24/11/18 ~~2~~. RESIDUE THEORY & TRANSFORMATIONS.

Taylor's theorem:-

Stmt:- Let $f(z)$ be analytic at all points in a $\bigcirc$le 'C_0' with centre at 'a' & radius 'r' then $f(z) = f(a) + \frac{(z-a)}{1!}f'(a) + \frac{(z-a)^2}{2!}f''(a) + \frac{(z-a)^3}{3!}f'''(a) + \frac{(z-a)^4}{4!}f''''(a) + \cdots$ is Taylor's series about the point $z=a$ (or) in powers of $(z-a)$.

Q:-1) Expand e^z as taylor's. ~~serious~~ series about $z=1$.

Sol. Given $f(z) = e^z$ & given pt $z=1$.

Using Taylor's series expansion about the pt $z=a$ is

$$f(z) = f(a) + \frac{(z-a)}{1!}f'(a) + \frac{(z-a)^2}{2!}f''(a) + \frac{(z-a)^3}{3!}f'''(a) + \cdots \longrightarrow (1)$$

Here $a=1$.

$$\therefore (1) \Rightarrow f(z) = f(1) + \frac{(z-1)}{1!}f'(1) + \frac{(z-1)^2}{2!}f''(1) + \frac{(z-1)^3}{3!}f'''(1) + \cdots \longrightarrow (2)$$

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$\cancel{z}/_{F}$ $f(z) = e^z \Rightarrow f(1) = e^1 = e$

$f'(z) = e^z \Rightarrow f'(1) = e^1 = e$

$f''(z) = e^z \Rightarrow f''(1) = e^1 = e$

$f'''(z) = e^z \Rightarrow f'''(1) = e^1 = e$

Sub these values in eqⁿ ②.

$$f(z) = e + \frac{(z-1)}{1!}(e) + \frac{(z-1)^2}{2!}(e) + \frac{(z-1)^3}{3!}(e) + \ldots$$

$$f(z) = e\left[1 + \frac{(z-1)}{1!} + \frac{(z-1)^2}{2!} + \frac{(z-1)^3}{3!} + \ldots\right]$$

2) Expand e^z as taylor's series about the pt $z=3$.

Sol. Given $f(z) = e^z$ & $z=3$.

Using Taylor's series expansion about $z=a$,

$$f(z) = f(a) + \frac{(z-a)^1}{1!}f'(a) + \frac{(z-a)^2}{2!}f''(a) + \frac{(z-a)^3}{3!}f'''(a) + \ldots \rightarrow ①$$

Here $a=3$.

$$① \Rightarrow f(z) = f(3) + \frac{(z-3)^1}{1!}f'(3) + \frac{(z-3)^2}{2!}f''(3) + \frac{(z-3)^3}{3!}f'''(3) + \ldots \rightarrow ②$$

$f(z) = e^z \Rightarrow f(3) = e^3$

$f'(z) = e^z \rightarrow f'(3) = e^3$

$f''(z) = e^z \Rightarrow f''(3) = e^3$

$f'''(z) = e^z \Rightarrow f'''(3) = e^3$

Sub these values in eq^n 2

$$f(z) = e^3 + (z-3)e^3 + \frac{(z-3)^2}{2!}e^3 + \frac{(z-3)^3}{3!}e^3 + \cdots$$

$$f(z) = e^3\left[1 + (z-3) + \frac{(z-3)^2}{2!} + \frac{(z-3)^3}{3!} + \cdots\right]$$

3) Expand $\sin z$ about $z = \pi/4$ using Taylor's theorem. ↳ $\cos z$

Sol Given f^n $f(z) = \sin z$, $z = \pi/4$.

Using Taylor's series expansion about $z = a$

$$f(z) = f(a) + (z-a)f'(a) + \frac{(z-a)^2}{2!}f''(a) + \frac{(z-a)^3}{3!}f'''(a) + \cdots \rightarrow 1$$

Here $a = \pi/4$

$$f(z) = f(\pi/4) + (z-\pi/4)f'(\pi/4) + \frac{(z-\pi/4)^2}{2!}f''(\pi/4) + \frac{(z-\pi/4)^3}{3!}f'''(\pi/4) + \cdots \longrightarrow 2$$

$f(z) = \sin z \Rightarrow f(\pi/4) = \sin \pi/4 = 1/\sqrt{2}$

$f'(z) = \cos z \Rightarrow f'(\pi/4) = \cos \pi/4 = 1/\sqrt{2}$

$f''(z) = -\sin z \Rightarrow f''(\pi/4) = -\sin \pi/4 = -1/\sqrt{2}$.

$f'''(z) = -\cos z \Rightarrow f'''(\pi/4) = -\cos\pi/4 = -1/\sqrt{2}$

Sub these values in eqn 2.

$$f(z) = \frac{1}{\sqrt{2}} + (z-\pi/4)\frac{1}{\sqrt{2}} + \frac{(z-\pi/4)^2}{2!}\left(\frac{-1}{\sqrt{2}}\right) + \frac{(z-\pi/4)^3}{3!}\left(\frac{-1}{\sqrt{2}}\right) + \cdots$$

$$= \frac{1}{\sqrt{2}}\left[1+(z-\pi/4) - \frac{(z-\pi/4)^2}{2!} - \frac{(z-\pi/4)^3}{3!} + \cdots\right].$$

4) Expand $f(z) = \cos z$ about $z = \pi/2$, using taylor's theorem $\sin z$

Sol Given $f(z) = \cos z$, given $z = \pi/2$.

Using Taylor's theorem about the pt $z=a$ is

$$f(z) = f(a) + (z-a)f'(a) + \frac{(z-a)^2}{2!}f''(a) + \frac{(z-a)^3}{3!}f'''(a) + \cdots \rightarrow (1)$$

Here $a = \pi/2$

$$f(z) = f\left(\frac{\pi}{2}\right) + \left(z-\frac{\pi}{2}\right)f'\left(\frac{\pi}{2}\right) + \frac{\left(z-\frac{\pi}{2}\right)^2}{2!}f''\left(\frac{\pi}{2}\right) + \frac{\left(z-\frac{\pi}{2}\right)^3}{3!}f'''\left(\frac{\pi}{2}\right) + \cdots \rightarrow (2)$$

$f(z) = \cos z \Rightarrow f(\pi/2) = \cos\pi/2 = 0$

$f'(z) = -\sin z \Rightarrow f'(\pi/2) = -\sin\pi/2 = -1$

$f''(z) = -\cos z \Rightarrow f''(\pi/2) = -\cos\left(\frac{\pi}{2}\right) = 0.$

$f'''(z) = \sin z \Rightarrow f'''(\pi/2) = \sin\pi/2 = 1$

Sub in 2

$\therefore f(z) = 0 + \left(z-\frac{\pi}{2}\right)(-1) + \frac{\left(z-\frac{\pi}{2}\right)^2}{2!}(0) + \frac{\left(z-\frac{\pi}{8}\right)^2}{3!}(1) + \ldots$

$= -(z-\pi/2) + \frac{(z-\pi/2)^3}{3!} - \frac{(z-\pi/2)^5}{5!} + \ldots$

i) Expand $f(z) = \sinh z$ about $z = \pi i$ using Taylor's theorem. $\cosh z$ $\left[\begin{array}{l}\sinh \pi i = i \sin \pi \\ \cosh \pi i = \cos \pi\end{array}\right]$.

Sol. Given $f(z) = \sinh z$ & $z = \pi i$.

Using Taylor's theorem about the pt $z = a$ is.

$$f(z) = f(a) + (z-a)f'(a) + \frac{(z-a)^2}{2!}f''(a) + \frac{(z-a)^3}{3!}f'''(a) + \ldots \quad \text{—(1)}$$

Here $a = \pi i$.

$$f(z) = f(\pi i) + (z-\pi i)f'(\pi i) + \frac{(z-\pi i)^2}{2!}f''(\pi i) + \frac{(z-\pi i)^3}{3!}f'''(\pi i) + \ldots \quad \rightarrow (2)$$

$f(z) = \sinh z \Rightarrow f(\pi i) = \sinh(\pi i) = i \sin \pi = 0$

$f'(z) = \cosh z \Rightarrow f(\pi i) = \cosh(\pi i) = \cos \pi = -1.$

$f''(z) = \sinh z \Rightarrow f''(\pi i) = \sinh(\pi i) = 0$

$f'''(z) = \cosh z \Rightarrow f'''(\pi i) = \cosh(\pi i) = \cos \pi = -1.$

Sub in (2)

$$f(z) = 0 + (z-\pi i)(-1) + \frac{(z-\pi i)^2}{2}(0) + \frac{(z-\pi i)^3}{3!}(-1) + \ldots$$

$$\therefore f(z) = -(z-\pi i) - \frac{(z-\pi i)^3}{3!} - \frac{(z-\pi i)^5}{5!}$$

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6.) Expand $\log(1-z)$ when $|z|<1$, using taylor's theorem.

Sol. Given $f(z) = \log(1-z)$ & $|z|<1 \Rightarrow z=0$

Using Taylor's series expansion about $z=a$

$$f(z) = f(a) + (z-a)f'(a) + \frac{(z-a)^2}{2!}f''(a) + \frac{(z-a)^3}{3!}f'''(a) + \dots \text{---}(1)$$

Here $a=0$.

$$f(z) = f(0) + zf'(0) + \frac{z^2}{2!}f''(0) + \frac{z^3}{3!}f'''(0) + \dots \text{---}(2)$$

$f(z) = \log(1-z) \Rightarrow f(0) = \log 1 = 0$

$f'(z) = \frac{1}{1-z}(-1) \Rightarrow f'(0) = \frac{-1}{1-0} = \frac{-1}{1} = -1$

$f''(z) = -\left(\frac{-1}{(1-z)^2}\right)(-1) \Rightarrow f''(0) = \frac{-1}{(1-0)^2} = -1.$

$f'''(z) = -\frac{(-2)}{(1-z)^3}(-1) \Rightarrow f'''(0) = \frac{-2}{(1-0)^3} = -2$

Sub in (2)

$\therefore f(z) = 0 + z(-1) + \frac{z^2}{2!}(-1) + \frac{z^3}{3!}(-2)$

$$= -\left[z + \frac{z^2}{2!} + \frac{2z^3}{3!} + \dots\right].$$

7.) Expand $f(z) = \log z$ about $z=1$, using taylor's theorem.

Sol Given $f(z) = \log z$ & $z=1$.

Using Taylor's series expansion about $z=a$ is

$$f(z) = f(a) + (z-a)f'(a) + \frac{(z-a)^2}{2!}f''(a) + \frac{(z-a)^3}{3!}f'''(a) + \ldots \quad \text{(1)}$$

Here $a=1$

$$\Rightarrow f(z) = f(1) + (z-1)f'(1) + \frac{(z-1)^2}{2!}f''(1) + \frac{(z-1)^3}{3!}f'''(1) + \ldots \quad \text{(2)}$$

$f(z) = \log z \Rightarrow f(1) = \log 1 = 0$

$f'(z) = 1/z \Rightarrow f'(1) = 1/1 = 1$

$f''(z) = \frac{-1}{z^2} \Rightarrow f''(1) = \frac{-1}{1^2} = -1$

$f'''(z) = (-1)\frac{(-2)}{z^3} = \frac{2}{z^3} \Rightarrow f'''(1) = \frac{2}{1^3} = 2$

Sub in eqn (2)

$$\therefore f(z) = 0 + (z-1)(1) + \frac{(z-1)^2}{2!}(-1) + \frac{(z-1)^3}{3!}(2) + \ldots$$

$$\therefore f(z) = (z-1)\left[1 - \frac{(z-1)}{2!} + \frac{2(z-1)^2}{3!} - \ldots\right].$$

Note:-

1) $\frac{1}{1+z} = (1+z)^{-1} = 1 - z + z^2 - z^3 + z^4 - \ldots + (-1)^n z^n + \ldots$

$$= \sum_{n=0}^{\infty} (-1)^n z^n \quad \text{if } |z| < 1.$$

2.) $\frac{1}{1-z} = (1-z)^{-1} = 1 + z + z^2 + z^3 + \ldots + z^n + \ldots$

$$= \sum_{n=0}^{\infty} z^n \quad \text{if } |z| < 1.$$

Laurent's Theorem:- (5M).

Let C_1 & C_2 be two circles given by $|z'-a| = r_1$ and $|z'-a| = r_2$ respectively, where $r_2 < r_1$ and z' is any point on C_1 or C.

Let $f(z)$ be analytic on C_1 and C_2 & throughout the region b/w the two circles. Let z be any pt in the ring shaped region b/w the circles C_1 and C_2,

then $$f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-a)^n}$$

where $$a_n = \frac{1}{2\pi i} \oint_{C_1} \frac{f(z')}{(z'-a)^{n+1}}\, dz'$$

$$b_n = \frac{1}{2\pi i} \oint_{C_2} \frac{f(z')}{(z'-a)^{-n+1}}\, dz'.$$

Where the integrals are taken around C_1 and C_2 in the anti-clockwise sense.

Eg:- 1) Expand the Laurent's series expansion $\frac{7z-2}{(z+1)z(z-2)}$ in the region $1 < |z+1| < 3$.

Sol Given fn $f(z) = \frac{7z-2}{(z+1)(z)(z-2)}$ &

region $1 < |z+1| < 3$.

Consider, $\frac{7z-2}{(z+1)(z)(z-2)} = \frac{A}{(z+1)} + \frac{B}{z} + \frac{C}{(z-2)}$ — (1)

$7z-2 = Az(z-2) + B(z+1)(z-2) + Cz(z+1)$ → (2)

$z=2 \Rightarrow 7(2)-2 = 0+0+C(2)(3)$

$\Rightarrow 12 = 6(C) \Rightarrow \boxed{C=2}$

$z=0 \Rightarrow 7(0)-2 = 0 + B(1)(-2) + 0$

$\Rightarrow -2 = B(-2) \Rightarrow \boxed{B=1}$

$z=-1 \Rightarrow 7(-1)-2 = A(-1)(-1-2) + 0 + 0$

$\Rightarrow -9 = A(3) \Rightarrow \boxed{A=-3}$

Sub A, B, C in eqn (1)

$$\frac{7z-2}{(z+1)(z)(z-2)} = \frac{-3}{z+1} + \frac{1}{z} + \frac{2}{(z-2)}$$

$$= \frac{1}{z} + \frac{2}{(z-2)} - \frac{3}{(z+1)} \rightarrow (3)$$

Given $1 < |z+1| < 3$.

Taking $1 < |z+1| \Rightarrow \frac{1}{|z+1|} < 1$

Taking $|z+1| < 3 \Rightarrow \frac{|z+1|}{3} < 1$

$$f(z) = \frac{1}{z} + \frac{2}{z-2} - \frac{3}{z+1}$$

$$= \frac{1}{(z+1)-1} + \frac{2}{(z+1)-3} - \frac{3}{(z+1)}$$

$$= \frac{1}{(z+1)\left[1-\frac{1}{z+1}\right]} + \frac{2}{(-3)\left[1-\frac{z+1}{3}\right]} - \frac{3}{z+1}.$$

$$= \frac{1}{z+1}\left[1-\left(\frac{1}{z+1}\right)\right]^{-1} - \frac{2}{3}\left[1-\left(\frac{z+1}{3}\right)\right]^{-1} - \frac{3}{z+1}.$$

$$f(z) = \frac{1}{z+1}\left[1+\frac{1}{z+1}+\left(\frac{1}{z+1}\right)^2+\left(\frac{1}{z+1}\right)^3+\cdots+\left(\frac{1}{z+1}\right)^n+\cdots\right]$$

$$-\frac{2}{3}\left[1+\left(\frac{z+1}{3}\right)+\left(\frac{z+1}{3}\right)^2+\cdots+\left(\frac{z+1}{3}\right)^n+\cdots\right]-\frac{3}{z+1}$$

if $\left|\frac{1}{z+1}\right|<1$ & $\left|\frac{z+1}{3}\right|<1$

$$= \frac{1}{z+1}\sum_{n=0}^{\infty}\left(\frac{1}{z+1}\right)^n - \frac{2}{3}\sum_{n=0}^{\infty}\left(\frac{z+1}{3}\right)^n - \frac{3}{(z+1)}$$

if $1<|z+1|$ and $|z+1|<3$.

$$\therefore f(z) = \sum_{n=0}^{\infty}\frac{1}{(z+1)^{n+1}} - 2\sum_{n=0}^{\infty}\frac{(z+1)^n}{3^{n+1}} - \frac{3}{z+1}$$

if $1<|z+1|<3$.

2) Expand the laurent's series exp of the fn $f(z) = \frac{z^2-6z-1}{(z-1)(z-3)(z+2)}$ in the region $3<|z+2|<5$

Sol. Given $f(z) = \frac{z^2-6z-1}{(z-1)(z-3)(z+2)}$ &

region $3<|z+2|<5$.

Consider $\dfrac{z^2-6z-1}{(z-1)(z-3)(z+2)} = \dfrac{A}{(z-1)} + \dfrac{B}{(z-3)} + \dfrac{C}{(z+2)}$ — ①

$z^2-6z-1 = A(z-3)(z+2) + B(z-1)(z+2) + C(z-1)(z-3)$ — ②

$z=-2 \Rightarrow (-2)^2-6(-2)-1 = C(-2-1)(-2-3)$

$\Rightarrow 4+12-1 = C(-3)(-5)$

$\Rightarrow C = \dfrac{15}{+15} = 1 \Rightarrow \boxed{C=1}$

$z=3 \Rightarrow 3^2-6(3)-1 = 0 + B(3-1)(3+2) + 0$

$\Rightarrow 9-18-1 = B(2)(5)$

$\Rightarrow -10 = B(10) \Rightarrow \boxed{B=-1}$

$z=1 \Rightarrow 1^2-6(1)-1 = A(1-3)(1+2)$

$\Rightarrow \cancel{1}-6-\cancel{1} = A(-2)(3)$

$\Rightarrow -6 = A(-6) \Rightarrow \boxed{A=1}$

$\therefore f(z) = \dfrac{z^2-6z-1}{(z-1)(z-3)(z+2)} = \dfrac{1}{(z-1)} - \dfrac{1}{(z-3)} + \dfrac{1}{(z+2)}$ — ③

Given $3 < |z+2| < 5$

Taking $3 < |z+2| \Rightarrow \dfrac{3}{|z+2|} < 1$

& $|z+2| < 5 \Rightarrow \dfrac{|z+2|}{5} < 1$

$f(z) = \dfrac{1}{(z-1)} - \dfrac{1}{(z-3)} + \dfrac{1}{(z+2)}$

$= \dfrac{1}{(z+2)-3} - \dfrac{1}{(z+2)-5} + \dfrac{1}{(z+2)}$

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$$= \frac{1}{(z+2)\left[1-\frac{3}{(z+2)}\right]} - \frac{1}{-5\left[1-\frac{(z+2)}{5}\right]} + \frac{1}{(z+2)}$$

$$= \frac{1}{(z+2)}\left[1-\frac{3}{(z+2)}\right]^{-1} + \frac{1}{5}\left[1-\frac{(z+2)}{5}\right]^{-1} + \frac{1}{(z+2)}$$

$$= \frac{1}{(z+2)}\left[1+\frac{3}{z+2}+\left(\frac{3}{z+2}\right)^2+\left(\frac{3}{z+2}\right)^3+\cdots+\left(\frac{3}{z+2}\right)^n+\cdots\right]$$

$$+ \frac{1}{5}\left[1+\left(\frac{z+2}{5}\right)+\left(\frac{z+2}{5}\right)^2+\left(\frac{z+2}{5}\right)^3+\cdots+\left(\frac{z+2}{5}\right)^n+\cdots\right]$$

$$+ \frac{1}{(z+2)}.$$

if $\frac{3}{|z+2|} < 1$ & $\frac{|z+2|}{5} < 1$.

$$= \frac{1}{z+2}\sum_{n=0}^{\infty}\left(\frac{3}{z+2}\right)^n + \frac{1}{5}\sum_{n=0}^{\infty}\left(\frac{z+2}{5}\right)^n + \frac{1}{z+2}$$

if $3 < |z+2|$ & $|z+2| < 5$

$$\therefore f(z) = \sum_{n=0}^{\infty}\frac{3^n}{(z+2)^{n+1}} + \sum_{n=0}^{\infty}\frac{(z+2)^n}{5^{n+1}} + \frac{1}{z+2}$$

if $3 < |z+2| < 5$.

3.) Expand the f^n $f(z) = \frac{z^2-1}{(z+2)(z+3)}$ if $2 < |z| < 3$ by using laurent's series expansion.

Sol. Given $f(z) = \dfrac{z^2-1}{(z+2)(z+3)}$ & $2<|z|<3$.

$$f(z) = 1 - \frac{(5z+7)}{(z+2)(z+3)} \longrightarrow (1)$$

$$z^2+5z+6\,\big)\,z^2-1\,\big(\,1$$
$$z^2+5z+6$$
$$-5z-7$$

Consider $\dfrac{5z+7}{(z+2)(z+3)} = \dfrac{A}{(z+2)} + \dfrac{B}{(z+3)} \rightarrow (2)$

$$5z+7 = A(z+3) + B(z+2). \longrightarrow (3)$$

$z=-3 \Rightarrow 5(-3)+7 = B(-3+2)$

$\cancel{\Rightarrow -18 = B(-3) \Rightarrow B =}$

$\Rightarrow -15+7 = B(-1) \Rightarrow \boxed{B=8}$

$z=-2 \Rightarrow 5(-2)+7 = A(-2+3)$

$\Rightarrow -10+7 = A(1) \Rightarrow \boxed{A=-3.}$

Sub A, B in eqn (2).

$$\frac{5z+7}{(z+2)(z+3)} = \frac{-3}{z+2} + \frac{8}{z+3} = \frac{8}{z+3} - \frac{3}{z+2} \rightarrow (4).$$

$$f(z) = 1 + \frac{3}{z+2} - \frac{8}{z+3} \longrightarrow (5)$$

Here given $2<|z|<3$

Taking $2<|z| \Rightarrow \dfrac{2}{|z|}<1$

Taking $|z|<3 \Rightarrow \dfrac{|z|}{3}<1$.

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$$f(z) = 1 + \frac{3}{z+2} - \frac{8}{z+3}.$$

$$= 1 + \frac{3}{z\left[1+\frac{2}{z}\right]} - \frac{8}{3\left[\frac{z}{3}+1\right]}.$$

$$= 1 + \frac{3}{z}\left[1+\frac{2}{z}\right]^{-1} - \frac{8}{3}\left[\frac{z}{3}+1\right]^{-1}.$$

$$= 1 + \frac{3}{z}\left[1 - \frac{2}{z} + \left(\frac{2}{z}\right)^2 - \left(\frac{2}{z}\right)^3 + \left(\frac{2}{z}\right)^4 - \cdots\right]$$

$$- \frac{8}{3}\left[1 - \left(\frac{z}{3}\right) + \left(\frac{z}{3}\right)^2 - \left(\frac{z}{3}\right)^3 + \left(\frac{z}{3}\right)^4 - \cdots\right]$$

$\cancel{= 1 + \frac{3}{z}}$ if $\left|\frac{2}{z}\right| < 1$ and $\left|\frac{z}{3}\right| < 1$

$$= 1 + \frac{3}{z}\left[\sum_{n=0}^{\infty} (-1)^n \left(\frac{2}{z}\right)^n\right] - \frac{8}{3}\left[\sum_{n=0}^{\infty} (-1)^n \left(\frac{z}{3}\right)^n\right]$$

if $2 < |z|$ and $|z| < 3$.

$$f(z) = 1 + 3\sum_{n=0}^{\infty} (-1)^n \frac{2^n}{z^{n+1}} - 8\sum_{n=0}^{\infty} (-1)^n \frac{z^n}{3^{n+1}},$$

if $2 < |z| < 3$. //

4) Expand $f(z) = \frac{1}{z^2 - 3z + 2}$ in the region (i) $2 < |z-1| < 1$ (ii) $1 < |z| < 2$ by Laurent's series expansion.

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Sol Given $f(z) = \dfrac{1}{z^2 - 3z + 2}$, if

$$f(z) = \frac{1}{(z-2)(z-1)} \rightarrow (1)$$

Consider $\dfrac{1}{(z-2)(z-1)} = \dfrac{A}{(z-2)} + \dfrac{B}{(z-1)}$

$$1 = A(z-1) + B(z-2)$$

$z=1 \Rightarrow 1 = B(1-2) \Rightarrow 1 = B(-1) \Rightarrow B = -1.$

$z=2 \Rightarrow 1 = A(2-1) \Rightarrow 1 = A(1) \Rightarrow A = 1$

$$\therefore f(z) = \frac{1}{(z-2)} - \frac{1}{(z-1)}$$

(i) Given $0 < |z-1| < 1$

~~Taking~~ $\cancel{0 < |z-1|} \Rightarrow \cancel{\frac{0}{|z-1|}} < 1$

$|z-1| < 1 \Rightarrow \dfrac{|z-1|}{\cancel{1}} < 1.$

$$f(z) = \frac{1}{(z-2)} - \frac{1}{(z-1)}$$

$$= \frac{1}{(z-1)-1} - \frac{1}{(z-1)}$$

$$\cancel{= \frac{1}{(z-1)\left[1 - \frac{1}{(z-1)}\right]} - \frac{1}{1[z-1]}}.$$

$$\cancel{= \frac{1}{(z-1)}} \quad (-1)\left[1-(z-1)\right]^{-1} - \frac{1}{z-1}.$$

$$= (-1)\left[1+(z-1)+(z-1)^2+(z-1)^3+\cdots\right] - \frac{1}{z-1}$$

if $|z-1|<1$.

$$= (-1)\sum_{n=0}^{\infty}(z-1)^n - \frac{1}{z-1} \quad \text{if } 0<|z-1|<1.$$

(ii) Given $1<|z|<2$

Taking $1<|z| \Rightarrow \frac{1}{|z|}<1$

& $|z|<2 \Rightarrow \frac{|z|}{2}<1$.

$$f(z) = \frac{1}{(z-2)} - \frac{1}{(z-1)}$$

$$\cancel{= \frac{1}{z\left(1-\frac{2}{z}\right)} - \frac{1}{}}$$

$$= \frac{1}{-2\left(1-\frac{z}{2}\right)} - \frac{1}{z\left(1-\frac{1}{z}\right)}$$

$$= \frac{-1}{2}\left(1-\frac{z}{2}\right)^{-1} - \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}$$

$$= \frac{-1}{2}\left[1+\left(\frac{z}{2}\right)+\left(\frac{z}{2}\right)^2+\left(\frac{z}{2}\right)^3+\cdots\right]$$

$$- \frac{1}{z}\left[1+\left(\frac{1}{z}\right)+\left(\frac{1}{z}\right)^2+\left(\frac{1}{z}\right)^3+\cdots\right],$$

if $\frac{1}{|z|}<1$ & $\frac{|z|}{2}<1$

$$= \frac{-1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n - \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n \quad \text{if } 1<|z| \;\&\; |z|<2$$

$$\Rightarrow f = -\sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}} - \sum_{n=0}^{\infty}\frac{1}{z^{n+1}} \quad \text{if } 1<|z|<2.$$

5) $f(z) = \dfrac{z^2-1}{(z+2)(z+3)}$. $|z|>3$.

$\cancel{|z|<3}$

$\cancel{3 > |z|}$

$$f(z) = 1 + \frac{3}{z+2} - \frac{8}{z+3}$$

$|z|>3$

$3<|z|$

$\frac{3}{|z|} < 1$

$2<|z|$

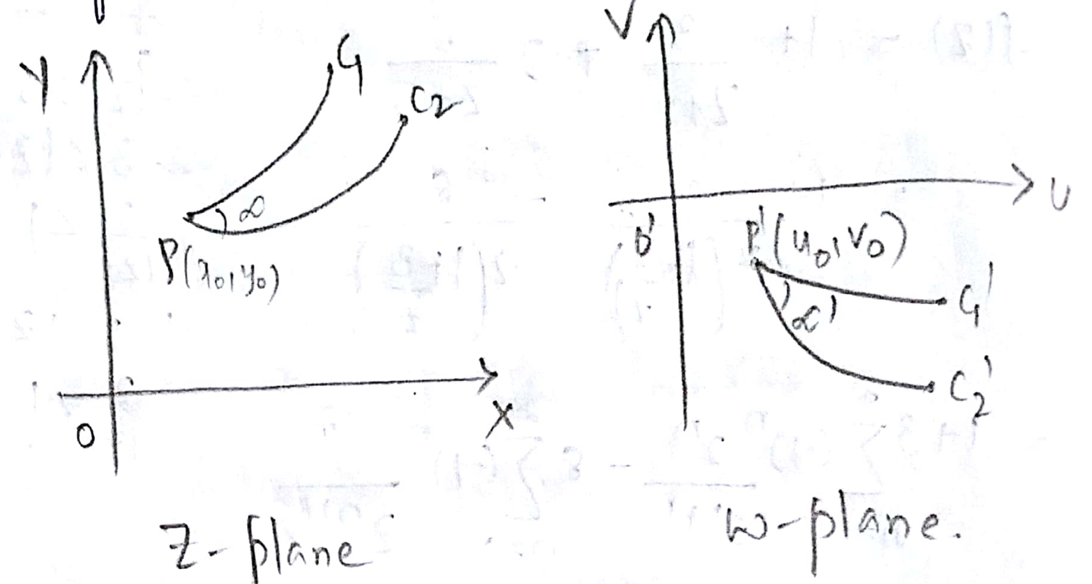
$\frac{2}{|z|} < 1$

$$= 1 + \frac{3}{z\left(1+\frac{2}{z}\right)} - \frac{8}{z\left(1+\frac{3}{z}\right)}$$

$$= 1 + 3\sum_{n=1}^{\infty} (-1)^n \frac{2^n}{z^{n+1}} - 8\sum_{n=0}^{\infty} (-1)^n \frac{3^n}{z^{n+1}}.$$

Conformal Mapping:

Suppose under the transformation, $w = f(z)$ (i.e) $u = u(x,y)$ $v = v(x,y)$, the point $P(x_0, y_0)$ of z-plane. is mapped into the point $P(u_0, v_0)$ of the w-plane. Suppose C_1 and C_2 are any two curves intersecting at the point $P(x_0, y_0)$. Suppose the mapping takes C_1 and C_2 into the curves C_1' and C_2' which are intersecting at the pt $P'(u_0, v_0)$. If the transformation is such that the angle b/w C_1 and C_2 at the pt (x_0, y_0) is equal, both in magnitude and sense to the angle b/w C_1' and C_2' at the pt (u_0, v_0) then it is said to be conformal ~~..~~ transformation at (x_0, y_0).



Bilinear transformation:-

The eqn of the form

$w = \dfrac{az+b}{cz+d}$, is called Bilinear transformation.

where a, b, c, d are complex constants.

It is also called Mobius transformation or fractional transformation.

bilinear transformation formula:

w plane pts – w_1, w_2, w_3.

z " " – z_1, z_2, z_3, then

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

is called Bilinear transformation formula.

1) Find the bilinear transformation which maps the points $(-1, 0, 1)$ in z-plane into $(-1, -i, 1)$ in w plane.

Sol. Given z-plane pts $(-1, 0, 1)$

w-plane pts $(-1, -i, 1)$

$z_1 = -1$, $z_2 = 0$, $z_3 = 1$ and

$w_1 = -1$, $w_2 = -i$, $w_3 = 1$.

w.k.t the bilinear transformation formula is

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{(w+1)(-i-1)}{(w-1)(-i+1)} = \frac{(z+1)(0-1)}{(z-1)(0+1)}.$$

$$\frac{-wi-w-i-1}{-wi+w+i-1} = \frac{-z-1}{z-1}.$$

$$(-wi-w-i-1)(z-1) = (-z-1)(-wi+w+i-1)$$

$$-wzi - \cancel{wz} - \cancel{iz} - z + \cancel{wi} + w + i + \cancel{1} = wzi - \cancel{wz} - \cancel{zi} + z + \cancel{wi} - w - i + \cancel{1}$$

$$-wzi - z + w + i = wzi + z - w - i.$$

$$-wzi + w - wzi + w = z - i + z - i$$

$$-2wzi + 2w = 2z - 2i$$

$$-wzi + w = z - i$$

$w(1-iz) = z-i$

$$\boxed{w = \frac{z-i}{1-iz}}.$$

2) Find the b.T. which maps the pts $(0,-i,-1)$ in z-plane into $(i,1,0)$ in w-plane.

Sol Given z-plane points $(0,-i,-1)$

w-plane points $(i,1,0)$.

~~w.k.t~~ If $z_1 = 0 \quad z_2 = -i \quad z_3 = -1$

$w_1 = i, \; w_2 = 1 \quad w_3 = 0$

w.k.t the bilinear transformation formula is

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{(w-i)(1-0)}{(w-0)(1-i)} = \frac{(z-0)(-i+1)}{(z+1)(-i-0)}$$

$$\frac{w-i}{w-wi} = \frac{-zi+z}{-zi-i}$$

$$(w-i)(-zi-i) = (w-wi)(-zi+z)$$

$$-zw\cancel{i} - wi - z - 1 = -w\cancel{zi} + \cancel{w}z - \cancel{w}z - wzi$$

$$-wi - z - 1 = -wzi$$

$$-wi + wzi = z+1$$

$$w(-i + zi) = z+1$$

$$\therefore\ w = \frac{z+1}{zi - i}$$ //

3.) Find the b.T which maps the pts $(0, i, 1)$ into the pts $(-1, 0, 1)$

Sol) Given z-plane pts $(0, i, 1)$

w-plane pts $(-1, 0, 1)$

$z_1 = 0 \quad z_2 = i \quad z_3 = 1$

$w_1 = -1 \quad w_2 = 0 \quad w_3 = 1.$

w.k.t the bTf is

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{(w+1)(0-1)}{(w-1)(0+1)} = \frac{(z-0)(i-1)}{(z-1)(i-0)}$$

$$\frac{-w-1}{w-1} = \frac{zi - z}{zi - i}$$

$$(-w-1)(zi - i) = (w-1)(zi - z)$$

$$-zwi + wi - zi + i = zwi - zw - zi + z$$

$$-zwi + wi - zwi + zw = -\cancel{zi} + \cancel{z}\ \cancel{i} + zi$$

$$-2zwi + zw + wi = \cancel{z+1}\ z$$

$$w(-2zi + zfi) = f\, z\bar{f}\, i$$

$$\therefore\ w = \frac{f\, z\bar{f}\, i}{-2zi + zfi}$$

4) Find B.T. which maps the pts $(0, 1, \infty)$ into the pts $(-1, i, 1)$

Sol Given z-plane pts $(0, 1, \infty)$

w-plane pts $(-1, i, 1)$

$\therefore\ z_1 = 0 \quad z_2 = 1 \quad z_3 = \infty$

$w_1 = -1 \quad w_2 = i \quad w_3 = 1.$

W.K.T the binary trans for. is

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{(w+1)(-i-1)}{(w-1)(-i+1)} = \frac{(z-0)(1-\infty)}{(z-\infty)(1-0)}$$

$$\frac{-wi - w - i - 1}{-wi + w + i - 1} = \frac{z(1-\infty)}{(z-\infty)(1)}$$

$$\frac{-wi - w - i - 1}{-wi + w + i - 1} = \frac{z \lim_{n\to\infty}(1-n)}{\lim_{n\to\infty}(z-n)}$$

$$= \frac{z\cancel{n} \lim_{n\to\infty}\left[\frac{1}{n} - 1\right]}{\cancel{n} \lim_{n\to\infty}\left[\frac{z}{n} - 1\right]}$$

$$= \frac{z(0-1)}{(0-1)}$$

$$= \frac{z(-1)}{(-1)} = z$$

$$\frac{-wi-w-i-1}{-wi+w+i-1} = z$$

$$-wi-w-i-1 = -wiz+wz+iz-z$$

$$-wi-w+wiz-wz = iz-z+i+1$$

$$+w(-i-1+iz-z) = \cancel{z(i-1)+(i+1)}\ iz-z+i+1.$$

$$w = \frac{iz-z+i+1}{-i-1+iz-z}$$

$$w = \frac{(1-z)+i(z+1)}{-(1+z)+i(z-1)}$$

5) z-plane pts $(\infty, i, 0)$

w-plane pts $(0, i, \infty)$

6) $(-1, i, 1)$ into $(0, i, \infty)$

7) $(\infty, i, 0)$ into $(-1, -i, 1)$.

Calculation of Residues :-

Zeros of analytic f^n:

A zero of analytic f^n $f(z)$ is a value of z $\ni$ $f(z)=0$

i.e, a point 'a' is called zero of an analytic f^n $f(z)$ if $f(a)=0$.

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Singular pt :- Def^n :- A singular pt of a f^n $f(z)$ is the pt at which the f^n $f(z)$ fails to be analytic.

Poles of an analytic f^n :- If $\exists$ a +ve integer n $\ni$ $\lim_{z \to a} (z-a)^n \cdot f(z) = A \neq 0$ then $z = A$ is pole of order n.

Residues : The co-efficient of $\frac{1}{(z-a)}$ in the Lawrent's series expansion of $f(z)$ about the isolated singularity $z = a$ is called the Residue of $f(z)$ at that pt.

Thus the residue of $f(z)$ at $z = a$ is b_1.

where $b_1 = \frac{1}{2\pi i} \int_c f(z)\,dz$

$$\int_c f(z)\,dz = 2\pi i b_1$$

$$= 2\pi i \left[\text{Residue of } f(z) \text{ at } z = a\right]$$

$$\int_c f(z)\,dz = 2\pi i \left[\operatorname{Res} f(z)\right]_{z=a}.$$

Cauchy's Residue Theorem :-

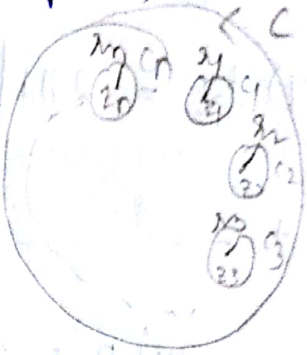
Stmt :- If $f(z)$ is analytic f^n, within a closed curve c. ~~If~~ ~~ff~~ except finite no. of poles $z_1, z_2, z_3, \ldots z_n$ within C and $R_1, R_2, R_3, \ldots R_n$ be the residues of

$f(z)$ at these poles. then.

$$\int_C f(z)\,dz = 2\pi i\,(R_1+R_2+R_3+\cdots+R_n).$$

(or) $$\int_C f(z)\,dz = 2\pi i\,(\text{Sum of residues of inside poles}).$$

Proof:-



Let $\gamma_1, \gamma_2, \gamma_3, \ldots \gamma_n$ be the circles with centres $z_1, z_2, z_3 \ldots z_n$ respectively ~~w.r.t~~ & their radius is so small that they lie entirely within a closed curve C.

$f(z)$ is an analytic within the region enclosed by the curve small c b/w these circles.

Therefore, by using cauchy's theorem for multi-connected region

$$\int_C f(z)\,dz = \int_{\gamma_1} f(z)\,dz + \int_{\gamma_2} f(z)\,dz + \int_{\gamma_3} f(z)\,dz + \cdots + \int_{\gamma_n} f(z)\,dz.$$

$\therefore$ By using the defn of residues,

$$\int_{\gamma_1} f(z)\,dz = 2\pi i\left[\operatorname{Res}(f(z))\right]_{z=z_1} \cancel{=R_1}$$

$$\int_{\gamma_2} f(z)\,dz = 2\pi i\left[\operatorname{Res}(f(z))\right]_{z=z_2} \cancel{=R_2}$$

$$\int_{\gamma_n} f(z)\,dz = 2\pi i\left[\operatorname{Res} f(z)\right]_{z=z_n} \rightarrow R_n$$

Sub these values in ①

$$\int_C f(z)\,dz = 2\pi i\left[\operatorname{Res} f(z)\right]_{z=z_1} + 2\pi i\left[\operatorname{Res} f(z)\right]_{z=z_2}$$

$$+ \cdots + 2\pi i\left[\operatorname{Res} f(z)\right]_{z=z_n}.$$

$$= 2\pi i R_1 + 2\pi i R_2 + \cdots 2\pi i R_n$$

$$= 2\pi i\left[R_1 + R_2 + \cdots + R_n\right]$$

$$= 2\pi i\,[\text{sum of residues of inside poles}]$$

Residue of $f(z)$ at a pole of order m:-

$\left[\text{Res of } f(z)\right]_{z=z_0}$ is of order m

$$= \lim_{z\to z_0} \frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}\left[(z-z_0)^m f(z)\right].$$

$$m=1 \Rightarrow \left[\text{Res of } f(z)\right]_{z=z_0} = \lim_{z\to z_0}\left[(z-z_0) f(z)\right].$$

$$m=2 \Rightarrow \left[\text{Res of } f(z)\right]_{z=z_0} = \lim_{z\to z_0}\frac{d}{dz}\left[(z-z_0)^2 f(z)\right].$$

$$m=3 \Rightarrow \left[\operatorname{Res} f(z)\right] = \lim_{z\to z_0}\frac{1}{2!}\frac{d^2}{dz^2}\left[(z-z_0)^3 f(z)\right].$$

1. Find zeroes & poles of $f(z) = \left(\frac{z+1}{z^2+1}\right)^2$.

Sol. Given f^n $f(z) = \left(\frac{z+1}{(z^2+1)}\right)^2 = \frac{(z+1)^2}{(z^2+1)^2}$.

Numerator equating to zero, we get the zero of $f(z) \Rightarrow (z+1)^2 = 0 \Rightarrow z = -1$.

Denominator equating to zero, we get poles of $f(z) \Rightarrow (z^2+1)^2 = 0$

$$(z^2 - i^2) = 0$$

$$(z+i)(z-i) = 0$$

$$\therefore z = -i, i.$$

2) Find poles & residues of $f(z) :- \frac{1}{(z+1)(z+3)}$.

Sol) Given $f(z) = \frac{1}{(z+1)(z+3)}$.

$z = -1, -3$ are the poles of $f(z)$

$[\text{Res } f(z)]_{z=z_0}$ of order 1 $= \lim_{z \to z_0} [(z-z_0) f(z)]$

Case i) :- of $z = -1$.

$$[\text{Res } f(z)]_{z=-1} = \lim_{z \to -1} [(z+1) f(z)].$$

$$= \lim_{z \to -1} \left[\cancel{(z+1)} \cdot \frac{1}{\cancel{(z+1)}(z+3)}\right]$$

$$= \frac{1}{-1+3} = 1/2.$$

(ii) If $z=-3$

$$[\text{Res} f(z)]_{z=-3} = \lim_{z\to -3} [(z+3) f(z)]$$

$$= \lim_{z\to -3} \cancel{(z+3)}\left[\frac{1}{(z+1)\cancel{(z+3)}}\right]$$

$$= \lim_{z\to -3} \frac{1}{z+1}.$$

$$= \frac{1}{-3+1} = -1/2$$

3.) find poles & residues of $f(z) = \frac{1}{(z-2)(z+1)}$

Sol Given $f(z) = \frac{1}{(z-2)(z+1)}$

$\therefore z = 2, -1$ are the poles of $f(z)$.

(i) $z=2$

$$[\text{Res} f(z)]_{z=2} = \lim_{z\to 2} [(z-2) \cdot f(z)]$$

$$= \lim_{z\to 2} \left[\cancel{(z-2)} \cdot \frac{1}{\cancel{(z-2)}(z+1)}\right]$$

$$= \frac{1}{2+1} = 1/3.$$

(ii) $z=-1$.

$$[\text{Res} f(z)]_{z=-1} = \lim_{z\to -1} \left[\cancel{(z+1)} \cdot \frac{1}{(z-2)\cancel{(z+1)}}\right]$$

$$= \frac{1}{-1-2} = -1/3.$$

4) Find poles & residues of $f(z) = \frac{z^2}{(z-1)(z-2)^2}$.

Sol) Given $f(z) = \frac{z^2}{(z-1)(z-2)^2}$.

$z=1$ is a pole of order 1

$z=2$ is a pole of order 2

(i) $z=1$

$$\left[\operatorname{Res} f(z)\right]_{z=1} = \lim_{z\to 1}\left[(z-1)f(z)\right]$$

$$= \lim_{z\to 1}\left[\cancel{(z-1)}\frac{z^2}{\cancel{(z-1)}(z-2)^2}\right]$$

$$= \frac{1}{(1-2)^2} = 1.$$

(ii) $$\left[\operatorname{Res} f(z)\right]_{z=2} = \lim_{z\to 1}\frac{d}{dz}\left[(z-2)^2 \cdot f(z)\right]$$

$$= \lim_{z\to 2}\frac{d}{dz}\left[\cancel{(z-2)^2}\cdot\frac{z^2}{(z-1)\cancel{(z-2)^2}}\right]$$

$$= \lim_{z\to 2}\frac{d}{dz}\left(\frac{z^2}{z-1}\right)$$

$$= \lim_{z\to 2}\frac{(z-1)(2z)-z^2(1)}{(z-1)^2}$$

$$= \frac{1(4)-4}{1} = 0.$$

5) Find poles & residues of $f(z) = \frac{ze^z}{(z-1)^3}$

Sol. Given $f(z) = \frac{ze^z}{(z-1)^3}$.

$z=1$ is a pole of order 3.

$$\left[\operatorname{Res} f(z)\right]_{z_0=1} = \lim_{z \to z_0} \frac{1}{2}\frac{d^2}{dz^2}\left[(z-z_0)^3 \cdot f(z)\right]$$

$$= \lim_{z \to z_0} \frac{1}{2}\frac{d^2}{dz^2}\left[\cancel{(z-1)^3} \cdot \frac{ze^z}{\cancel{(z-1)^3}}\right]$$

$$= \lim_{z \to z_0} \frac{1}{2}\frac{d}{dz}\left(ze^z + e^z\right)$$

$$= \lim_{z \to z_0=1} \frac{1}{2}\left(ze^z + e^z + e^z\right)$$

$$= \frac{1(e^1)+e^1+e^1}{2} = \frac{3e}{2}.$$

6) Find poles & residues of $f(z) = \frac{(z+1)}{z^2(z-2)}$.

Sol Given $f(z) = \frac{z+1}{z^2(z-2)}$.

$z=0$ is a pole of order 2

$z=2$ is a pole of order 1

i) $z_0 = 2$

$$\therefore \left[\operatorname{Res}(f(z))\right]_{z=2} = \lim_{z \to 2}\left[(z-2)\, f(z)\right]$$

$$= \lim_{z \to 2}\left[\cancel{(z-2)}\frac{z+1}{z^2\cancel{(z-2)}}\right]$$

$$= \lim_{z \to 2} \frac{z+1}{z^2}$$

$$= \frac{2+1}{4} = 3/4.$$

(ii) $z_0 = 0$.

$$\left[\text{Res } f(z)\right]_{z=0} = \lim_{z \to 0} \frac{d}{dz}\left[(z-z_0)^2 \cdot f(z)\right]$$

$$= \lim_{z \to 0} \frac{d}{dz}\left[\cancel{(z)^2} \cdot \frac{(z+1)}{\cancel{z^2}(z-2)}\right]$$

$$= \lim_{z \to 0} \frac{d}{dz}\left(\frac{z+1}{z-2}\right)$$

$$= \lim_{z \to 0} \frac{(z-2)(1)-(z+1)(1)}{(z-2)^2}.$$

$$= \frac{(-2)-(1)}{(-2)^2}$$

$$= -3/4$$

7) Find poles & residues of $f(z) = \frac{1}{(z^2+4)^2}$

Sol Given $f(z) = \frac{1}{(z^2+4)^2} = \frac{1}{(z+2i)^2(z-2i)^2}$

$[z^2-(2i)^2]^2$

$z = -2i, 2i$ are poles of order 2.

(i) $\left[\text{Res } f(z)\right]_{z=2i} = \lim_{z \to 2i} \frac{d}{dz}\left[(z-2i)^2 f(z)\right]$

$$= \lim_{z \to 2i} \frac{d}{dz}\left[\cancel{(z-2i)^2} \cdot \frac{1}{(z+2i)^2\cancel{(z-2i)^2}}\right]$$

$$= \lim_{z \to 2i} \frac{d}{dz} \cdot \frac{1}{(z+2i)^2}$$

$$= \lim_{z \to 2i} \frac{(-2)}{(z+2i)^3}$$

$$= \frac{-2}{(4i)^3} = \frac{-2}{4^3 i^3}$$

$$= \frac{-2}{64(-i)} = \frac{1}{32i}$$

(ii) $$[\text{Res } f(z)]_{z=-2i} = \lim_{z \to -2i} \frac{d}{dz}\left[\cancel{(z+2i)^2} \cdot \frac{1}{\cancel{(z+2i)^2}(z-2i)^2}\right]$$

$$= \lim_{z \to -2i} \frac{d}{dz} \cdot \frac{1}{(z-2i)^2}$$

$$= \lim_{z \to -2i} \frac{(-2)}{(z-2i)^3}$$

$$= \frac{(-2)}{(-4i)^3}$$

$$= \frac{-2}{-4^3 \cdot i^3}$$

$$= \frac{2}{64(-i)} = \frac{-1}{32i}.$$

8) $f(z) = \dfrac{e^z}{(1+z)^2}$

9) $f(z) = z/z^2-4$

10) $f(z) = \dfrac{z \cdot \sin z}{(z-\pi)^3}$

11) $f(z) = z/z^2+1$

12) $f(z) = \dfrac{e^{iz}}{z^2+1}.$

1) Evaluate $\oint_C \frac{4-3z}{z(z-1)(z-2)} dz$ where c is the circle $|z| = 3/2$. by using Cauchy's residue theorem.

Sol. Given $\oint_C \frac{4-3z}{z(z-1)(z-2)} dz$ & $|z| = 3/2$.

Let $f(z) = \frac{4-3z}{z(z-1)(z-2)}$

$z = 0, 1, 2$ are the poles of order 1.

$z = 0, 1$ are inside the circle $|z| = 3/2$

$z = 2$ is outside the circle $|z| = 3/2$.

$$[\text{Res} f(z)]_{z=0} = \lim_{z \to 0} [(z-0) \cdot f(z)]$$

$$= \lim_{z \to 0} \left[z \cdot \frac{4-3z}{z(z-1)(z-2)} \right]$$

$$= \frac{4}{(-1)(-2)} = 2 = R_1$$

$$[\text{Res} f(z)]_{z=1} = \lim_{z \to 1} [(z-1) f(z)]$$

$$= \lim_{z \to 1} \left[(z-1) \cdot \frac{4-3z}{z(z-1)(z-2)} \right]$$

$$= \frac{1}{1(-1)} = -1 = R_2.$$

Using Cauchy's residues theorem,

$$\oint_C f(z)\,dz = 2\pi i\,(R_1 + R_2)$$

$$\therefore \oint_C \frac{4-3z}{z(z-1)(z-2)}\,dz = 2\pi i\,(2-1) = 2\pi i$$

2) Evaluate $\oint_C \frac{e^{2z}}{(z-1)(z-2)}\,dz$ where C is the o^{le} $|z| = 3$ using Cauchy's Residues theorem

Sol. Given $\oint_C \frac{e^{2z}}{(z-1)(z-2)}\,dz$ & $|z| = 3$.

Let $f(z) = \frac{e^{2z}}{(z-1)(z-2)}$

$z = 1, 2$ are the poles of order 1

$z = 1, 2$ are inside the o^{le} $|z| = 3$.

$$\left[\operatorname{Res} f(z)\right]_{z=1} = \lim_{z \to 1}\left[(z-1)\cdot f(z)\right]$$

$$= \lim_{z \to 1}\left[\cancel{(z-1)}\cdot \frac{e^{2z}}{\cancel{(z-1)}(z-2)}\right]$$

$$= \frac{e^{2(1)}}{(-1)} \quad \cancel{\frac{e^{2t}}{2}} = -e^2.$$

$$\left[\operatorname{Res} f(z)\right]_{z=2} = \lim_{z \to 2}\left[\cancel{(z-2)}\cdot \frac{e^{2z}}{(z-1)\cancel{(z-2)}}\right]$$

$$= \lim_{z \to 2} \frac{e^{2z}}{(z-1)}$$

$$= \frac{e^{2(2)}}{(2-1)} = e^4$$

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Using Cauchy's Residues theorem,

$$\oint_C f(z)\,dz = 2\pi i(R_1+R_2)$$

$$\therefore \oint_C \frac{e^{2z}}{(z-1)(z-2)} = 2\pi i(e^4-e^2).$$

3) Evaluate $\oint_C \frac{2e^z}{z(z-3)}dz$ where C is the ok $|z|=2$ using Cauchy's residues theorem.

Sol) Given $\oint_C \frac{2e^z}{z(z-3)}$ & $|z|=2$.

Let $f(z) = \frac{2e^z}{z(z-3)}$

$z=0,3$ are the poles of Order 1.

$z=0$ is inside the ok $|z|=3$

& $z=3$ is outside the ok $|z|=3$.

$$[\text{Res } f(z)]_{z=0} = \lim_{z\to 0}\left[(z-0)f(z)\right]$$

$$= \lim_{z\to 0}\left[\cancel{(z-0)}\cdot\frac{2e^z}{\cancel{(z)}(z-3)}\right]$$

$$= \frac{2e^0}{(0-3)} = -\frac{2}{3}.$$

Using Cauchy's residues theorem,

$$\oint_C f(z)\,dz = 2\pi i(R_1+R_2)$$

$$\therefore \oint_C \frac{2e^z}{z(z-3)} = 2\pi i\left(-\frac{2}{3}\right) = \frac{-4\pi i}{3}.$$

4) Evaluate $\oint_c \frac{3z-4}{z(z-3)} dz$ where c is the
cle $|z|=2$ using Cauchy's Residue theorem

Sol Given $\oint_c \frac{3z-4}{z(z-3)} dz$. & $|z|=2$.

Let $f(z) = \frac{3z-4}{z(z-3)}$

$z=0, 3$ are the poles of order 1.

$z=0$ is inside the cle $|z|=2$ But $z=3$ is outside $|z|=2$

$$\left[\text{Res } f(z)\right]_{z=0} = \lim_{z\to 0}\left[\cancel{(z-0)} \cdot \frac{3z-4}{\cancel{z}(z-3)}\right]$$

$$= \frac{-4}{-3} = 4/3.$$

Using C. R. th,

$$\oint_c f(z)\,dz = 2\pi i(R_1+R_2)$$

$$\therefore \oint_c \cancel{\frac{4\;3z}{(z)(z-1)}} \frac{3z-4}{z(z-3)} = 2\pi i\left(\frac{4}{3}\right) = \frac{8\pi i}{3}$$

5) Evaluate $\oint_c \frac{(2z+1)^2}{4z^3+z} dz$ where $|z|=1$.

Sol Given $\oint_c \frac{(2z+1)^2}{4z^3+z}$ & $|z|=1$

Let $f(z) = \frac{(2z+1)^2}{4z^3+z}$

$z=0$, $z=i/2$, $z=-i/2$ are poles of order 1 and 2

$z=0$ is inside $|z|=1$.

$z=i/2$

$x+iy=0+\frac{i}{2}$

$x=0, y=1/2$

$|z|=\sqrt{x^2+y^2}=1$

$\sqrt{0+(\frac{1}{2})^2}=1/2<1$

$z=i/2, -i/2$ are inside the c/e $|z|=1$.

6) $\int \frac{z^3}{(z-1)^2(z-3)} dz$ $\quad |z|=2$

7) $\int_c \frac{\cos\pi z^2}{(z-1)(z-2)} dz$ $\quad |z|=3/2$

8) $\int_c \frac{(z)}{(z-1)(z-2)^2} dz$ $\quad |z-2|=1/2$

9) $\int_c \frac{ze^z}{(z^2+9)} dz$ $\quad |z|=5$.

10) $\int_c \frac{z^2+2z-2}{z(z-4)(z-1)} dz$ $\quad |z|=1.5$.

no Ms paper without this

Evaluation of Integrals of the type

$$\int_0^{2\pi} F(\cos\theta, \sin\theta)\, d\theta.$$

Given eqn is of the form $\int_0^{2\pi} F(\cos\theta, \sin\theta) d\theta$.

Replacing $z = e^{i\theta}$

$dz = i e^{i\theta} d\theta$

$iz\, d\theta = dz \Rightarrow d\theta = \frac{dz}{iz}$.

~~$\sin\theta = e^{i\theta} -$~~

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \qquad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

$$= \frac{z + \frac{1}{z}}{2} \qquad = \frac{z - \frac{1}{z}}{2i}$$

$$= \frac{z^2+1}{2z} \qquad = \frac{z^2-1}{2zi} \qquad |z| = 1.$$

1. S.T $\int_0^{2\pi} \frac{d\theta}{2+\cos\theta} = \frac{2\pi}{\sqrt{3}}$ by using Cauchy's Residues theorem.

Sol. Given $\int_0^{2\pi} \frac{1}{2+\cos\theta} d\theta \longrightarrow (1)$

Put $z = e^{i\theta} \Rightarrow d\theta = dz/iz$

$\cos\theta = \frac{z^2+1}{2z}$ and $|z| = 1$.

Sub these values in (1)

$$\int_0^{2\pi} \frac{1}{2+\cos\theta} d\theta = \int_C \frac{1}{2 + \left(\frac{z^2+1}{2z}\right)} \frac{dz}{iz}$$

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$$= \int_C \frac{1}{\frac{4z+z^2+1}{2z}} \frac{dz}{iz}$$

$$= \int_C \frac{2\cancel{z}}{z^2+4z+1} \frac{dz}{i\cancel{z}}$$

$$= \frac{2}{i} \int_C \frac{1}{z^2+4z+1} dz$$

$$\int_0^{2\pi} \frac{1}{2+\cos\theta} d\theta = \frac{2}{i} \int_C f(z)\, dz \rightarrow (2)$$

where $f(z) = 1/z^2+4z+1$

$z^2+4z+1=0$ (same sign - inside poles / opp sign - outside poles)

$z = -2+\sqrt{3}, -2-\sqrt{3}$ are poles of $f(z)$.

$z = -2+\sqrt{3}$ is inside the cle $|z|=1$

$z = -2-\sqrt{3}$ is outside the cle $|z|=1$.

Put $\alpha = -2+\sqrt{3}$, $\beta = -2-\sqrt{3}$

$$\left[\operatorname{Res}_{z=\alpha} f(z)\right] = \lim_{z\to\alpha} \left[(z-\alpha) f(z)\right]$$

$$= \lim_{z\to\alpha} \left[\cancel{(z-\alpha)} \frac{1}{\cancel{(z-\alpha)}(z-\beta)}\right]$$

$$= \frac{1}{\alpha-\beta}$$

$$= \frac{1}{\cancel{-2}+\sqrt{3}+\cancel{2}+\sqrt{3}} = \frac{1}{2\sqrt{3}} = R$$

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By using Cauchy's Residue theorem

$$\oint_C f(z)\,dz = 2\pi i\, R$$

$$\oint_C f(z)\,dz = 2\pi i \times \frac{1}{2\sqrt{3}} = \frac{\pi i}{\sqrt{3}} \quad -③$$

Sub eq^n ③ ~~and~~ in ②

$$\int_0^{2\pi} \frac{1}{2+\cos\theta}\,d\theta = \frac{2}{i}\,\frac{\pi i}{\sqrt{3}}$$

$$\Rightarrow \frac{2\pi}{\sqrt{3}} \;//$$

2. Evaluate $\int_0^{2\pi} \frac{d\theta}{1+a^2-2a\cos\theta}$, $0<a<1$, by using Cauchy's Residue theorem.

Sol Given $\int_0^{2\pi} \frac{1}{1+a^2-2a\cos\theta}\,d\theta$

Put $z = e^{i\theta} \Rightarrow d\theta = dz/iz$

$\cos\theta = \frac{z^2+1}{2z}$ and $|z|=1$

Sub these values in ①

$$\int_0^{2\pi} \frac{1}{1+a^2-2a\cos\theta}\,d\theta = \int_C \frac{1}{1+a^2-\cancel{2}a\left(\frac{z^2+1}{\cancel{2}z}\right)}\,\frac{dz}{iz}$$

$$= \int_C \frac{1}{1+a^2-\frac{(az^2+a)}{z}}\,\frac{dz}{iz}$$

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$$= \int_C \frac{1}{\frac{z + a^2 z - az^2 - a}{z}} \cdot \frac{dz}{iz}$$

$$= \frac{1}{i} \int_C \frac{1}{-az^2 + (1+a^2) z - a} \, dz,$$

$$= \frac{1}{i} \int_C \frac{1}{-a\left[z^2 - \left(\frac{1+a^2}{a}\right) z + 1\right]} \, dz.$$

$$= \frac{1}{-ai} \int_C \frac{1}{z^2 - \left(\frac{1+a^2}{a}\right) z + 1} \, dz$$

$$= \frac{-1}{ai} \int_C f(z) \, dz \longrightarrow (1)$$

where $f(z) = \dfrac{1}{z^2 - \left(\frac{1+a^2}{a}\right) z + 1}$

$$z^2 - \left(\frac{1+a^2}{a}\right) \cdot z + 1 = 0.$$

$$z = \frac{\left(\frac{1+a^2}{a}\right) \pm \sqrt{\left(\frac{1+a^2}{a}\right)^2 - 4}}{2}$$

$$= \frac{\left(\frac{1+a^2}{a}\right)^2 \pm \sqrt{\frac{1 + (a^2)^2 + 2a^2 - 4a^2}{a^2}}}{2}$$

$$= \frac{\left(\frac{1+a^2}{a}\right) \pm \sqrt{\frac{1 + (a^2)^2 - 2a^2}{a^2}}}{2}$$

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$$= \frac{1}{a}\left[\frac{(1+a^2) \pm \sqrt{(1-a^2)^2}}{2}\right].$$

$$= \frac{1}{2a}\left[(1+a^2) \pm (1-a^2)\right]$$

$$= \frac{1}{2a}\left[1+\cancel{a^2}+1-\cancel{a^2}\right], \quad \frac{1}{2a}\left[\cancel{1}+a^2-\cancel{1}+a^2\right]$$

$$= \frac{1}{2a}(2), \quad \frac{1}{2a}(2a^2)$$

$\therefore z = 1/a, a.$

$z = a, 1/a$ are poles of $f(z)$

Given $0 < a < 1$

$\therefore z = a$ is inside of $|z| = 1$

& $z = 1/a$ is outside of $|z| = 1$.

(i.) $$\left[\operatorname{Res} f(z)\right]_{z=a} = \lim_{z \to a}\left[(z-a)\, f(z)\right]$$

$$= \lim_{z \to a}\left[\cancel{(z-a)} \cdot \frac{1}{\cancel{(z-a)}\left(z-\frac{1}{a}\right)}\right]$$

$$= \frac{1}{a-\frac{1}{a}} = \frac{a}{a^2-1} = R.$$

Using Cauchy's residue theorem,

$$\oint_C f(z)\,dz = 2\pi i R = 2\pi i \cdot \frac{a}{a^2-1}. \rightarrow (2)$$

Sub (2) in (1).

$$\therefore \int_0^{2\pi} \frac{1}{1+a^2-2a\cos\theta}\,d\theta = \frac{-1}{\cancel{a i}}\left(\frac{2\pi \cancel{i}\, \cancel{a}}{a^2-1}\right) = \frac{-2\pi}{a^2-1}$$

$$= \frac{2\pi}{1-a^2}$$

3.) Evaluate $\int_0^{2\pi} \frac{\cos 2\theta}{1-2a\cos\theta+a^2} d\theta$; $a^2<1$ by using Cauchy's Residues theorem.

Sol. Given $\int_0^{2\pi} \frac{\cos 2\theta}{1+a^2-2a\cos\theta} d\theta$

Put $z=e^{i\theta} \Rightarrow d\theta = dz/iz$.

$\cos\theta = \frac{z^2+1}{2z}$ and $|z|=1$.

4.) $\int_0^{2\pi} \frac{d\theta}{(5-3\cos\theta)^2}$, 5.) $\int_0^{2\pi} \frac{d\theta}{(5-3\sin\theta)^2}$

→ Integral of the type $\int_{-\infty}^{\infty} f(x)\,dx$.

$$\int_C f(z)\,dz = \int_{-R}^{R} f(x)\,dx + \int_{C_R} f(x)\,dx.$$

~~$R\to\infty$, $C_R\to 0$.~~

$$\therefore \cancel{\int_C f(z)\,dz = \int_{-\infty}^{\infty} f(x)\,dx}$$

where C is a closed curve containing the real axis $-R$ to R together with semicircle C_R

If $R\to\infty$ then semicircle $C_R\to 0$

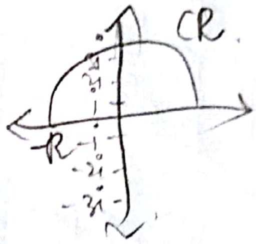
$$\therefore \int_C f(z)\,dz = \int_{-\infty}^{\infty} f(x)\,dx.$$

5 times
2

Eg:-1 Evaluate $\int_0^{\infty} \frac{1}{x^2+1}\,dx$. by using Cauchy's Residues theorem.

Sol. Given $\int_0^{\infty} \frac{1}{1+x^2}\,dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx.$

$$= \frac{1}{2}\int_{-\infty}^{\infty} f(x)\,dx \quad \text{where } f(x) = 1/1+x^2.$$



$$f(z) = \frac{1}{1+z^2} = \frac{1}{(z+i)(z-i)}$$

$z = i, -i$ are poles of $f(z)$.

$z = i$ is inside pole

$z = -i$ is outside pole

$$\left[\operatorname{Res} f(z)\right]_{z=i} = \lim_{z\to i}\left[(z-i)f(z)\right]$$

$$= \lim_{z\to i}\left[(z-i)\frac{1}{(z-i)(z-i)}\right]$$

$$= \frac{1}{i+i} = \frac{1}{2i} = R.$$

Using C-R theorem, $\int_C f(z)\,dz = 2\pi i R$.

$$\int_C f(z)\,dz = 2\pi i \frac{1}{2i} = \pi \longrightarrow (2)$$

Consider $\int_C f(z)\,dz = \int_{-R}^{R} f(x)\,dx + \int_{CR} f(x)\,dx.$

$$\cancel{f(z) = \frac{1}{1+z}}$$

where c is a closed curve containing the real axis $-R$ to R and the semi circle C_R.

If $R \to \infty$, then $C_R \to 0$.

$$\therefore \int_C f(z)\,dz = \int_{-\infty}^{\infty} f(x)\,dx \longrightarrow (3)$$

From ② & ③ $\int_{-\infty}^{\infty} f(x)\,dx = \pi \longrightarrow$ ④

sub ④ in ①

$\therefore \int_0^{\infty} \frac{1}{1+x^2}\,dx = \frac{1}{2}\pi = \pi/2$

2) Evaluate $\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)(x^2+4)}\,dx$ by using cauchy's residue theorem.

Sol) Given $\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)(x^2+4)}\,dx = \int_{-\infty}^{\infty} f(x)\,dx \longrightarrow$ ①,

$$f(x) = \frac{x^2}{(x^2+1)(x^2+4)}$$

$$f(z) = \frac{z^2}{(z^2+1)(z^2+4)} = \frac{z^2}{(z+i)(z-i)(z+2i)(z-2i)}$$

$z = \pm i, \pm 2i$, are the poles.

$z = i, 2i$ are inside

$$[\text{Res } f(z)]_{z=i} = \lim_{z\to i} [(z+i) f(z)]$$

$$= \lim_{z\to i}\left[\cancel{(z-i)} \cdot \frac{z^2}{(z+i)\cancel{(z-i)}(z+2i)(z-2i)}\right]$$

$$= \frac{i^2}{(2i)(3i)(-i)} = \frac{-1}{6i} = R_1.$$

$$[\text{Res } f(z)]_{z=2i} = \lim_{z\to 2i}\left[\cancel{(z-2i)} \cdot \frac{z^2}{(z+i)(z-i)(z+2i)\cancel{(z-2i)}}\right]$$

$$= \frac{i(2i)^2}{3i \cdot i \cdot 4i}$$

$$= \frac{-\cancel{4}}{3(-1)\cancel{4}i} = \frac{1}{3i} = R_2$$

By using Cauchy's Residue theorem,

$$\int_C f(z)\,dz = 2\pi i\,(R_1 + R_2)$$

$$= 2\pi i\left(\frac{-1}{6i} + \frac{1}{3i}\right)$$

$$= \frac{2\pi \cancel{i}}{\cancel{i}}\left(\frac{1}{3} - \frac{1}{6}\right)$$

$$= 2\pi\left(\frac{1}{6}\right) = \frac{\pi}{3} \longrightarrow (2)$$

Consider,

$$\int_C f(z)\,dz = \int_{-R}^{R} f(x)\,dx + \int_{C_R} f(x)\,dx$$

where C is a closed curve containing the real axis $-R$ to R together with the semicircle C_R.

If $R \to \infty$ then $C_R \to 0$.

$$\therefore \int_C f(z)\,dz = \int_{-\infty}^{\infty} f(x)\,dx \longrightarrow (3)$$

from (2) & (3).

$$\int_{-\infty}^{\infty} f(x)\,dx = \pi/3.$$

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3) Evaluate $\int_{-\infty}^{\infty} \frac{1}{(1+x^2)^2} dx$ using Cauchy's Residues theorem.

Sol) Given $\int_{-\infty}^{\infty} \frac{1}{(1+x^2)^2} dx = \int_{-\infty}^{\infty} f(x)\, dx \rightarrow (1)$.

$$f(x) = \frac{1}{(1+x^2)^2}.$$

$$f(z) = \frac{1}{(1+z^2)^2} = \frac{1}{(z+i)^2(z-i)^2}.$$

$z = -i, i$ are the poles.

$z = i$ is inside pole.

$z = -i$ is outside pole.

$$\left[\operatorname{Res} f(z)\right]_{z=i} = \lim_{z\to i} \frac{d}{dz}\left[(z-i)^2 f(z)\right]$$

$$= \lim_{z\to i} \frac{d}{dz}\left[\cancel{(z-i)^2} \cdot \frac{1}{(z+i)^2\cancel{(z-i)^2}}\right]$$

$$= \lim_{z\to i} \frac{d}{dz}\left[\frac{1}{(z+i)^2}\right]$$

$$= \lim_{z\to i} \frac{-2}{(z+i)^3}$$

$$\cancel{\lim_{z\to}} = \frac{-2}{8i^3} = \frac{-2}{-i8}$$

$$= \frac{1}{4i} = R$$

By using C-R th, $\int_C f(z)\, dz = 2\pi i R$.

$$\int_C f(z)\, dz = 2\pi i\left(\frac{1}{4i}\right) = \pi/2 \rightarrow (2)$$

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Consider,

$$\int_C f(z)\,dz = \int_{-R}^{R} f(x)\,dx + \int_{CR} f(x)\,dx$$

Where c is a closed curve containing the real axis $-R$ to R together with the semicircle CR.

If $R \to \infty$ then $CR \to 0$

$$\therefore \int_C f(z)\,dz = \int_{-\infty}^{\infty} f(x)\,dx \to (3)$$

From (2) and (3),

$$\therefore \int_{-\infty}^{\infty} f(x)\,dx = \pi/2.$$

4) Evaluate $\int_{-\infty}^{\infty} \frac{1}{(x^2+a^2)^2}\,dx$ using Cauchy's Residue theorem.

Sol Given $\int_{-\infty}^{\infty} \frac{1}{(x^2+a^2)^2}\,dx = \int_{-\infty}^{\infty} f(x)\,dx. \to (1)$

$$f(x) = \frac{1}{(x^2+a^2)^2}.$$

$$f(z) = \frac{1}{(z^2+a^2)^2} = \frac{1}{(z+ai)^2(z-ai)^2}.$$

$z = -ai, ai$ are the poles.

$z = ai$ is inside pole

$z = -ai$ is outside pole.

$$\left[\operatorname{Res} f(z)\right]_{z=ai} = \lim_{z \to ai} \frac{d}{dz}\left[(z-ai)^2 \cdot f(z)\right]$$

$$= \lim_{z \to ai} \frac{d}{dz}\left[\cancel{(z-ai)^2} \cdot \frac{1}{(z+ai)^2 \cancel{(z-ai)^2}}\right]$$

$$= \lim_{z \to ai} \frac{-2}{(z+ai)^3}$$

$$= \frac{-2}{(2ai)^3} = \frac{-2}{8a^3(-i)} \quad \& R = \frac{+1}{4ia^3}.$$

By using C.R th, $\int_C f(z)\,dz = 2\pi i R$

$$\cancel{\int_C f(z)\,dz = 2\pi i\left(\frac{2}{a^4}\right) = \frac{-4\pi i}{a^4}}.$$

$$\int_C f(z)\,dz = \cancel{2}\pi \cancel{i}\left(\frac{+1}{\cancel{4}^{2}\cancel{i}a^3}\right) = \frac{+\pi}{2a^3} \longrightarrow (2)$$

Consider,

$$\int_C f(z)\,dz = \int_{-R}^{R} f(x)\,dx + \int_{CR} f(x)\,dx$$

where c is a closed curve containing the real axis $-R$ to R together with the semicircle CR.

If $R \to \infty$ Then $CR \to 0$

$$\therefore \int_C f(z)\,dz = \int_{-\infty}^{\infty} f(x)\,dx \longrightarrow (3)$$

from (2) & (3)

$$\int_{-\infty}^{\infty} f(x)\,dx = \frac{\pi}{2a^3}.$$