Unit 5: Applications to Partial Differential Equations

Linear PDE with constant co-effs:-

1. Homogeneous Linear PDE:

An eq^n of the form

$$\frac{\partial^n z}{\partial x^n} + k_1 \frac{\partial^n z}{\partial x^{n-1}\partial y} + k_2 \frac{\partial^n z}{\partial x^{n-2}\partial y^2} + \cdots +$$

$$k_n \frac{\partial^n z}{\partial y^n} = F(x,y) \longrightarrow (1)$$

where k_1, k_2, k_3 are constants &c is called a a n^{th} order linear PDE with constant co-efficients.

Let $\frac{\partial}{\partial x} = D$, $\frac{\partial^2}{\partial x^2} = D^2$, $\frac{\partial^3}{\partial x^3} = D^3 \cdots \frac{\partial^n}{\partial x^n} = D^n$.

$\frac{\partial}{\partial y} = D'$, $\frac{\partial^2}{\partial y^2} = D'^2$, $\frac{\partial^3}{\partial y^3} = D'^3 \cdots \frac{\partial^n}{\partial y^n} = D'^n$.

Sub. these values in eq^n (1),

$$[D^n + k_1 D^{n-1} D' + k_2 D^{n-2} D'^2 + k_3 D^{n-3} D'^3 + \cdots$$

$$+ k_n D'^n] z = F(x,y) \longrightarrow (2)$$

is called operator which can also be written as

$$f(D, D') z = F(x,y)$$

Complementary f^n (CF):-

Suppose the given eq^n is of the form

$$\left[D^n + k_1 D^{n-1} D' + k_2 D^{n-2} D'^2 + \dots + k_n D'^n\right] z = 0 \rightarrow (3)$$

Case (i):- Suppose the roots are real and distinct & say $m_1, m_2, m_3 \dots m_n$, then Complementary function

$$z = f_1(y+m_1x) + f_2(y+m_2x) + f_3(y+m_3x) + \dots$$

Case (ii):- Suppose roots are real & equal.

i.e, $m_1 = m_2$ then C.F. is

$$z = f_1(y+m_1x) + x f_2(y+m_1x).$$

Case (iii):- Suppose the three roots are equal.

i.e, $m_1 = m_2 = m_3$ then C.F

$$z = f_1(y+m_1x) + x f_2(y+m_1x) + x^2 f_3(y+m_1x).$$

Eg:-1. Solve $2\dfrac{\partial^2 z}{\partial x^2} + 5\dfrac{\partial^2 z}{\partial x \cdot \partial y} + 2\cdot\dfrac{\partial^2 z}{\partial y^2} = 0.$

Sol. Given $2\cdot\dfrac{\partial^2 z}{\partial x^2} + 5\cdot\dfrac{\partial^2 z}{\partial x \cdot \partial y} + 2\cdot\dfrac{\partial^2 z}{\partial y^2} = 0 \rightarrow (1)$

$$2D^2 + 5DD' + 2D'^2 = 0 \rightarrow (2)$$

Put $D = m$ & $D' = 1$.

$\therefore$ Auxiliary eqn is $2m^2 + 5m + 2 = 0$

$$2m^2 + 4m + m + 2 = 0$$

$$2m(m+2) + 1(m+2) = 0$$

$$(m+2)\cdot(2m+1) = 0$$

$$\therefore m = -1/2, -2.$$

$\therefore$ Roots are real & distinct.

$\therefore CF = z = f_1(y-2x) + f_2\left(y-\frac{1}{2}x\right)$ (case(i))

2.) Solve $r+6s+9t=0$

Sol Given $r+6s+9t=0 \rightarrow (1)$

$$\frac{\partial^2 z}{\partial x^2} + 6\frac{\partial z}{\partial x.\partial y} + 9\frac{\partial^2 z}{\partial y^2} = 0 \rightarrow (1)$$

$$D^2 + 6DD' + 9D'^2 = 0 \rightarrow (2)$$

Put $D=m$ & $D'=1$.

A E $\Rightarrow m^2+6m+9=0$.

$m^2+3m+3m+9=0$

$m(m+3)+3(m+3)=0$

$(m+3)(m+3)=0$.

$\therefore m=-3,-3$ are real & equal

$\therefore$ CF is $z = f_1(y-3x) + xf_2(y-3x)$ (case ii)

3) Solve $(D_x^2 - 2D_x D_y + 2D_y^2)z = 0$.

Sol. Given $(D_x^2 - 2D_xD_y + 2D_y^2)z = 0$.

$$D_x^2 z - 2D_xD_y z + 2D_y^2 z = 0$$

$$\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial^2 z}{\partial x.\partial y} + 2.\frac{\partial^2 z}{\partial y^2} = 0. \rightarrow (1)$$

$$D^2 - 2DD' + 2D'^2 = 0 \rightarrow (2)$$

Put $D=m$ & $D'=1$

A E $\Rightarrow m^2-2m+2=0$

$$m = \frac{2 \pm \sqrt{4-4.1.2}}{2} = \frac{2\pm 0}{2},$$

$m = \frac{2}{2} = 1, 1.$

$\therefore$ CF is $z = f_1(y+x) + x f_2(y+x)$.

Particular Integrals Computation :-

1) The particular integral of $f(D, D')z = e^{ax+by}$

is $\frac{1}{g(a,b)} \cdot \frac{x^r}{r!} e^{ax+by}$, $g(a,b) \neq 0$ (r = degree)

If $g(a,b) = 0$, then $PI = \frac{x^r}{r!} e^{ax+by}$.

Replacing $D = a$ and $D' = b$ in given eq^n.

2) The P.I of $f(D, D')z = \sin(ax+by)$ (or) $\cos(ax+by)$ is

$$P.I = \frac{1}{f(D^2, DD', D'^2)} \sin(ax+by)$$

Replacing $D^2 = -a^2$, $DD' = -ab$, $D'^2 = -b^2$

$$\therefore PI = \frac{1}{g(-a^2, -ab, -b^2)} \cdot \frac{x^{2r}}{(2r)!} \sin(ax+by).$$

Eg:- 1. Solve $4r + 12s + 9t = e^{3x-2y}$.

Sol. Given $4r + 12s + 9t = e^{3x-2y} \rightarrow (1)$.

$$4 \cdot \frac{\partial^2 z}{\partial x^2} + 12 \cdot \frac{\partial^2 z}{\partial x . \partial y} + 9 \frac{\partial^2 z}{\partial y^2} = e^{3x-2y} \rightarrow (2)$$

$$4D^2 + 12DD' + 9D'^2 = e^{3x-2y} \rightarrow (3)$$

Replacing $D = m$, $D' = 1$ in (3).

Auxiliary eq^n is $4m^2 + 12m + 9 = \cancel{e^{3x-2y}}\ 0$

$$4m^2+6m+6m+9=0$$

$$2m(2m+3)+3(2m+3)=0$$

$$(2m+3)(2m+3)=0$$

$m=-3/2\,,\ -3/2$ real & equal.

$$\therefore CF = f_1\left(y-\frac{3x}{2}\right)+x f_2\left(y-\frac{3x}{2}\right) \rightarrow (4)$$

Given $4D^2+12DD'+9D'^2 = e^{3x-2y}$.

Here $a=3,\ b=-2$

Replacing $D=a=3$

$D'=b=-2$

$$P.I = \frac{1}{4D^2+12DD'+9D'^2}\cdot e^{3x-2y}.$$

$$= \frac{1}{4(3)^2+12(3)(-2)+9(-2)^2}e^{3x-2y}.$$

$$= \frac{1}{36-72+36}\cdot e^{3x-2y}.$$

$$= \frac{x^2}{2!}\,e^{3x-2y}.\ \left[\because \text{If } g(a,b)=0 \text{ then } P.I.\ \frac{x^r}{r!}e^{ax+by}\right].$$

$\therefore$ General $sol^n = CF+PI$

$$= f_1\left(y-\frac{3x}{2}\right)+x f_2\left(y-\frac{3x}{2}\right)+\frac{x^2}{2!}e^{3x-2y}.$$

2.) Solve $\dfrac{\partial^3 z}{\partial x^3} - 3.\dfrac{\partial^3 z}{\partial x^2.\partial y} + 4.\dfrac{\partial^3 z}{\partial y^3} = e^{x+2y}$.

Sol Given $\dfrac{\partial^3 z}{\partial x^3} - \dfrac{3\partial^3 z}{\partial x^2\partial y} + 4\dfrac{\partial^3 z}{\partial y^3} = e^{x+2y} \rightarrow (1)$

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$D^3 - 3D^2D' + 4D'^3 = e^{x+2y}$.

Replacing $D = m$, $D' = 1$.

A.E $\Rightarrow$ $m^3 - 3m^2 + 4 = 0$

$m = 2, 2, -1$.

$\therefore CF = f_1(y+2x) + x f_2(y+2x) + f_3(y-x)$

Given $D^3 - 3D^2D' + 4D'^3 = e^{x+2y}$.

$a = 1 \quad b = 2$

Replacing $D = a = 1$

$D' = b = 2 \Rightarrow PI = \frac{1}{D^3 - 3D^2D' + 4D'^3} = e^{x+2y}$

$$P.I = \frac{1}{1 - 3 \cdot 1 \cdot 2 + 4(8)} \cdot e^{x+2y}$$

$$= \frac{1}{27} e^{x+2y}.$$

Ans:

General soln $=$ CF + PI

$\therefore$ General soln $= f_1(y+2x) + x f_2(y+2x)$

$+ f_3(y-x) + \frac{e^{x+2y}}{27}$. //

3.) Solve $(D^3 - 4D^2D' + 5DD'^2 - 2D'^3)z = e^{2x+y}$

4.) $r - 4s + 4t = e^{2x+y}$.

5.) Solve $(D^2 + 4DD' - 5D'^2)z = \sin(2x+3y)$

Sol: Given $(D^2 + 4DD' - 5D'^2)z = \sin(2x+3y) \rightarrow (1)$

Put $D = m$, $D' = 1$ in (1), then

A.E is $m^2+4m-5=0$

$$m^2-m+5m-5=0$$
$$m(m-1)+5(m-1)=0$$
$$(m-1)(m+5)=0$$
$$m=1,-5$$

$$C.F = f_1(y+x)+f_2(y-5x) \rightarrow (2)$$

$$P.I = \frac{1}{D^2+4DD'-5D'^2}\sin(2x+3y)$$

$a=2 \quad b=3$

Replacing $D^2=-a^2=-4$

$$D'^2=-b^2=-9$$
$$DD'=-ab=-6$$

$$P.I = \frac{1}{-4+24+45}\sin(2x+3y) = \frac{1}{17}\sin(2x+3y)$$

General sol^n $= CF+PI$

$$= f_1(y+x)+f_2(y-5x)+\frac{1}{17}\sin(2x+3y)$$

2. Solve $\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial x\partial y}-6\frac{\partial^2 z}{\partial y^2}=\cos(3x+2y)$.

Sol Given

$$(D^2+DD'-6D'^2)z=\cos(3x+2y)$$

$D=m \quad D'=1$

$$m^2+m-6=0$$
$$(m+3)(m-2)=0$$
$$m=2,-3$$

$$CF = f_1(y+2x)+f_2(y-3x)$$

$$P.I = \frac{1}{D^2+DD'-6D'^2}\cos(3x+2y)$$

$a=3 \quad b=2$

$$PI = \frac{1}{-9-6-6(-4)}\cos(3x+2y) = \frac{1}{9}\cos(3x+2y)$$

General sol^n = CF + PI

$$= f_1(y+2x) + f_2(y-3x) + \frac{1}{9}\cos(3x+2y).$$

Classification of P.D.E:

The PDE is of the form:

$$A(x,y)\frac{\partial^2 u}{\partial x^2} + B(x,y)\frac{\partial^2 u}{\partial x \partial y} + C(x,y)\frac{\partial^2 u}{\partial y^2} +$$

$$F(x,y,u,u_x,u_y)=0 \longrightarrow (1)$$

Eq^n (1) can also be written as

$$A \cdot u_{xx} + B \cdot u_{xy} + C \cdot u_{yy} + F(x,y,u,u_x,u_y)=0 \rightarrow (2)$$

where A, B, C are f^ns of x, y

eq^n (2) is said to be

i) Elliptic if $B^2-4AC<0$ at a pt in (x,y) plane. (Laplace eq^n)

ii) Parabolic if $B^2-4AC=0$ at a pt in (x,y) plane (wave eq^n)

iii) hyperbolic if $B^2-4AC>0$ at a pt in (x,y) plane (heat eq^n)

1.) Classify the partial diff eq^n $f_{xx}+2f_{xy}+4f_{yy}=0$

Sol) Given $f_{xx}+2f_{xy}+4f_{yy}=0 \rightarrow (1)$

is of the form $f_{xx}+B \cdot f_{xy}+C \cdot f_{yy}=0 \rightarrow (2)$

$A=1, \quad B=2, \quad C=4$

1. ~~Classify (x+1)~~.

$B^2-4AC = (2)^2-4(1)(1) = -12 < 0$

$\therefore$ It is elliptic.

2.) Classify PDE $U_{xx}+4U_{xy}+4U_{yy}-U_x+2U_y=0$

Sol Given $U_{xx}+4U_{xy}+4U_{yy}-U_x+2U_y=0 \rightarrow (1)$

It is of the form

$AU_{xx}+BU_{xy}+CU_{yy}+F(x,y,u,U_x,U_y)=0 \rightarrow (2)$

$A=1, \quad B=4, \quad C=4$

$B^2-4AC = 16-4(1)(4) = 0.$

$\therefore$ It is parabolic.

Method of seperation of variables:

When we have a partial differential eq involving two independent variables say x, y we seek a soln in the form

$X(x), Y(y)$

1. Solve $\frac{\partial u}{\partial x} = 2 \cdot \frac{\partial u}{\partial t} + u$. where $u(x,0) = 6e^{-3x}$ by the method of seperation of variables.

Sol. Given eqn $\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u \rightarrow (1)$

Given $u(x,0) = 6e^{-3x} \rightarrow (2)$

Let $u(x,t) = X(x) \cdot T(t) \rightarrow (3)$ be the solution of eqn (1)

P.d eqn (3) w.r.to x and t.

$$\frac{\partial u}{\partial x} = X'(x)\,T(t) \rightarrow (4)$$

$$\frac{\partial u}{\partial t} = X(x)\,T'(t) \longrightarrow (5)$$

Sub (4) and (5) in eqn (1)

$$X'(x) \cdot T(t) = 2\,X(x) \cdot T'(t) + X(x)T(t).$$

$$X'(x) \cdot T(t) = X(x) \cdot (2T'(t) + T(t)).$$

$$\frac{X'(x)}{X(x)} =$$

Take first & last terms of (6).

$$\frac{X'(x)}{X(x)} = \lambda \Rightarrow X'(x) = \lambda X(x).$$

$$\Rightarrow X'(x) - \lambda X(x) = 0$$

$$\Rightarrow Dx - \lambda x = 0$$

$$(D-\lambda)x = 0.$$

$$m - \lambda = 0 \Rightarrow m = \lambda.$$

$$X(x) = Ae^{\lambda x} \longrightarrow (7)$$

Take second & third term of (6).

$$\frac{2T'(t) + T(t)}{T(t)} = \lambda.$$

$$2T'(t) + T(t) = \lambda T(t).$$

$$2T'(t) + (1-\lambda)T(t) = 0$$

$$2Dt + (1-\lambda)t = 0$$

$$2D + (1-\lambda) = 0$$

A.E is $2m + (1-\lambda) = 0$

$$2m = (\lambda - 1)$$

$$m = \left(\frac{\lambda-1}{2}\right).$$

Solution is $T(t) = Be^{-\left(\frac{1-\lambda}{2}\right)t} \longrightarrow (8)$

Sub (7) and (8) in (3)

$$U(x,t) = ABe^{\lambda x}e^{-\left(\frac{1-\lambda}{2}\right)t} \longrightarrow (9).$$

Given $U(x,0) = 6e^{-3x} \longrightarrow (10).$

Taking $t=0$ in (9)

$U(x,0) = Ae^{\lambda x} Be^{(\lambda - \frac{1}{2})0}$

$6e^{-3x} = ABe^{\lambda x}$

$AB = 6 \qquad \lambda = -3.$

Sub $AB = 6,\ \lambda = -3$ in (9)

$U(x,t) = 6e^{-3x} e^{\left(\frac{-3-1}{2}\right)t}$

$= 6 \cdot e^{-3x} \cdot e^{-2t}.$

$U(x,t) = 6e^{-(3x+2t)}.$

2.) Solve $3U_x + 2U_y = 0$ where $U(x,0) = 4e^{-x}$ by the method of seperation of variables.

Sol. Given $3U_x + 2U_y = 0 \rightarrow (1)$

$U(x,0) = 4e^{-x} \rightarrow (2)$

Let $U(x,y) = X(x) \cdot Y(y) \rightarrow (3)$ be the solⁿ of eqⁿ (1)

P.d eqⁿ (3) w.r.to x and t

$\frac{\partial U}{\partial x} = X'(x)\, Y(y) \rightarrow (4) \qquad \frac{\partial U}{\partial y} = X(x) \cdot Y'(y) \rightarrow (5)$

Sub (4) and (5) in eqⁿ (1).

$3X'(x) Y(y) + 2X(x) Y'(y) = 0$

$3X'(x) Y(y) = -2X(x) Y'(y)$

$\frac{3X'(x)}{X(x)} = -\frac{2Y'(y)}{Y(y)} = \lambda \rightarrow (6).$

Take first and last terms of (6).

$$\frac{3X'(x)}{X(x)} = \lambda \Rightarrow 3X'(x) = \lambda X(x)$$

$$\Rightarrow 3X'(x) - \lambda X(x) = 0$$

$$\Rightarrow 3Dx - \lambda x = 0$$

$$(3D-\lambda)x = 0$$

$$3m - \lambda = 0 \Rightarrow m = \lambda/3.$$

$$X(x) = A e^{(\lambda/3)x} \rightarrow (7)$$

Take second and third term of (6)

$$\frac{-2Y'(y)}{Y(y)} = \lambda \Rightarrow -2Y'(y) = \lambda Y(y)$$

$$\rightarrow 2Y'(y) - \lambda Y(y) = 0.$$

$$\Rightarrow \cancel{2Y'(y)}\ 2Dy + \lambda y = 0$$

A·E is $(2m+\lambda) = 0 \Rightarrow m = -\lambda/2$

∴ Solution is $Y(y) = Be^{\left(-\frac{\lambda}{2}\right)y} \rightarrow (8)$

Sub (7) and (8) in (3):

$$U(x,y) = A e^{\frac{\lambda x}{3}} \cdot B e^{\frac{-\lambda y}{2}}$$

$$U(x,y) = ABe^{\frac{\lambda x}{3}} \cdot e^{\frac{-\lambda y}{2}} \rightarrow (9)$$

Given $U(x,0) = 4e^{-x} \rightarrow (10)$

Put $y=0$ in (9) $\quad U(x,0) = ABe^{\frac{\lambda x}{3}}$

$$4e^{-x} = ABe^{\frac{\lambda}{3}x}.$$

$$AB = 4;\ \frac{\lambda}{3} = -1 \Rightarrow \lambda = -3$$

Sub AB, $\lambda = -3$ in (9).

$$\therefore U(x,y) = 4 \cdot e^{\frac{-3x}{3}} e^{\frac{3}{2}y}$$

$$= 4e^{-x} e^{\frac{3y}{2}}.$$

3.) Solve $U_x - 4U_y = 0$ where $u(0,y) = 8e^{-3y}$ by the method of seperation of variables.

Sol) Given $U_x - 4U_y = 0 \rightarrow (1)$

$u(0,y) = 8e^{-3y} \rightarrow (2).$

Let $u(x,y) = X(x) \cdot Y(y) \rightarrow (3)$ be the solⁿ of eqⁿ (1).

P.d eqⁿ (3) w.r.to x and t.

$\frac{\partial u}{\partial x} = X'(x)\, Y(y) \rightarrow (4)$ $\quad \frac{\partial u}{\partial y} = X(x) \cdot Y'(y) \rightarrow (5)$

The solⁿ of wave eqⁿ is

One-Dimensional Wave Equation :-

The one-dimensional wave eq^n is.

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2} \cdot \frac{\partial^2 u}{\partial t^2} \rightarrow (1)$$

where $u(0,t) = 0$ for all $t \rightarrow (2)$

$u(l,t) = 0$ for all $t \rightarrow (3)$

$u(x,0) = f(x),\ (0 \leq x \leq l) \rightarrow (4)$

$\left(\frac{\partial u}{\partial t}\right)_{t=0} = g(x) = 0 \rightarrow (5)$

The solution of eq^n (1) is,

$$u(x,t) = \sum_{n=1}^{\infty} \left(C_n \cdot \cos\left(\frac{n\pi ct}{l}\right) + D_n \sin\left(\frac{n\pi ct}{l}\right) \right) \sin\left(\frac{n\pi x}{l}\right)$$

1.) A tightly stretched string at a fixed end pt $x=0$ and $x=l$ is initially in a position given by $y = y_0 \sin^3 \frac{\pi x}{l}$. If it is released from this position. find the displacement $y(x,t)$.

Given $f(x) = y_0 \sin^3 \frac{\pi x}{l}$

By using one-dimensional wave eqⁿ, we have.

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2} \cdot \frac{\partial^2 y}{\partial t^2} \rightarrow (1)$$

Subject to $y(0,t) = 0$ for all $t \rightarrow (2)$

$y(l,t) = 0 \quad \forall \; t \rightarrow (3)$

$y(x,0) = \cancel{f(x)}\; y_0 \frac{\sin^3 \pi x}{l}, \; 0 \leq x \leq l \rightarrow (4)$

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0 \quad 0 \leq x \leq l \longrightarrow (5)$$

Solⁿ of one dimensional wave eqⁿ is

$$y(x,t) = \sum_{n=1}^{\infty} \left(C_n \cdot \cos\left(\frac{n\pi c t}{l}\right) + D_n \sin\left(\frac{n\pi c t}{l}\right)\right) \sin\left(\frac{n\pi x}{l}\right) \rightarrow (6)$$

Partially differentiating eqⁿ (6) with respect to 't' on both sides.

$$\left(\frac{\partial y}{\partial t}\right) = \sum \left(-C_n \left(\frac{n\pi c}{l}\right) \sin\left(\frac{n\pi c t}{l}\right) + D_n \cdot \cos\left(\frac{n\pi c t}{l}\right)\left(\frac{n\pi c}{l}\right)\right) \cdot \sin\left(\frac{n\pi x}{l}\right).$$

Using eqⁿ (5), take $t=0$ in above eqⁿ,

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = \sum 0 + D_n\left(\frac{n\pi c}{l}\right) \cdot \sin\left(\frac{n\pi x}{l}\right)$$

$$\sum D_n \left(\frac{n\pi c}{l}\right) \cdot \sin\left(\frac{n\pi x}{l}\right) = 0 \quad [\text{Using } (5)].$$

but $\sin\left(\frac{n\pi x}{l}\right) \neq 0$, $D_n = 0$..

Taking $D_n = 0$ in (6).

$$y(x,t) = \sum_{n=1}^{\infty} C_n \cos\left(\frac{n\pi ct}{l}\right) \sin\left(\frac{n\pi x}{l}\right) \rightarrow (7)$$

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Taking $t=0$ in (7)

$$y(x,0) = \sum_{n=1}^{\infty} C_n \cdot \sin\left(\frac{n\pi x}{l}\right)$$

$$\left[\begin{aligned} &\sin 3\theta = 3\sin\theta - 4\sin^3\theta \\ &\& \\ &\sin^3\theta = \frac{3}{4}\sin\theta - \frac{1}{4}\sin 3\theta \end{aligned}\right]$$

Using eqn (4)

$$\sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{l}\right) = f(x) = y_0 \sin^3\left(\frac{\pi x}{l}\right)$$

$$C_1 \cdot \sin\left(\frac{\pi x}{l}\right) + C_2 \sin\left(\frac{2\pi x}{l}\right) + C_3 \sin\left(\frac{3\pi x}{l}\right) + \cdots$$

$$= y_0\left[\frac{3}{4}\sin\left(\frac{\pi x}{l}\right) - \frac{1}{4}\sin\left(\frac{3\pi x}{l}\right)\right]$$

~~$\sin^3\theta = 3\sin\theta - 4\sin^3\theta$ & $\sin^3\theta = \frac{3}{4}$~~

$$C_1 \sin\left(\frac{\pi x}{l}\right) + C_2 \sin\left(\frac{2\pi x}{l}\right) + C_3 \cdot \sin\left(\frac{3\pi x}{l}\right) + \cdots$$

$$= \frac{3y_0}{4} \sin\left(\frac{\pi x}{l}\right) - \frac{y_0}{4} \sin\left(\frac{3\pi x}{l}\right).$$

$$C_1 = 3y_0/4 \quad C_2 = 0, \; C_3 = -y_0/4, \; C_4 = 0$$

Sub C_1, C_2, C_3 & C_4 in (7)

$$\therefore y(x,t) = \frac{3y_0}{4} \cos\left(\frac{\pi ct}{4}\right) \sin\left(\frac{\pi x}{l}\right) + 0 - \frac{y_0}{4} \cos\left(\frac{3\pi ct}{l}\right) \sin\left(\frac{3\pi x}{l}\right)$$

2.) If a string of length l is initially at rest in equilibrium position & each of its points is given the velocity $V_0 \sin^3 \frac{\pi x}{l}$. Find displacement $y(x,t)$

Sol. Given velocity $= V_0 \sin^3 \frac{\pi x}{l} = \left(\frac{\partial y}{\partial t}\right)$

By using one-dimensional wave eqⁿ,

$$\frac{\partial^2 v}{\partial x^2} = \frac{1}{c^2}\cdot\frac{\partial^2 v}{\partial t^2} \rightarrow (1).$$

Subject to,

$$y(0,t) = 0 \quad \forall t \rightarrow (2).$$

$$y(l,t) = 0 \quad \forall t \rightarrow (3)$$

$$y(x,0) = 0 \quad , 0 \le x \le l \rightarrow (4)$$

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = V_0 \sin^3\frac{\pi x}{l} \quad , 0 \le x \le l \rightarrow (5)$$

Solⁿ: $y(x,t) = \sum_{n=1}^{\infty}\left(C_n \cdot \cos\left(\frac{n\pi ct}{l}\right) + D_n \sin\left(\frac{n\pi ct}{l}\right)\right) \cdot \sin\left(\frac{n\pi x}{l}\right) \rightarrow (6).$

~~Partially diff~~ eqⁿ (6) ~~w.r.t 't' on b.s.~~

$\cancel{\left(\frac{\partial y}{\partial t}\right) = \sum\left(-C_n\left(n\pi\right.\right.}$ Taking $t=0$ in eqⁿ (2) [∵ (4)]

$$y(x,0) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{l}\right);$$ Now, using (4).

$$\sum_{n=1} C_n.\sin\left(\frac{n\pi x}{l}\right) = 0.$$

$$\sin\left(\frac{n\pi x}{l}\right) \ne 0 \quad \Rightarrow C_n = 0$$

$$(6) \Rightarrow y(x,t) = \sum_{n=1}^{\infty} D_n \cdot \sin\left(\frac{n\pi ct}{l}\right) \sin\left(\frac{n\pi x}{l}\right) \rightarrow (7)$$

P-diff eqⁿ (7) w.r.t to on b.s.

$$\cancel{\left(\frac{\partial y}{\partial t}\right) = \sum\left[-C_n\left(\frac{n\pi c}{l}\right)\sin\left(\frac{n\pi ct}{l}\right) + D_n\cdot\right.}$$

$$\left(\frac{\partial y}{\partial t}\right) = \sum_{n=1}^{\infty}\left[D_n \cdot \cos\left(\frac{n\pi ct}{l}\right)\cdot\left(\frac{n\pi c}{l}\right)\cdot\sin\left(\frac{n\pi x}{l}\right)\right].$$

Taking $t=0$ in above eqⁿ

$$\therefore \left(\frac{\partial y}{\partial t}\right)_{t=0} = \sum_{n=1}^{\infty}\left[D_n\left(\frac{n\pi c}{l}\right)\sin\left(\frac{n\pi x}{l}\right)\right].$$

Using eqⁿ (5)

$$\sum_{n=1}^{\infty} D_n\left(\frac{n\pi c}{l}\right)\sin\left(\frac{n\pi x}{l}\right) = V_0 \sin^3\frac{\pi x}{l}.$$

$$D_1\left(\frac{\pi c}{l}\right)\sin\left(\frac{\pi x}{l}\right) + D_2\left(\frac{2\pi c}{l}\right)\cdot\sin\left(\frac{2\pi x}{l}\right) + D_3\cdot\left(\frac{3\pi c}{l}\right)\cdot \sin 3\left(\frac{\pi x}{l}\right) + \ldots = V_0\left[\frac{3}{4}\sin\left(\frac{\pi x}{4}\right) - \frac{1}{4}\sin\left(\frac{3\pi x}{l}\right)\right]$$

Comparing LHS & RHS.

$$D_1\left(\frac{\pi c}{l}\right) = \frac{3V_0}{4} \Rightarrow D_1 = \frac{3V_0 l}{4\pi c}.$$

$$D_2\left(\frac{2\pi c}{l}\right) = 0 \Rightarrow D_2 = 0.$$

$$D_3\left(\frac{3\pi c}{l}\right) = -\frac{V_0}{4} \Rightarrow D_3 = -\frac{V_0 l}{12\pi c}.$$

Sub $D_1, D_2, D_3 \ldots$ in eqⁿ (7).

$$\therefore y(x,t) = \frac{3V_0 l}{4\pi c}\sin\left(\frac{\pi c t}{l}\right)\cdot\sin\left(\frac{\pi x}{l}\right) - \frac{V_0 l}{12\pi c}\sin\left(\frac{3\pi c t}{l}\right)\cdot\sin\left(\frac{3\pi x}{l}\right).$$

One-Dimensional Heat Eqⁿ :-

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\cdot\frac{\partial u}{\partial t} \rightarrow (1)$$

Subject to $u(0,t) = 0 \;\forall t \rightarrow (2)$

$u(l,t) = 0 \quad \forall\, t \rightarrow (3)$

$u(x,0) = f(x), \quad 0 \le x \le l \longrightarrow (4).$

soln of one dimensional heat eqn is,

$$u(x,t) = (A\cos px + B\sin px)\, e^{-p^2c^2t}.$$