Hz$

3) Damped frequency $f = f_n\sqrt{1-\varepsilon^2}$

$$= 4\sqrt{1-(0.232)^2} = 3.885\,Hz$$

4) Stiffness of spring $S = ?$

$$f_n = \frac{1}{2\pi}\sqrt{S/m}$$

$$3.885 = \frac{1}{2\pi}\sqrt{\frac{S}{5}}$$

$S = 3158$ N/m

5) Critical damping coeff (C_c)

$$C_c = 2m\omega_n = 2m\sqrt{\frac{S}{m}}$$

$$= 2\times 5\sqrt{\frac{3158}{5}}$$

$= 251.3$ N s/m

6) Actual damping coeff

$$\varepsilon = \frac{C}{C_c}$$

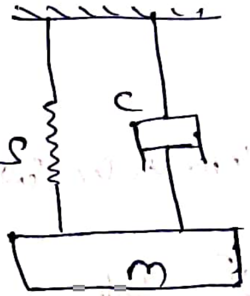
$$0.232 = \frac{C}{251.3}$$

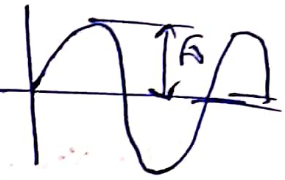
$C = 58.3$ N s/m

Forced vibrations

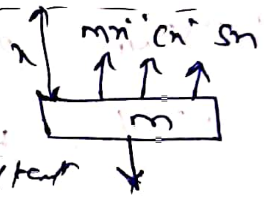
The vibration takes place under the excitation of external force are called forced vibration.

Forced vibration of single degree of freedom system under Harmonic motion





Consider a spring mass system having a viscous damping excited by a harmonic force $F = F_0 \sin\omega t$ as shown in fig.



The different forces acting on body at any instant are given as

1) Oscillating force $F = F_0 \sin\omega t$ (downwards)
2) Inertia force $= m\ddot{x}$ (upwards)
3) Damping force $= c\dot{x}$ (↑)
4) Spring force $= Sx$ (↑)

From Newton 2nd law of motion

$$m\ddot{x} + c\dot{x} + Sx = F_0 \sin\omega t \quad \text{— (1)}$$

Eq (1) is differential eq of motion, the solution of this eq consists two parts: complementary function (CF) & particular integral (PI)

The first solution is transient solution & can be obtained from differential eq $m\ddot{x}+c\dot{x}+sx=0$

Second solution is steady state solution & is of form

$x = X\sin(\omega t - \phi)$

X = steady state amplitude ϕ is phase diff between external force & displacement.

Solution of complementary part $CF = Xe^{-\xi\omega_n t}\sin(\omega_d t+\phi_1)$

To obtain P.I let

$a = c/m$ $b = s/m$ $d = \frac{f_0}{m}$

$(D^2+aD+b)x = \sin\omega t$

$$P.I = \frac{d\sin\omega t}{D^2+aD+b}$$

$$= \frac{d\sin\omega t}{-\omega^2+aD+b}$$

$$= \frac{1}{(b-\omega^2)+aD} \times \frac{b-\omega^2-aD}{b-\omega^2-aD}\, d\sin\omega t$$

$$P.I = d\left[\frac{\sin\omega t(b-\omega^2) - a\omega\cos\omega t}{(b-\omega^2)^2+(a\omega)^2}\right]$$

Assume $b-\omega^2 = R\cos\phi$

$a\omega = R\sin\phi$

Resultant $R = \sqrt{(b-\omega^2)^2 + (a\omega)^2}$ R, ϕ are constant

$\phi = \tan^{-1}\left(\frac{a}{b-\omega^2}\right)$

$\therefore PI = \frac{dR(\sin\omega t \cos\phi - \cos\omega t \sin\phi)}{(b-\omega^2)^2 + (a\omega)^2}$

$= \frac{d\sqrt{(b-\omega^2)^2 + (a\omega)^2}}{\sqrt{(b-\omega^2)^2 + (a\omega)^2}} \sin(\omega t - \phi)$

$= \frac{d}{\sqrt{(b-\omega^2)^2 + (a\omega)^2}} \sin(\omega t - \phi)$

$= \frac{F_0/m}{\sqrt{\left(\frac{S}{m} - \omega^2\right)^2 + \left(\frac{c}{m}\omega\right)^2}} \sin(\omega t - \phi)$

$PI = \frac{F_0}{\sqrt{(S - m\omega^2)^2 + (c\omega)^2}} \sin(\omega t - \phi)$

$\therefore x = CF + PI$

$= x e^{-\xi\omega_n t} \sin(\omega_d t - \phi_1) + \frac{F_0}{\sqrt{(S - m\omega^2)^2 + (c\omega)^2}} \sin(\omega t)$

In above Eq the damped vibration represented by (CF), becomes negligible $e^{-\infty} = 0$

$\therefore$ The amplitude of steady state response is given by

$$\boxed{A = \frac{F_0}{(S - m\omega^2)^2 + (C\omega)^2}}$$

$$= \frac{F_0/S}{\sqrt{\left(1 - \frac{m\omega^2}{S}\right)^2 + \left(\frac{C}{S}\omega\right)^2}} = \frac{F_0/S}{\sqrt{\left(1 - \frac{\omega^2}{S/m}\right)^2 + \left(2\xi \frac{\omega}{\omega_n}\right)^2}}$$

$$A = \frac{F_0/S}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left(2\xi \frac{\omega}{\omega_n}\right)^2}}$$

$\tan\phi = \frac{a\omega}{b - \omega^2}$

$$\therefore \tan\phi = \frac{a\omega}{b - \omega^2} = \frac{\frac{c}{m}\omega}{\frac{S}{m} - \omega^2} = \frac{C\omega}{S - m\omega^2}$$

$$\therefore \tan\phi = \frac{2\xi\, \omega/\omega_n}{1 - \left(\frac{\omega}{\omega_n}\right)^2}$$

Max displacement

$$x_{max} = \frac{F/S}{\sqrt{\frac{C^2\omega^2}{k^2} + \left[1 - \frac{m\omega^2}{k}\right]}} = \frac{x_0}{\sqrt{\frac{C^2\omega^2}{S^2} + \left(1 - \frac{\omega^2}{\omega_n^2}\right)^2}}$$

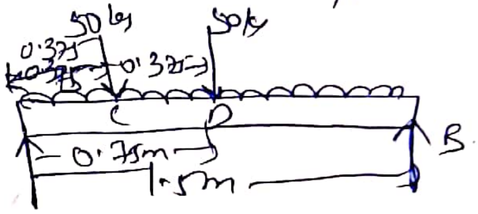
Scanned by CamScanner

1) A shaft 1.5 m long supported in flexible bearings at the ends carries two wheels each of 50 kg mass. One wheel at centre of shaft & other at a distance of 375 mm from the centre towards left. The shaft is hollow of OD 75 mm & ID 40 mm. $\rho = 7700\ kg/m^3$, $E = 200\ GN/m^2$. Find the lowest whirling speed of shaft taking into account the mass of shaft.

Sol: $l = 1.5\ m$ $m_1 = m_2 = 50\ kg$ $d_1 = 75\ mm = 0.075\ m$

$d_2 = 40\ mm = 0.04\ m$ $\rho = 7700\ kg/m^3$

$E = 200\ GN/m^2$



WKT $I = \frac{\pi}{64}(d_1^4 - d_2^4) = 1.4\times10^{-6}\ m^4$

∴ Mass of shaft / m length $m_s = A \times l \times \rho$

$= \frac{\pi}{4}(0.075^2 - 0.04^2)\,1 \times 7700$

$= 24.34\ kg/m$

WKT δ due to load W

$$= \frac{W a^2 b^2}{3EIl} = \frac{m g a^2 b^2}{3EIl}$$

∴ Static deflection due to mass of 50 kg at C

$$\delta_1 = \frac{m_1 g a^2 b^2}{3EIl} = \frac{50\times9.81\times0.375^2(1.125)^2}{3\times200\times10^9\times1.4\times10^{-6}\times1.5} = 70\times10^{-6}\ m$$

δ_2 due to mass 50 kg at D

$$\delta_2 = \frac{m_2 g a^2 b^2}{3EIl} = 50\times9.81\times(0.75)^2(0.75)^2/3\times200\times10^9\times1.4\times10^{-6}\times1.5$$

δ_s = deflection due to UDL of mass of shaft

$$\delta_s = \frac{5}{384} \times \frac{wl^4}{EI} = \qquad w = m_s \times g$$

$$= \frac{5}{384} \times \frac{24.34 \times 9.81 (1.5)^4}{200 \times 10^9 \times 1.4 \times 10^{-6}} = 56 \times 10^{-6} m$$

$$f_n = \frac{0.4985}{\sqrt{\delta_1 + \delta_2 + \frac{\delta_s}{1.27}}} = \frac{0.4985}{\sqrt{70 \times 10^{-6} + 123 \times 10^{-6} + \frac{56 \times 10^{-6}}{1.27}}} = 32.4 \text{ Hz}$$

N_c = 32.4 rps = 32.4 × 60 = 1944 rpm

∵ whirling speed of shaft (N_c) in rps = frequency of transverse vibration

(3) A vertical shaft of 5mm dia 200mm long supported at 2 ends by bearings. A disc of mass 50kg is attached at centre of shaft. Neglect any ↑ in stiffness due to disc. Find critical speed of rotation & max bending stress when shaft is rotating at 75% of N_c. The centre of disc is 0.25 mm from geometric axis of shaft $E = 200 G N/m^2$

Scanned by CamScanner